ARITHMETIC OVER FUNCTION FIELDS

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Abstract. These notes accompany lectures presented at the Clay Mathematics Institute 2006 Summer School on Arithmetic Geometry. The lectures summarize some recent progress on existence of rational points of projective varieties defined over a function field over an algebraically closed field.

1. Introduction

These notes accompany lectures presented at the Clay Mathematics Institute 2006 Summer School on Arithmetic Geometry. They are more complete than the lectures themselves. Exercises assigned during the lectures are proved as lemmas or propositions in these notes. Hopefully this makes the notes useful to a wider audience than the original participants of the summer school.

This report describes some recent progress on questions in the interface between arithmetic geometry and algebraic geometry. In fact the questions come from arithmetic geometry: what is known about existence and "abundance" of points on algebraic varieties defined over a non-algebraically closed field K. But the answers are in algebraic geometry, i.e., they apply only when the field K is the function field of an algebraic variety over an algebraically closed field. For workers in number theory, such answer are of limited interest. But hopefully the techniques will be of interest, perhaps as simple analogues for more advanced techniques in arithmetic. With regards to this hope, the reader is encouraged to look at two articles on the arithmetic side, [GHMS04a] and [GHMS04b]. Also, of course, the answers have interesting consequences within algebraic geometry itself.

There are three sections corresponding to the three lectures I delivered in the summer school. The first lecture proves the classical theorems of Chevalley-Warning and Tsen-Lang: complete intersections in projective space of sufficiently low degree defined over finite fields or over function fields always have rational points. These theorems imply corollaries about the Brauer group and Galois cohomology of these fields, which are also described.

The second section introduces rationally connected varieties and presents the proof of Tom Graber, Joe Harris and myself of a conjecture of Kollár, Miyaoka and Mori: every rationally connected fibration over a curve over an algebraically closed field of characteristic 0 has a section. The proof presented here incorporates simplifications due to A. J. de Jong. Some effort is made to indicate the changes necessary to prove A. J. de Jong's generalization to separably rationally connected fibrations over curves over fields of arbitrary characteristic. In the course of the proof, we give a thorough introduction to the "smoothing combs" technique of Kollár, Miyaoka and

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Mori and its application to weak approximation for "generic jets" in smooth fibers of rationally connected fibrations. This has been significantly generalized to weak approximation for *all* jets in smooth fibers by Hassett and Tschinkel, cf. [HT06]. Some corollaries of the Kollár-Miyaoka-Mori conjecture to Mumford's conjecture, fixed point theorems, and fundamental groups are also described (these were known to follow before the conjecture was proved).

Finally, the last section hints at the beginnings of a generalization of the Kollár-Miyaoka-Mori conjecture to higher-dimensional function fields (not just function fields of curves). A rigorous result in this area is a second proof of A. J. de Jong's *Period-Index Theorem*: for a division algebra D whose center is the function field K of a surface, the index of D equals the order of $[D]$ in the Brauer group of K. This also ties together the first and second sections. Historically the primary motivation for the theorems of Chevalley, Tsen and Lang had to do with Brauer groups and Galois cohomology. The subject has grown beyond these first steps. But the newer results do have consequences for Brauer groups and Galois cohomology in much the same vein as the original results in this subject.

2. The Tsen-Lang theorem

A motivating problem in both arithmetic and geometry is the following.

Problem 2.1. Given a field K and a K-variety X find sufficient, resp. necessary, conditions for existence of a K-point of X.

The problem depends dramatically on the type of K: number field, finite field, p-adic field, function field over a finite field, or function field over an algebraically closed field. In arithmetic the number field case is most exciting. However the geometric case, i.e., the case of a function field over an algebraically closed field, is typically easier and may suggest approaches and conjectures in the arithmetic case.

Two results, the Chevalley-Warning theorem and Tsen's theorem, deduce a sufficient condition for existence of K-points by "counting". More generally, counting leads to a relative result: the Tsen-Lang theorem that a strong property about existence of k-points for a field k propagates to a weaker property about K-points for certain field extensions K/k. The prototype result, both historically and logically, is a theorem of Chevalley and its generalization by Warning. The counting result at the heart of the proof is Lagrange's theorem together with the observation that a nonzero single-variable polynomial of degree $\leq q-1$ cannot have q distinct zeroes.

Lemma 2.2. *For a finite field K with q elements, the polynomial $1-x^{q-1}$ vanishes on K^* and $x^q - x$ vanishes on all of K. For every integer $n \geq 0$, for the K-algebra homomorphism*

$$ev_n : K[X_0, \dots, X_n] \to Hom_{Sets}(K^{n+1}, K),$$
$$ev_n(p(X_0, \dots, X_n)) = ((a_0, \dots, a_n) \mapsto p(a_0, \dots, a_n)),$$

the kernel equals the ideal

$$I_n = \langle X_0^q - X_0, \dots, X_n^q - X_n \rangle.$$

Finally, the collection $(X_i^q - X_i)_{i=0,\dots,n}$ is a Gröbner basis with respect to every monomial order refining the grading of monomials by total order. In particular, for every p in I_n some term of p of highest degree is in the ideal $\langle X_0^q, \dots, X_n^q \rangle$.

Proof. Because K^* is a group of order $q-1$, Lagrange's theorem implies $a^{q-1} = 1$ for every element a of K^*, i.e., $1 - x^{q-1}$ vanishes on K^*. Multiplying by x shows that $x^q - x$ vanishes on K. Thus the ideal I_n is at least contained in the kernel of ev_n.

Modulo $X_n^q - X_n$, every element of $K[X_0, \dots, X_n]$ is congruent to one of the form

$$p(X_0, \dots, X_n) = p_{q-1} \cdot X_n^q + \cdots + p_0 X_n^0, \quad p_0, \dots, p_{q-1} \in K[X_0, \dots, X_{n-1}].$$

(Of course K^n is defined to be $\{0\}$ and $K[X_0, \dots, X_{n-1}]$ is defined to be K if n equals 0.) Since K has q elements and since a nonzero polynomial of degree $\leq q-1$ can have at most $q-1$ distinct zeroes, for every $(a_0, \dots, a_{n-1}) \in K^n$ the polynomial $p(a_0, \dots, a_{n-1}, X_n)$ is zero on K if and only if

$$p_0(a_0, \dots, a_{n-1}) = \cdots = p_{q-1}(a_0, \dots, a_{n-1}).$$

Thus $\mathrm{ev}_n(p)$ equals 0 if and only if each $\mathrm{ev}_{n-1}(p_i)$ equals 0. In that case, by the induction hypothesis, each p_i is in I_{n-1} (in case $n = 0$, each p_i equals 0). Then, since $I_{n-1}K[X_0, \dots, X_n]$ is in I_n, p is in I_n. Therefore, by induction on n, the kernel of ev_n is precisely I_n.

Finally, Buchberger's algorithms applied to the set $(X_0^q - X_0, \dots, X_n^q - X_n)$ produces S-polynomials

$$S_{i,j} = X_j^q(X_i^q - X_i) - X_i^q(X_j^q - X_j) = X_j(X_i^q - X_i) - X_i(X_j^q - X_j)$$

which have remainder 0. Therefore this set is a Gröbner basis by Buchberger's criterion. □

Theorem 2.3. [Che35],[War35] *Let K be a finite field. Let n and r be positive integers. Let $F_1, \dots, F_r$ be nonconstant, homogeneous polynomials in $K[X_0, \dots, X_n]$. If*

$$deg(F_1) + \cdots + deg(F_r) \leq n$$

then there exists $(a_0, \dots, a_n) \in K^{n+1} - \{0\}$ such that for every $i = 1, \dots, r$, $F_i(a_0, \dots, a_n)$ equals 0. Stated differently, the projective scheme $\mathbb{V}(F_1, \dots, F_r) \subset \mathbb{P}^n_K$ has a K-point.

Proof. Denote by q the number of elements in K. The polynomial

$$G(X_0, \dots, X_n) = 1 - \prod_{i=0}^{n}(1 - X_i^{q-1})$$

equals 0 on $\{0\}$ and equals 1 on $K^{n+1} - \{0\}$. For the same reason, the polynomial

$$H(X_0, \dots, X_n) = 1 - \prod_{j=1}^{r}(1 - F_j(X_0, \dots, X_n)^{q-1})$$

equals 0 on

$$\{(a_0, \dots, a_n) \in K^{n+1} | F_1(a_0, \dots, a_n) = \cdots = F_r(a_0, \dots, a_n) = 0\}$$

and equals 1 on the complement of this set in K^{n+1}. Since each F_i is homogeneous, 0 is a common zero of $F_1, \dots, F_r$. Thus the difference $G - H$ equals 1 on

$$\{(a_0, \dots, a_n) \in K^{n+1} - \{0\} | F_1(a_0, \dots, a_n) = \cdots = F_r(a_0, \dots, a_n) = 0\}$$

and equals 0 on the complement of this set in K^{n+1}. Thus, to prove that $F_1, \dots, F_r$ have a nontrivial common zero, it suffices to prove the polynomial $G - H$ does not lie in the ideal I_n.

Since

$$\deg(F_1) + \cdots + \deg(F_r) \leq n,$$

H has strictly smaller degree than G. Thus the leading term of $G - H$ equals the leading term of G. There is only one term of G of degree $\deg(G)$. Thus, for every monomial ordering refining the grading by total degree, the leading term of G equals

$$(-1)^{n+1} X_0^{q-1} X_1^{q-1} \cdots X_n^{q-1}.$$

This is clearly divisible by none of X_i^q for $i = 0, \ldots, n$, i.e., the leading term of $G - H$ is not in the ideal $\langle X_0^q, \ldots, X_n^q \rangle$. Because $(X_0^q - X_0, \ldots, X_n^q - X_n)$ is a Gröbner basis for I_n with respect to the monomial order, $G - H$ is not in I_n. $\square$

On the geometric side, an analogue of Chevalley's theorem was proved by Tsen, cf. [Tse33]. This was later generalized independently by Tsen and Lang, cf. [Tse36], [Lan52]. Lang introduced a definition which simplifies the argument.

Definition 2.4. [Lan52] Let m be a nonnegative integer. A field K is called C_m, or said to have *property* C_m, if it satisfies the following. For every positive integer n and every sequence of positive integers $(d_1, \ldots, d_r)$ satisfying

$$d_1^m + \cdots + d_r^m \leq n,$$

every sequence $(F_1, \ldots, F_r)$ of homogeneous polynomials $F_i \in K[X_0, \ldots, X_n]$ with $\deg(F_i) = d_i$ has a common zero in $K^{n+1} - \{0\}$.

Remark 2.5. In fact the definition in [Lan52] is a little bit different than this. For fields having normic forms, Lang proves the definition above is equivalent to his definition. And the definition above works best with the following results.

With this definition, the statement of the Chevalley-Warning theorem is quite simple: every finite field has property C_1. The next result proves that property C_m is preserved by algebraic extension.

Lemma 2.6. *For every nonnegative integer m, every algebraic extension of a field with property C_m has property C_m.*

Proof. Let K be a field with property C_m and let L'/K be an algebraic extension. For every sequence of polynomials $(F_1, \ldots, F_r)$ as in the definition, the coefficients generate a finitely generated subextension L/K of L'/K. Thus clearly it suffices to prove the lemma for finitely generated, algebraic extensions L/K.

Denote by e the finite dimension $\dim_K(L)$. Because multiplication on L is K-bilinear, each homogeneous, degree d_i, polynomial map of L-vector spaces,

$$F_i : L^{\oplus(n+1)} \to L,$$

is also a homogeneous, degree d_i, polynomial map of K-vector spaces. Choosing a K-basis for L and decomposing F_i accordingly, F_i is equivalent to e distinct homogeneous, degree d_i, polynomial maps of K-vector spaces,

$$F_{i,j} : L^{\oplus(n+1)} \to K, \quad j = 1, \ldots, e.$$

The set of common zeroes of the collection of homogeneous polynomial maps $(F_i | i = 1, \ldots, r)$ equals the set of common zeroes of the collection of homogeneous polynomial functions $(F_{i,j} | i = 1, \ldots, r, j = 1, \ldots, e)$. Thus it suffices to prove there is a nontrivial common zero of all the functions $F_{i,j}$.

By hypothesis,

$$\sum_{i=1}^{r} \deg(F_i)^m \text{ is no greater than } n.$$

Thus, also

$$\sum_{i=1}^{r}\sum_{j=1}^{e} \deg(F_{i,j})^m = e\sum_{i=1}^{r} \deg(F_i)^m \text{ is no greater than } en.$$

Since K has property C_m and since

$$\dim_K(L^{\oplus(n+1)}), \text{ i.e., } (n+1)\dim_K(L) = e(n+1),$$

is larger than en, the collection of homogeneous polynomials $F_{i,j}$ has a common zero in $L^{\oplus(n+1)} - \{0\}$. □

The heart of the Tsen-Lang theorem is the following proposition.

Proposition 2.7. *Let K/k be a function field of a curve, i.e., a finitely generated, separable field extension of transcendence degree 1. If k has property C_m then K has property C_{m+1}.*

This is proved in a series of steps. Let n, r and $d_1, \dots, d_r$ be positive integers such that

$$d_1^{m+1} + \cdots + d_r^{m+1} \leq n.$$

For every collection of homogeneous polynomials

$$F_1, \dots, F_r \in K[X_0, \dots, X_n], \quad \deg(F_i) = d_i,$$

the goal is to prove that the collection of homogeneous, degree d_i, polynomial maps of K-vector spaces

$$F_1, \dots, F_r : K^{\oplus(n+1)} \to K$$

has a common zero. Of course, as in the proof of Lemma 2.6, this is also a collection of homogeneous polynomial maps of k-vector spaces. Unfortunately both of these k-vector spaces are infinite dimensional. However, using geometry, these polynomial maps can be realized as the colimits of polynomial maps of finite dimensional k-vector spaces. For these maps there is an analogue of the Chevalley-Warning argument replacing the counting argument by a *parameter counting argument* which ultimately follows from the Riemann-Roch theorem for curves. The first step is to give a *projective model* of K/k.

Lemma 2.8. *For every separable, finitely generated field extension K/k of transcendence degree 1, there exists a smooth, projective, connected curve C over k and an isomorphism of k-extensions $K \cong k(C)$. Moreover the pair $(C, K \cong k(C))$ is unique up to unique isomorphism.*

Proof. This is essentially the *Zariski-Riemann surface* of the extension K/k. For a proof in the case that k is algebraically closed, see [Har77, Theorem I.6.9]. The proof in the general case is similar. □

The isomorphism $K \cong k(C)$ is useful because the infinite dimensional k-vector space $k(C)$ has a plethora of naturally-defined finite dimensional subspaces. For every Cartier divisor D on C, denote by V_D the subspace

$$V_D := H^0(C, \mathcal{O}_C(D)) = \{f \in k(C) | \text{div}(f) + D \geq 0\}.$$

The collection of all Cartier divisors D on C is a partially ordered set where

$$D' \geq D \text{ if and only if } D' - D \text{ is effective.}$$

The system of subspaces V_D of $k(C)$ is compatible for this partial order, i.e., if $D' \geq D$ then $V_{D'} \supset V_D$. And K is the union of all the subspaces V_D, i.e., it is the colimit of this compatible system of finite dimensional k-vector spaces. Thus for all k-multilinear algebra operations which commute with colimits, the operation on $k(C)$ can be understood in terms of its restrictions to the finite dimensional subspaces $k(C)$. The next lemma makes this more concrete for the polynomial map F.

Lemma 2.9. *Let C be a smooth, projective, connected curve over a field k and let*

$$F_i \in k(C)[X_0, \dots, X_n]_{d_i}, \quad i = 1, \dots, r$$

be a collection of polynomials in the spaces $k(C)[X_0, \dots, X_n]_{d_i}$ of homogeneous, degree d_i polynomials. There exists an effective, Cartier divisor P on C and for every $i = 1, \dots, r$ there exists a global section $F_{C,i}$ of the coherent sheaf $\mathcal{O}_C(P)[X_0, \dots, X_n]_{d_i}$ such that for every $i = 1, \dots, r$ the germ of $F_{C,i}$ at the generic point of C equals F_i.

Remark 2.10. In particular, for every Cartier divisor D on C and for every $i = 1, \dots, r$ there is a homogeneous, degree d, polynomial map of k-vector spaces

$$F_{C,D,i} : V_D^{\oplus(n+1)} \to W_{d_i,P,D}, \quad W_{d_i,P,D} := V_{d_i D + P},$$

such that for every $i = 1, \dots, r$ the restriction of F_i to $V_D^{\oplus(n+1)}$ equals $F_{C,D,i}$ considered as a map with target K (rather than the subspace V_{dD+P}).

Proof. The coefficients of each F_i are rational functions on C. Each such function has a polar divisor. Since there are only finitely many coefficients of the finitely many polynomials $F_1, \dots, F_r$, there exists a single effective, Cartier divisor P on C such that every coefficient is a global section of $\mathcal{O}_C(P)$. □

Because of Lemma 2.9, the original polynomial maps $F_1, \dots, F_r$ can be understood in terms of their restrictions to the subspaces V_D. The dimensions of these subspaces are determined by the Riemann-Roch theorem.

