

COHOMOLOGICAL INVARIANTS OF CENTRAL SIMPLE ALGEBRAS OF DEGREE 4

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ABSTRACT. In this paper, we prove a result of Rost, which describes the cohomological invariants of central simple algebras of degree 4 with values in μ_2 when the base field contains a square root of -1 .

INTRODUCTION

In [6], Rost, Serre and Tignol defined a cohomological invariant of central simple algebras of degree 4 with values in μ_2 when the base field contains a square root of -1 . On the other hand, taking the Brauer class of the tensor square of a central simple algebra of degree 4 yields a cohomological invariant of degree 2 with values in μ_2 . In this paper, we prove a result of Rost (unpublished), which asserts that these two invariants are essentially the only ones (see Section 3 for a more precise statement).

This paper is organized as follows. After proving some preliminary results in Section 1, we construct a generic central simple algebra of degree 4 in Section 2. Finally, in Section 3, we prove that a cohomological invariant which vanishes on symbol algebras and biquaternion algebras is identically zero (which another result due to Rost). As a corollary, we obtain a complete description of cohomological invariants of central simple algebras of degree 4. The main arguments presented in the proofs are due to Rost, and are extracted from Merkurjev's lecture notes on Rost's theorem. These arguments may be also found in the first author PhD thesis [2].

The proofs of all the results of this paper rely heavily on the use of valuations and residue maps. We let the reader refer to [4] for the basic definitions and results on these topics.

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1. PRELIMINARIES

Let F be a field of characteristic different from 2. We will denote by **Sets** the category of sets, by **Rings** the category of commutative rings, by **Fields_F** the category of fields extensions of F and by **Alg_F** the category of commutative F -algebras.

For any field extension K/F , we will denote by $H^*(K)$ the cohomology ring of K with coefficients in μ_2 . We then get a functor $H^* : \mathbf{Fields}_F \rightarrow \mathbf{Rings}$.

If K/F is a field extension, then $H^*(K)$ carries a natural structure of a $H^*(F)$ -module, given by the external law

$$\begin{aligned} H^*(F) \times H^*(K) &\longrightarrow H^*(K) \\ (a, \xi) &\longmapsto a \cdot \xi = \text{Res}_{K/F}(a) \cup \xi. \end{aligned}$$

We start with some results on the cohomology of rational extensions.

Let $n \geq 1$ be an integer, let t_1, \dots, t_n be algebraically independent indeterminates over F , and let $K_n = F(t_1, \dots, t_n)$. Let us denote by \mathcal{P}_n the set of subsets of $\{1, \dots, n\}$. If $I = \{i_1, \dots, i_r\} \in \mathcal{P}_n$, we set

$$(\mathbf{t}_I) = (t_{i_1}) \cup \dots \cup (t_{i_r}).$$

Notice that this definition does not depend on the numbering of the elements of I , since the cup-product in $H^*(K)$ is commutative.

If $I = \emptyset$, then $(\mathbf{t}_I) = 1$ (and thus $a \cdot (\mathbf{t}_I) = \text{Res}_{K/k}(a)$ in this case).

The following two results are classical.

Lemma 1.1. *The cohomology classes $(\mathbf{t}_I), I \in \mathcal{P}_n$ are linearly independent over $H^*(F)$.*

Proposition 1.2. *Let $\xi \in H^*(K_n)$ be a cohomology class which is unramified at every discrete k -valuation $v \neq v_{t_i}, i = 1, \dots, n$. Then there exist unique cohomology classes $a_I \in H^*(F)$ such that $\xi = \sum_I a_I \cdot (\mathbf{t}_I)$.*

Definition 1.3. Let $\mathbf{F} : \mathbf{Fields}_F \longrightarrow \mathbf{Sets}$ be a covariant functor. A *cohomological invariant* of \mathbf{F} over F is a natural transformation $\mathbf{F} \longrightarrow H^*$ of functors \mathbf{Fields}_F from \mathbf{Sets} .

Cohomological invariants of \mathbf{F} form an $H^*(F)$ -module, that we will denote by $\text{Inv}(\mathbf{F}, H^*)$.

We will need in the sequel the notion of a classifying pair for a functor \mathbf{F} .

Definition 1.4. Let $\mathbf{F} : \mathbf{Alg}_F \longrightarrow \mathbf{Sets}$ be a covariant functor. Let R be an F -algebra, and let $a \in \mathbf{F}(R)$. We say that the pair (R, a) is classifying for \mathbf{F} , if the following conditions hold:

- (1) The ring R is a noetherian domain;
- (2) For every field extension L/F , and every $a' \in \mathbf{F}(L)$, there exists a maximal ideal \mathfrak{m} of R and a morphism of F -algebras $f : R \longrightarrow L$ such that $\ker(f) \supset \mathfrak{m}$ and $\mathbf{F}(f)$ maps a onto a' .

