KATOK'S THEOREM ON SURFACE DIFFEOMORPHISMS ECOLE D'ETE – GRENOBLE JULY 2006

par

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Résumé. — Katok's theorem on $C^{1+\epsilon}$ smooth surface diffeomorphisms shows that their dynamics can be approximated in entropy by uniformly hyperbolic invariant sets in contrast to the case of homeomorphisms in dimension 2 or diffeomorphisms in higher dimensions. The proof which we explain in some details, rests on Pesin's theory of non-uniformly hyperbolic dynamics which allows the application of results about (sequences of) uniformly hyperbolic diffeomorphisms.

There are no formal prerequisites beyond the most basic knowledge of dynamical system theory up to the Birkhoff ergodic theorem but some prior contact with entropy theory and uniform hyperbolic theory will certainly help.

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Table des matières

The goal of these lectures is to present the proof of Katok's theorem on $C^{1+\epsilon}$-smooth surface diffeomorphisms. This theorem says that the entropy (or "complexity") of such dynamical systems is essentially explained by (uniformly) hyperbolic dynamics. In particular,

Mots clefs. — surface diffeomorphisms; entropy; periodic points; Lyapunov exponents; Pesin theory; shadowing.

***Theorem 1* (Katok (1980)).** — *A $C^{1+\epsilon}$ diffeomorphism of a compact surface with nonzero topological entropy has infinitely many periodic points. More precisely, the topological entropy*[(1)] *provides a lower bound for their numbers:*

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\{x : f^n x = x\} \geq h_{top}(f).$$

One can compare with the much simpler situation on the interval - see Appendix. Remark that this is obviously false in dimension 3 or higher. It is also false for surface homeomorphisms by a celebrated construction of Rees [**16**]. For now let us just state that a main open problem is to decide whether the above result holds for C^1 diffeomorphisms.

Remark 1. — *Though we emphasize here the fundamental case of surfaces, Katok's theorem is in fact* **a statement in arbitrary dimension about hyperbolic measures***, i.e., invariant and ergodic probability measures with no zero Lyapunov exponents.*

We present a proof which is a very slight variant of the original proof of Katok (we construct true stable/unstable manifold instead of considering the maybe more general case of s, u-admissible manifolds). The ingredients are:

1. the computation of topological and measure-theoretic entropy *à la* Bowen, i.e., through counts of (ϵ, n)-separated or covering sets;
2. linear non-uniform hyperbolic theory due to Oseledets;
3. non-linear Pesin theory (especially Pesin construction of Lyapunov charts);
4. shadowing of pseudo-orbits by pseudo-orbits for globally hyperbolic diffeomorphisms.

We give complete proofs for points (3) and (4) on surfaces.

We conclude the lectures by stating the main corollaries of Katok's theorem and presenting various open problems and counter-examples.

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The author will be grateful for any comments and/or corrections.

1. Entropy theory

We recall some well-known fact about so-called topological and metric entropies. See, e.g., [**20**] or [**9**].

Topological entropy was introduced for continuous maps of compact spaces by Adler, MacAndrew and Konheim [**1**] by mimicking the earlier measure-theoretic notion recalled below. We use Bowen's formulation.

(1) The definition of entropy is recalled in section 1.

Definition 2. — *For a map T of a metric space (X,d) one considers the following. The (ϵ,n)*-**Bowen ball** *centered at $x \in X$ is:*

$$B(x,\epsilon,n) := \{y \in X : \forall 0 \le k < n \; d(T^k y, T^k y) < \epsilon\}$$

The $r(\epsilon,n,Y)$ **covering number** *of $Y \subset X$ is:*

$$r(\epsilon,n,Y) := \min\{\#R : Y \subset \bigcup_{x \in C} B(x,\epsilon,n)\}$$

The **entropy** *of T wrt to d is:*

$$h_d(T) := \lim_{\epsilon \to 0^+} h_d(T,\epsilon) \text{ where } h_d(T,\epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log r(\epsilon,n,X)$$

Lemma 3. —
1. *One can replace the* lim sup *by a* lim inf *in the definition of $h_d(T)$ without modifying the values of $h_d(T)$;*
2. *If T is Lipschitz and M is a compact d-dimensional manifold, then $h_d(T) \le d \cdot \log \mathrm{lip}(T)$;*
3. *$h_d(T)$ is an invariant of uniform topological conjugacy ($T \circ \Psi = \Psi \circ T'$ with Ψ a bijection which is, together with its inverse, uniformly continuous). Thus one can indeed speak meaningfully of topological entropy in the setting of continuous self-maps of compact spaces and write $h_{\mathrm{top}}(f) := h_d(T)$ for any distance d.*
4. *If T is uniformly continuous, $h_d(T^k) = |k| h_d(T)$ for $k \in \mathbb{N}$ (or $k \in \mathbb{Z}$ if T is invertible)*

Exercise 1. — *Prove this Lemma.* Hint for (i): *Remark that:*

$$r(\epsilon,n+m,X) \le r(\epsilon/2,n,X) r(\epsilon/2,m,X)$$

Find an example of a continuous map for which $h_d(T^2) \ne 2h_d(T)$.