Theorem 2.11 (Riemann-Roch for smooth, projective curves)**.** *Let k be a field. Let C be a smooth, projective, connected curve over k. Denote by $\omega_{C/k}$ the sheaf of relative differentials of C over k and denote by $g(C) = genus(C)$ the unique integer such that $deg(\omega_{C/k}) = 2g(C) - 2$. For every invertible sheaf $\mathcal{L}$ on C,*

$$h^0(C, \mathcal{L}) - h^0(C, \omega_C \otimes_{\mathcal{O}_C} \mathcal{L}^\vee) = deg(\mathcal{L}) + 1 - g(C).$$

Remark 2.12. In particular, if $\deg(\mathcal{L}) > \deg(\omega_C) = 2g(C) - 2$ so that $\omega_C \otimes_{\mathcal{O}_C} \mathcal{L}^\vee$ has negative degree, then $h^0(C, \omega_C \otimes_{\mathcal{O}_C} \mathcal{L}^\vee)$ equals zero. And then

$$h^0(C, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g(C).$$

For a Cartier divisor D satisfying

$$\deg(D) > 2g(C) - 2 \text{ and for each } i = 1, \dots, r, d_i\deg(D) + \deg(P) > 2g(C) - 2,$$

the Riemann-Roch theorem gives that $V_D^{\oplus(n+1)}$ and $W_{d_i,P,D}$ are finite dimensional k-vector spaces of respective dimensions,

$$\dim_k(V_D^{\oplus(n+1)}) = (n+1)h^0(C,\mathcal{O}_C(D)) = (n+1)(\deg(D)+1-g)$$

and

$$\dim_k(W_{d_i,P,D}) = \dim(V_{d_iD+P}) = d_i\deg(D)+\deg(P)+1-g.$$

In this case, choosing a basis for $W_{d_i,P,D}$ and decomposing

$$F_{C,D,i} : V_D^{\oplus(n+1)} \to W_{d_i,P,D}$$

into its associated components, there exist $\dim_k(W_{d_i,P,D})$ homogeneous, degree d, polynomial functions

$$(F_{C,D,i})_j : V_D^{\oplus(n+1)} \to k, \quad j = 1,\ldots,\dim_k(W_{d_i,P,D})$$

such that a zero of $F_{C,D,i}$ is precisely the same as a common zero of all the functions $(F_{C,D,i})_j$.

Proof of Proposition 2.7. By hypothesis, each d_i and $n+1-\sum_{i=1}^r d_i^{m+1}$ are nonzero so that the fractions

$$\frac{2g(C)-2-\deg(P)}{d_i} \text{ for each } i=1,\ldots,r, \quad \frac{(n+1-\sum_{i=1}^r d_i^m)(g-1)+\sum_{i=1}^r d_i^m\deg(P)}{n+1-\sum_{i=1}^r d_i^{m+1}}$$

are all defined. Let D be an effective, Cartier divisor on C such that

$$\deg(D) > 2g(C)-2, \ \deg(D) > \frac{2g(C)-2-\deg(P)}{d_i}, i=1,\ldots,r, \text{ and}$$

$$\deg(D) > \frac{(n+1-\sum_{i=1}^r d_i^m)(g-1)+\sum_{i=1}^r d_i^m\deg(P)}{n+1-\sum_{i=1}^r d_i^{m+1}}.$$

Because $\deg(D) > 2g(C)-2$, the Riemann-Roch theorem states that

$$\dim_k(V_D^{\oplus(n+1)}) = (n+1)\dim_k(V_D) = (n+1)(\deg(D)+1-g).$$

For every $i=1,\ldots,r$, because d_i is positive and because $\deg(D) > (2g(C)-2-\deg(P))/d_i$, also

$$\deg(d_iD+P) = d_i\deg(D)+\deg(P) \text{ is greater than } 2g(C)-2.$$

Thus the Riemann-Roch theorem states that

$$\dim_k(W_{d_i,P,D}) = \dim_k(V_{d_iD+P}) = d_i\deg(D)+\deg(P)+1-g(C).$$

Thus for the collection of polynomial functions $(F_{C,D,i})_j$,

$$\dim_k(V_D^{\oplus(n+1)}) - \sum_{i=1}^r\sum_j \deg((F_{C,D,i})_j)^m$$

equals

$$(n+1)(\deg(D)+1-g) - \sum_{i=1}^r (d_i\deg(D)+\deg(P)+1-g(C))d_i^m =$$

$$(n+1-\sum_{i=1}^r d_i^{m+1})\deg(D) - [(n+1-\sum_{i=1}^r d_i^m)(g-1)+\sum_{i=1}^r d_i^m\deg(P)].$$

Because

$$\deg(D) > \frac{(n+1-\sum_{i=1}^r d_i^m)(g-1) + \sum_{i=1}^r d_i^m \deg(P)}{n+1-\sum_{i=1}^r d_i^{m+1}}$$

and because $n+1-\sum_{i=1}^r d_i^{m+1}$ is positive, also

$$(n+1-\sum_{i=1}^r d_i^{m+1})\deg(D) > [(n+1-\sum_{i=1}^r d_i^m)(g-1) + \sum_{i=1}^r d_i^m \deg(P)].$$

Therefore

$$\dim_k(V_D^{\oplus(n+1)}) \text{ is greater than } \sum_{i=1}^r \sum_j \deg((F_{i,C,D})_j)^m.$$

Because of the inequality above, and because k has property C_m, there is a nontrivial common zero of the collection of homogeneous polynomial functions $(F_{C,D,i})_j$, $i = 1,\dots,r$, $j = 1,\dots,\dim_k(W_{d_i,P_i,D})$. Therefore there is a nontrivial common zero of the collection of homogeneous polynomial maps $F_{C,D,i}$, $i = 1,\dots,r$. By Lemma 2.9, the image of this nonzero element in $K^{\oplus(n+1)}$ is a nonzero element which is a common zero of the polynomials $F_1,\dots,F_r$. □

Proposition 2.7 is the main step in the proof of the Tsen-Lang theorem.

Theorem 2.13 (The Tsen-Lang Theorem)**.** [Lan52] *Let K/k be a field extension with finite transcendence degree, tr.deg.$(K/k) = t$. If k has property C_m then K has property C_{m+t}.*

Proof. The proof of the theorem is by induction on t. When $t = 0$, i.e., when K/k is algebraic, the result follows from Lemma 2.6. Thus assume $t > 0$ and the result is known for $t-1$. Let $(b_1,\dots,b_t)$ be a transcendence basis for K/k. Let E_t, resp. E_{t-1}, denote the subfield of K generated by k and $b_1,\dots,b_t$, resp. generated by k and $b_1,\dots,b_{t-1}$. Since E_{t-1}/k has transcendence degree $t-1$, by the induction hypothesis E_{t-1} has property C_{m+t-1}. Now E_t/E_{t-1} is a purely transcendental extension of transcendence degree 1. In particular, it is finitely generated and separable. Since E_{t-1} has property C_{m+t-1}, by Proposition 2.7 E_t has property C_{m+t}. Finally by Lemma 2.6 again, since K/E_t is algebraic and E_t has property C_{m+t}, also K has property C_{m+t}. □

The homogeneous version of the Nullstellensatz implies a field k has property C_0 if and only if k is algebraically closed. Thus one corollary of Theorem 2.13 is the following.

Corollary 2.14. *Let k be an algebraically closed field and let K/k be a field extension of finite transcendence degree t. The field K has property C_t.*

In particular, the case $t = 1$ is the historically the first result in this direction.

Corollary 2.15 (Tsen's theorem)**.** [Tse36] *The function field of a curve over an algebraically closed field has property C_1.*

Chevalley and Tsen recognized that property C_1, which they called *quasi-algebraic closure*, has an important consequence for division algebras. Lang recognized that property C_2 also has an important consequence for division algebras, cf. [Lan52, Theorem 13]. Let K be a field. A *division algebra with center* K is a K-algebra D with center K such that every nonzero element of D has a (left-right) inverse. Although this is not always the case, we will also demand that $\dim_K(D)$ is finite.

Denote by $\overline{K}$ the separable closure of K. Every division algebra with center K is an example of a *central simple algebra over* K, i.e., a K-algebra A with center K and $\dim_K(A)$ finite such that $A \otimes_K \overline{K}$ is isomorphic as a $\overline{K}$-algebra to the algebra $\mathrm{Mat}_{n\times n}(\overline{K})$ of $n \times n$ matrices with entries in $\overline{K}$ for some integer n. In particular, $\dim_K(A) = n^2$ for a unique integer n. For a division algebra D with center K, the unique integer n is called the *index* of D.

Let $\phi : A \otimes_K \overline{K} \to \mathrm{Mat}_{n\times n}(\overline{K})$ be an isomorphism of $\overline{K}$-algebras. There is an induced homogeneous, degree n, polynomial map of $\overline{K}$-vector spaces

$$\det \circ \phi : A \to \mathrm{Mat}_{n\times n}(\overline{K}) \to \overline{K}.$$

By the Skolem-Noether theorem, every other isomorphism $\phi' : A \otimes_K \overline{K} \to \mathrm{Mat}_{n\times n}(\overline{K})$ is of the form $\mathrm{conj}_a \circ \phi$ where $a \in \mathrm{Mat}_{n\times n}(\overline{K})$ is an invertible element and

$$\mathrm{conj}_a : \mathrm{Mat}_{n\times n}(\overline{K}) \to \mathrm{Mat}_{n\times n}(\overline{K}), \quad \mathrm{conj}_a(b) = aba^{-1}$$

is conjugation by a. But $\det \circ \mathrm{conj}_a$ equals det. Thus the map $\det \circ \phi$ is independent of the particular choice of ϕ. Since the Galois group of $\overline{K}/K$ acts on the polynomial map through its action on ϕ, the polynomial map is also Galois invariant. Therefore there exists a unique homogeneous, degree n, polynomial map of K-vector spaces

$$\mathrm{Nrm}_{A/K} : A \to K$$

such that for every isomorphism of $\overline{K}$-algebras ϕ, $\det \circ \phi$ equals $\mathrm{Nrm}_{A/K} \otimes 1$.

The homogeneous, polynomial map of K-vectors spaces $\mathrm{Nrm}_{A/K}$ is the *reduced norm* of A. It is multiplicative, i.e.,

$$\forall a, b \in A, \quad \mathrm{Nrm}_{A/K}(ab) = \mathrm{Nrm}_{A/K}(a)\mathrm{Nrm}_{A/K}(b).$$

And the restriction to the center K is the polynomial map $\lambda \mapsto \lambda^n$. These properties characterize the reduced norm. By the same type of Galois invariance argument as above, and using Cramer's rule, an element a of A has a (left and right) inverse if and only if $\mathrm{Nrm}_{A/K}(a)$ is nonzero. In particular, if D is a division algebra the only zero of $\mathrm{Nrm}_{A/K}$ is $a = 0$.

Proposition 2.16. *Let K be a field*

(i) *If K has property C_1, then the only division algebra with center K is K itself.*
(ii) *If K has property C_2 then for every division algebra D with center K the reduced norm map*

$$Nrm_{D/K} : D \to K$$

is surjective.

Proof. Let D be a division algebra with center K. Denote by n the index of D. Because $\mathrm{Mat}_{n\times n}(\overline{K})$ has dimension n^2 as a $\overline{K}$-vector space, also D has dimension n^2 as a K-vector space. If K has property C_1, then since the homogeneous polynomial map $\mathrm{Nrm}_{D/K}$ has only the trivial zero,

$$n = \deg(\mathrm{Nrm}_{D/K}) \geq \dim_K(D) = n^2,$$

i.e., $n = 1$. Thus for a field K with property C_1, the only finite dimensional, division algebra with center K has dimension 1, i.e., D equals K.

Next suppose that K has property C_2. Clearly $\text{Nrm}_{D/k}(0)$ equals 0. Thus to prove that

$$\text{Nrm}_{D/K} : D \to K$$

is surjective, it suffices to prove that for every nonzero $c \in K$ there exists b in D with $\text{Nrm}_{D/K}(b) = c$. Consider the homogeneous, degree n, polynomial map

$$F_c : D \oplus K \to K, \quad (a, \lambda) \mapsto \text{Nrm}_{D/K}(a) - c\lambda^n.$$

Since

$$\dim_K(D \oplus K) = n^2 + 1 > \deg(F_c)^2,$$

by property C_2 the map F_c has a zero $(a, \lambda) \neq (0, 0)$, i.e., $\text{Nrm}_{D/k}(a) = c\lambda^n$. In particular, λ must be nonzero since otherwise a is a nonzero element of D with $\text{Nrm}_{D/K}(a) = 0$. But then $b = (1/\lambda)a$ is an element of D with $\text{Nrm}_{D/k}(b) = c$. □

It was later recognized, particularly through the work of Merkurjev and Suslin, that these properties of division algebras are equivalent to properties of Galois cohomology. The *cohomological dimension* of a field K is the smallest integer $\text{cd}(K)$ such that for every Abelian, discrete, torsion Galois module A and for every integer $m > \text{cd}(K)$,

$$H^m(\overline{K}/K, A) = \{0\}.$$

Theorem 2.17. [Ser02, Proposition 5, §I.3.1], [Sus84, Corollary 24.9] *Let K be a field.*

(i) *The cohomological dimension of K is ≤ 1 if and only if for every finite extension L/K, the only division algebra with center L is L itself.*

(ii) *If K is perfect, the cohomological dimension of K is ≤ 2 if and only if for every finite extension L/K, for every division algebra D with center L, the reduced norm map $Nrm_{D/L}$ is surjective.*

3. Rationally connected varieties

The theorems of Chevalley-Warning and Tsen-Lang are positive answers to Problem 2.1 for a certain class of fields. It is natural to ask whether these theorems can be generalized for such fields.

Problem 3.1. Let r be a nonnegative integer. Give sufficient geometric conditions on a variety such that for every C_r field K (or perhaps every C_r field satisfying some additional hypotheses) and for every K-variety satisfying the conditions, X has a K-point.

As with Problem 2.1, this problem is quite vague. Nonetheless there are important partial answers. One such answer, whose proof was sketched in the lectures of Hassett in this same Clay Summer School, is the following.

Theorem 3.2. [Man86] [CT87] *Let K be a C_1 field and let X be a projective K-variety. If $X \otimes_K \overline{K}$ is birational to $\mathbb{P}^2_{\overline{K}}$ then X has a K-point.*

This begs the question: What (if anything) is the common feature of rational surfaces and of the varieties occurring in the Chevalley-Warning and Tsen-Lang theorems, i.e., complete intersections in $\mathbb{P}^n$ of hypersurfaces of degrees $d_1, \dots, d_r$ with $d_1 + \cdots + d_r \leq n$? One answer is *rational connectedness*. This is a property that was studied by Kollár-Miyaoka-Mori and Campana, cf. [Kol96].

Definition 3.3. Let k be an algebraically closed field. An integral (thus nonempty), separated, finite type, k-scheme X is *rationally connected*, resp. *separably rationally connected*, if there exists an integral, finite type k-scheme M and a morphism of k-schemes

$$u : M \times_k \mathbb{P}^1_k \to X, \quad (m,t) \mapsto u(m,t)$$

such that the induced morphism of k-schemes

$$u^{(2)} : M \times_k \mathbb{P}^1_k \times_k \mathbb{P}^1_k \to X \times_k X, \quad (m, t_1, t_1) \mapsto (u(m,t_1), u(m,t_2))$$

is surjective, resp. surjective and generically smooth.

In a similar way, X is *rationally chain connected*, resp. *separably rationally chain connected*, if for some integer $n \geq 1$, the analogous property holds after replacing $\mathbb{P}^1_k$ by the proper, connected, nodal, reducible curve C_n which is a chain of n smooth rational curves.

The definition of rational connectedness, resp. rational chain connectedness, mentions a particular parameter space M. However, using the general theory of Hilbert schemes, it suffices to check that every pair (x_1, x_2) of K-points of $X \otimes_k K$ is contained in some rational K-curve, resp. a chain of rational K-curves, (not necessarily from a fixed parameter space) for one *sufficiently large*, algebraically closed, field extension K/k, i.e., for an algebraically closed extension K/k such that for every countable collection of proper closed subvarieties $Y_i \subsetneq X$, there exists a K-point of X contained in none of the sets Y_i. For instance, K/k is sufficiently large if K is uncountable or if K/k contains the fraction field $k(X)/k$ as a subextension.

A very closely related property is the existence of a *very free rational curve*, i.e., a morphism

$$f : \mathbb{P}^1_k \to X_{\text{smooth}}$$

into the smooth locus of X such that f^*T_X is ample, i.e.,

$$f^*T_X \cong \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_n), \ a_1, \dots, a_n > 0.$$

Definition 3.4. The *very free locus* $X_{\text{v.f.}}$ of X is the union of the images in X of all very free rational curves to $X \otimes_k K$ as K/k varies over all algebraically closed extensions.

The next theorem explains the relation of these different properties.

Theorem 3.5. [Kol96, §IV.3], [HT06]*, Unless stated otherwise, all varieties below are quasi-projective over an algebraically closed field k.*

(0) *In characteristic* 0*, every rationally connected variety is separably rationally connected.*

(i) *For every flat, proper morphism $\pi : X \to B$ (not necessarily of quasi-projective varieties over a field), the subset of B parameterizing points whose geometric fiber is rationally chain connected is stable under specialization. (If one bounds the degree of the chains with respect to a relatively ample invertible $\mathcal{O}_X$-module, then it is a closed subset.)*

(ii) *The very free locus of a quasi-projective variety is open. More generally, for every flat, quasi-projective morphism $\pi : X \to B$ (not necessarily of quasi-projective varieties over a field) having irreducible geometric fibers, there exists an open subset $X_{\pi,v.f.}$ of X such that for every geometric fiber X_b of π, $(X_b)_{v.f.}$ equals $X_{\pi,v.f.} \times_B X_b$. In particular, the subset of B parameterizing points whose geometric fiber contains a very free rational curve is an open subset, $\pi(X_{\pi,v.f.})$.*

(iii) *The very free locus $X_{v.f.}$ of a quasi-projective variety is (separably) rationally connected in the following strong sense. For every positive integer N, for every positive integer m, and for every positive integer a, for every collection of distinct closed points $t_1, \dots, t_N \in \mathbb{P}^1_k$, for every collection of closed points $x_1, \dots, x_N \in X_{v.f.}$, and for every specification of an m-jet of a smooth curve in X at each point x_i, there exists a morphism*

$$f : \mathbb{P}^1_k \to X_{v.f.}$$

such that for every $i = 1, \dots, n$, f is unramified at t_i, $f(t_i)$ equals x_i and the m-jet of t_i in $\mathbb{P}^1_k$ maps isomorphically to the specified m-jet at x_i, and

$$f^*T_X \cong \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_n),\ a_1, \dots, a_n \geq a.$$

(iv) *Every rational curve in X_{smooth} intersecting $X_{v.f.}$ is contained in $X_{v.f.}$. Thus for every smooth, rationally chain connected variety, if X contains a very free rational curve then $X_{v.f.}$ equals all of X.*

(v) *A proper, irreducible variety X is rationally chain connected if it is* generically *rationally chain connected, i.e., if there exists a morphism u as in the definition such that $u^{(2)}$ is dominant (but not necessarily surjective).*

(vi) *For the morphism $u : M \times_k \mathbb{P}^1_k \to X$, let m be a closed point of M such that $u_m : \mathbb{P}^1 \to X$ has image in X_{smooth} and such that $u^{(2)}$ is smooth at (m, t_1, t_2) for some $t_1, t_2 \in \mathbb{P}^1_k$. Then the morphism u_m is very free. Thus an irreducible, quasi-projective variety X contains a very free curve if and only if there is a separably rationally connected open subset of X_{smooth}. Also, a smooth, quasi-projective variety X in characteristic 0 which is generically rationally connected contains a very free morphism.*

(vii) *For a surjective morphism $f : X \to Y$ of varieties over an algebraically closed field, if X is rationally connected, resp. rationally chain connected, then also Y is rationally connected, resp. rationally chain connected.*

(viii) *For a birational morphism $f : X \to Y$ of proper varieties over an algebraically closed field, if Y is rationally connected then X is rationally chain connected. If the characteristic is zero, then X is rationally connected.*

Remark 3.6. Item (ii) is proved in Proposition 3.18. The *generic case* of Item (iii), which is all we will need, is proved in Proposition 3.19. The complete result was proved by Hassett and Tschinkel, [HT06]. Item (iv) follows from Corollary 3.20. The remaining items are not proved, nor are they used in the proof of the main theorem. For the most part they are proved by similar arguments; complete proofs are in [Kol96, §IV.3].