The Specialization Theorem (cf. [4, Theorem 12.2]) then immediately gives:

Lemma 1.5. *Let $\mathbf{F} : \mathbf{Alg}_F \longrightarrow \mathbf{Sets}$ be a subfunctor of $H^1(-, G)$, where G is an algebraic group over F . Assume that (R, a) is classifying for \mathbf{F} , and let $\alpha, \beta \in \text{Inv}(\mathbf{F}, H^*)$. Let K be the quotient field of R . If $\alpha(a_K) = \beta(a_K)$, then $\alpha = \beta$.*

2. CENTRAL SIMPLE ALGEBRAS OF DEGREE 4

We start this section with a parametrization of central simple algebras of degree 4.

Let R be a commutative ring such that $2 \in R^\times$. If $a \in R^\times$, we will denote by $R[\sqrt{a}]$ the étale quadratic R -algebra $R[X]/(X^2 - a)$.

Let L be a biquadratic étale R -algebra, that is an R -algebra L generated by two elements α, β subject to the relations

$$\alpha^2 = d, \beta^2 = d', \alpha\beta = \beta\alpha,$$

for some $d, d' \in R^\times$. Such an algebra will be denoted by $R[\sqrt{d}, \sqrt{d'}]$.

The group G of automorphisms of the R -algebra $L = R[\sqrt{d}, \sqrt{d'}]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by the automorphisms σ and τ which are uniquely determined by the following relations:

$$\sigma(\alpha) = \alpha, \sigma(\beta) = -\beta, \tau(\alpha) = -\alpha, \tau(\beta) = \beta.$$

Let $a \in R[\sqrt{d}]^\times, b \in R[\sqrt{d'}]^\times, c \in R[\sqrt{dd'}]^\times$, and set

$$n_a = N_{R[\sqrt{d}]/R}(a), n_b = N_{R[\sqrt{d'}]/R}(b), n_c = N_{R[\sqrt{dd'}]/R}(c).$$

Notice that $n_a, n_b, n_c \in R^\times$ and that we have

$$n_a = a\tau(a), n_b = b\sigma(b), n_c = c\sigma(c) = c\tau(c).$$

Assume that $n_c = n_a n_b$, and let $1, e_\sigma, e_\tau, e_{\sigma\tau}$ be the canonical basis of the L -vector space $\text{Map}(G, L)$. We define a product law on $\text{Map}(G, L)$ by imposing the relations

$$e_\sigma^2 = a, e_\tau^2 = b, e_{\sigma\tau}^2 = c, e_\sigma e_\tau = e_{\sigma\tau},$$

$$e_\sigma \lambda = \sigma(\lambda) e_\sigma, e_\tau \lambda = \tau(\lambda) e_\tau, \text{ for all } \lambda \in L,$$

and extending by distributivity.

By definition, the elements $e_\rho, \rho \in G$ are invertible with respect to this product law. Notice that the missing products $e_\rho e_{\rho'}, \rho, \rho' \in G$ may be obtained using the relations above, so that the product law on $\text{Map}(G, L)$ is completely determined. For example, we have

$$e_\tau e_{\sigma\tau} = e_\sigma^{-1} e_\sigma e_\tau e_{\sigma\tau} = e_\sigma^{-1} (e_{\sigma\tau})^2 = a^{-1} e_\sigma c = a^{-1} \sigma(c) e_\sigma.$$

One can check that we obtain an Azumaya R -algebra of rank 4, that we will denote by $(a, b, c, L/R)$. Notice that if we set $e = e_\sigma$ and $f = e_\tau$, then we have $(a, b, c, L/R) = L \oplus Le \oplus Lf \oplus Lef$, and

$$e^2 = a, f^2 = b, (ef)^2 = c, e\lambda = \sigma(\lambda)e, f\lambda = \tau(\lambda)f, \text{ for all } \lambda \in L.$$

Notice also for later use that we have $fe = a^{-1}\sigma(cb^{-1})ef$, since

$$fe = e^{-1}(ef)^2 f^{-1} = a^{-1} ecb^{-1} f = a^{-1} \sigma(cb^{-1})ef.$$

Lemma 2.1. *Let F be a field of characteristic different from 2. Then every central simple F -algebra of degree 4 is isomorphic to some $(a, b, c, L/F)$. Moreover, one may assume that $a, b, c \notin F^\times$.*

Proof. Assume first that A is a division F -algebra. By a theorem of Albert (see [1, Theorem 11.9] for example), A contains a biquadratic étale F -algebra L . By Skolem-Noether's theorem, σ and τ extend to inner automorphisms $\text{Int}(e)$ and $\text{Int}(f)$ of A , for some $e, f \in A^\times$. This rewrites

$$e\lambda = \sigma(\lambda)e, f\lambda = \tau(\lambda)f, \text{ for all } \lambda \in L.$$

Since $\text{Int}(e)|_L = \sigma$, we have

$$\text{Int}(e^2)|_L = (\text{Int}(e)|_L)^2 = \sigma^2 = \text{Id}_L,$$

so $a = e^2 \in C_A(L) = L$ (since L is a maximal subfield of A). Since $e \in A^\times$, we have $a \in L^\times$. Similarly, $b = f^2 \in L^\times$ and $c = (ef)^2 \in L^\times$.