Exercise 2. — *Show that (1) in the space of continuous self-maps of $[0,1]$, the entropy is generically infinite; (2) in the space of homeomorphisms of $[0,1]$, it is always zero.*

Question 1. — *If one replaces (ϵ,n)-Bowen balls by subsets $S \subset X$ such that $\forall 0 \le k < n \; \mathrm{diam}(T^k S) < \epsilon$, it is easy to see that $n \mapsto r(\epsilon,n,X)$ becomes sub-multiplicative and that therefore the* lim sup *becomes a limit in the definition of $h_d(T,\epsilon)$. But is the original* lim sup *a true limit in fact?*

Instead of considering covering sets one can use separated sets: S is (ϵ,n)-separated if

$$x \ne y \in S \implies \exists 0 \le k < n \; d(T^k x, T^k y) \ge \epsilon$$

$s(\epsilon,n,Y)$ is the maximum cardinality of an (ϵ,n)-separated subset of Y.

Lemma 4. — *Substituting $s(\epsilon,n,Y)$ for $r(\epsilon,n,Y)$ in the above definitions of entropy does not change $h_d(T)$. More precisely,*

$$s(2\epsilon,n,Y) \le r(\epsilon,n,Y) \le s(\epsilon,n,Y)$$

PROOF: To see the first inequality, let C be an (ϵ,n)-cover of Y with $\#C=r(\epsilon,n,Y)$. Let S be a $(2\epsilon,n)$-separated subset of Y. Let $c:S\to C$ be such that for every $x\in S$, $x\in B(c(x),\epsilon,n)$. If $c(x)=c(x')$ then $d(T^kx,T^kx')\le d(T^kx,T^kc(x))+d(T^kc(x'),T^kx')<2\epsilon$, a contradiction. Hence $c:S\to C$ is one-to-one, proving the first inequality.

For the second, observe that if S is an (ϵ,n)-separated subset of Y with $\#S=s(\epsilon,n,Y)$, then for any $x\in Y\setminus S$, $Y\cup\{x\}$ is not separated, i.e., $x\in B(y,\epsilon,n)$ for some $y\in S$. Thus, S is an (ϵ,n)-cover of Y, proving the second inequality. □

We turn to the measure-theoretic entropy defined by Kolmogorov in 1958 as a new invariant for classification problems in abstract ergodic theory. We use the formulation given by Katok in a metric setting.

Definition 5. — *Let T be a Borel map of the metric space (X,d). Let μ be a probabily measure. Let $0<\lambda<1$. The (ϵ,n,μ,λ)-**cover number** is:*

$$r(\epsilon,n,\mu,\lambda):=\min\left\{\#R:\mu\left(\bigcup_{x\in C}B(x,\epsilon,n)\right)>\lambda\right\}$$

*If μ is a T-invariant and ergodic probability measure, the **entropy** of (T,μ) wrt to d is:*

$$h_d(T,\mu):=\lim_{\epsilon\to0^+}h_d(T,\mu,\epsilon)\ \textit{where}\ h_d(T,\mu,\epsilon):=\limsup_{n\to\infty}\frac{1}{n}\log r(\epsilon,n,\mu,\lambda)$$

Lemma 6. — *1. $h_d(T,\mu)$ is independent of $0<\lambda<1$;*
2. *One can replace the $\limsup$ by $\liminf$ in the definition of $h_d(T,\epsilon)$ without modifying $h_d(T)$;*
3. *$h_d(T,\mu,\epsilon)\le h_d(T,\epsilon)$;*
4. *$h_d(T)$ is an invariant of measure-preserving conjugacy;*
5. *$h_d(T^k,\mu)=|k|h_d(T)$ for $k\in\mathbb{N}$ (or $k\in\mathbb{Z}$ if T is invertible).*

As for topological entropy, we can use s-numbers defined as follows:

$$s(\epsilon,n,\mu,\lambda):=\inf_{\mu(Y)>\lambda}s(\epsilon,n,Y)$$

Indeed, $s(\epsilon,n,Y)\le r(\epsilon/2,n,Y)\le s(\epsilon/2,n,Y)$ (see above) and $r(\epsilon/2,n,Y)=\inf_{\mu(E)>\lambda}r(\epsilon/2,n,Y)$ so that

$$s(\epsilon,n,\mu,\lambda)\le r(\epsilon/2,n,\lambda)\le s(\epsilon/2,n,\lambda)$$

proving the claim.