Rational connectedness is analogous to path connectedness in topology, and satisfies the analogues of many properties of path connectedness. One property of path connectedness is this: for a fibration of CW complexes, if the base space and the fibers are path connected, then also the total space is path connected. This led to two conjectures by Kollár, Miyaoka and Mori.

Conjecture 3.7. [Kol96, Conjecture IV.5.6] Let $\pi : X \to B$ be a surjective morphism of smooth, projective schemes over an algebraically closed field of characteristic 0. If both B and a general fiber of π are rationally connected, then X is also rationally connected.

Conjecture 3.7 is implied by the following conjecture about rationally connected fibrations over curves.

Conjecture 3.8. [Kol96, Conjecture IV.6.1.1] Let $\pi : X \to B$ be a surjective morphism of projective schemes over an algebraically closed field of characteristic 0. If B is a smooth curve and if a general fiber of f is rationally connected, then there exists a morphism $s : B \to X$ such that $\pi \circ s$ equals Id_B, i.e., s is a section of π.

Our next goal is to prove the following result.

Theorem 3.9. [GHS03] *Conjecture 3.8 of Kollár-Miyaoka-Mori is true. Precisely, let k be an algebraically closed field of characteristic 0 and let $\pi_X : X \to B$ be a surjective morphism from a normal, projective k-scheme X to a smooth, projective, connected k-curve B. If the geometric generic fiber $X_{\overline{\eta}_B}$ is a normal, integral scheme whose smooth locus contains a very free curve, then there exists a morphism $s : B \to X$ such that $\pi_X \circ s$ equals Id_B.*

This was generalized by A. J. de Jong to the case that k is algebraically closed of arbitrary characteristic, [dJS03]. The key difference has to do with extensions of valuation rings in characteristic 0 and in positive characteristic. Given a flat morphism of smooth schemes in characteristic 0, $\pi : U \to B$, and given codimension 1 points η_D of U and η_Δ of B with $\pi(\eta_D) = \eta_\Delta$, the induced local homomorphism of stalks $\pi_U^* : \widehat{\mathcal{O}}_{B,\eta_\Delta} \to \widehat{\mathcal{O}}_{U,\eta_D}$, is equivalent to

$$k(\Delta)\,[\![t]\!] \to k(D)\,[\![r]\!]\,, \quad t \mapsto ur^m$$

for a unit u and a positive integer m, cf. the proof of Lemma 3.23 below. In particular, it is *rigid* in the sense that $t \mapsto ur^m + vr^{m+1} + \dots$ is equivalent to $t \mapsto ur^m$. However, extensions of positive characteristic valuation rings are not rigid, e.g., $t \mapsto r^p + v_1 r^{p+1}$ is equivalent to $t \mapsto r^p + v_2 r^{p+1}$ only if $v_1 = v_2$. But there is a weak rigidity of local homomorphisms, Krasner's lemma in the theory of non-Archimedean valuations. This is a key step in the generalization to positive characteristic.

Of course when k has characteristic 0, then since X is normal the fiber $X_{\overline{\eta}_B}$ is automatically normal. If X is also smooth (which can be achieved thanks to resolution of singularities in characteristic 0), then also $X_{\overline{\eta}_B}$ is smooth. Then the hypothesis on $X_{\overline{\eta}_B}$ is equivalent to rational connectedness.

3.1. **Outline of the proof.** The proof that follows is based on a proof by T. Graber, J. Harris and myself (not quite the version we chose to publish) together with several major simplifications due to A. J. de Jong. The basic idea is to choose a smooth curve $C \subset X$ such that $\pi_X|_C : C \to B$ is finite, and then try to deform C as a curve in X until it specializes to a reducible curve in X, one component of which is the image of a section s of π_X. Here are some definitions that make this precise.

Definition 3.10. Let $\pi_C : C \to B$ be a finite morphism of smooth, projective k-curves. A *linked curve with handle* C is a reduced, connected, projective curve C_{link} with irreducible components

$$C_{\text{link}} = C \cup L_1 \cup \cdots \cup L_m$$

together with a morphism

$$\pi_{C,\text{link}} : C_{\text{link}} \to B$$

such that

(i) $\pi_{C,\text{link}}$ restricts to π_C on the component C,
(ii) the restriction of $\pi_{C,\text{link}}$ to each *link* component L_i is a constant morphism with image b_i, where $b_1, \dots, b_m$ are distinct closed points of B,
(iii) and each link L_i is a smooth, rational curve intersecting C in a finite number of nodes of C_{link}.

If every link L_i intersects C in a single node of C_{link}, then $(C_{\text{link}}, \pi_{C,\text{link}})$ is called a *comb* and the links L_i are called *teeth*. For combs we will use the notation C_{comb} rather than C_{link}.

A *one-parameter deformation* of a linked curve $(C_{\text{link}}, \pi_{C,\text{link}})$ is a datum of a smooth, connected, pointed curve $(\Pi, 0)$ and a projective morphism

$$(\rho, \pi_{\mathcal{C}}) : \mathcal{C} \to \Pi \times_k B$$

such that ρ is flat and such that $\mathcal{C}_0 := \rho^{-1}(0)$ together with the restriction of $\pi_{\mathcal{C}}$ equals the linked curve $(C_{\text{link}}, \pi_{C,\text{link}})$.

A one-parameter deformation *specializes to a section curve* if there exists a closed point $\infty \in \Pi$ and an irreducible component B_i of $\mathcal{C}_\infty := \rho^{-1}(\infty)$ such that

(i) $\mathcal{C}_\infty$ is reduced at the generic point of B_i
(ii) and the restriction of $\pi_{\mathcal{C}}$ to B_i is an isomorphism

$$\pi_{\mathcal{C}}|_{B_i} : B_i \xrightarrow{\cong} B.$$

Given a linked curve, a one-parameter deformation of the linked curve and a B-morphism $j : C_{\text{link}} \to X$, an *extension* of j is an open neighborhood of 0, $0 \in N \subset B$ and a B-morphism

$$j_N : \mathcal{C}_N \to X, \quad \mathcal{C}_N := \rho^{-1}(N)$$

restricting to j on $\mathcal{C}_0 = C_{\text{link}}$.

For the purposes of producing a section, the particular parameter space $(\Pi, 0)$ of the one-parameter deformation is irrelevant. Thus, it is allowed to replace the one-parameter deformation by the new one-parameter deformation obtained from a finite base change $(\Pi', 0') \to (\Pi, 0)$. The following lemma is straightforward.

Lemma 3.11. *Let $(\Pi, 0, \infty)$ together with $(\rho, \pi_{\mathcal{C}}) : \mathcal{C} \to \Pi \times_k B$ be a one-parameter deformation of $(C_{link}, \pi_{C,link})$ specializing to a section curve B_i. For every morphism of 2-pointed, smooth, connected curves*

$$(\Pi', 0', \infty') \to (\Pi, 0, \infty),$$

the base change morphism

$$\Pi' \times_\Pi \mathcal{C} \to \Pi' \times_k B$$

is also a one-parameter deformation of $(C_{link}, \pi_{C,link})$ specializing to the section curve B_i.

The usefulness of these definitions is the following simple consequence of the valuative criterion of properness.

Lemma 3.12. *Let (C_{link}, π_{link}) be a linked curve together with a B-morphism $j : C_{link} \to X$. If there exists a one-parameter deformation of the linked curve specializing to a section curve and if there exists an extension of j, then there exists a section $s : B \to X$ of π_X.*

Proof. Let R denote the stalk $\mathcal{O}_{\mathcal{C},\eta_{B_i}}$ of $\mathcal{O}_{\mathcal{C}}$ at the generic point η_{B_i} of B_i. By the hypotheses on $\mathcal{C}$ and B_i, R is a discrete valuation ring with residue field $\kappa = k(B_i)$ and fraction field $K = k(\mathcal{C})$. The restriction of j_N to the generic point of $\mathcal{C}$ is a B-morphism

$$j_K : \text{Spec } K \to X.$$

Because $\pi_X : X \to B$ is proper, by the valuative criterion of properness the B-morphism j_K extends to a B-morphism

$$j_R : \text{Spec } R \to X,$$

which in turn gives a B-morphism from the residue field Spec κ to X, i.e., a rational B-map

$$j_{B_i} : B_i \supset U \to X, \quad U \subset B_i \text{ a dense, Zariski open.}$$

Finally, because B_i is a smooth curve, the valuative criterion applies once more and this rational transformation extends to a B-morphism

$$j_{B_i} : B_i \to X.$$

Because $\pi_{\mathcal{C}}|_{B_i} : B_i \to B$ is an isomorphism, there exists a unique B-morphism

$$s : B \to X$$

such that $j_{B_i} = s \circ \pi_{\mathcal{C}}|_{B_i}$. The morphism s is a section of π_X. □

Thus the proof of the theorem breaks into three parts:

(i) find a "good" linked curve $j : C_{\text{link}} \to X$,

(ii) find a one-parameter deformation of the linked curve specializing to a section curve,

(iii) and find an extension of j to the one-parameter deformation.

The first step in finding $j : C_{\text{link}} \to X$ is to form a curve C_{init} which is an intersection of X with $\dim(X)-1$ general hyperplanes in projective space. By Bertini's theorem, if the hyperplanes are sufficiently general, then C_{init} will satisfy any reasonable transversality property. Moreover, there is a technique due to Kollár-Miyaoka-Mori – the smoothing combs technique – for improving C_{init} to another curve $C \subset X$ still satisfying the transversality property and also satisfying a positivity property with respect to the vertical tangent bundle of $\pi : X \to B$.

Unfortunately, even after such an improvement, there may be no one-parameter deformation of $\pi_X|_C : C \to B$ specializing to a section curve. However, after attaching sufficiently many link components over general closed points of B, there does exist a one-parameter deformation of C_{link} specializing to a section curve. This is one aspect of the well-known theorem that for a fixed base curve B and for a fixed degree d, if the number β of branch points is sufficiently large the Hurwitz scheme of degree d covers of B with β branch points is irreducible. (This was proved by Hurwitz when $g(B) = 0$, [Hur91], proved by Richard Hamilton for arbitrary genus in his thesis, and periodically reproved ever since, cf. [GHS02].) Because the general fibers of $\pi : X \to B$ are rationally connected, the inclusion $C \subset X$ extends to a B-morphism $j : C_{\text{link}} \to X$.

Finally the positivity property mentioned above implies j extends to the one-parameter deformation, at least after base change by a morphism $\Pi' \to \Pi$.

3.2. **Hilbert schemes and smoothing combs.** The smoothing combs technique of Kollár-Miyaoka-Mori depends on a result from the deformation theory of Hilbert schemes. Here is the setup for this result. Let $Y \to S$ be a flat, quasi-projective morphism and let

$$(\rho_{\text{Hilb}} : \text{Hilb}(Y/S) \to S, \text{Univ}(Y/S) \subset \text{Hilb}(Y/S) \times_S T)$$

be universal among pairs $(\rho : T \to S, Z \subset T \times_S Y)$ of an S-scheme T and a closed subscheme $Z \subset T \times_S Y$ such that $Z \to T$ is proper, flat and finitely presented. In other words, $\text{Hilb}(Y/S)$ is the *relative Hilbert scheme* of Y over S.

In particular, for every field K the K-valued points of $\text{Hilb}(Y/S)$ are naturally in bijection with pairs (s, Z) of a K-valued point s of S and a closed subscheme Z of $Y_s := \{s\} \times_S Y$. The closed immersion $Z \to Y_s$ is a *regular embedding* if at every point of Z the stalk of the ideal sheaf $\mathcal{I}_{Z/Y_s}$ is generated by a regular sequence of elements in the stalk of $\mathcal{O}_{Y_s}$. In this case the *conormal sheaf* $\mathcal{I}_{Z/Y_s}/\mathcal{I}^2_{Z/Y_s}$ is a locally free $\mathcal{O}_Z$-module, and hence also the *normal sheaf*

$$\mathcal{N}_{Z/Y_s} := \mathit{Hom}_{\mathcal{O}_Z}(\mathcal{I}_{Z/Y_s}/\mathcal{I}^2_{Z/Y_s}, \mathcal{O}_S)$$

is a locally free $\mathcal{O}_Z$-module.

Proposition 3.13. [Kol96, Theorem I.2.10, Lemma I.2.12.1, Proposition I.2.14.2] *If $Z \subset Y_s$ is a regular embedding and if $h^1(Z, \mathcal{N}_{Z/Y_s})$ equals 0, then Hilb(Y/S) is smooth over S at (s, Z).*

There is a variation of this proposition which is also useful. There is a *flag Hilbert scheme* of Y over S, i.e., a universal pair

$$(\rho_{\text{fHilb}} : \text{fHilb}(Y/S) \to S, \text{Univ}_1(Y/S) \subset \text{Univ}_2(Y/S) \subset \text{Hilb}(Y/S) \times_S T)$$

among all pairs $(\rho : T \to S, Z_1 \subset Z_2 \subset T \times_S Y)$ of an S-scheme T and a nested pair of closed subschemes $Z_1 \subset Z_2 \subset T \times_S Y$ such that for $i = 1, 2$, the projection $Z_i \to T$ is proper, flat and finitely presented. There are obvious forgetful morphisms

$$F_i : \text{fHilb}(Y/S) \to \text{Hilb}(Y/S), \quad F_i(s, Z_1, Z_2) = (s, Z_i).$$

Proposition 3.14. *Let K be a field and let (s, Z_1, Z_2) be a K-point of fHilb(Y/S). If each closed immersion $Z_1 \subset Z_2$ and $Z_2 \subset Y_s$ is a regular embedding and if*

$$h^1(Z_2, \mathcal{N}_{Z_2/Y_s}) = 0, h^1(Z_1, \mathcal{N}_{Z_1/Z_2}) = 0, \text{ and if } h^i(Z_2, \mathcal{I}_{Z_1/Z_2} \cdot \mathcal{N}_{Z_2/Y_2}) = 0 \text{ for } i = 1, 2,$$

fHilb(Y/S) is smooth over S at (s, Z_1, Z_2), for each $i = 1, 2$, Hilb(Y/S) is smooth over S at (s, Z_i), and each forgetful morphism F_i : fHilb$(Y/S) \to$ Hilb(Y/S) is smooth at (s, Z_1, Z_2).

Proof. Since $h^1(Z_2, \mathcal{N}_{Z_2/Y_s})$ equals 0, Hilb(Y/S) is smooth at (s, Z_2) by Proposition 3.13. It is easy to see that the forgetful morphism F_2 is equivalent to the relative Hilbert scheme Hilb(Univ(Y/S)/Hilb(Y/S)) over Hilb(Y/S). Thus, applying Proposition 3.13 to this Hilbert scheme, the vanishing of $h^1(Z_1, \mathcal{N}_{Z_1/Z_2})$ implies F_2 is smooth at (s, Z_1, Z_2). Since a composition of smooth morphisms is smooth, also fHilb(Y/S) is smooth over S at (s, Z_1, Z_2). The long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{I}_{Z_1/Z_2} \cdot \mathcal{N}_{Z_2/Y_s} \longrightarrow \mathcal{N}_{Z_2/Y_s} \longrightarrow \mathcal{N}_{Z_2/Y_s}|_{Z_1} \longrightarrow 0$$

implies that $h^1(Z_1, \mathcal{N}_{Z_2/Y_s}|_{Z_1})$ equals $h^1(Z_2, \mathcal{N}_{Z_2/Y_s})$, which is 0. Thus, the long exact sequence of cohomology associated to

$$0 \longrightarrow \mathcal{N}_{Z_1/Z_2} \longrightarrow \mathcal{N}_{Z_1/Y_s} \longrightarrow \mathcal{N}_{Z_2/Y_s}|_{Z_1} \longrightarrow 0$$

implies that $h^1(Z_2, \mathcal{N}_{Z_1/Y_2})$ equals 0. So again by Proposition 3.13, Hilb(Y/S) is smooth over S at (s, Z_1). Finally, F_1 is a morphism of smooth S-schemes at (s, Z_1, Z_2). Thus, to prove F_1 is smooth, it suffices to prove it is surjective on Zariski tangent vector spaces. This follows from the vanishing of $h^1(Z_2, \mathcal{I}_{Z_1/Z_2} \cdot \mathcal{N}_{Z_2/Y_s})$. □

Another ingredient in the smoothing combs technique is a a simple result about *elementary transforms* of locally free sheaves on a curve: the higher cohomology of the sheaf becomes zero after applying elementary transforms at sufficiently many points.