Associativity of the product law of A implies that

$$\sigma(a) = a, \tau(b) = b, \sigma\tau(c) = c, n_c = n_a n_b.$$

Indeed, we have

$$e^3 = ee^2 = ea = \sigma(a)e = e^2e = ae,$$

and since e is invertible, we get $\sigma(a) = a$. Similar arguments show that $\tau(b) = b$ and $\sigma\tau(c) = c$. Straightforward computations show that $f(fe) = n_c n_a^{-1} \sigma(b)^{-1} e$, which implies

$$be = f^2e = f(fe) = n_c n_a^{-1} \sigma(b)^{-1} e.$$

Since e is invertible, we get $b = n_c n_a^{-1} \sigma(b)^{-1}$, that is $n_c = n_a n_b$. The proof of the linear independence of $1, e, f, ef$ over L is straightforward and left to the reader. Hence A contains a subalgebra isomorphic to $(a, b, c, L/F)$, and therefore $A \simeq (a, b, c, L/F)$ since they have same dimension over F .

Assume now that A is not a division algebra. Then $A \simeq M_2(Q)$, where $Q = (d, d')$ is a (not necessarily division) quaternion F -algebra.

Let $L = F[\sqrt{d}, \sqrt{d'}]$ be a biquadratic extension with generators α and β , and let us consider the F -algebra $(d, 1, d, L/F)$. We then have

$$e^2 = d, f^2 = 1, (ef)^2 = d, e\alpha = \alpha e, e\beta = -\beta e, f\alpha = -\alpha f, f\beta = \beta f.$$

Notice that we also have $fe = d\sigma(d^{-1})ef = ef$.

Set $i = e, j = \beta f, i' = f, j' = \alpha e$. It is straightforward to check that the F -algebra B generated by i and j is isomorphic to Q , and that the F -algebra B' generated by i' and j' is isomorphic to $(1, d^2) \simeq M_2(F)$. Moreover one may check that B' is the centralizer of B . Since B' is a central simple F -algebra, we get

$$(d, 1, d, L/F) \simeq B \otimes_F B' \simeq Q \otimes_F M_2(F) \simeq M_2(Q) \simeq A.$$

We now prove the last part. Write $A \simeq (a, b, c, L/F)$, for some biquadratic étale F -algebra $L = F[\sqrt{d}, \sqrt{d'}]$. Replacing f by $(1 + \alpha + \beta)f$, one may assume that $b \notin F^\times$. Indeed, if $b \in F^\times$, we then have

$$[(1 + \alpha + \beta)f]^2 = (1 + \alpha + \beta)(1 - \alpha + \beta)f^2 = b(1 + d' - d + 2\beta) \notin F^\times.$$

Write $a = a_0 + a_1\alpha$. Notice that, for all $s \in F^\times$, we have

$$\begin{aligned} [(s + \alpha + \beta)e]^2 &= (s^2 + d - d' + 2s\alpha)(a_0 + a_1\alpha) \\ &= (a_0(s^2 + d' - d) + 2sa_1d) + (a_1(s^2 + d - d') + 2a_0s)\alpha, \end{aligned}$$

and

$$[(s + \alpha + \beta)ef]^2 = (s + \alpha + \beta)(s - \alpha - \beta)c = c(s^2 - d - d' - 2\alpha\beta).$$

Thus, after replacing e by $(s + \alpha + \beta)e$ for a suitable $s \in F^\times$, one may also assume that $a, c \notin F^\times$. This concludes the proof. \square

Remark 2.2. For any biquadratic F -algebra L , we have $M_4(F) \simeq (1, 1, 1, L/F)$.

Indeed, the subalgebra B generated by e and β is isomorphic to $(1, d') \simeq M_2(F)$, and its centralizer C is the subalgebra generated by f and αe , which is isomorphic to $(1, d^2) \simeq M_2(F)$. Since B is a central simple F -algebra, we have

$$(1, 1, 1, L/F) \simeq B \otimes_F C \simeq M_2(F) \otimes_F M_2(F) \simeq M_4(F).$$

Proposition 2.3. *Let F be a field of characteristic different from 2, and let $A = (a, b, c, L/F)$. Then*

$$2[A] = [(d, n_c)] \in \text{Br}(F).$$

Proof. Let $M = F(\sqrt{d'})$. Since $M \subset A$, we have $[A_M] = [C_A(M)] \in \text{Br}(M)$ by the Centralizer Theorem. Now it is easy to check that $C_A(M)$ has M -basis $1, \sqrt{d}, f, f\sqrt{d}$. It follows that $C_A(M)$ is isomorphic to the quaternion M -algebra (d, b) . Now, applying corestriction and the transfer formula, we get

$$2[A] = [(d, n_b)] = [(d, n_a n_b)] = [(d, n_c)] \in \text{Br}(F),$$

the second equality coming from the fact that (d, n_a) is split. \square

We now define a classifying pair for \mathbf{CSA}_4 (see Section 3 for its definition). Assume that F contains a primitive 4-th root of 1 (denoted by i). Let us consider the affine variety $\mathcal{V} \subset \mathbb{A}^6$ defined by the equation