Question 2. — *One can make the same remark and the same question here as we made for $h_d(T)$.*

***Theorem 2* (Variational principle).** — *If T is continuous and X is compact then*

$$h_d(T)=\sup_{\mu\in\mathrm{Prob}(T)}h_d(T,\mu)=\sup_{\mu\in\mathrm{Prob}_e(T)}h_d(T,\mu)$$

where $\mathrm{Prob}(T)$ is the set of T-invariant probability measures on X and $\mathrm{Prob}_e(T)$ is the subset of those who are ergodic.

Remark 7. — *Measure-theoretic entropy can be (and was first) defined in a purely measure-theoretic setting. The variational principle then indicates the relevant "topological" notion (which in fact depends only on the Borel structure), eg, the right distance to use to compute the topological entropy in the above formalism (see the important example of Markov shifts with countably many states* [**5**]*).*

2. Non-uniform hyperbolic theory

2.1. Linear theory. — **Note:** Complete proofs can be found in [**11**] for the Kingman and Oseledets theorems and [**9**] for the ϵ-reduction theorem (which we prove in the main case of interest to us).

Theorem 3 **(Kingman's subadditive ergodic theorem)**
Let T be an endomorphism of a probability space $(X, \mathcal{A}, \mu)$. Let $a : \mathbb{Z} \times X \to \mathbb{R}$ be subadditive:

$$a(m+n, x) \leq a(m, x) + a(n, T^m x)$$

with $\max(a(1,\cdot), 0) \in L^1(\mu)$. *Then*

$$a_*(x) := \lim_{n\to\infty} \frac{1}{n} a(n, x)$$

exists a.e. and satisfies: $a_*(Tx) = a_*(x)$ *and* $\int a_*\, d\mu = \inf_{n\geq 1} \frac{1}{n} \int a(n, x)\, d\mu(x)$.

Exercise 3. — *Show that this theorem easily implies the existence of the Lyapunov exponents for a diffeomorphism f of a surface M:*

$$\lambda^{\pm}(x) := \lim_{n\to\pm\infty} \frac{1}{n} \log \|Tf_x^n\| \text{ exists a.e.}$$

More precisely, for a.e. $x \in M$, there exists non-zero vectors $v^{\pm} \in \mathbb{R}$ such that:

$$\lambda^{\pm}(x) := \lim_{n\to\pm\infty} \frac{1}{n} \log \|T_x f^n.v^{\pm}\| \text{ exists.}$$

Exercise 4. — *Show that if f is a C^1 map on a compact manifold and μ is an invariant and ergodic probability measure with only strictly negative exponents, then μ is carried by a periodic orbit.*

In fact, much more is true. First, going forward in time, one gets:

Theorem 4 **(Oseledets)**. — *Let T be an endomorphism of a probability space $(X, \mathcal{A}, \mu)$. Let $A : X \to M(d, \mathbb{R})$ be a measurable map such that:*

$$\max(\log \|A\|, 0) \in L^1(\mu)$$

Then there exists a measurable flag:

$$\mathbb{R}^d = F_x^1 \supsetneq F_x^2 \supsetneq \cdots \supsetneq F_x^{r(x)} = \{0\}$$

and measurable functions $\lambda_x^1 > \cdots > \lambda_x^{r(x)}$ with the following properties for all x in a subset of X of full measure:

- *the functions $r(x)$, $\dim F_x^i$, λ_x^i are T-invariant;*
- *$A(x).F_x^i \subset F_x^i$;*

– *for all* $v \in F_x^i \setminus F_x^{i+1}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A(T^{n-1}x)\dots A(x).v\| = \lambda_x^i$$

Going both ways, it yields:

***Theorem 5* (Oseledets).** — *Let T be an automorphism of a probability space $(X, \mathcal{A}, \mu)$. Let $A : X \to GL(d, \mathbb{R})$ be a measurable map such that:*

$$\max(\log \|A\|, \log \|A^{-1}\|, 0) \in L^1(\mu)$$

Then there exists a measurable decomposition:

$$\mathbb{R}^d = E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^{r(x)} = \{0\}$$

and measurable functions $\lambda_x^1 > \cdots > \lambda_x^{r(x)}$ with the following properties for x in a set of full measure:

– *the functions $r(x)$, $\dim F_x^i$, λ_x^i are T-invariant;*
– $A(x).E_x^i = E_x^i$;
– *for all* $v \in E_x^i \setminus \{0\}$,

$$\limsup_{n\to\pm\infty} \frac{1}{n} \log \|A(n,x).v\| = \lambda_x^i$$

where $A(n,x) := A(T^{n-1}x)\dots A(x)$ *and* $A(-n,x) := A(n, T^{-n}x)^{-1}$ *for* $n \geq 0$.
– *the angles don't close too fast: for all* $i \neq j$,

$$\lim_{n\to\infty} \frac{1}{n} \log \angle(E^i_{T^n x}, E^j_{T^n x}) = 0$$

***Exercise 5*.** — *Deduce Theorem 5 from Theorem 4 (in dimension 2). Hint: Apply the previous theorem to both (T, A) and $(\tilde{T}, \tilde{A}) = (T^{-1}, A^{-1} \circ T^{-1})$. First show that the Lyapunov exponents of $\tilde{A}$ are the opposite of those of A: $\lambda_x^i = \tilde{\lambda}_x^{r(x)-i+1}$, $\tilde{r}(x) = r(x)$. Then consider $E_x^i = F_x^i \cap \tilde{F}_x^{r(x)-i+1}$. Finally compute $\lim_{n\to\infty} \frac{1}{n} \log \det A(n,x)$ both by using these results and Birkhoff's ergodic theorem.*

2.2. Ruelle-Margulis Inequality. — The following is the Ruelle-Margulis inequality [**18**].

***Theorem 6*.** — *Let $f : M \to M$ be a C^1-map of a compact d-dimensional manifold and let μ be a f-invariant and ergodic probability measure with a.e. Lyapunov exponents $\mu^1 \geq \mu^2 \geq \cdots \geq \mu^d$ (repeated according to multiplicities). Then*

$$h(f, \mu) \leq \sum_{i=1}^{d} \max(\mu^i, 0)$$

We need the following consequence of Oseledets theorem:

Fact 8. — *Let $f : M \to M$ be a C^1 map and μ an invariant and ergodic probability measure with $\log^+ \|T_x f\| \in L^1(\mu)$ and Lyapunov exponents $\mu^1 \geq \mu^2 \geq \cdots \geq \mu^d$ (repeated according to multiplicities). For any $\epsilon > 0$, there exists $N < \infty$ and a measurable subset M_N with $\mu(M_n) > 1 - \epsilon$ such that for all $x \in M_N$,*

$$T_x f^N B_1 \subset K \begin{pmatrix} e^{(\mu^1 \pm \epsilon)N} & 0 & \dots & & & \\ 0 & e^{(\mu^2 \pm \epsilon)N} & 0 & \dots & & \\ \dots & 0 & e^{(\mu^3 \pm \epsilon)N} & 0 & \dots & \\ & & \dots & & & \\ & & & \dots & 0 & e^{(\mu^d \pm \epsilon)N} \end{pmatrix} B_1$$

where B_1 is the unit ball and $K(x)$ is some orthogonal map, depending on x.

Proof: Let $A_n := T_x f^n$ (we assume $M = \mathbb{T}^d$ to avoid charts, etc.). The polar decomposition says that $A_n = O_n P_n$ where O_n is orthogonal and P_n is the positive matrix $(A_n^* A_n)^{1/2}$. $P_n = Q_n^{-1} D_n Q_n$ with Q_n orthogonal and D_n diagonal, so that $A_n = K_n D_n Q_n$ with $K_n := O_n Q_n^{-1}$. In particular, $A_n(B_1) = K_n D_n(B_1)$.

Let $d_n^1 \geq d_n^2 \geq \cdots \geq d_n^d$ be the diagonal coefficients of D_n. We have to show that, for μ-a.e. $x \in M$, $\frac{1}{n} \log d_n^i \to \lambda^i$.

We do it for $d = 2$. Observe that $d_n^1 = \|A_n\|$, which grows like $e^{(\lambda^1 \pm \epsilon)n}$. Now $|\det A_n| = d_n^1 d_n^2$ from the above formula but it is equal to $e^{(\lambda^1 + \lambda^2 \pm 2\epsilon)n}$ by the Oseledets theorem 4. This solves the problem for $i = 2$.

The general case ($d > 2$) can be solved similarly by using $\|A_n \wedge A_n\| = d_n^1 d_n^2$, $\|A_n \wedge A_n \wedge A_n\| = d_n^1 d_n^2 d_n^3$, etc. □

Remark 9. — *The above is in fact the first step of the proof of Oseledets theorem (see* [**11**]*), rather than its consequence.*

Proof of Theorem 6: For simplicity, we assume that $M = \mathbb{T}^d$. Let $\epsilon > 0$.

Let $N < \infty$ and M_N be given by the previous Fact.

As f is C^1, there exists $\rho_1 > 0$ such that, for every $x \in M$, $0 < r < \rho_1$, every $y \in B(x, r)$,

$$f^N(B(y, r)) \subset T_x f^N B(y, 2r)$$

By Birkhoff ergodic theorem, there exist $\Lambda \subset M$ with $\mu(\Lambda) > 0$ and $n_0 < \infty$ such that for all $x \in \Lambda$, all $n \geq n_0$,

$$\frac{1}{n} \#\{0 \leq k < n : f^k x \in M_N\} > 1 - \epsilon.$$

Thus, for each $x \in \Lambda$, there exists $0 \leq i(x) < N$ such that for all $n \geq n_0$,

$$\frac{1}{n} \#\{0 \leq k < n : f^{kN + i(x)} x \in M_N\} > 1 - \epsilon.$$

Note that $i(x)$ is necessary as μ is not necessarily ergodic for f^N.