Lemma 3.15. *Let C be projective, at-worst-nodal, connected curve over a field k and let $\mathcal{E}$ be a locally free $\mathcal{O}_C$-module.*

(i) *There exists a short exact sequence of coherent sheaves,*

$$0 \to \mathcal{F}^\vee \to \mathcal{E}^\vee \to \mathcal{T} \to 0$$

such that $\mathcal{T}$ is a torsion sheaf with support in C_{smooth} and such that $h^1(C, \mathcal{F})$ equals 0.

(ii) *Inside the parameter space of torsion quotients $q : \mathcal{E}^\vee \twoheadrightarrow \mathcal{T}$ with support in C_{smooth}, denoting*

$$\mathcal{F}^\vee := Ker(\mathcal{E}^\vee \to \mathcal{T}) \text{ and } \mathcal{F} := Hom_{\mathcal{O}_C}(\mathcal{F}^\vee, \mathcal{O}_C),$$

the subset parameterizing quotients for which $h^1(C, \mathcal{F}) = 0$ is an open subset.

(iii) *If $h^1(C, \mathcal{F})$ equals 0, then for every short exact sequence of coherent sheaves*

$$0 \to \mathcal{G}^\vee \to \mathcal{E}^\vee \xrightarrow{q'} \mathcal{S} \to 0$$

admitting a morphism $r : \mathcal{S} \to \mathcal{T}$ of torsion sheaves with support in C_{smooth} for which $q = r \circ q'$, $h^1(C, \mathcal{G})$ equals 0.

Proof. **(i)** By Serre's vanishing theorem, there exists an effective, ample divisor D in the smooth locus of C such that $h^1(C, \mathcal{E}(D))$ equals 0. Define $\mathcal{F} = \mathcal{E}(D)$, define $\mathcal{E} \to \mathcal{F}$ to be the obvious morphism $\mathcal{E} \to \mathcal{E}(D)$, and define $\mathcal{T}$ to be the cokernel of $\mathcal{F}^\vee \to \mathcal{E}^\vee$.

(ii) This follows immediately from the semicontinuity theorem, cf. [Har77, §III.12].

(iii) There exists an injective morphism of coherent sheaves $\mathcal{F} \to \mathcal{G}$ with torsion cokernel. Because $h^1(C, \mathcal{F})$ equals 0 and because h^1 of every torsion sheaf is zero, the long exact sequence of cohomology implies that also $h^1(C, \mathcal{G})$ equals 0. □

It is worth noting one interpretation of the sheaf $\mathcal{F}$ associated to a torsion quotient $\mathcal{T}$. Assume that $\mathcal{T}$ is isomorphic to a direct sum of skyscraper sheaves at n distinct points $c_1, \dots, c_n$ of C_{smooth}. (Inside the parameter space of torsion quotients, those with this property is a dense, open subset.) For each point c_i, the linear functional $\mathcal{E}^\vee|_{c_i} \twoheadrightarrow \mathcal{T}|_{c_i}$ gives a one-dimensional subspace $\text{Hom}_k(\mathcal{T}|_{c_i}, k) \hookrightarrow \mathcal{E}|_{c_i}$. The sheaf $\mathcal{F}$ is precisely the sheaf of rational sections of $\mathcal{E}$ having at worst a simple pole at each point c_i in the direction of this one-dimensional subspace of $\mathcal{E}|_{c_i}$. This is often called an *elementary transform up* of $\mathcal{E}$ at the point c_i in the specified direction. So Lemma 3.15 says that h^1 becomes zero after sufficiently many elementary transforms up at general points in general directions.

This interpretation is useful because the normal sheaf of a reducible curve can be understood in terms of elementary transforms up. To be precise, let Y be a k-scheme, let C be a proper, nodal curve, let C_0 be a closed subcurve (i.e., a union of irreducible components of C), and let $j : C \to Y$ be a regular embedding such that Y is smooth at every node $p_1, \dots, p_n$ of C which is contained in C_0 and which is not a node of C_0. Then $j_0 : C_0 \to Y$ is also a regular embedding and both $\mathcal{N}_{C/Y}|_{C_0}$ and $\mathcal{N}_{C_0/Y}$ are locally free sheaves on C_0. For each i, there is a branch C_i of C at p_i other than C_0. Denote by T_{C_i,p_i} the tangent direction of this branch in T_{Y,p_i}.

Lemma 3.16. [GHS03, Lemma 2.6] *The restriction $\mathcal{N}_{C/Y}|_{C_0}$ equals the sheaf of rational sections of $\mathcal{N}_{C_0/Y}$ having at most a simple pole at each point p_i in the normal direction determined by T_{C_i,p_i}.*

Proof. The restrictions of $\mathcal{N}_{C/Y}|_{C_0}$ and $\mathcal{N}_{C_0/Y}$ to the complement of $\{p_1, \dots, p_n\}$ are canonically isomorphic. The lemma states that this canonical isomorphism is the restriction of an injection $\mathcal{N}_{C_0/Y} \hookrightarrow \mathcal{N}_{C/Y}|_{C_0}$ which identifies $\mathcal{N}_{C/Y}|_{C_0}$ with the sheaf of rational sections, etc. This local assertion can be verified in a formal neighborhood of each node p_i.

Locally near p_i, $C \to Y$ is formally isomorphic to the union of the two axes inside a 2-plane inside an n-plane, i.e., the subscheme of $\mathbb{A}^n_k$ with ideal $I_{C/Y} = \langle x_1x_2, x_3, \dots, x_n \rangle$. The branch C_0 corresponds to just one of the axes, e.g., the subscheme of $\mathbb{A}^n_k$ with ideal $I_{C_0/Y} = \langle x_2, x_3, \dots, x_n \rangle$. The tangent direction of the other branch C_i is spanned by $(0, 1, 0, \dots, 0)$. Thus it is clear that $I_{C/Y}/I_{C_0/Y} \cdot I_{C/Y}$ is the submodule of $I_{C_0/Y}/I^2_{C_0/Y}$ of elements whose fiber at 0 is contained in the annihilator of T_{C_i,p_i}. Dualizing gives the lemma. □

The final bit of deformation theory needed has to do with deforming nodes. Let C be a proper, nodal curve and let $jC \to Y$ be a regular embedding. Let p be a node of C and assume that Y is smooth at p. There are two branches C_1 and C_2 of C at p (possibly contained in the same irreducible component of C). The sheaf

$$\mathcal{T} := \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$$

is a skyscraper sheaf supported at p and with fiber canonically identified to

$$\mathcal{T}|_p = T_{C_1,p} \otimes_k T_{C_2,p}.$$

The following lemma is as much definition as lemma.

Lemma 3.17. *There exists a quotient of coherent sheaves*

$$\mathcal{N}_{C/Y} \twoheadrightarrow \mathcal{T}$$

such that for both $i = 1, 2$ the quotient $\mathcal{N}_{C/Y}|_{C_i}/\mathcal{N}_{C_i/Y}$ equals $\mathcal{T}$. A first-order deformation of $C \subset Y$, i.e., a global section of $\mathcal{N}_{C/Y}$ is said to smooth the node p to first-order *if the image of the section in $T_{C_1,p} \otimes_k T_{C_2,p}$ is nonzero. For a deformation*

$$\mathcal{C} \subset \Pi \times_k Y$$

of $C \subset Y$ over a smooth pointed curve $(\Pi, 0)$ (i.e., $\mathcal{C}_0 = C$), if the associated first-order deformation of $C \subset Y$ smooths the node p to first-order, then p is not contained in the closure of the singular locus of the projection,

$$(\Pi - \{0\}) \times_\Pi \mathcal{C} \to (\Pi - \{0\})$$

i.e., a general fiber $\mathcal{C}_t$ of the deformation smooths the node.

This is a well-known result. A good reference for this result, and many other results about deformations of singularities, is [Art76], particularly §I.6. Here is a brief remark on the proof. Because $C \subset Y$ is a regular embedding, the conormal sequence is exact on the left, i.e.,

$$0 \longrightarrow \mathcal{I}_{C/Y}/\mathcal{I}^2_{C/Y} \longrightarrow \Omega_Y|_C \longrightarrow \Omega_C \longrightarrow 0$$

is a short exact sequence. Applying global Ext, there is a connecting map

$$\delta : H^0(C, \mathcal{N}_{C/Y}) \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C).$$

There is also a local-to-global sequence for global Ext inducing a map

$$\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to H^0(C, \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = H^0(C, \mathcal{T}) = T_{C_1,p} \otimes_k T_{C_2,p}.$$

The composition of these two maps is precisely the map on global sections associated to $\mathcal{N}_{C/Y} \to \mathcal{T}$. induced map to $H^0(C, \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = \mathcal{T}$ is precisely the map above. The global Ext group is identified with the first-order deformations of C as an abstract scheme, and the Ext term is identified with the first-order deformations of the node. It is worth noting that even if the first-order deformation does not smooth the node, the full deformation $\mathcal{C} \subset \Pi \times_k Y$ may smooth the node if the total space $\mathcal{C}$ is singular at $(0, p)$.

The first result using the smoothing combs technique is the following.

Proposition 3.18. *Let Y be a smooth, irreducible, quasi-projective scheme over an algebraically closed field k. The very free locus $Y_{v.f.}$ is an open subset of Y. More generally, for a smooth, quasi-projective morphism $\pi : Y \to B$, the relative very free locus $Y_{\pi,v.f.}$ is an open subset of Y.*

Again in the absolute case, denote by t_1, resp. t_2, the closed point of $\mathbb{P}^1_k$, $t_1 = 0$, resp. $t_2 = \infty$. Let y_1 and y_2 be closed points of $Y_{v.f.}$, let a and k be nonnegative integers, and let there be given curvilinear k-jets in Y at each of y_1 and y_2. If the given k-jets are general among all curvilinear k-jets at y_1 and y_2, then there exists a morphism

$$f : (\mathbb{P}^1_k, t_1, t_2) \to (Y_{v.f.}, y_1, y_2)$$

mapping the k-jet of $\mathbb{P}^1$ at t_1 isomorphically to the given k-jet at y_i for $i = 1, 2$ and such that

$$f^* T_Y \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n),\ a_1, \ldots, a_n \geq a.$$

Proof. By the definition of $Y_{\text{v.f.}}$, for each $i = 1, 2$ there exists a very free morphism

$$f_i : (\mathbb{P}^1, 0) \to (Y_{\text{v.f.}}, y_i), \quad f_i^* T_Y \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n), a_1, \ldots, a_n \geq 1.$$

In particular, for each $i = 1, 2$, $h^1(\mathbb{P}^1, f_i^* T_Y(-\underline{0} - \underline{\infty}))$ equals 0, where $\underline{0}$, resp. $\underline{\infty}$, is the Cartier divisor of the point 0, resp. ∞, in $\mathbb{P}^1$. Since the normal sheaf of f_i is a quotient of $f_i^* T_Y$, also $h^1(\mathbb{P}^1, \mathcal{N}_{f_i}(-\underline{0} - \underline{\infty}))$ equals 0. Thus, applying Proposition 3.14 where $Z_1 = \{0, \infty\}$ and $Z_2 = \mathbb{P}^1$, there exist deformations of the morphism f_i such that $f_i(0)$ equals y_i and $f_i(\infty)$ is any point in a nonempty Zariski open subset of Y. The same argument holds in the relative case.

Since Y is irreducible, the open for $i = 1$ intersects the open for $i = 2$. Thus there exist very free morphisms f_1 and f_2 such that $f_1(\infty) = f_2(\infty)$. Let C be the union of two copies C_1 and C_2 of $\mathbb{P}^1$ attached at ∞ and let $f : C \to Y$ be the morphism whose restriction to each C_i equals f_i. Denote by

$$0 \longrightarrow \mathcal{N}'_{C/Y} \longrightarrow \mathcal{N}_{C/Y} \longrightarrow \mathcal{T} \longrightarrow 0$$

the short exact sequence coming from Lemma 3.17. Using Lemma 3.16, there is an exact sequence

$$0 \longrightarrow \mathcal{N}_{C/Y}|_{C_1}(-\underline{0} - \underline{\infty}) \longrightarrow \mathcal{N}'_{C/Y}(-y_1 - y_2) \longrightarrow \mathcal{N}_{C_2/Y}(-\underline{0}) \longrightarrow 0$$

and an exact sequence

$$0 \longrightarrow \mathcal{N}_{C_1/Y}(-\underline{0} - \underline{\infty}) \longrightarrow \mathcal{N}_{C/Y}|_{C_1}(-\underline{0} - \underline{\infty}) \longrightarrow \kappa_\infty \longrightarrow 0$$

where κ_∞ is the skyscraper sheaf on C_1 supported at ∞. Applying the long exact sequence of cohomology, using that $h^1(C_i, \mathcal{N}_{C_i/Y}(-\underline{0} - \underline{\infty}))$ equals 0 for $i = 1, 2$, and chasing diagrams, this finally gives that $h^1(C, \mathcal{N}'_{C/Y}(-y_1 - y_2))$ also equals 0.

This has two consequences. First, this implies $h^1(C, \mathcal{N}_{C/Y}(-y_1 - y_2))$ equals 0, and thus the space of deformations of C containing y_1 and y_2 is smooth by Proposition 3.14. And second, the map

$$H^0(C, \mathcal{N}_{C/Y}(-y_1 - y_2)) \to T_{C_1,\infty} \otimes T_{C_2,\infty}$$

is surjective. Thus there exist first-order deformations of C containing y_1 and y_2 and smoothing the node at ∞. Since the space of deformations containing y_1 and y_2 is smooth, this first-order deformation is the one associated to a one-parameter deformation

$$\mathcal{C} \subset \Pi \times_k Y$$

of $[C]$ over a smooth, pointed curve $(\Pi, 0)$ (e.g., choose Π to be a general complete intersection curve in the smooth deformation space containing the given Zariski tangent vector). By Lemma 3.17, for a general point t of Π, $\mathcal{C}_t$ is a smooth, connected curve containing y_1 and y_2. Since the arithmetic genus of C is 0, the arithmetic genus of $\mathcal{C}_t$ is also 0, i.e., $\mathcal{C}_t \cong \mathbb{P}^1_k$. Let

$$f_1 : \mathbb{P}^1_k \to \mathcal{C}_t$$

be an isomorphism with $f_1(t_i) = y_i$ for $i = 1, 2$. Because $h^1(C, \mathcal{N}_{C/Y}(-y_1 - y_2))$ equals 0, by the semicontinuity theorem also $h^1(\mathcal{C}_t, \mathcal{N}_{\mathcal{C}_t/Y}(-y_1 - y_2))$ equals 0. This implies that

$$f_1^* T_Y \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n), \text{ for integers } a_1, \ldots, a_n \geq 1.$$

Next, for every integer a, let $g_a : (\mathbb{P}^1 \to \mathbb{P}^1$ be the morphism $z \mapsto z^a$. Then the composition $f_a = f_1 \circ g_1$ is a morphism

$$f_a : (\mathbb{P}^1_k, t_1, t_2) \to (Y_{\text{v.f.}}, y_1, y_2)$$

with

$$f_a^* T_Y = g_a^*(f_1^* T_Y) \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n) \text{ for integers } a_1, \dots, a_n \geq a,$$

namely the new integer $a_i(f_a)$ equals $a \cdot a_i(f_1)$. Next, choosing $a \geq 2k+1$, this implies that

$$h^1(\mathbb{P}^1, f_a^* T_Y(-(k+1)(t_1+t_2))) = 0.$$

Applying Proposition 3.14 with $\mathbb{P}^1 \times_k Y$ in the place of Y, with the graph of f_a in the place of Z_2 and with $Z_1 = (k+1)(t_1+t_2)$ in the place of Z, deformations of f_a map the k-jet of $\mathbb{P}^1$ at t_1, resp. at t_2, isomorphically to a general k-jet at y_1, resp. at y_2. □

The following proposition is the strongest generalization of Proposition 3.18 we will need. It is stated as a theorem about finding new sections of a rationally connected fibration under the hypothesis that one such section exists. In this sense it may seem premature (and dangerously close to circular logic), since Theorem 3.9 is not yet proved. In fact the proposition is used in the proof of Theorem 3.9 not for the original fibration, but only for a *constant* fibration

$$\text{pr}_{\mathbb{P}^1} : \mathbb{P}^1_k \times_k Y \to \mathbb{P}^1_k$$

which obviously admits sections (constant sections). So there is nothing circular in the application of the proposition to the proof of Theorem 3.9.

Proposition 3.19 (Generic weak approximation)**.** [KMM92], [HT06] *Let B be a smooth, connected, projective curve over an algebraically closed field k. Let U be a smooth, quasi-projective k-scheme and let $\pi : U \to B$ be a good morphism (in the sense of Definition 3.22). Assume there exists a section $s : B \to U$. Let $(b_1, \dots, b_M, b'_1, \dots, b'_{M'})$ be distinct closed points of B such that $s(b_i)$ is in the very free locus $U_{b_i,v.f.}$ of the fiber U_{b_i} for each $i = 1, \dots, M$. Let k and a be nonnegative integers. For each i, let x_i be a closed point of $U_{b_i,v.f.}$ and let there be given an curvilinear k-jet in U at x_i. Assuming each of these k-jets is a general k-jet at x_i, there exists a section $\sigma : B \to U$ such that*

(i) *for each $i = 1, \dots, M$, $\sigma(b_i)$ equals x_i,*
(ii) *for each $i = 1, \dots, M'$, $\sigma(b'_i)$ equals $s(b'_i)$,*
(iii) *for every invertible $\mathcal{O}_B$-module $\mathcal{L}$ of degree $\leq a$, $h^1(B, \mathcal{N}_{\sigma(B)/U} \otimes_{\mathcal{O}_B} \mathcal{L}^\vee)$ equals 0,*
(iv) *and σ maps the k-jet of b_i in B isomorphically to the given k-jet at x_i for each i.*

Proof. Choose a large integer N and enlarge the set of pairs $((b_i, x_i))|_{i=1,\dots,M}$ to a set $((b_i, x_i))_{i=1,\dots,N}$ having the same properties above and such that the collection $(b_i)_{i=M+1,\dots,N}$ is a general collection of $N-M$ points in B (this is possible because for all but finitely many closed points of B, $s(b)$ is contained in $U_{b,\text{v.f.}}$). By Prop 3.18, applied with $k=1$, i.e., in the case that k-jets are simply tangent directions, for every $i = 1, \dots, N$ there exists a morphism

$$f_i : (\mathbb{P}^1, 0, \infty) \to (U_{b_i,\text{v.f.}}, s(b_i), x_i)$$

such that

$$f_i^* T_{\pi,\log} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n), \quad a_1, \ldots, a_n \geq 1$$

and the tangent direction of $f_i(\mathbb{P}^1)$ at $s(b_i)$ is a general tangent direction in $T_{U_{b_i},s(b_i)}$. But of course the tangent space $T_{U_{b_i},s(b_i)}$ equals the normal space $N_{s(B)/U}|_{s(b_i)}$. Thus the tangent direction of $f_i(\mathbb{P}^1)$ at $s(b_i)$ gives a general normal direction to $s(B)$ in U at $s(b_i)$.