$$x^2 - uy^2 - vz^2 + uvt^2 + uv = 0.$$

This is a rational variety, with coordinate ring

$$F[\mathcal{V}] = F[X, Y, Z, T, U, V]/(X^2 - UY^2 - VZ^2 + UVT^2 + UV).$$

Let us denote by $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}, \mathbf{u}, \mathbf{v}$ the images of X, Y, Z, T, U, V in $F[\mathcal{V}]$ respectively. Notice that \mathbf{u} and \mathbf{v} are algebraically independent over F . Indeed, we have a surjective F -algebra morphism

$$\begin{aligned} F[\mathcal{V}] &\longrightarrow F[V, Z] \\ P(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}, \mathbf{u}, \mathbf{v}) &\longmapsto P(0, i, Z, i, VZ^2, V) \end{aligned}$$

which maps \mathbf{u} and \mathbf{v} onto VZ^2 and V respectively. Since VZ^2 and V are algebraically independent over F , so are \mathbf{u} and \mathbf{v} .

Now let us consider the open subset \mathcal{U} of \mathcal{V} defined by the equations

$$u \neq 0, v \neq 0, t \neq 0, y^2 - vt^2 \neq 0, z^2 - u \neq 0.$$

Then \mathcal{U} is also a rational variety, whose coordinate ring $F[\mathcal{U}]$ is the localization of $F[\mathcal{V}]$ at $\mathbf{u}, \mathbf{v}, \mathbf{t}, \mathbf{y}^2 - \mathbf{v}\mathbf{t}^2$ and $\mathbf{z}^2 - \mathbf{u}$. Set $R_0 = F[\mathcal{U}]$. Then R_0 is a noetherian ring, whose quotient field F_0 is a rational extension of F . Moreover, $\mathbf{u}, \mathbf{v} \in R_0$ are algebraically independent over F . In particular, there exists a transcendence basis of F_0/F containing \mathbf{u} and \mathbf{v} .

Set $\mathbf{w} = \mathbf{u}(\mathbf{y}^2 - \mathbf{v}\mathbf{t}^2) \in R_0^\times$ and let $L_0 = R_0[\sqrt{\mathbf{u}}, \sqrt{\mathbf{uw}}]$. We now define three elements $\mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0 \in L_0^\times$ by

$$\mathbf{a}_0 = \mathbf{z} + \sqrt{\mathbf{u}} \in R_0[\sqrt{\mathbf{u}}]^\times, \mathbf{b}_0 = \frac{\mathbf{y}}{\mathbf{t}} + \frac{\sqrt{\mathbf{uw}}}{\mathbf{t}\mathbf{u}} \in R_0[\sqrt{\mathbf{uw}}]^\times, \mathbf{c}_0 = \mathbf{x} + \sqrt{\mathbf{w}} \in R_0[\sqrt{\mathbf{w}}]^\times.$$

Let us check that $n_{\mathbf{a}_0}n_{\mathbf{b}_0} = n_{\mathbf{c}_0}$. We have

$$n_{\mathbf{a}_0}n_{\mathbf{b}_0} = \frac{(\mathbf{z}^2 - \mathbf{u})(\mathbf{y}^2\mathbf{u}^2 - \mathbf{uw})}{\mathbf{t}^2\mathbf{u}^2} = \frac{(\mathbf{z}^2 - \mathbf{u})(\mathbf{y}^2\mathbf{u} - \mathbf{w})}{\mathbf{t}^2\mathbf{u}}.$$

Since $\mathbf{y}^2\mathbf{u} - \mathbf{w} = \mathbf{u}\mathbf{v}\mathbf{t}^2$, we get $n_{\mathbf{a}_0}n_{\mathbf{b}_0} = (\mathbf{z}^2 - \mathbf{u})\mathbf{v}$. Now $\mathbf{z}^2\mathbf{v} - \mathbf{u}\mathbf{v} = \mathbf{x}^2 - \mathbf{u}\mathbf{y}^2 + \mathbf{u}\mathbf{v}\mathbf{t}^2 = \mathbf{x}^2 - \mathbf{w} = n_{\mathbf{c}_0}$, and we are done. Hence we may consider the Azumaya algebra

$$\mathcal{A}_0 = (\mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0, L_0/R_0).$$

Lemma 2.4. *The pair (R_0, \mathcal{A}_0) is classifying for \mathbf{CSA}_4 .*

Proof. Let E/F be a field extension, and let A be a central simple E -algebra of degree 4. By Lemma 2.1, we have $A \simeq (a, b, c, L/E)$, with $a, b, c \notin E^\times$. Write $L = E[\sqrt{d}, \sqrt{d'}]$, with generators α, β . Write

$$a = \lambda_0 + \lambda_1\alpha, b = \mu_0 + \mu_1\beta, c = \gamma_0 + \gamma_1\alpha\beta.$$

We set

$$\bar{x} = \gamma_0, \bar{y} = \frac{\mu_0\gamma_1}{\lambda_1\mu_1}, \bar{z} = \lambda_0, \bar{t} = \frac{\gamma_1}{\lambda_1\mu_1}, \bar{u} = \lambda_1^2d, \bar{v} = \mu_0^2 - \mu_1^2d'.$$

