THE ISING MODEL
Exercises

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Correlation inequalities - GKS and applications

QUESTION 1. *Griffith Kelly Sherman inequalities [GKS]*

Let $\Lambda \Subset \mathbb{Z}^d$ and $K = (K_C)_{C\subset\Lambda}$ be a family of real numbers. Consider the following probability distribution on $\Omega_\Lambda = \{-1,+1\}^\Lambda$:

$$\mathbb{P}_{\Lambda,K}(\omega) = \frac{1}{Z_{\Lambda,K}} \exp(\sum_{C\subset\Lambda} K_C \omega_C)$$

where $\omega_C = \prod_{i\in C} \omega_i$.

1. Show that the Ising measures in Λ with $+$, free, and periodic boundary conditions can be written in this form with $K_C \geq 0$ for all $C \subset \Lambda$ if the magnetic field h is positive.
2. If $K_C \geq 0$ for all $C \subset \Lambda$, show that for any $A \subset \Lambda$,
$$\mathbb{E}_{\Lambda,K}(\sigma_A) \geq 0. \tag{1}$$
Hint : expand each $e^{K_C\omega_C}$ in Taylor series, and recall that $\sum_{\omega=\pm 1}\omega^m$ is equal to 2 if m is even or to 0 if m is odd.
3. If $K_C \geq 0$ for all $C \subset \Lambda$, show that for any $A, B \subset \Lambda$,
$$\mathbb{E}_{\Lambda,K}(\sigma_A\sigma_B) \geq \mathbb{E}_{\Lambda,K}(\sigma_A)\mathbb{E}_{\Lambda,K}(\sigma_B). \tag{2}$$
Hint : the "duplicated variables trick".
   - *Write $\mathbb{E}_{\Lambda,K}(\sigma_A\sigma_B) - \mathbb{E}_{\Lambda,K}(\sigma_A)\mathbb{E}_{\Lambda,K}(\sigma_B)$ as the expectation of some random variable under the product measure $\mathbb{P}_{\Lambda,K} \otimes \mathbb{P}_{\Lambda,K}(\omega,\omega') = \mathbb{P}_{\Lambda,K}(\omega)\mathbb{P}_{\Lambda,K}(\omega')$.*
   - *Rewrite the expectation using the change of variables $\omega''_i = \omega_i\omega'_i = \omega_i/\omega'_i$.*
   - *Apply the inequalities (1) for each fixed configuration ω''.*

QUESTION 2. *Consequences of the GKS inequalities.* Consider the Ising model in a box $\Lambda \Subset \mathbb{Z}^d$, with the following Hamiltonian:

$$\mathcal{H}^\eta_{\Lambda,\boldsymbol{J},\boldsymbol{h}}(\boldsymbol{\sigma}) := -\sum_{\substack{i,j\in\Lambda\\ i\sim j}} J_{ij}\sigma_i\sigma_j - \sum_{i\in\Lambda} h_i\sigma_i - \sum_{\substack{i\in\Lambda, j\notin\Lambda\\ i\sim j}} J_{ij}\sigma_i\eta_j$$

We write:

$$\mu^\eta_{\Lambda,\boldsymbol{J},\boldsymbol{h}}(\boldsymbol{\sigma}) = \frac{e^{-\mathcal{H}^\eta_{\Lambda,\boldsymbol{J},\boldsymbol{h}}(\boldsymbol{\sigma})}}{Z^\eta_{\Lambda,\boldsymbol{J},\boldsymbol{h}}} \quad \text{and} \quad \langle f\rangle^\eta_{\Lambda,\boldsymbol{J},\boldsymbol{h}} = \sum_{\boldsymbol{\sigma}\in\Omega_\Lambda} f(\boldsymbol{\sigma})\mu^\eta_{\Lambda,\boldsymbol{J},\boldsymbol{h}}(\boldsymbol{\sigma})$$

Show that

1. $\langle \sigma_A \rangle^{+}_{\Lambda, \boldsymbol{J}, \boldsymbol{h}}$ is an non decreasing function of $\boldsymbol{J}$ and of $\boldsymbol{h}$.

2. for all $\boldsymbol{J} \geq 0$, $\boldsymbol{h} \geq 0$ and $A \subset \Lambda_1 \subset \Lambda_2 \Subset Z^d$ we have

$$\langle \sigma_A \rangle^{\varnothing}_{\Lambda_1, \boldsymbol{J}, \boldsymbol{h}} \leq \langle \sigma_A \rangle^{\varnothing}_{\Lambda_2, \boldsymbol{J}, \boldsymbol{h}}$$

deduce the existence of the free state

$$\mu^{\varnothing}_{\boldsymbol{J}, \boldsymbol{h}} := \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu^{\varnothing}_{\Lambda, \boldsymbol{J}, \boldsymbol{h}},$$

3. for all $\boldsymbol{J} \geq 0$, $\boldsymbol{h} \geq 0$, $\langle \sigma_A \rangle^{+}_{\boldsymbol{J}, \boldsymbol{h}}$ is an increasing function of the dimension d of the lattice.

4. Recall the definition of the critical inverse temperature $\beta_c(d)$ of the ferromagnetic Ising model, and show that it is a non increasing function of d.

5. for $\boldsymbol{J} \equiv \beta J$ with $J > 0$ and $\boldsymbol{h} \geq 0$,

   (a) $\langle \sigma_A \rangle^{\varnothing}_{\beta, h} := \langle \sigma_A \rangle^{\varnothing}_{\boldsymbol{J}, \boldsymbol{h}}$ is left-continuous as a function of β.

   (b) $\langle \sigma_A \rangle^{+}_{\beta, h} := \langle \sigma_A \rangle^{+}_{\boldsymbol{J}, \boldsymbol{h}}$ is right-continuous as a function of β.

   Hint : Use the following lemma :
   Let $a_{m,n}$ be a non decreasing double sequence bounded above. Then,

$$\lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} = \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = \sup_{m,n} \{a_{m,n}\}$$

   The same result holds if the double sequence is nonincreasing and bounded below.

QUESTION 3. *On the article "GHS and other inequalities" by Lebowitz, CMP 35, 87—92 (1974)*

1. Explain why $u_2(i, j) \geq 0$ for arbitrary field h, while $u_1(i) \geq 0$ only holds for $h \geq 0$.

2. Prove the relation (2.2).

3. Prove the relation (2.6).

4. We will now focus on the proof of the inequalities (2.4) of the Theorem:

   (a) Prove the relation (2.7) for $\phi(q) = q_C$ and $\Psi(t) = 1$.

   (b) Explain why $P(A) \in [0, 1]$ and $f(A) \geq 0$.

   (c) Deduce that $\langle q_C \rangle' \geq 0$ for any $C \subset \Lambda$.

   (d) Explain why this implies $u_2(i, j) \geq 0$ for arbitrary field h without the use of the FKG inequalities.