Form the comb $j : C_{\text{comb}} \to U$ with handle $s(B)$ and with each morphism f_i being a tooth L_i attached at $s(b_i)$. By Lemma 3.16, $\mathcal{N}_{C_{\text{comb}}/U}|_{s(B)}$ equals the sheaf of rational sections of $\mathcal{N}_{s(B)/U}$ having at most a simple pole at each point $s(b_i)$ in a general normal direction at $s(b_i)$. Assuming the integer N is sufficiently large, Lemma 3.15 then implies that $h^1(B, s^*\mathcal{N}_{C_{\text{comb}}/U})$ equals 0. Moreover, fixing an auxiliary invertible sheaf $\mathcal{M}$ on B of degree $g(B)+1$ and applying Lemma 3.15 to $s^*\mathcal{N}_{s(B)/U}(-(b'_1 + \cdots + b'_{M'})) \otimes_{\mathcal{O}_B} \mathcal{M}^\vee$, for N sufficiently large also $h^1(B, s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1 + \cdots + b'_{M'})) \otimes_{\mathcal{O}_B} \mathcal{M})^\vee$ equals 0.

For every i, there is a short exact sequence

$$0 \longrightarrow f_i^*\mathcal{N}_{L_i/U_{b_i}} \longrightarrow f_i^*\mathcal{N}_{L_i/U} \longrightarrow f_i^*\mathcal{N}_{U_{b_i}/U} \longrightarrow 0.$$

Of course the normal sheaf $\mathcal{N}_{U_{b_i}/U}$ is just $\mathcal{O}_{U_{b_i}}$ since U_{b_i} is a smooth fiber of a morphism to a curve. Also the normal direction of $s(B)$ at $s(b_i)$ surjects onto the fiber of $\mathcal{N}_{U_{b_i}/U}$ at $s(b_i)$. Thus the elementary transform up of $\mathcal{N}_{L_i/U}$ at $s(b_i)$ in this normal direction surjects onto the elementary transform up of $\mathcal{O}^1_{\mathbb{P}}$ at ∞, i.e., it surjects onto $\mathcal{O}_{\mathbb{P}^1}(1)$. Thus, by Lemma 3.16, there is a short exact sequence

$$0 \longrightarrow f_i^*\mathcal{N}_{L_i/U_{b_i}} \longrightarrow f_i^*\mathcal{N}_{C_{\text{comb}}/U} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^1}(-2)$ and applying the long exact sequence of cohomology associated to the short exact sequence, $h^1(\mathbb{P}^1, f_i^*\mathcal{N}_{C_{\text{comb}}/U}(-\underline{0} - \underline{\infty}))$ equals 0. Combined with the result of the previous paragraph and joining the two types of normal sheaf via the short exact sequence

$$0 \to \oplus_{i=1}^N \mathcal{N}_{C_{\text{comb}}/U}|_{L_i}(-x_i - s(b_i)) \to \mathcal{N}_{C_{\text{comb}}/U}(-(x_1 + \cdots + x_N) - (b'_1 + \cdots + b'_{M'}))$$
$$\to \mathcal{N}_{C_{\text{comb}}/U}|_{s(B)}(-(b'_1 + \cdots + b'_{M'})) \to 0,$$

the long exact sequence of cohomology implies both that

$$h^1(C_{\text{comb}}, \mathcal{N}_{C_{\text{comb}}/U}(-(x_1 + \cdots + x_N) - (b'_1 + \cdots + b'_{M'}))) \text{ equals } 0,$$

and that the map

$$H^0(C_{\text{comb}}, \mathcal{N}_{C_{\text{comb}}/U}(-(x_1+\cdots+x_N)-(b'_1+\cdots+b'_{M'}))) \to H^0(B, s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1+\cdots+b'_{M'})))$$

is surjective.

Thus, by Proposition 3.14, the space of deformations of C_{comb} containing $x_1, \ldots, x_N$ and $b'_1, \ldots, b'_{M'}$ is smooth. And, by Lemma 3.17, to prove there exists a deformation smoothing every node of C_{comb}, it suffices to prove for every i there exists a section of $s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1 + \cdots + b'_{M'}))$ whose image in $T_{s(B),s(b_i)} \otimes_k T_{L_i,s(b_i)}$ is nonzero. Of course this skyscraper sheaf $\mathcal{T}_{s(b_i)}$ is a quotient of the fiber of $s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1 + \cdots + b'_{M'}))$ at b_i. Thus it suffices to prove for every i that $h^1(B, s^*\mathcal{N}_{C_{\text{comb}}/U}(-b_i - (b'_1+\cdots+b'_{M'})))$ equals 0. Recall the auxiliary invertible sheaf $\mathcal{M}$ of degree $g(B)+1$.

Because the invertible sheaf $\mathcal{M}(-b_i)$ has degree $g(B)$, it is effective, say $\mathcal{O}_B(\Delta_i)$. Thus there exists an injective $\mathcal{O}_B$-module homomorphism

$$s^*\mathcal{N}_{C_{\mathrm{comb}}/U}(-(b'_1+\dots+b'_{M'}))\otimes_{\mathcal{O}_B}\mathcal{M}^\vee \hookrightarrow$$

$$s^*\mathcal{N}_{C_{\mathrm{comb}}/U}(-(b'_1+\dots b'_{M'}))\otimes_{\mathcal{O}_B}\mathcal{M}^\vee(\Delta_i) = s^*\mathcal{N}_{C_{\mathrm{comb}}/U}(-b_i-(b'_1+\dots+b'_{M'}))$$

with torsion cokernel. Since $h^1(B, s^*\mathcal{N}_{C_{\mathrm{comb}}/U}(-(b'_1+\dots+b'_{M'}))\otimes_{\mathcal{O}_B}\mathcal{M}^\vee)$ equals 0, and since every torsion sheaf has h^1 equal to 0, also $h^1(B, s^*\mathcal{N}_{C_{\mathrm{comb}}/U}(-b_i-(b'_1+\dots+b'_{M'})))$ equals 0 for every i. Therefore there exist a one-parameter family of deformations $(\mathcal{C}_t)_{t\in\Pi}$ of C_{comb} containing each of $x_1,\dots,x_M$, containing each of $s(b'_1),\dots,s(b'_{M'})$ and smoothing every node of C_{comb}, i.e., for t general, $\mathcal{C}_t$ is smooth.

Because π_U maps $s(B)$ to B with degree 1, also π_U maps $\mathcal{C}_t$ to B with degree 1. Because $\mathcal{C}_t$ is smooth, this means the projection $\mathcal{C}_t\to B$ is an isomorphism. Therefore there exists a section $\sigma_t: B\to U$ of π_U with image $\mathcal{C}_t$. In particular, $\sigma_t(b_i)=x_i$ for every $i=1,\dots,M$ and $\sigma_t(b'_i)=s(b'_i)$ for every $i=1,\dots,M'$. Because $h^1(C_{\mathrm{comb}},\mathcal{N}_{C_{\mathrm{comb}}/U}(-(x_1+\dots+x_N)))$ equals 0, by semicontinuity also $h^1(B,\sigma_t^*\mathcal{N}_{\sigma_t(B)/U}(-(x_1+\dots+x_N)))$ equals 0 for t general. In particular, if $N\geq a+g(B)$, then for every invertible sheaf $\mathcal{L}$ of degree $\leq a$, $\mathcal{L}^\vee(x_1+\dots+x_N)$ has degree $\geq g(B)$ and thus is effective, say $\mathcal{O}_B(\Delta)$. Therefore there exists an injective sheaf homomorphism

$$\sigma_t^*\mathcal{N}_{\sigma_t(B)/U}(-(x_1+\dots+x_N))\hookrightarrow\sigma_t^*\mathcal{N}_{\sigma_t(B)/U}(-(x_1+\dots+x_N)+\Delta)=\sigma_t^*\mathcal{N}_{\sigma_t(B)/U}\otimes_{\mathcal{O}_B}\mathcal{L}^\vee$$

with torsion cokernel. So, by the same type of argument as above, $h^1(B,\sigma_t^*\mathcal{N}_{\sigma_t(B)/U}\otimes_{\mathcal{O}_B}\mathcal{L}^\vee)$ equals 0 for every invertible sheaf $\mathcal{L}$ of degree $\leq a$.

Finally, applying the last result when $a=(k+1)(M+M')$ and $\mathcal{L}=\mathcal{O}_B((k+1)(b_1+\dots+b_M+b'_1+\dots+b'_{M'}))$, there exists a section $\sigma: B\to U$ of π_U as above and satisfying $h^1(B,\sigma^*\mathcal{N}_{\sigma(B)/U}(-(k+1)(b_1+\dots+b_M+b'_1+\dots+b'_{M'})))$ equals 0. Therefore, by Proposition 3.14 once more, for a general deformation of $\sigma(B)$ containing $x_1,\dots,x_M$ and $s(b'_1),\dots,s(b'_{M'})$, the k-jet of the curve at each point x_i and $s(b'_i)$ is a general curvilinear k-jet in U at that point. ☐

The main application is to the case when $U=\mathbb{P}^1\times_k Y$ where Y is a smooth, irreducible, quasi-projective k-scheme whose very free locus $Y_{\mathrm{v.f.}}$ is nonempty.

Corollary 3.20. *Every rational curve in Y intersecting $Y_{v.f.}$ is contained in $Y_{v.f.}$. For every integer k, for every integer a, for every collection of closed points $b_1,\dots,b_M$ of $\mathbb{P}^1$, for every collection of closed points $y_1,\dots,y_M$ of $Y_{v.f.}$, and for every choice of a curvilinear k-jet in Y at each point y_i, if each k-jet is general among curvilinear k-jets at y_i, then there exists a morphism*

$$f:(\mathbb{P}^1,b_1,\dots,b_M)\to(Y,y_1,\dots,y_M)$$

mapping the k-jet of $\mathbb{P}^1$ at b_i isomorphically onto the given k-jet at y_i and such that

$$f^*T_Y\cong\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\dots\oplus\mathcal{O}_{\mathbb{P}^1}(a_n),\ a_1,\dots,a_n\geq a.$$

Proof. Let $B=\mathbb{P}^1$, let $U=B\times_k Y$ and let π_B be the obvious projection. The sections of π_B are precisely the graphs of morphisms $f:\mathbb{P}^1\to Y$. In particular, if f is a morphism whose image intersects $Y_{\mathrm{v.f.}}$, then the section $s=(\mathrm{Id}_B,f)$ satisfies the hypotheses of Proposition 3.19. Thus, for every point $b'=b'_1$ of $\mathbb{P}^1$, there exists a section $\sigma=(\mathrm{Id}_{\mathbb{P}^1},\phi)$ with $\sigma(b')=s(b')$ and $h^1(B,\sigma^*\mathcal{N}_{\sigma(B)/U}(-2))$

equals 0. In other words, $\phi : \mathbb{P}^1_k \to Y$ is a morphism with $\phi(b') = f(b')$ and with $h^1(\mathbb{P}^1, \phi^* T_Y(-2))$ equals 0. Thus ϕ is a very free morphism whose image contains $f(b')$. Therefore every point in the image of f is contained in the very free locus, i.e., every rational curve in Y intersecting $Y_{\text{v.f.}}$ is contained in $Y_{\text{v.f.}}$.

The rest of the corollary is just a straightforward translation of Proposition 3.19 to this context. ☐

There is one more result in this direction which is useful. The proof is similar to the arguments above.

Lemma 3.21. [Kol96, Lemma II.7.10.1] *Let C_{comb} be a comb with handle C and teeth $L_1, \dots, L_n$. Let $\rho : \mathcal{C} \to \Pi$ be a one-parameter deformation of C_{comb} over a pointed curve $(\Pi, 0)$ whose general fiber $\mathcal{C}_t$ is smooth. Let $\mathcal{E}$ be a locally free sheaf on $\mathcal{C}$. If $\mathcal{E}|_{L_i}$ is ample for every i and if $h^1(C, (\mathcal{E}|_C) \otimes_{\mathcal{O}_C} \mathcal{M})$ equals 0 for every invertible $\mathcal{O}_C$-module $\mathcal{M}$ of degree $\geq n$, then $h^1(\mathcal{C}_t, \mathcal{E}|_{\mathcal{C}_t})$ equals 0 for general t in Π.*

3.3. **Ramification issues.** The argument sketched in Subsection 3.1 and the powerful smoothing combs technique from Subsection 3.2 form the core of the proof of Theorem 3.9. However there is a technical issue complicating matters. There may be codimension 1 points of X at which $\pi_X : X \to B$ is not smooth. In other words, finitely many scheme-theoretic fibers of π_X may have irreducible components occurring with multiplicity ≥ 1. This is a well-known issue when working with fibrations. Although there are sophisticated ways to deal with this (using log structures or Deligne-Mumford stacks), for the purposes of this proof it suffices to deal with this in a more naive manner.

Definition 3.22. Let B be a smooth k-curve, let X be a reduced, finite type k-scheme and let $\pi : X \to B$ be a flat morphism. The *good locus* of π is the maximal open subscheme U of X such that U is smooth and such that for every point b of B the reduced scheme of the fiber $\pi_X^{-1}(b) \cap U$ is smooth. Denote the restriction of π_X to U by π_U. The morphism π is *good* if the good locus equals all of X.

The *log divisor* of π is the Cartier divisor $D_{\pi,\log}$ of U given by

$$D_{\pi,\log} := \sum_{b \in B(k)} \pi_U^*(b) - \pi_U^*(b)_{\text{red}},$$

where $\pi_U^*(b)_{\text{red}}$ is the reduced Cartier divisor. Notice this is actually a finite sum over those closed points b of B for which $\pi_U^*(b)$ is nonreduced.

Lemma 3.23. *The complement of U in X has codimension ≥ 2. If char(k) equals 0, then the pullback map on relative differentials*

$$\pi_U^* : \pi_U^* \Omega_{B/k} \to \Omega_{U/k}$$

factors uniquely through the inclusion

$$\pi_U^* \Omega_{B/k} \hookrightarrow \pi_U^* \Omega_{B/k}(D_{\pi,log})$$

and the cokernel

$$\Omega_{\pi,log} := Coker(\pi_U^* \Omega_{B/k}(D_{\pi,log}) \to \Omega_{U/k})$$

is locally free.

Proof. To construct U, first remove the closure of the singular locus of the geometric generic fiber of π_X and next remove the singular locus from the reduced scheme of the finitely many singular fibers. Both of these sets have codimension 2 in X (the first by hypothesis).

The proof of the second part uses that $\mathrm{char}(k) = 0$. It can be checked formally locally near every closed point x of U. Denote by b the image $\pi_X(x)$ in B and denote by D the reduced structure on the irreducible component of $\pi^{-1}(b)$ containing x. Since x is in U, D is a smooth Cartier divisor in U. Let r be a defining equation for D in U and let t be a defining equation for b in B. Near x, $\pi^*(b) = mD+$ other terms. Thus, in $\widehat{\mathcal{O}}_{U,x}$,

$$\pi_X^* t = a_m r^m + a_{m+1} r^{m+1} + \dots$$

where a_m is a unit. Because $\mathrm{char}(k) = 0$, the power series

$$u = \sqrt[m]{a_m + a_{m+1} r + \dots}$$

is a well-defined unit in $\widehat{\mathcal{O}}_{U,x}$. Thus, after replacing r by ur, there exists a regular system of parameters $r, r_2, \dots, r_n$ for $\widehat{\mathcal{O}}_{U,x}$ such that π_X^* is given by

$$\pi_X^* t = r^m, \quad \pi_X^*(dt) = m r^{m-1} dr.$$

Thus π^* locally factors through $\pi_X^* \Omega_B((e-1)D) = \pi_X^* \Omega_B(D_{\pi,\log})$. Moreover it is clear that locally the cokernel is the locally free sheaf generated by $dr_2, \dots, dr_n$. □

The locally free quotient $\Omega_{\pi,\log}$ of Ω_π is called the sheaf of *log relative differentials*. Of course it equals the torsion-free quotient of Ω_π. But its true importance comes from the following lemma: given a base change $V \to B$ for which the normalized fiber product $\widetilde{U \times_B} V$ is smooth over V, the sheaf of relative differentials of $\widetilde{U \times_B V}/V$ equals the pullback of $\Omega_{\pi,\log}$. Thus the relative deformation theory of the $\widetilde{U \times_B} V$ over V is already captured by the sheaf $\Omega_{\pi,\log}$ on U. Before stating the lemma precisely, there is some setup.