Using the equality

$$(\lambda_0^2 - \lambda_1^2d)(\mu_0^2 - \mu_1^2d') = (\gamma_0^2 - \gamma_1^2dd'),$$

and the fact that $\lambda_1, \mu_1, \gamma_1 \in F^\times$, one may check that $\mathbf{p} = (\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{u}, \bar{v}) \in \mathcal{U}(E)$. Then evaluation at \mathbf{p} yields an F -algebra morphism $f : R_0 \rightarrow E$ such that

$$\mathcal{A}_0 \otimes_{R_0} E \simeq A.$$

Now it suffices to notice that $\ker(f) = (\mathbf{x} - \bar{x}, \mathbf{y} - \bar{y}, \mathbf{z} - \bar{z}, \mathbf{t} - \bar{t}, \mathbf{u} - \bar{u}, \mathbf{v} - \bar{v})$ is a maximal ideal to conclude. \square

3. COHOMOLOGICAL INVARIANTS OF CENTRAL SIMPLE ALGEBRAS OF DEGREE 4

For $n \geq 1$, we will denote by $\mathbf{CSA}_n : \mathbf{Alg}_F \rightarrow \mathbf{Sets}$ the functor of isomorphism classes of Azumaya algebras of rank n .

Let R be a commutative ring. Assume that $n \nmid \text{char}(R)$ and that R contains a primitive n -th root of 1, that we will denote by ζ_n . For all $u, v \in R^\times$, the R -algebra $\{u, v\}_{n,R}$ generated by two elements e and f subject to the relations

$$e^n = u, f^n = v, fe = \zeta_n ef$$

is an Azumaya R -algebra of rank n , called a symbol algebra.

For $r \geq 1$, we will denote by $\mathbf{MS}_{n,r}$ the subfunctor of \mathbf{CSA}_{nr} of isomorphism classes of tensor products of r symbol algebras of degree n .

We will denote by 1 the constant cohomological invariant of \mathbf{CSA}_n (where \mathbf{CSA}_n is now viewed as a functor from \mathbf{Fields}_F to \mathbf{Sets}). If $n = 2m$, then for every field extension K/F and every central simple K -algebra A of exponent dividing n , the

class $m[A]$ is killed by 2, and therefore defines a cohomology class of $H^2(K)$ via the usual isomorphism. This defines a cohomological invariant of \mathbf{CSA}_n , as well as a cohomological invariant of $\mathbf{MS}_{n,r}$ for all $r \geq 1$, that we will denote by f_m in both cases. In particular, we have

$$f_m(\{a_1, b_1\}_{n,K} \otimes_F \cdots \otimes_F \{a_r, b_r\}_{n,K}) = (a_1) \cup (b_1) + \cdots + (a_r) \cup (b_r)$$

for every field extension K/F and all $a_i, b_i \in K^\times$.

We then have the following lemma.

Lemma 3.1. *Let F be a field of characteristic different from 2, and let $n = 2m$. Assume that $n \nmid \text{char}(F)$, and that $\zeta_n \in F$, where ζ_n is a primitive n^{th} -root of 1. Then $\text{Inv}(\mathbf{MS}_{n,1}, H^*)$ is a free $H^*(F)$ -module with basis $1, f_m$.*

Proof. Let $\alpha \in \text{Inv}(\mathbf{MS}_{n,1}, H^*)$. Let t_1, t_2 be two indeterminates over F , and let $K = F(t_1, t_2)$. Set $A = \{t_1, t_2\}_{n,K}$, and let $v \neq v_{t_i}, i = 1, 2$ be a discrete F -valuation. By assumption on $v, t_1, t_2 \in \mathcal{O}_v^\times$, and we have $A \simeq \{t_1, t_2\}_{n, \mathcal{O}_v} \otimes_{\mathcal{O}_v} K$. It follows from [4, Theorem 11.7] that the class $\alpha_K(\{t_1, t_2\}_{n,K})$ is unramified at v . By Proposition 1.2, there exist $a_0, a_1, a_2, a_3 \in H^*(F)$ such that

$$\alpha_K(\{t_1, t_2\}_{n,K}) = a_0 \cdot 1 + a_1 \cdot (t_1) + a_2 \cdot (t_2) + a_3 \cdot (t_1) \cup (t_2).$$