Let $0 < \rho_0 < \rho_1$ be arbitrarily small. Let $0 < \rho < \rho_0/2$ be small enough so that

$$d(x, y) < \rho \implies \forall 0 \leq i < N \; d(f^k x, f^k y) < \rho_0/2$$

Notice the chain of dependence:

$$\epsilon \rightsquigarrow N, M_N \rightsquigarrow \rho_1 \rightsquigarrow \rho_0 \rightsquigarrow \rho$$

We are going to bound $r(\rho, n, \Lambda)$ and hence $h(f, \mu, \rho)$.

Let M_ρ satisfy: for all $x \in M$, $1 \leq \#(B(x, \rho/2) \cap M_\rho) \leq C(M)$, with $C(M)$ depending only on M. We give each $x \in \Lambda$ the following description. Let $c(x) = y \in M_\rho$ such that $x \in B(y, \rho)$ and

$$c_n(x) := (i(x), c(x); c(x'), c(f^N x'), \dots, c(f^{(n-1)N} x')) \text{ where } x' = f^{i(x)} x$$
$$I_n(x) := (i_0(x), \dots, i_n(x)) \text{ with } i_k(x) = 1_{M_N}(f^{kN} x')$$

Observe that

$$\{y \in M : c_n(y) = c_n(x)\} \subset B(x, \rho_0, nN)$$

Hence it is enough to bound the number of distinct descriptions for $n \geq 1$, $\underline{i} \in \{0,1\}^m$, $m \geq n$:

$$\mathcal{C}(n, \underline{i}) = \{c_n(x) : x \in \Lambda \text{ and } I_n(x) = \underline{i}\}$$

The previous Fact is easily seen to imply:

Claim 10. — *Let $C_\rho(S)$ be a set of $\rho/2$-balls from M_ρ covering $S \subset M$ with $\#C_\rho(S)$ minimum.*

If B is a ball of radius ρ and $x \in M_N$:

$$\#C_1(T_x f^N(B)) \leq C(M) \exp N \sum_{i=1}^{d} \max(\mu^i, 0).$$

where $C(M)$ is a number depending only on M.

On the other hand, for any $x \in M$,

$$\#C_1(f^N(B)) \leq C(M) \mathrm{lip}(f)^{dN}$$

Therefore for $\underline{i} \in \{0,1\}^n$, for all $x \in M$:

$$\#\mathcal{C}(k+1, \underline{i}) \leq \#\mathcal{C}(k, \underline{i}) C(M) \mathrm{lip}(f)^N$$

If $i_k = 1$, then

$$\#\mathcal{C}(k+1, \underline{i}) \leq \#\mathcal{C}(k, \underline{i}) C(M) \exp N \sum_{j=1}^{d} \max(\mu^j, 0).$$

For $x \in \Lambda$, $\sum_{k=0}^{n-1}(1 - i_k) \leq \epsilon n$, therefore:

$$\#\mathcal{C}(n, \underline{i}) \leq N(\#M_\rho)^2 \cdot C(M)^n \cdot \mathrm{lip}(f)^{\epsilon d N n} \cdot \exp nN \sum_{j=1}^{d} \max(\mu^j, 0).$$

Observe that $\#\{\underline{i}(x) : x \in \Lambda\} \leq \epsilon n \binom{n}{\epsilon n}$. Therefore, taking $\frac{1}{nN}\log$ and letting $n \to \infty$:

$$h_d(f, \mu, \rho_0) \leq \sum_{j=1}^{d} \max(\mu^j, 0) + \epsilon d \log \mathrm{lip}(f) + \frac{1}{N} \log C(M) + \frac{1}{N} \phi(\epsilon)$$

with ϕ depending only on ϵ with $\lim_{\epsilon\to 0}\phi(\epsilon)=0$. We then let $\rho_0\to 0$, then $N\to\infty$, then $\epsilon\to 0$. The theorem is proved. □

2.3. Lyapunov charts. — By making a non-autonomous (linear) change of coordinates, we can make the hyperbolicity appear uniform:

***Theorem 7* (Pesin ϵ-reduction).** — *Assume the hypothesis of Theorem 5: Let T be a measure-preserving self-map of a probability space $(X,\mathcal{A},\mu)$. Let $A:X\to GL(d,\mathbb{R})$ be a measurable map such that:*

$$\max(\log\|A\|,\log\|A^{-1}\|,0)\in L^1(\mu)$$

For any $\epsilon>0$, there exists a measurable map $C_\epsilon: X\to GL(n,\mathbb{R})$ with the following properties:

- $\lim_{n\to\pm\infty}\frac{1}{n}\log(\|C_\epsilon(T^n)\|+\|C_\epsilon(T^nx)^{-1}\|)=0$;
- $C_\epsilon(Tx)^{-1}A(x)C_\epsilon(x)$ *has the following form:*

$$\begin{pmatrix} A^1_\epsilon(x) & 0 & \dots & & & \\ 0 & A^2_\epsilon(x) & 0 & \dots & & \\ & & & \dots & & \\ & & & \dots & 0 & A^{r(x)}_\epsilon(x) \end{pmatrix}$$

where $A^i_\epsilon(x)$ is a $\dim E^i_x\times\dim E^i_x$-matrix satisfying:

$$e^{\lambda^i_x-\epsilon}\|v\|\le\|A^i_\epsilon(x).v\|\le e^{\lambda^i_x+\epsilon}\|v\|$$

- *writing $(e_1,\dots,e_d)$ for the canonical basis of $\mathbb{R}^d$,*

$$C_\epsilon(x)\left\langle e_k,e_{k+1},\dots,e_{k+\dim E^i_x-1}\right\rangle=E^i_x$$

if $k=\dim E^1_x+\dots+\dim E^{i-1}_x$.

Problem 1. — *Can one have $\log\|C_\epsilon(x)^{-1}\|\notin L^1(\mu)$?*

The following simple reasoning will be used repeatedly for uniformization:

Exercise 6. — *Let $(a_n)_{n\in\mathbb{Z}}$ be some sequence of numbers satisfying: $\lim_{n\to\infty}\frac{1}{n}a_n=0$. Then for every $\epsilon>0$, setting $A_m:=\sup_{n\in\mathbb{Z}}a_{m+n}-\epsilon|n|<\infty$, we have:*

$$a_{m+n}\le A_m+\epsilon|n| \text{ and } -\epsilon\le A_{m+1}-A_m\le\epsilon$$

Proof of Theorem 7: We restrict ourselves to the case $r=d=2$, $\dim E^1_x=\dim E^2_x=1$. In this case, the Oseledets theorem gives two measurable unit vectors v^1_x, v^2_x (unique up to sign). For $i=1,2$, define

$$M^i_{\epsilon,x}:=\sum_{n\in\mathbb{Z}}\|A(n,x).v^i_x\|e^{-\lambda^i n-\epsilon|n|}$$

and let:

$$C_\epsilon(x).e_i=\frac{v^i_x}{M^i_{\epsilon,x}}$$

So that

$$C_\epsilon(T(x))^{-1}A(x)C_\epsilon(x).e_i = C_\epsilon(T(x))^{-1}.\frac{\pm\|A(x).v_x^i\|v_{T(x)}^i}{M_{\epsilon,x}^i} = \pm\frac{M_{\epsilon,T(x)}^i}{M_{\epsilon,x}^i}\|A(x)v_x^i\|e_i$$

But (using $A(x).v_x^i = \|A(x).v_x^i\|v_{T(x)}^i$):

$$\|A(x).v_x^i\|\cdot M_{\epsilon,T(x)}^i = \sum_{n\in\mathbb{Z}}\|A(x).v_x^i\|\cdot\|A(n,T(x)).v_{T(x)}^i\|e^{-\lambda^i n-\epsilon|n|}$$

$$= \sum_{n\in\mathbb{Z}}\|A(n+1,x).v_x^i\|e^{-\lambda^i n-\epsilon|n|} = e^{\lambda^i\pm\epsilon}\sum_{n\in\mathbb{Z}}\|A(n,x).v_x^i\|e^{-\lambda^i n-\epsilon|n|} = e^{\lambda^i\pm\epsilon}M_{\epsilon,x}^i$$

So that the above yields:

$$C_\epsilon(T(x))^{-1}.A(x).C_\epsilon(x).e_i =: A_\epsilon^i(x).e_i \text{ with } A_\epsilon^i(x) = e^{\lambda^i\pm\epsilon}.$$

Let us prove that $C_\epsilon(x)$ is tempered. Observing that $M_{\epsilon,x}^i \geq 1$ yields:

$$\|C_\epsilon(x)\| \leq 1$$

On the other hand,

$$C_\epsilon(x)^{-1} = \begin{pmatrix} M_{\epsilon,x}^1 & 0 \\ 0 & M_{\epsilon,x}^2 \end{pmatrix} W(x)^{-1}$$

where $W(x).(ae_1 + be_2) = av_x^1 + bv_x^2$. So

$$\|C_\epsilon(x)^{-1}\| \leq \|W(x)^{-1}\|\max(M_{\epsilon,x}^1, M_{\epsilon,x}^2)$$

Setting $w(x) := \angle(v_x^1, v_x^2)$, we have

$$\|av_x^1 + bv_x^2\|^2 = a^2 + b^2 + 2ab\cos w(x) = (a - b\cos w(x))^2 + b^2\sin^2 w(x)$$

$$= (b - a\cos w(x))^2 + a^2\sin^2 w(x) \geq \frac{a^2+b^2}{2}\sin^2 w(x)$$

Hence, $\|W(x)^{-1}\| \leq \frac{1}{\sin w(x)}$, and, by Theorem 5

$$\lim_{|m|\to\infty}\frac{1}{m}\log w(f^m x) = 0$$

So it is enough to prove that $M_{\epsilon,x}^i$ is tempered. This is a delicate point[(2)], which requires another application of Oseledets theorem (and of Poincaré recurrence) as follows.