Let

$$\pi : U \to B, \quad \varpi : V \to B$$

be two good morphisms with respective log divisors $D_{\pi,\log}$ and $E_{\varpi,\log}$. Let b be a closed point of B. Let D be a prime divisor of U in $\mathrm{Supp}(D_{\pi,\log}) \cap \pi^{-1}(b)$, and let E be a prime divisor of V in $\mathrm{Supp}(E_{\varpi,\log}) \cap \varpi^{-1}(b)$. Denote by $m_D - 1$, resp. $m_E - 1$, the coefficient of D in $D_{\pi,\log}$, resp. the coefficient of E in $E_{\varpi,\log}$. The *normalized fiber product* of U and V along D and E is the normalization $\widetilde{U \times_B} V$ of $U \times_B V$ along $D \times_{\{b\}} E$. Denote by

$$\mathrm{pr}_U : U \times_B V \to U, \quad \mathrm{pr}_V : U \times_B V \to V$$

the two projections, and denote by

$$\widetilde{\mathrm{pr}}_U : \widetilde{U \times_B} V \to U, \quad \widetilde{\mathrm{pr}}_V : \widetilde{U \times_B} V \to V$$

the compositions with the blowdown morphism. Denote by Exc the exceptional locus of the morphism, i.e.,

$$\mathrm{Exc} := (\widetilde{\mathrm{pr}}_U^{-1}(D) \cap \widetilde{\mathrm{pr}}_V^{-1}(E))_{\mathrm{reduced}}.$$

From this point forward we explicitly assume that $\mathrm{char}(k)$ equals 0.

Hypothesis 3.24. The algebraically closed ground field k has characteristic 0.

The sheaves Ω_π and $\Omega_{\pi,\log}$ agree on a dense open subset of U, namely $U - \mathrm{Supp}(D_{\pi,\log})$. Because $\widetilde{\mathrm{pr}}_V$ and pr_V are isomorphic over a dense open subset of V (namely $V - E$) also $\Omega_{\widetilde{\mathrm{pr}}_V}$ agrees with $\widetilde{\mathrm{pr}}_V^*\Omega_\pi$ on a dense open subset of $\widetilde{U \times_B V}$. Therefore also Ω_{ϖ} agrees with $\widetilde{\mathrm{pr}}_V^*\Omega_{\pi,\log}$ on a dense open subset of $\widetilde{U \times_B V}$.

Lemma 3.25. *The morphism*

$$\widetilde{pr}_V : \widetilde{U \times_B V} \to V$$

is smooth at every point of Exc if and only if m_D divides m_E. In this case the reduced normalization equals the blowing up of $U \times_B V$ along the closed subscheme $pr_U^{-1}(D) \times pr_V^{-1}((m_E/m_D)E)$ and Exc is contained in the maximal open neighborhood of $\widetilde{U \times_B V}$ on which $\Omega_{\widetilde{\pi}}$ agrees with $(\widetilde{pr_V})^*\Omega_{\pi,log}$.

Proof. This is proved in much the same way as the second part of Lemma 3.23. For every closed point x of U and y of V with common image point $b = \pi(x) = \varpi(y)$, there exist a regular system of parameters $(r, r_2, \dots, r_n)$ for $\widehat{\mathcal{O}}_{U,x}$, resp. $(s, s_2, \dots, s_p)$ for $\widehat{\mathcal{O}}_{V,y}$, and a regular parameter t for $\widehat{\mathcal{O}}_{B,b}$ such that

$$\pi^* t = r^{m_D} \text{ and } \varpi^* t = s^{m_E},$$

and thus,

$$\widehat{\mathcal{O}}_{U\times_B V,(x,y)} = k\,[\![r, r_2, \dots, r_n, s, s_2, \dots, s_p]\!]\,/\langle r^{m_D} - s^{m_E}\rangle.$$

Denoting by m the greatest common factor of m_D and m_E, the stalk of the normalization equals

$$k\,[\![u, r, r_2, \dots, r_n, s, s_2, \dots, s_p]\!]\,/\langle r - u^{m_E/m}, s - u^{m_D/m}\rangle.$$

Thus it is formally smooth as a $k\,[\![s, s_2, \dots, s_p]\!]$-algebra if and only if m_D/m equals 1, i.e., if and only if m_D divides m_E. In this case it is easy to see that the normalization is the blowing up at the ideal $\langle s, t^{m_E/m_D}\rangle$ and it is easy to see that the module of relative differentials is the free module generated by $dr_2, \dots, dr_n$, i.e., it is the pullback of $\Omega_{\pi,\log}$. $\square$

Definition 3.26. A *log preflexible curve* is a connected, smooth, proper curve $C \subset U$ such that

(i) C intersects the very free locus $U_{\pi,\text{v.f.}}$ of π_U,
(ii) $\pi_U(C)$ equals B,
(iii) and every intersection point of C with $\mathrm{supp}(D_{\pi,\log})$ is transverse, i.e., the tangent direction of C at the intersection point is not contained in the tangent space of $\mathrm{supp}(D_{\pi,\log})$.

A *linked log preflexible curve* is a B-morphism from a linked curve $j : C_{\text{link}} \to U$ such that the handle C is log preflexible and for every link L_i the image in B of L_i is disjoint from the image in B of $D_{\pi,\log}$.

A log preflexible curve C is a *log flexible curve* if

$$h^1(C, T_{\pi,\log}|_C) \text{ equals } 0, \text{ where } T_{\pi,\log} := \mathit{Hom}_{\mathcal{O}_U}(\Omega_{\pi,\log}, \mathcal{O}_U).$$

A linked log preflexible curve is a *linked log flexible curve* if

$$h^1(C_{\text{link}}, j^* T_{\pi,\log}) \text{ equals } 0.$$

Lemma 3.27. *There exists a log preflexible curve C. In fact, every intersection of X with $dim(X) - 1$ general hyperplanes is a log preflexible curve.*

Proof. Because $X - U$ has codimension 2 in X, a general complete intersection curve in X is disjoint from $X - U$, i.e., it is contained in U. By hypothesis, $U_{\pi,\text{v.f.}}$ is a dense open subset of U and thus a general complete intersection curve intersects this open. Finally, by Bertini's theorem a general complete intersection curve in U is smooth and intersects $\text{supp}(D_{\pi,\log})$ transversally. □

An important consequence of the smoothing combs technique is the following result.

Proposition 3.28. *There exists a log flexible curve in X. In fact, for every comb in X with log preflexible handle C and with sufficiently many very free teeth in fibers of π_U attached at general points of C and with general tangent directions, there exists a one-parameter deformation of the comb whose general member is a log flexible curve.*

Proof. By hypothesis, C intersects the very free locus $U_{\pi,\text{v.f.}}$ of the morphism π_U. By the same argument as in the proof of Proposition 3.18, $U_{\pi,\text{v.f.}}$ is open. Therefore all but finitely many points of C are contained in $U_{\pi,\text{v.f.}}$. By Proposition 3.18 applied to 1-jets, i.e., to tangent directions, for each such point c there exists a very free rational curve in $U_{\pi_U(c)}$ containing c and whose tangent direction at c is a general tangent direction in $U_{\pi_U(c)}$.

Let C_{comb} be a comb obtained by attaching to C a number of teeth $L_1, \dots, L_N$ as in the previous paragraph at general points of C (in particular, points where $C \to B$ is unramified) and with general tangent directions in $U_{\pi_U(c)}$. These tangent directions are the same as normal directions to C in U. By the same argument as in the proof of Proposition 3.19, if N is sufficiently large there is a one-parameter deformation

$$\mathcal{C} \subset \Pi \times_k U$$

of C_{comb} such that $\mathcal{C}_t$ is smooth for general t in Π. The properties (i), (ii) and (iii) of Definition 3.26 are all open properties and hold for $\mathcal{C}_0 = C_{\text{comb}}$, thus also hold for $\mathcal{C}_t$ so long as t is general.

For each tooth L_i in a fiber U_{b_i}, $T_{\pi,\log}|_{L_i}$ equals $T_{U_{b_i}}|_{L_i}$. Since L_i is very free, this is an ample locally free sheaf. Thus, by Lemma 3.21 with the pullback of $T_{\pi,\log}$ in the place of $\mathcal{E}$, the restriction $h^1(\mathcal{C}_t, T_{\pi,\log}|_{\mathcal{C}_t})$ equals 0 for t a general point of Π. Therefore, for t a general point of Π, $\mathcal{C}_t$ is a log flexible curve. □

Because the fibers of π are rationally connected, every log preflexible curve, resp. log flexible curve, extends to a linked log preflexible curve, resp. linked log flexible curve.

Lemma 3.29. *For every linked curve C_{link} such that each point $b_i = \pi_{C,link}(L_i)$ is disjoint from $\pi_U(D_{\pi,log})$, and for every B-morphism $j_0 : C \to X$ mapping C isomorphically to a log preflexible curve, resp. log flexible curve, and mapping each fiber C_{b_i} into the very free locus $U_{\pi,v.f.}$ of π_U, there exists a B-morphism $j : C_{link} \to X$ which is linked log preflexible, resp. linked log flexible, and restricting to j_C on C.*

Proof. Let L_i be a link of C_{link}. Let L_i intersect C in m points $t_1, \dots, t_m$ contained in the fiber over a general point b_i of B. Let $x_1, \dots, x_m$ be the images $j(t_1), \dots, j(t_m)$ in $U_{b_i,\text{v.f.}}$. By Corollary 3.20, there exists a morphism

$$j_i : (L_i, t_1, \dots, t_m) \to ((U_{b_i,\text{v.f.}}, x_1, \dots, x_m)$$

such that

$$j_i^* T_{U_{b_i}} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n) \text{ for integers } a_1, \dots, a_n \geq m-1.$$

Because of this,

$$h^1(L_i, j_i^* T_{U_{b_i}}(-(t_1 + \cdots + t_m))) \text{ equals } 0.$$

Define $j : C_{\text{link}} \to U$ to be the unique morphism such restricting to j_0 on C and restricting to j_i on each link L_i. Because $j_i(t_k) = j_0(t_k)$ for every link L_i and for every node t_k contained in L_i, this morphism is defined. It is clearly log preflexible.

Next assume that j_0 is log flexible. The claim is that j is also log flexible. To see this, consider the short exact sequence

$$0 \longrightarrow \oplus_i j_i^* T_{U_{b_i}}(-C_{b_i}) \longrightarrow j^* T_{\pi,\log} \longrightarrow j_0^* T_{\pi,\log} \longrightarrow 0.$$

By the hypothesis that j_0 is log flexible, the third term has vanishing h^1. And by the construction of j_i, $j_i^* T_{U_{b_i}}(-C_{b_i})$, i.e., $j_i^* T_{U_{b_i}}(-(t_1 + \cdots + t_m))$, has vanishing h^1. Thus, by the long exact sequence of cohomology, also $h^1(C_{\text{link}}, j^* T_{\pi,\log})$ equals 0. Therefore $j : C_{\text{comb}} \to U$ is a linked log flexible curve. □

3.4. **Existence of log deformations.** There is a definition of one-parameter deformation that takes the divisor $D_{\pi,\log}$ into account. Unfortunately, not every curve over B admits a log deformation specializing to a section curve, e.g., étale covers of B are rigid. However, after attaching a sufficient number of links, the linked curve does admit a log deformation specializing to a section curve.

Definition 3.30. Let $(C_{\text{link}}, \pi_{C,\text{link}})$ be a linked curve with handle C. Let $D_C \subset C$ be an effective, reduced, Cartier divisor contained in the smooth locus of C_{link}. A *one-parameter log deformation* of $(C_{\text{link}}, \pi_{C,\text{link}}, D_C)$ is a one-parameter deformation of $(C_{\text{link}}, \pi_{C,\text{link}})$,

$$(\rho, \pi_{\mathcal{C}}) : \mathcal{C} \to \Pi \times_k B$$

together with an effective Cartier divisor $D_{\mathcal{C}} \subset \mathcal{C}$ such that

(i) the pullback of $D_{\mathcal{C}}$ to $\mathcal{C}_0 = C_{\text{link}}$ equals D_C
(ii) and $\pi_{\mathcal{C}}(D_{\mathcal{C}})$ equals $\pi_C(D_C)$, i.e., $D_{\mathcal{C}}$ is vertical over B.

Lemma 3.31. *For every finite morphism of smooth, projective curves $\pi_C : C \to B$ and for every effective, reduced, Cartier divisor D_C of C, after attaching sufficiently many links to C over general points of B, there exists a one-parameter log deformation specializing to a section curve.*

Proof. For all sufficiently positive integers e, for a general morphism $g : C \to \mathbb{P}^1$ of degree e, the induced morphism $(\pi_C, g) : C \to B \times_k \mathbb{P}^1$ is unramified and is injective except for finitely many double points, none of which intersect the image of D_C. Denote by $\Sigma \to B \times_k \mathbb{P}^1$ the blowing up along the finitely many double points of $(\pi_C, g)(C)$. Then there is a B-morphism $h : C \to \Sigma$ which is an embedding.

For each point p of D_C, denote by m_p the multiplicity of p in the Cartier divisor $\pi_C^*(\pi_C(p))$. Denote by $\nu_p : \Sigma'_p \to \Sigma$ the m_p-fold iterated blowup of Σ first at p, then at the image of p in the strict transform of $h(C)$, etc. Denote by E_p the final exceptional divisor of this sequence of blowups. The point of this construction is that the strict transform of $h(C)$ intersects E_p at p, and E_p occurs with multiplicity m_p in the Cartier divisor $\Sigma'_p \times_B \{\pi_C(p)\}$. Denote by $\nu : \Sigma' \to \Sigma$ the fiber product over all points p in D_C of $\nu_p : \Sigma'_p \to \Sigma$. Denote by E the Cartier divisor in Σ' being the sum over all p of the pullback of E_p from Σ'_p. Denote by $\pi_{\Sigma'} : \Sigma' \to B$ the composition of $\Sigma' \to \Sigma \to B \times_k \mathbb{P}^1$ with pr_B. Denote by $h' : C \to \Sigma'$ the strict transform of $h(C)$. The point of this construction is that E is a Cartier divisor in Σ' which is vertical over B and such that h^*E equals D_C.

Denote by d the degree of π_C and let $t_1, \dots, t_d$ be closed points of $\mathbb{P}^1$ such that the Cartier divisor $B \times_k \{t_1, \dots, t_d\}$ of $B \times_k \mathbb{P}^1$ is disjoint from all double points of $(\pi_C, g)(C)$ and disjoint from $(\pi_C, g)(D)$. Denote by T the strict transform of $B \times_k \{t_1, \dots, t_d\}$ in Σ'. Form the invertible sheaf $\mathcal{O}_{\Sigma'}(h'(C) - T)$ and the pushforward $\mathcal{E} := \pi_{\Sigma',*}\mathcal{O}_{\Sigma'}(h'(C) - T)$ on B. Because $\pi_{\Sigma'}$ is flat and because $\mathcal{O}_{\Sigma'}(h'(C) - T)$ is locally free, $\mathcal{E}$ is torsion-free. For every point b $B - \pi_C(D_C)$, Σ'_b is isomorphic to $\mathbb{P}^1$ (via the projection $\Sigma' \to B \times_k \mathbb{P}^1 \to \mathbb{P}^1$). And $\Sigma'_b \cap h'(C)$ and $\Sigma'_b \cap T$ are divisors of the same degree d. Thus $\mathcal{O}_{\Sigma'}(h'(C) - T)|_{\Sigma'_b}$ is isomorphic to $\mathcal{O}_{\Sigma'_b} \cong \mathcal{O}_{\mathbb{P}^1}$. Therefore $\mathcal{E}|_b$ is isomorphic to $h^0(\Sigma'_b, \mathcal{O}_{\Sigma'_b})$, which is one-dimensional. Therefore $\mathcal{E}$ is an invertible sheaf.

By Riemann-Roch and Serre duality, for every sufficiently large degree, for a general effective divisor Δ on B of that degree, $\mathcal{E} \otimes_{\mathcal{O}_B} \mathcal{O}_B(\Delta)$ is globally generated. Choose Δ to be disjoint from $\pi_C(D_C)$ and from the image in B of the finitely many intersection points of $h'(C)$ and T. Since $\mathcal{E} \otimes_{\mathcal{O}_B} \mathcal{O}_B(\Delta)$ is globally generated, there exists a section which is nonzero at every point of Δ. Of course a nonzero section of this sheaf (up to scaling) is precisely the same thing as a divisor V on Σ' such that

$$h'(C) + \pi^*_{\Sigma'}\Delta \sim T + V.$$

For b in $B - \pi_C(D_C)$, if the section is nonzero at b then V does not intersect Σ'_b. The same does not necessarily hold for points b of $\pi_C(D_C)$ since b may lie in the support of $R^1\pi_{\Sigma',*}\mathcal{O}_{\Sigma'}(h'(C) - T)$. Therefore V is a sum of finitely many irreducible components of fibers of $\pi_{\Sigma'}$ (possibly with multiplicity) lying over points not in Δ.

The linked curve $(C_{\mathrm{link}}, \pi_{C,\mathrm{link}})$ is $h'(C) + \pi^*_{\Sigma'}\Delta$ together with the restriction of $\pi_{\Sigma'}$. Denote by Π the pencil of divisors in Σ' spanned by the divisors $h'(C) + \pi^*_{\Sigma'}\Delta$ and $T + V$, with these two divisors marked as 0 and ∞ respectively. Denote by $\mathcal{C} \subset \Pi \times_k \Sigma'$ the corresponding family of divisors. By Bertini's theorem, the general member $\mathcal{C}_t$ is smooth away from the base locus. Now the only singular points of $h'(C) + \pi^*_{\Sigma'}\Delta$ are the points $h'(\pi_C^{-1}(\Delta))$. Since V does not intersect $\pi^*_{\Sigma'}\Delta$, these singular points are not in the base locus. Since $\mathcal{C}_0$ is nonsingular at every basepoint, the same is true for $\mathcal{C}_t$ for t general. Thus a general member $\mathcal{C}_t$ is smooth everywhere.

Define $D_{\mathcal{C}}$ to be the pullback to $\mathcal{C}$ of the Cartier divisor E in Σ'. Because E is vertical over B and because h^*E equals D_C, the deformation $\mathcal{C}$ together with the effective Cartier divisor $D_{\mathcal{C}}$ is a one-parameter log deformation of $(C_{\mathrm{link}}, \pi_{C,\mathrm{link}}, D_C)$. And it specializes at $t = \infty$ to a union of section curves and vertical curves. □

3.5. **Completion of the proof.** We are finally prepared for the proof of Theorem 3.9.