Now let v be the $(t_1 - 1)$ -adic valuation on $F(t_2)$. Applying [4, Theorem 11.7] shows by specialization that

$$\alpha_K(\{1, t_2\}_{n, F(t_2)}) = a_0 \cdot 1 + a_2 \cdot (t_2),$$

that is

$$\alpha_K(M_n(F(t_2))) = a_0 \cdot 1 + a_2 \cdot (t_2).$$

Now $M_n(F(t_2))$ is unramified at v_{t_2} , and therefore, so is $\alpha_K(M_n(F(t_2)))$. Taking residues then yields $a_2 = 0$. Similar arguments show that $a_1 = 0$, and thus

$$\alpha_K(\{t_1, t_2\}_{n,K}) = a_0 \cdot 1 + a_3 \cdot (t_1) \cup (t_2).$$

Let $R = F[t_1, t_1^{-1}, t_2, t_2^{-1}]$. It is clear that the pair $(R, \{t_1, t_2\}_{n,R})$ is classifying for $\mathbf{MS}_{n,1}$. The previous equality shows that the invariants α and $a_0 \cdot 1 + a_3 \cdot f_m$ coincide on a classifying pair, hence they are equal by Lemma 1.5. Moreover, if $a_0 \cdot 1 + a_3 \cdot f_m = 0$, then applying this equality to $\{t_1, t_2\}_{n,K}$ shows that $a_0 \cdot 1 + a_3 \cdot (t_1) \cup (t_2) = 0$. Lemma 1.1 then yields $a_0 = a_3 = 0$. This concludes the proof. \square

We now prove the following theorem, due to Rost (unpublished):

Theorem 3.2 (Rost). *Let F be a field of characteristic different from 2. Assume that $-1 \in F^{\times 2}$. Then the map*

$$\text{Inv}(\mathbf{CSA}_4, H^*) \longrightarrow \text{Inv}(\mathbf{MS}_{2,2}, H^*) \times \text{Inv}(\mathbf{MS}_{4,1}, H^*)$$

is injective. In other words, a cohomological invariant which is zero on biquaternion algebras and cyclic algebras is identically zero.

Proof. Let $\alpha \in \text{Inv}(\mathbf{CSA}_4, H^*)$. Let (R_0, A_0) be the classifying pair of \mathbf{CSA}_4 defined in the previous section, and let F_0 be the quotient field of R_0 . Finally, set

$$A_0 = \mathcal{A}_0 \otimes_{R_0} F_0 = (\mathbf{z} + \sqrt{\mathbf{u}}, \frac{\mathbf{y}}{\mathbf{t}} + \frac{\sqrt{\mathbf{uw}}}{\mathbf{tu}}, \mathbf{x} + \sqrt{\mathbf{w}}, F_0[\sqrt{\mathbf{u}}, \sqrt{\mathbf{uw}}]/F_0).$$

Notice that we have $2[A_0] = [(\mathbf{u}, \mathbf{v})]$ in $\text{Br}(F_0)$. Indeed, by Proposition 2.3, we have

$$\begin{aligned} 2[A_0] &= [(\mathbf{u}, \mathbf{x}^2 - \mathbf{w})] \\ &= [(\mathbf{u}, \mathbf{x}^2 - \mathbf{u}\mathbf{y}^2 + \mathbf{u}\mathbf{v}\mathbf{t}^2)] \\ &= [(\mathbf{u}, \mathbf{v}\mathbf{z}^2)] \\ &= [(\mathbf{u}, \mathbf{v})]. \end{aligned}$$

Let v be a discrete F -valuation on F_0 . If $v \neq v_\pi$, with $\pi = \mathbf{u}, \mathbf{v}, \mathbf{t}, \mathbf{y}^2 - \mathbf{v}\mathbf{t}^2, \mathbf{z}^2 - \mathbf{u}$, then $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{t} \in \mathcal{O}_v^\times$ and $\mathbf{z} + \sqrt{\mathbf{u}}, \frac{\mathbf{y}}{\mathbf{t}} + \frac{\sqrt{\mathbf{u}\mathbf{w}}}{\mathbf{t}\mathbf{u}}, \mathbf{x} + \sqrt{\mathbf{w}} \in \mathcal{O}_v[\sqrt{\mathbf{u}}, \sqrt{\mathbf{u}\mathbf{w}}]^\times$. We then get that

$$A_0 \simeq (\mathbf{z} + \sqrt{\mathbf{u}}, \frac{\mathbf{y}}{\mathbf{t}} + \frac{\sqrt{\mathbf{u}\mathbf{w}}}{\mathbf{t}\mathbf{u}}, \mathbf{x} + \sqrt{\mathbf{w}}, \mathcal{O}_v[\sqrt{\mathbf{u}}, \sqrt{\mathbf{u}\mathbf{w}}]/\mathcal{O}_v) \otimes_{\mathcal{O}_v} F_0,$$

and therefore $\alpha_{F_0}(A_0)$ is unramified at v by [4, Theorem 11.7].

Assume now that $v = v_\pi$, with $\pi = \mathbf{t}, \mathbf{z}^2 - \mathbf{u}$ or $\mathbf{y}^2 - \mathbf{t}\mathbf{v}^2$. Then $\mathbf{u}, \mathbf{v} \in \mathcal{O}_v^\times$, and thus we have

$$(\mathbf{u}) \cup (\mathbf{v}) = j_v((\bar{\mathbf{u}}) \cup (\bar{\mathbf{v}})),$$

where $j_v : H^2(\kappa(v)) \hookrightarrow H^2((F_0)_v)$ is the canonical injection from the residue field to the completion of F_0 with respect to v . Now in all three cases, the quadratic form $\langle 1, -\bar{\mathbf{u}}, -\bar{\mathbf{v}}, \bar{\mathbf{u}}\bar{\mathbf{v}} \rangle$ is isotropic over $\kappa(v)$. It is clear if $\pi = \mathbf{z}^2 - \mathbf{u}$ or $\mathbf{y}^2 - \mathbf{t}\mathbf{v}^2$, and it follows from the equalities