Observe that the cocycle A_ϵ^i (which satisfies the assumptions of Oseledets theorem as it is bounded with bounded inverse) must also have exponent λ^i since:

$$\lambda^i = \lim_{|n|\to\infty}\frac{1}{n}\log\|A(x,n).v_x^i\| = \lim_{|n|\to\infty}\frac{1}{n}\log\left(M_{\epsilon,x}^i(M_{\epsilon,T^n x}^i)^{-1}A_\epsilon^i(n,x)\right)$$

and, by Poincaré recurrence $M_{\epsilon,x}^i \approx M_{\epsilon,T^n x}^i$ for arbitrarily large n. Now the same identity gives:

$$M_{\epsilon,f^n x}^i = \|A(x,n).v_x^i\|^{-1}A_\epsilon^i(n,x)^{-1}M_{\epsilon,x}$$

(2) Though $M_{\epsilon,T(x)}^i = e^{\pm\epsilon}M_{\epsilon,x}^i$ is immediate.

so that

$$\lim_{n\to\infty} \frac{1}{n} \log M^i_{\epsilon,f^n x} = -\lambda^i_x + \lambda^i_x = 0$$

□

Pesin's Lyapunov charts extends the above linear uniformization to the non-linear map.

***Theorem 8* (Pesin's Lyapunov Charts).** — *Let M be a a compact d-dimensional manifold with a Riemannian structure defining $\exp : TM \to M$. Let $f : M \to M$ be a $C^{1+\alpha}$-diffeomorphism, $\alpha > 0$. Let μ be an ergodic and invariant probability measure.*

For all $\rho_0 > 0$ and $\epsilon > 0$ small enough, there exist a measurable function $\rho : M \to (0, \rho_0)$ and a measurable family of linear isomorphisms $C(x) : \mathbb{R}^d \to T_x M$ satisfying the following properties. Setting $\chi_x : B(0, \rho(x)) \to M$, $z \mapsto \exp_x(C(x).z)$,

1. *$\lim_{n\to\infty} \frac{1}{n} \log \rho(f^n x) = 0$ and $\rho(fx) = e^{\pm\epsilon}\rho(x)$;*
2. *$\chi^{-1}_{fx} \circ f \circ \chi_x : B(0, \rho(x)/2 \max e^{\mu_i}) \to B(0, \rho(fx))$ coincides with the restriction to $B(0, \rho(x)/2)$ of a diffeomorphism of $\mathbb{R}^2$ of the following form:*

$$z \mapsto A_\epsilon(x).z + \beta_x(z) \tag{2.1}$$

with $A_\epsilon(x)$ as in the ϵ-reduction theorem and $\beta_x(0) = 0$ and for some constant $K(f)$:

$$\|T_z \beta_x\| \leq K \|C(Tx)\|^{-1} \rho(x)^\alpha \leq \epsilon$$

Proof: For simplicity, we assume that $M = \mathbb{T}^2$. In particular, $\exp(x, v) = x + v$. We fix some invariant and ergodic probabilty measure $\mu \in \mathrm{Prob}_e(T)$ with two Lyapunov exponents of distinct signs (as it is the most interesting case), i.e., $r = 2$ and

$$\lambda := e^{\lambda^1} > 1 > \mu := e^{\lambda^2}$$

and let

$$\epsilon := \frac{1}{100} \min\{\lambda - 1, 1 - \mu\}$$

The ϵ-reduction theorem 7 applied to $A(x) = T_x f$ yields a linear change of coordinates $C_\epsilon : M \to GL(n, \mathbb{R})$ inducing $\chi_x : \mathbb{R}^2 \to \mathbb{R}^2$ by $\chi_x(z) = x + C_\epsilon(x)z$. Let

$$F_x(z) = C_\epsilon(fx)^{-1}(f(x + C_\epsilon(x)z) - f(x))$$

$F'_x(0) = C_\epsilon(fx)^{-1} \circ T_x f \circ C_\epsilon(x)$ is therefore of the form $A_\epsilon(x)$ in the notations of Theorem 7. Thus

$$F_x(u, v) = ((\lambda \pm \epsilon/10)u + \tilde\alpha_1(u, v), (\mu \pm \epsilon/10)v + \tilde\alpha_2(u, v))$$

where $\tilde\alpha_i(0, 0) = 0$, $T_0\tilde\alpha_i = 0$ and $(u, v) \mapsto T_{(u,v)}\tilde\alpha_i$ is α-Hölder with constant bounded by

$$k(x) := K\|C_\epsilon(Tx)^{-1}\| \cdot \|C_\epsilon(x)\|^\alpha$$

where K is the α-Hölder constant of $x \mapsto T_x f$ on M. Observe that $k(x)$ is tempered ($\lim_{n\to\infty} \frac{1}{n} \log k(T^n x) = 0$).