Proof of Theorem 3.9. By Proposition 3.28, there exists a log flexible curve $j_0 : C \to U$. Denote by D_C the reduced scheme of the $C \cap D_{\pi,\log}$. By Lemma 3.31, after attaching finitely many links to C over the points of a general divisor Δ of B, the linked curve C_{link} together with D_C admits a one-parameter log deformation

$$(\rho, \pi_{\mathcal{C}}) : \mathcal{C} \to \Pi \times_k B, \quad D_{\mathcal{C}} \subset \mathcal{C}$$

of (C_{link}, D_C) specializing to a section curve (in fact $\mathcal{C}_\infty$ is a union of section curves and vertical curves).

By Proposition 3.18, the relative very free locus $U_{\pi,\text{v.f.}}$ is open in U. Thus $C \cap U_{\pi,\text{v.f.}}$ is open in C. So its complement is finitely many points in C. Thus a general divisor Δ is disjoint from the finite set $\pi_U(D_{\pi,\log})$ and from the finite set $\pi_C(C - C \cap U_{\pi,\text{v.f.}})$. Then, by Lemma 3.29, there exists an extension of j_0 to a linked log flexible curve

$$j : C_{\text{link}} \to U.$$

Form the fiber product

$$U_{\mathcal{C}} := \mathcal{C} \times_{\pi_{\mathcal{C}}, B, \pi_U} U.$$

Since π_U is flat, also the projection

$$\text{pr}_{\mathcal{C}} : U_{\mathcal{C}} \to \mathcal{C}$$

is flat. Since $\pi_{\mathcal{C}}$ is surjective, the geometric generic fiber of $\text{pr}_{\mathcal{C}}$ equals the geometric generic fiber of π_U, which is integral. Since $\text{pr}_{\mathcal{C}}$ is flat with integral geometric generic fiber, $U_{\mathcal{C}}$ is integral. Define

$$\nu : \widetilde{U}_{\mathcal{C}} \to U_{\mathcal{C}}$$

to be the blowing up of $U_{\mathcal{C}}$ along the closed subscheme $D_{\mathcal{C}} \times_B D_{\pi,\log}$. Since $U_{\mathcal{C}}$ is integral, also $\widetilde{U}_{\mathcal{C}}$ is integral. And the composition

$$\widetilde{U}_{\mathcal{C}} \to U_{\mathcal{C}} \to \mathcal{C} \to \Pi$$

is surjective. Since Π is a smooth curve, the morphism

$$\widetilde{\rho} : \widetilde{U}_{\mathcal{C}} \to \Pi$$

is flat.

Consider the graph,

$$\Gamma_j : C_{\text{link}} = \mathcal{C}_0 \to \mathcal{C}_0 \times_B U = U_{\mathcal{C},0}.$$

Because the links of C_{link} do not intersect $D_{\pi,\log}$, the image of Γ_j is smooth at every point of intersection with $D_C \times_B D_{\pi,\log}$. Since ν is birational, Γ_j gives a rational transformation from C_{link} to $\widetilde{U}_{\mathcal{C},0}$. Since ν is proper, and since C_{link} is smooth at every point of intersection with $D_C \times_B D_{\pi,\log}$, the valuative criterion of properness implies this rational transformation is actually a regular morphism

$$\widetilde{\Gamma}_j : \mathcal{C}_0 \to \widetilde{U}_{\mathcal{C},0}.$$

Clearly this is a section of the projection morphism

$$\text{pr}_{\mathcal{C}_0} : \widetilde{U}_{\mathcal{C},0} \to \mathcal{C}_0.$$

For every point t in $C_{\text{link}} - D_C$, the morphism $\pi_U : U \to B$ is smooth at $j(t)$. Therefore also $U_{\mathcal{C},0} \to \mathcal{C}_0$ is smooth at $\Gamma_j(t)$. And since ν is an isomorphism over

$\Gamma_j(t)$, also $\mathrm{pr}_{\mathcal{C}_0} : \widetilde{U}_{\mathcal{C},0} \to \mathcal{C}_0$ is smooth at $\widetilde{\Gamma}_j(t)$. Also the vertical tangent bundle equals the pullback of the vertical tangent bundle of $\pi_U : U \to B$, which also equals $T_{\pi,\log}$ (since $j(t)$ is not in $D_{\pi,\log}$).

Let t be a point of D_C and let D_t be the unique irreducible component of $D_{\pi,\log}$ containing $j(t)$. Give D_t the reduced structure. Because $j_0(C)$ is transverse to D_t at $j_0(t)$, the ramification index $m_C - 1$ of $\pi_C : C \to B$ at t equals the ramification index $m_D - 1$ of π_U along D_t. Therefore, by Lemma 3.25, the projection

$$\mathrm{pr}_{\mathcal{C}_0} : \widetilde{U}_{\mathcal{C},0} \to \mathcal{C}_0$$

is smooth over the preimage of $\{t\} \times D_t$ for every t and the vertical tangent bundle equals the pullback of $T_{\pi,\log}$. Since $\Gamma_j(t)$ is in $\{t\} \times D_t$, this implies that $\mathrm{pr}_{\mathcal{C}_0}$ is smooth at every point of the image of $\widetilde{\Gamma}_j$ and the vertical tangent bundle of $\mathrm{pr}_{\mathcal{C}_0}$ equals the pullback of $T_{\pi,\log}$.

Since $\widetilde{\Gamma}_j$ is a section with image in the smooth locus of $\mathrm{pr}_{\mathcal{C}_0}$, the normal sheaf $\mathcal{N}$ equals the restriction of the vertical tangent bundle. Therefore $\widetilde{\Gamma}_j^*\mathcal{N}$ equals $j^*T_{\pi,\log}$. Since $j : C_{\mathrm{link}} \to U$ is log flexible, $h^1(C_{\mathrm{link}}, j^*T_{\pi,\log})$ equals 0. Therefore, by Proposition 3.13, the relative Hilbert scheme $\mathrm{Hilb}(\widetilde{U}_{\mathcal{C}}/\Pi)$ is smooth over Π at the point $0' := [\mathrm{Image}(\widetilde{\Gamma}_j)]$. Thus for a general complete intersection curve Π' containing $0'$, the morphism $\Pi' \to \Pi$ is smooth at $0'$.

Replace Π' by the unique irreducible component containing $0'$, and then replace this by its normalization. The result is that Π' is a smooth, projective, connected curve together with a morphism $\Pi' \to \mathrm{Hilb}(\widetilde{U}_{\mathcal{C}}/\Pi)$ so that the induced morphism $\Pi' \to \Pi$ is smooth at $0'$. In particular it is flat, so surjective. Let ∞' denote a closed point of Π' mapping to ∞. Then $(\Pi', 0', \infty') \to (\Pi, 0, \infty)$ is a flat morphism of 2-pointed smooth curves. Thus, by Lemma 3.11, the base change $\Pi' \times_\Pi \mathcal{C}$ is a one-parameter deformation of C_{link} over $(\Pi', 0', \infty')$ specializing to a section curve.

Denote by

$$Z \subset \Pi' \times_\Pi \widetilde{U}_{\mathcal{C}}$$

the pullback of the universal closed subscheme $\mathrm{Univ}(\widetilde{U}_{\mathcal{C}}/\Pi)$ by the morphism $\Pi' \to \mathrm{Hilb}(\widetilde{U}_{\mathcal{C}}/\Pi)$. The composition with $\mathrm{pr}_{\mathcal{C}}$ is a projective morphism

$$Z \subset \Pi' \times_\Pi \widetilde{U}_{\mathcal{C}} \to \Pi' \times_\Pi \mathcal{C}$$

of flat Π'-schemes. Moreover, the fiber over $0' \in \Pi'$ is an isomorphism since the projection $\widetilde{\Gamma}_j(C_{\mathrm{flex}}) \to C_{\mathrm{flex}}$ is an isomorphism. Therefore the morphism is an isomorphism over $N \times_\Pi \mathcal{C}$ for some open neighborhood N of $0'$ in Π'. (This is well-known; a complete proof is given in [dJS03, Lemma 4.7].) Invert this isomorphism and compose it with the morphism

$$\Pi' \times_\Pi \widetilde{U}_{\mathcal{C}} \to \widetilde{U}_{\mathcal{C}} \to U_{\mathcal{C}} \to U.$$

The result is precisely an extension

$$j_N : N \times_\Pi \mathcal{C} \to X$$

of j for the one-parameter deformation $\Pi' \times_\Pi \mathcal{C}$. Therefore, by Lemma 3.12, there exists a section $s : B \to X$ of π_X. $\square$

3.6. **Corollaries.** There are a number of consequences of Theorem 3.9 and its generalization to positive characteristic in [dJS03]. Many of these consequences were recognized before Conjecture 3.8 was proved.

Corollary 3.32. [Kol96, Conjecture IV.5.6] *Conjecture 3.7 is true. Moreover, for every smooth, projective, irreducible variety X over an algebraically closed field of characteristic 0, there exists a dense open $X^0 \subset X$ and a projective, smooth morphism $q_0 : X^0 \to Q^0$ such that every fiber of q_0 is rationally connected, and every projective closure of Q^0 is nonuniruled.*

Corollary 3.33. [GHS03, Corollary 1.7] *The uniruledness conjecture implies Mumford's conjecture. To be precise, assume that for every smooth, projective, irreducible variety X over an algebraically closed field k of characteristic 0, if X is nonuniruled then $h^0(X, \omega_X^{\otimes n})$ is nonzero for some $n > 0$. Then for every smooth, projective, irreducible variety X over k, if X is not rationally connected then $h^0(X, \Omega_X^{\otimes n})$ is nonzero for some $n > 0$.*

The next corollary is a fixed point theorem. In characteristic 0 it can be proved using the Atiyah-Bott fixed point theorem. But in positive characteristic it is a new result. There are examples due to Shioda proving one cannot replace "separably rationally connected" by "rationally connected", cf. [Shi74].

Corollary 3.34. [Kol03] *Let Y be a smooth, projective, separably rationally connected variety over a field k and let $f : Y \to Y$ be a k-automorphism. If char(k) is positive, say p, assume in addition that f has finite order n not divisible by p^2. Then the fixed locus of f is nonempty.*

Of course it suffices to prove the case when k is algebraically closed, since the fixed locus of the base change equals the base change of the fixed locus. First assume f has finite order n. If n is prime to char(k), let B' denote $\mathbb{P}^1$ and let $\mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{P}^1$ by multiplication by a primitive n^{th} root of unity. Note this action fixes ∞ and has trivial generic stabilizer. If char$(k) = p$ is positive and if $n = pm$ where m is prime to p, let B' be the normal, projective completion of the affine curve

$$\mathbb{V}(y^m - (x^p - x)) \subset \mathbb{A}^2_k.$$

Let ζ be a primitive m^{th} root of unity, and let a generator of $\mathbb{Z}/m\mathbb{Z}$ act by $(x, y) \mapsto (x, \zeta y)$. Similarly, let a generator of $\mathbb{Z}/p\mathbb{Z}$ act by $(x, y) \mapsto (x + 1, y)$. Clearly these actions commute, thus define an action of $\mathbb{Z}/n\mathbb{Z}$ on B'. Note this action fixes the unique point ∞ not in the affine chart above, and the action has trivial generic stabilizer.

Let $\mathbb{Z}/n\mathbb{Z}$ act diagonally on $Y \times_k B'$, and let X be the quotient. Also let B the quotient of the $\mathbb{Z}/n\mathbb{Z}$-action on B'. The projection $\pi : X \to B$ satisfies the hypotheses of Theorem 3.9 (or its generalization in [dJS03]). Therefore there exists a section. This is the same as $\mathbb{Z}/n\mathbb{Z}$-equivariant k-morphism $f : B' \to Y$. In particular, since ∞ is a fixed point in B', $f(\infty)$ is a fixed point in Y.

Next assume k has characteristic 0. By general limit arguments there exists an integral, finitely generated $\mathbb{Z}$-algebra R, a ring homomorphism $R \hookrightarrow k$, a smooth, projective morphism $Y_R \to \text{Spec } R$ whose relative very free locus is all of Y_R, and an R-automorphism $f_R : Y_R \to XYR$ such that the base change $Y_R \otimes_R k$ equals Y and the base change of f_R equals f. Since the intersection of the graph of f_R and the diagonal of $Y_R \times_R Y_R$ is a proper scheme over Spec R, the image in Spec R is a closed subscheme of Spec R. To prove this closed subscheme equals all of Spec R, and thus contains the image of Spec k, it suffices to prove it contains a Zariski dense set of closed points.

Choose an f-invariant very ample sheaf, choose a basis for the space of global sections, and let A be the $N \times N$ matrix with entries in R giving the action of f on global sections with respect to this basis. The set of maximal ideal in Spec R with residue field of characteristic $p > N$ are Zariski dense in Spec R. Every invertible matrix over a characteristic p field with order divisible by p^2 has a Jordan block with eigenvalue 1 and size divisible by p. Thus, since $p > N$, the finite order of f_R modulo the prime is not divisible by p^2. Therefore, by the previous case, the reduction of f_R modulo the prime has nonempty fixed locus. Therefore the original automorphism f has nonempty fixed locus.

Corollary 3.34 is particularly interesting in combination with a result which is attributed to Campana.

Proposition 3.35 (Campana)**.** *Let Y be a quasi-projective, rationally chain connected variety over an algebraically closed field k. The algebraic fundamental group of Y is finite. Moreover, if $k = \mathbb{C}$, then the topological fundamental group of Y is finite.*

The basic point is this: there exists a choice of $u : M \times_k C_n \to Y$ such that the induced morphism $u^{(2)} : M \times_k C_n \times_k C_n \to Y \times_k Y$ is generically finite. For a general point y of Y, the fiber over $\{y\} \times Y$ has an irreducible component N which is generically finite over Y. Thus there exists a morphism

$$v : N \times_k C_n \to Y$$

and two points $0, \infty \in C_n$ such that

$$v_0 : N = N \times \{0\} \to Y$$

the constant map with image $\{Y\}$ and

$$v_\infty : N = N \times \{\infty\} \to Y$$

is dominant and generically finite. Let $w : \widetilde{Y} \to Y$ be a connected, unramified, cover in either the algebraic category or in the complex analytic category. Let $\widetilde{y}$ be a point over y. The morphism v_0 lifts to a constant morphism $\widetilde{v}_0$. Since C_n is simply connected, this implies that there is a unique morphism

$$\widetilde{v} : N \times_k C_n \to \widetilde{Y}$$

whose restriction to $N \times \{0\}$ equals $\widetilde{v}_0$ and which is compatible with v. In particular, v factors through w. Therefore the degree of w is bounded by the degree of v so that Y has fundamental group bounded by the degree of v.

Corollary 3.36. [Cam91] [Kol03] *Let X be a smooth, projective, separably rationally connected variety over an algebraically closed field k. The algebraic fundamental group of X is trivial. If $k = \mathbb{C}$, then the topological fundamental group of X is also trivial.*

This was proved over $\mathbb{C}$ by Campana. The general case was proved by Kollár. By Proposition 3.35, the universal cover $\widetilde{X} \to X$ is finite. Moreover, by the proof of the proposition, $\widetilde{X}$ is also smooth, projective and separably rationally connected. If the fundamental group of X is nonzero, then it contains a cyclic subgroup $\mathbb{Z}/n\mathbb{Z}$ such that p^2 does not divide n. Of course the action of this group on $\widetilde{X}$ is fixed-point-free. But Corollary 3.34 implies there exists a fixed point. Thus X is simply connected. Kollár has generalized this considerably to prove a result for open subschemes of rationally connected varieties.

Theorem 3.9 also plays an important role in the proof of a "converse" to Theorem 3.9.

Theorem 3.37. [GHMS05] *Let $\pi : X \to B$ be a surjective morphism of normal, projective, irreducible varieties over an algebraically closed field k of characteristic zero. Assume that for some sufficiently large, algebraically closed field extension K/k, for every k-morphism $C \to B$ from a smooth, projective, K-curve to X, the pullback $\pi_C : C \times_B X \to C$ has a section. Then there exists a closed subvariety $Y \subset X$ such that the geometric generic fiber of $\pi|_Y : Y \to B$ is nonempty, irreducible and rationally connected.*

One corollary of this theorem, in fact the motivation for proving it, was to answer a question first asked by Serre and left unresolved by Theorem 3.9: could it be that a smooth, projective variety X over the function field of a curve has a rational point if it is *$\mathcal{O}$-acyclic*, i.e., if $h^i(X, \mathcal{O}_X)$ equals 0 for all $i > 0$? One reason to ask this is that the corresponding question has a positive answer if "function field" is replaced by "finite field" (thanks to N. Katz's positive characteristic analogue of the Atiyah-Bott fixed point theorem).

Corollary 3.38. [GHMS05] *There exists a surjective morphism $\pi : X \to B$ of smooth, projective varieties over $\mathbb{C}$ such that B is a curve and the geometric generic fiber of π is an Enriques surface, but π has no section. Thus, to guarantee a fibration over a curve has a section, it is not sufficient to assume the geometric generic fiber is $\mathcal{O}$-acyclic.*

In fact G. Lafon found an *explicit* morphism π as in Corollary 3.38 where B is $\mathbb{P}^1_{\mathbb{C}}$, or in fact $\mathbb{P}^1_k$ for any field k with $\text{char}(k) \neq 2$, and there does not even exist a power series section near $0 \in \mathbb{P}^1_k$, cf. [Laf04].

4. The Period-Index theorem

Theorem 3.9 is a generalization of Tsen's theorem, Corollary 2.15, because a sufficiently general complete intersection $\mathbb{V}(F_1, \dots, F_r) \subset \mathbb{P}^n$ with $d_1 + \cdots + d_r \leq n$ is smooth, projective and separably rationally connected (the proof of this is nontrivial, as is the specialization argument reducing Tsen's theorem to the case of complete intersections which are sufficiently general). Is there a similar generalization of the Tsen-Lang theorem?