$$0 = \mathbf{x}^2 - \mathbf{u}\mathbf{y}^2 - \mathbf{v}\mathbf{z}^2 + \mathbf{u}\mathbf{v}\mathbf{t}^2 + \mathbf{u}\mathbf{v} = \mathbf{x}^2 - \mathbf{u}\mathbf{y}^2 - \mathbf{v}\mathbf{z}^2 + \mathbf{u}\mathbf{v} \pmod{\mathbf{t}},$$

if $\pi = \mathbf{t}$. Thus $(\bar{\mathbf{u}}) \cup (\bar{\mathbf{v}}) = 0$, and therefore $(\mathbf{u}) \cup (\mathbf{v}) = 0$ as well. Hence the quaternion algebra (\mathbf{u}, \mathbf{v}) splits over $(F_0)_v$, and thus $2[A_0]$ splits over $(F_0)_v$. By a theorem of Albert, $A_0 \otimes_{F_0} (F_0)_v$ is a biquaternion algebra. By assumption, we get

$$\text{Res}_{(F_0)_v/F_0}(\alpha_{F_0}(A_0)) = \alpha_{(F_0)_v}(A_0 \otimes_{F_0} (F_0)_v) = 0.$$

Consequently, $r_v(\alpha_{F_0}(A_0)) = 0$ and $\alpha_{F_0}(A_0)$ is once again unramified at v .

Finally, $\alpha_{F_0}(A_0)$ is unramified at every discrete F -valuation $v \neq v_{\mathbf{u}}, v_{\mathbf{v}}$. In particular, it is unramified at every F_1 -valuation, where $F_1 = F(\mathbf{u}, \mathbf{v})$. Since there exists a transcendence basis of F_0/F containing \mathbf{u} and \mathbf{v} , the field extension F_0/F_1 is rational, and we deduce that $\alpha_{F_0}(A_0) = \text{Res}_{F_0/F_1}(\beta)$, for some $\beta \in H^*(F_1)$. Let v_1 be a discrete F -valuation on F_0 , which different from the \mathbf{u} -adic and the \mathbf{v} -adic valuations. Let us show that β is unramified at v_1 .

Let $\mathbf{u}, \mathbf{v}, t_1, t_2, t_3$ be a transcendence basis of F_0/F , and let us extend v_1 to a discrete F -valuation v on F_0 by setting

$$v_{F_1} = v_1, v(t_i) = 0, i = 1, 2, 3.$$

Then the corresponding ramification index is 1 and $\kappa(v_1) = \kappa(v)(t_1, t_2, t_3)$, so that $\kappa(v)/\kappa(v_1)$ is a rational extension. Since $\alpha_{F_0}(A_0)$ is unramified at v , we get

$$0 = r_v(\alpha_{F_0}(A_0)) = \text{Res}_{\kappa(v)/\kappa(v_1)}(r_{v_1}(\beta)).$$

Since $\kappa(v)/\kappa(v_1)$ is rational, the map $\text{Res}_{\kappa(v)/\kappa(v_1)}$ is injective, and we get that β is unramified at v_1 . By Proposition 1.2, we have

$$\beta = a_0 \cdot 1 + a_1 \cdot (\mathbf{u}) + a_2 \cdot (\mathbf{v}) + a_3 \cdot (\mathbf{u}) \cup (\mathbf{v}), a_i \in H^*(F).$$

Hence we get

$$\alpha_{F_0}(A_0) = a_0 \cdot 1 + a_1 \cdot \text{Res}_{F_0/F_1}((\mathbf{u})) + a_2 \cdot \text{Res}_{F_0/F_1}((\mathbf{v})) + a_3 \cdot \text{Res}_{F_0/F_1}((\mathbf{u}) \cup (\mathbf{v})).$$

Let us consider the $(\mathbf{v} - 1)$ -adic valuation on F_0 , and let F_2 its residue field. Since A_0 is unramified at this valuation, one may specialize A_0 to a central simple F_2 -algebra B . Then $2[A_0]$ specializes to $2[B]$. But $2[A_0] = [(\mathbf{u}, \mathbf{v})]$ specializes to 0, and therefore B is a biquaternion algebra. Then $\alpha_{F_2}(B) = 0$ by assumption, and therefore

$$a_0 \cdot 1 + a_1 \cdot \text{Res}_{F_2/F_1}((\mathbf{u})) = 0.$$

In other words, $\text{Res}_{F_2/F_1}(a_0 \cdot 1 + a_1 \cdot (\mathbf{u})) = 0$. Since $F_1 \subset F_2 \subset F_0$, and F_0/F_1 is rational, the map Res_{F_2/F_1} is injective and thus

$$a_0 \cdot 1 + a_1 \cdot (\mathbf{u}) = 0 \text{ in } H^*(F_1).$$

By Lemma 1.1, we get $a_0 = a_1 = 0$. Considering the $(\mathbf{u} - 1)$ -adic valuation yields that $a_2 = 0$ in a similar way, so that

$$\alpha_{F_0}(A_0) = a_3 \cdot \text{Res}_{F_0/F_1}((\mathbf{u}) \cup (\mathbf{v})).$$