Joint work with Harris, [HS05], proves the spaces of rational curves on general low degree hypersurfaces are rationally connected. This was later generalized in joint work with A. J. de Jong: complete intersections $X = \mathbb{V}(F_1, \dots, F_r) \subset \mathbb{P}^n$ with $d_1^2 + \cdots + d_r^2 \leq n+1$ are rationally simply connected in the sense that the space of "good" rational curves in X containing two fixed, general points is itself a rationally connected variety. This is analogous to simple connectedness in topology: a path connected topological space is simply connected if the space of paths connecting two fixed points is itself path connected.

Moreover, de Jong gave a heuristic argument suggesting that for a rationally simply connected fibration over a surface, the only obstruction to existence of a rational section is the elementary obstruction. Given a geometrically integral scheme X defined over a field K, the *elementary obstruction* to existence of a K-point is the existence of a $\mathrm{Gal}(\overline{K}/K)$-invariant splitting of the homomorphism of Abelian Galois modules,

$$\overline{K}^* \hookrightarrow \mathrm{Frac}(X \otimes_K \overline{K})^*,$$

where $\overline{K}$ is the separable closure of K and Frac is the function field. If there exists a K-point of X, evaluation at this point gives a Galois-invariant splitting. The elementary obstruction was introduced by Colliot-Thélène and Sansuc, [CTS87]. Its vanishing implies the vanishing of other known obstructions. In particular, it implies the vanishing of a Brauer obstruction

$$\delta : \mathrm{Pic}(X \otimes_K \overline{K})^{\mathrm{Gal}(\overline{K}/K)} \to \mathrm{Br}(K)$$

measuring whether or not a Galois-invariant invertible sheaf $\overline{\mathcal{L}}$ on $X \otimes_K \overline{K}$ is the pullback of an invertible sheaf $\mathcal{L}$ on X.

At the moment, in order to give a rigorous proof, de Jong's heuristic argument requires several additional hypotheses on the rationally simply connected fibration. One case where the hypotheses hold is when all fibers of the fibration are Grassmannian varieties. Although this is very special, it is also quite interesting since it gives a second proof of de Jong's *period-index theorem.*

Theorem 4.1. [dJS05] *Let K be the function field of a surface over an algebraically closed field k. Let $(X, \mathcal{L})$ be a pair of a K-scheme and an invertible $\mathcal{O}_X$-module $\mathcal{L}$. If $(X \otimes_K \overline{K}, \mathcal{L} \otimes_K \overline{K})$ is isomorphic to $(Grass(r, \overline{K}^n), \mathcal{O}(1))$, then X has a K-point.*

Corollary 4.2 (de Jong's Period-Index theorem)**.** [dJ04] *Let A be a central simple K-algebra with $A \otimes_K \overline{K} \cong Mat_{n\times n}(\overline{K})$. Let $r < n$ be an integer such that $r[A]$ equals 0 in $Br(K)$. Then there exists a left ideal $I \subset A$ such that $dim_K(I) = rn$. In particular, if $A = D$ is a division algebra then $[D]$ has order n in $Br(K)$, i.e., the period of D equals the index of D.*

Corollary 4.2 follows from Theorem 4.1 by setting X to be the *generalized Brauer-Severi variety* parameterizing left ideals in A of rank rn. Since $A \otimes_K \overline{K}$ equals $\mathrm{Mat}_{n\times n}(\overline{K})$, $X \otimes_K \overline{K}$ equals $\mathrm{Grass}(r, \overline{K}^n)$. The Brauer obstruction to the existence of an invertible sheaf $\mathcal{L}$ with $\mathcal{L} \otimes_K \overline{K} \cong \mathcal{O}(1)$ is precisely the element $r[A]$ in $\mathrm{Br}(K)$.

The first reduction is "discriminant avoidance", i.e., reduction to the case that the variety X is the generic fiber of a smooth, projective morphism over a smooth, projective surface. Let T be a quasi-compact, integral scheme and let $\mathcal{G}$ be a smooth, affine group scheme over T whose geometric fibers are reductive.

Lemma 4.3. *For every integer c there exists a datum $(U, \overline{U}, \mathcal{T}_U)$ of a projective, flat T-scheme $\overline{U}$ with integral geometric fibers, an open subset U of $\overline{U}$ and a $\mathcal{G}$-torsor $\mathcal{T}_U$ over U such that*

(i) *U is smooth over T,*

(ii) *the complement* $\overline{U} - U$ *has codimension* $\geq c$ *in* $\overline{U}$,

(iii) *and for every* $\mathcal{G}$*-torsor* $\mathcal{T}_k$ *over an infinite field* K *over* T*, there exists a* T*-morphism* i : *Spec* $K \to U$ *and an isomorphism of* $\mathcal{G}$*-torsors over* K, $i^*\mathcal{T}_U \cong \mathcal{T}_K$.

The idea is to form the GIT quotient $\overline{U}$ of a linear action of $\mathcal{G}$ on $(\mathbb{P}^N_T, \mathcal{O}(1))$. If the linear representation is "sufficiently large", then $\mathcal{G}$ acts properly and freely on an open subset V of $\mathbb{P}^N_T$ of codimension $\geq c$. Take U to be the quotient of V. Then U is smooth over T and $\overline{U} - U$ has codimension $\geq c$. Finally, for every field K over $\mathcal{O}_T$ and for every $\mathcal{G}$-torsor $\mathcal{T}_K$ over K, the twist $\mathbb{P}^N_T \times_T \mathcal{T}_K/\mathcal{G}$ is isomorphic to $\mathbb{P}^N_K$. Thus there exists a K-point. If K is infinite, then the set of K-points is Zariski dense so that there exists a point in the image of V. This point is only well-defined up to the action of $\mathcal{G}$, but the associated morphism i : Spec $K \to U$ is well-defined. Chasing definitions, $i^*\mathcal{T}_U$ is isomorphic to $\mathcal{T}_K$.

Proposition 4.4. *Let* k *be an algebraically closed field of characteristic* 0*, let* S *be a smooth, projective surface over* k*, let* $\pi : \mathcal{X} \to S$ *be a smooth, projective morphism, and let* $\mathcal{L}$ *be an invertible* $\mathcal{O}_{\mathcal{X}}$*-module. Let* K/k *be the fraction field of* S *and let* X *be the generic fiber of* π.

(i) *If* $(X \otimes_K \overline{K}, \mathcal{L} \otimes_K \overline{K})$ *is isomorphic to* $(Grass(r, \overline{K}^n), \mathcal{O}(1))$*, then* X *has a* K*-point.*

(ii) *Item (i) implies Theorem 4.1*

The point of this proposition is Item (ii), i.e., to prove Theorem 4.1 it suffices to assume that X is the generic fiber of a proper and *everywhere smooth* morphism. The point is that for every algebraically closed field k, there exists the spectrum of a DVR, T, whose residue field is k and whose fraction field has characteristic 0. Now let $\mathcal{G}$ be the automorphism group scheme of $(\text{Grass}(r, \mathcal{O}^n_T), \mathcal{O}(1))$. This satisfies the hypotheses of Lemma 4.3. Taking $c = 3$, there exists a datum $(U, \overline{U}, \mathcal{T}_U)$ as in Lemma 4.3 such that $\overline{U} - U$ has codimension ≥ 3. For the original field K/k, there exists a morphism i : Spec $K \to U$ inducing the pair $(X, \mathcal{L})$. Because tr.deg.$(K/k) = 2$, the closure of Image(i) in $\overline{U}$ has dimension ≤ 2. Because $\overline{U} - U$ has codimension ≥ 3, there exists a locally closed subscheme $\mathcal{S}$ of $\overline{U}$ such that

(i) $\mathcal{S} \to T$ is flat,

(ii) the closed fiber S_0 of $\mathcal{S}$ is irreducible with generic point $i(\text{Spec } K)$,

(iii) and the generic fiber $\mathcal{S}_\eta$ of $\mathcal{S}$ is a closed subscheme of $\overline{U}_\eta$ completely contained in U_η.

Using specialization arguments, to prove Theorem 4.1 for the restriction of $\mathcal{T}_U$ to the generic point of the closed fiber S_0, it suffices to prove Theorem 4.1 for the restriction of $\mathcal{T}_U$ to the generic point of the geometric generic fiber $S_{\overline{\eta}}$. By construction, this satisfies the additional hypotheses in Proposition 4.4.

Thus, assume the additional hypotheses of Proposition 4.4 are satisfies. After replacing S by the blowing up at the base locus of a Lefschetz pencil of divisors, and replacing $\mathcal{X}$ by its base change, assume there exists a flat, proper morphism $\rho : S \to B$ with smooth, connected generic fiber. Denote by B^0 the maximal open subscheme of B over which ρ is smooth, and denote $S^0 = B^0 \times_B S$ and $\mathcal{X}^0 = B^0 \times_B \mathcal{X}$. There is a pair

$$(\rho_{\text{Sect}} : \text{Section}(\mathcal{X}^0/S^0/B^0) \to B^0, \sigma : \text{Section}(\mathcal{X}^0/S^0/B^0) \times_{B^0} S^0 \to \mathcal{X}^0)$$

which is universal among all pairs (T, σ_T) of a B^0-scheme T and an S_0-morphism $\sigma_T : T \times_{B^0} S^0 \to \mathcal{X}^0$. The universal pair can be constructed in terms of the relative Hilbert scheme. In Grothendieck's terminology, it is $\Pi_{S^0/B^0}\mathcal{X}_0$, cf. [Gro62, p. 195-13].

There is a relative Picard scheme $\text{Pic}(S^0/B^0)$ of S^0 over B^0. Associated to the invertible sheaf $\mathcal{L}$ on $\mathcal{X}^0$, there is an invertible sheaf $\sigma^*\mathcal{L}$ on $\text{Section}(\mathcal{X}^0/S^0/B^0)\times_{B^0} S^0$. This induces an *Abel morphism*

$$\alpha : \text{Section}(\mathcal{X}^0/S^0/B^0) \to \text{Pic}(S^0/B^0).$$

Of course $\text{Pic}(S^0/B^0)$ breaks up according the relative degree of the line bundle,

$$\text{Pic}(S^0/B^0) = \sqcup_{d\in\mathbb{Z}}\text{Pic}^d(S^0/B^0).$$

Pulling this back via the Abel morphism gives a decomposition

$$\text{Section}(\mathcal{X}^0/S^0/B^0) = \sqcup_{d\in\mathbb{Z}}\text{Section}^d(\mathcal{X}^0/S^0/B^0)$$

together with Abel morphisms

$$\alpha_d : \text{Section}^d(\mathcal{X}^0/S^0/B^0) \to \text{Pic}^d(S^0/B^0).$$

For all $d \geq 0$, there are sections of the projection $\text{Pic}^d(S^0/B^0) \to B^0$. The image of this section is a curve B_0' isomorphic to the smooth curve B^0. If the generic fiber of α_d is a dense open subset of a rationally connected variety, then Theorem 3.9 together with the generic version of weak approximation, Proposition 3.19, implies there exists a rational section of the restriction of α_d over B_0' (this uses a slight specialization argument, because the restriction of α_d may not be a rationally connected fibration). Thus it suffices to prove that for $d \gg 0$,

(i) the fiber of α_d over the geometric generic point of $\text{Pic}^d(S^0/B^0)$ is nonempty,
(ii) the fiber is also irreducible,
(iii) the fiber is also isomorphic to an open subset of a rationally connected variety.

Moreover, and this will be important, it suffices to prove there exists a canonically defined open subscheme $W \subset \text{Section}^d(\mathcal{X}^0/S^0/B^0)$ such that (i)–(iii) hold for $\alpha_d|_W$.

Note that (i)–(iii) are really statements about the morphism $\mathcal{X}_{\overline{\eta_B}} \to S_{\overline{\eta_B}}$ of fibers over the geometric generic point of B. Thus, it is again a question about a fibration over a projective curve, namely the curve $C = S_{\overline{\eta_B}}$ over the algebraically closed field $\kappa = \overline{k(B)}$. So Proposition 4.4 (i) follows from.

Proposition 4.5. *Let κ be an algebraically closed field of characteristic 0. Let C be a smooth, projective, connected curve over κ. Let $\pi : \mathcal{X}_C \to C$ be a smooth, projective morphism and let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}_C$. Assume the fiber of $(\mathcal{X}_C, \mathcal{L})$ over the geometric generic point of C is isomorphic to $(\text{Grass}(r, \overline{\kappa(C)}), \mathcal{O}(1))$. Then for $d \gg 0$ there exists a canonically defined open subset $W_d \subset \text{Section}^d(\mathcal{X}_C/C/\text{Spec } \kappa)$ such that the geometric generic fiber of*

$$\alpha_d : W_d \hookrightarrow \text{Section}^d(\mathcal{X}_C/C/\text{Spec } \kappa) \to \text{Pic}^d(C/\text{Spec } \kappa)$$

satisfies (i), (ii) and (iii) above.

In order to prove Corollary 4.2, it suffices to prove this in the special case that $\mathcal{X}_C \to C$ is the parameter scheme for rank rn left ideals in an Azumaya algebra $\mathcal{A}$ over C with $\mathcal{A} \otimes_{\mathcal{O}_C} \overline{\kappa(C)} \cong \text{Mat}_{n\times n}(\overline{\kappa(C)})$. An *Azumaya algebra* over a scheme T is a coherent $\mathcal{O}_T$-algebra which is étale locally isomorphic to $\text{Mat}_{n\times n}(\mathcal{O}_T)$ for some integer n.

Because of Tsen's theorem, Corollary 2.15, and Proposition 2.16(i), there exists a locally free $\mathcal{O}_C$-module $\mathcal{E}$ of rank n such that $\mathcal{A} \cong \mathit{End}(\mathcal{E})$. The locally free sheaf $\mathcal{E}$ is only well-defined up to the operation $\mathcal{E} \mapsto \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{N}$, for any invertible sheaf $\mathcal{N}$. The choice of a locally free sheaf $\mathcal{E}$ and an isomorphism of algebras gives an isomorphism of $\text{Sect}(\mathcal{X}_C/C/\text{Spec }\kappa)$ with the parameter scheme of locally free quotients $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ of rank r. To see this, associate to each quotient the left ideal of endomorphisms that factor as

$$\mathcal{E} \twoheadrightarrow \mathcal{Q} \xrightarrow{\phi} \mathcal{E}$$

as ϕ varies over all $\mathcal{O}_C$-module homomorphisms. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{N}$ gives a new isomorphism sending the original quotient to the twist $\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{N} \twoheadrightarrow \mathcal{Q} \otimes_{\mathcal{O}_C} \mathcal{N}$. The real effect of this change has to do with the Abel map. Up to a constant translation by a point of $\text{Pic}(C/\text{Spec }\kappa)$, which does not change (i)–(iii), the Abel map is identified with the map sending a quotient to $\det(\mathcal{Q})$ in $\text{Pic}(C/\text{Spec }\kappa)$. After replacing $\mathcal{E}$ by $\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{N}$, the constant changes by adding $[\mathcal{N}^{\otimes r}]$. Thus, it is clear that the Abel map really should only be considered as well-defined up to an additive constant.

If d is sufficiently large, there exist quotients such that $\mathcal{Q}$ is stable. Note that $\mathcal{Q}$ is stable if and only if $\mathcal{Q} \otimes_{\mathcal{O}_C} \mathcal{N}$ is stable. Therefore the open subset W_d of $\text{Section}(\mathcal{X}_C/C/\text{Spec }\kappa)$ parameterizing stable quotients is well-defined and canonical. Fix an integer d_0. For every integer e, the moduli space of stable, rank r locally free sheaves on C of degree d_0 is isomorphic to the moduli space for degree $d_0 + re$ via the map sending $\mathcal{Q}$ to $\mathcal{Q}(D)$, where D is any fixed Cartier divisor of degree e. For each fixed locally free sheaf $\mathcal{Q}$ of rank $r < n$, if e is sufficiently large, there exists a surjection $\mathcal{E} \twoheadrightarrow \mathcal{Q}(D)$ for all Cartier divisors of degree e. Because the moduli space of stable bundles is quasi-compact, it follows that there exists a single integer e_0 such that for every $e \geq e_0$ and every stable locally free sheaf $\mathcal{Q}$ of rank r and degree d_0, for every Cartier divisor D of degree e there exists a surjection $\mathcal{E} \twoheadrightarrow \mathcal{Q}(D)$. By the same sort of argument, if e is sufficiently large then $h^1(C, \mathit{Hom}_{\mathcal{O}_C}(\mathcal{E}, \mathcal{Q}(D)))$ equals 0.

Repeating this argument with d_0 replaced by each of $d_0, d_0+1, d_0+2, \dots, d_0+r-1$, there exists an integer d_1 such that for every $d \geq d_1$ (i.e., $d = d_0 + re$, etc.), for every stable, locally free sheaf $\mathcal{Q}$ of rank r and degree d, there exists a surjection

$$\mathcal{E} \twoheadrightarrow \mathcal{Q}$$

and also $h^1(C, \mathit{Hom}_{\mathcal{O}_C}(\mathcal{E}, \mathcal{Q}))$ equals 0. In other words, the forgetful morphism from the space of quotients $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ to the space of stable sheaves $\mathcal{Q}$ is smooth and surjective, and the geometric fibers are each isomorphic to an open subset of an affine space $\text{Hom}_{\mathcal{O}_C}(\mathcal{E}, \mathcal{Q})$. Moreover, the fiber of the Abel map α_d is the inverse image of the space of stable sheaves with fixed determinant. As is well-known, the moduli space of stable sheaves over C of fixed rank r and fixed determinant is a unirational variety of dimension $(r^2-1)(g(C)-1)$. Thus the fiber of the Abel map fibers over a rationally connected variety and the fibers are rationally connected. By Corollary 3.32, it follows that a general fiber of

$$\alpha_d|_W : W_d \hookrightarrow \text{Section}^d(\mathcal{X}_C/C/\text{Spec }\kappa) \to \text{Pic}^d(C/\text{Spec }\kappa)$$

is isomorphic to an open subset of a rationally connected variety, i.e., (i), (ii) and (iii) hold. Therefore Corollary 4.2 is true.

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