Since $\text{Res}_{F_0/F_1}((\mathbf{u}) \cup (\mathbf{v}))$ corresponds to $[(\mathbf{u}, \mathbf{v})] = 2[A_0]$ in $\text{Br}_2(F_0)$ via the usual isomorphism, the previous equality rewrites

$$\alpha_{F_0}(A_0) = a_3 \cdot f_{2, F_0}(A_0).$$

By Lemma 1.5, we get $\alpha = a_3 \cdot f_2$. By assumption on α , we then have

$$0 = \alpha_{F_1}(\{\mathbf{u}, \mathbf{v}\}_{4, F_1}) = a_3 \cdot (\mathbf{u}) \cup (\mathbf{v}) \in H^*(F_1).$$

By Lemma 1.1, we get $a_3 = 0$, and thus $\alpha = 0$. This concludes the proof. \square

We now describe the cohomological invariants of \mathbf{CSA}_4 . Let F be a field of characteristic different from 2. Assume that $-1 \in F^{\times 2}$, and let K/F be a field extension. By [6], for every central simple K -algebra A of degree 4, there exists a unique 2-fold Pfister form q_2 and a unique 4-fold Pfister form q_4 such that

$$q_A \sim q_2 + q_4 \in W(K).$$

Taking the cohomology class $e_4(q_4) \in H^4(K)$ yields a cohomological invariant of $\text{Inv}(\mathbf{CSA}_4, H^*)$. This invariant restricts to zero on cyclic algebras. Its restriction to $\mathbf{MS}_{2,2}$ will still be denoted by e_4 . Notice that, for every field extension L/F , and all $a, b, c, d \in F^\times$, we have

$$e_4((a, b) \otimes_F (c, d)) = (a) \cup (b) \cup (c) \cup (d).$$

Corollary 3.3. *Let F be a field of characteristic different from 2. Assume that $-1 \in F^{\times 2}$. Then $\text{Inv}(\mathbf{CSA}_4, H^*)$ is a free $H^*(F)$ -module with basis $1, f_2, e_4$.*

Proof. We first prove that $1, f_2, e_4$ are linearly independent. Assume that we have

$$a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4 = 0 \text{ for some } a_i \in H^*(F).$$

Since the restriction of e_4 to $\mathbf{MS}_{4,1}$ is zero, it follows that $a_0 + a_1 \cdot f_2 = 0 \in \text{Inv}(\mathbf{MS}_{4,1}, H^*)$. By Lemma 3.1, we get $a_0 = a_1 = 0$. Thus, we get $a_2 \cdot e_4 = 0 \in \text{Inv}(\mathbf{CSA}_4, H^*)$. In particular, $a_2 \cdot e_4 = 0 \in \text{Inv}(\mathbf{MS}_{2,2}, H^*)$. Applying this equality to the biquaternion algebra $(t_1, t_2) \otimes_K (t_3, t_4)$, where $K = F(t_1, t_2, t_3, t_4)$, we get $a_2 = 0$ using Lemma 1.1.

We now prove that $1, f_2, e_4$ span $\text{Inv}(\mathbf{CSA}_4, H^*)$ as an $H^*(F)$ -module. Let $\alpha \in \text{Inv}(\mathbf{CSA}_4, H^*)$. By [3, Theorem 2.6], $1, f_2$ and e_4 form a basis of the $H^*(F)$ -module $\text{Inv}(\mathbf{MS}_{2,2}, H^*)$, so we have

$$\alpha|_{\mathbf{MS}_{2,2}} = a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4,$$

for some $a_i \in H^*(F)$. By Lemma 3.1, we also have

$$\alpha|_{\mathbf{MS}_{4,1}} = b_0 \cdot 1 + b_1 \cdot f_2,$$

for some $b_i \in H^*(F)$. Since $M_4(F) \simeq \{1, 1\}_{4,F} \simeq (1, 1) \otimes_F (1, 1)$, we may apply $M_4(F)$ to both equalities to get $a_0 = b_0$.

Let u, v be two independent indeterminates over F . We have

$$M_2((u, v)) \simeq (1, 1) \otimes_{F(u,v)} (u, v) \simeq \{u^2, v^2\}_{4,F(u,v)},$$

and applying $M_2((u, v))$ to both equalities yields

$$a_1 \cdot (u) \cup (v) = b_1 \cdot (u) \cup (v) \in H^*(F(u, v)).$$

It follows from Lemma 1.1 that $a_1 = b_1$. Since e_4 is zero on $\mathbf{MS}_{4,1}$, we conclude that α and $a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4$ coincide on $\mathbf{MS}_{4,1}$ and $\mathbf{MS}_{2,2}$. By the previous theorem, we get $\alpha = a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4$, and this concludes the proof. \square

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