We have to show that F_x extends to a globally hyperbolic diffeomorphism of $\mathbb{R}^2$ in the sense of (2.1).

Observe first that it is the case for F_x restricted to $B(0,r(x))$ with

$$r(x) := \min(\rho_0, (\epsilon/2k(x))^{1/\alpha}))$$

has the form (2.1). Let $b : \mathbb{R}^2 \to [0,1]$ be a C^∞ function with $b(t) = 1$ for $\|t\| \leq 1$, $b(t) = 0$ for $\|t\| \geq 2$. Let $\rho(x) := \inf_{n\in\mathbb{Z}} r(f^n x)e^{\epsilon|n|/10}$ and define

$$\text{Global}(f,x,f(x),\rho(x))(z) := b(z/\rho(x))F_x(z) + (1 - b(z/\rho(x))T_0F_x.z \tag{2.2}$$

It is enough to see that $T_z\text{Global}(f,x,f(x),\rho(x)) - T_0F_x$ has small norm (as a linear operator in $\mathbb{R}^2$). We have

$$\begin{aligned} T_z\text{Global}(f,x,f(x),\rho(x)) - T_0F_x = b(z/\rho(x))T_zF_x + \\ (1 - b(z/\rho(x))T_0F_x + \rho(x)^{-1}(F_x(u,v) - T_0F_x(u,v))b'(z/\rho(x)) - T_0F_x \\ = -b(z/\rho(x))T_z\alpha_x + \rho(x)^{-1}\alpha_x(z)b'(z/\rho(x)) \end{aligned}$$

The first term is smaller than $k(x)\cdot(2\rho(x))^\alpha$. The second term is bounded by $r(x)^{-1} \times k(x)r(x)^{1+\alpha} \times C \leq Ck(x)r(x)^\alpha$. Both bounds are less than $\epsilon/2$ by the definition of $\rho(x) \leq r(x)$. □

Remark 11. — *Observe that what forces the Pesin charts to be small is the linear distortion $\|C(x)^{-1}\|$. This is only known to be subexponential along orbits. A modulus of continuity weaker than Hölder would not allow to control this with $r(x)$ also subexponentially small. Thus the $C^{1+\alpha}$ is essential to the techniques involved.*

The known counter-examples [**15**] *are however very restrictive, thus it is not impossible that a similar theory would exist for less smooth diffeomorphisms.*

3. Shadowing by Hyperbolic Sequences

3.1. Statement. — Recall that $\|\cdot\|_0$ is the sup norm in the relevant space of functions.

Definition 12. — *Let $0 < \mu < 1 < \lambda$, $\epsilon > 0$ and $d > 0$. A* **(λ,μ,ϵ,d)-hyperbolic sequence** *is a sequence of C^1-diffeomorphisms $f_m : \mathbb{R}^2 \to \mathbb{R}^2$, $m \in \mathbb{Z}$, of the form:*

$$f_m(x,y) = (x_m^0 + \lambda x + \alpha(x,y), y_m^0 + \mu y + \beta(x,y))$$

with $\alpha(0,0) = \beta(0,0) = 0$ and $\|\alpha'\|_0$ and $\|\beta'\|_0$ bounded by ϵ and $\max(|x_m^0|,|y_m^0|) \leq d$. Let $\mathcal{F}(\lambda,\mu,\epsilon,d)$ be the space of such sequences endowed with the product topology.

Theorem 9. — *For all $0 < \mu < 1 < \lambda < \infty$ and any $0 < \epsilon < \epsilon_0(\lambda,\mu)$ there exists*

$$\gamma_0 := \frac{3\epsilon}{\lambda - \mu - 2\epsilon} < \gamma_1 := \sqrt{\frac{3\epsilon}{\lambda - 10\epsilon}} < 1 \text{ and } \delta_0 := \frac{2}{1 - \mu - 2\epsilon}$$

with the following property.

Let $\gamma_0 < \gamma < \gamma_1$, $d > 0$, $\delta \geq \delta_0 d$ and $\Delta \geq \frac{1+\epsilon(\delta/d)}{\lambda-1-2\epsilon}d$. Let $f = (f_m)_{m\in\mathbb{Z}} \in \mathcal{F}(\lambda,\mu,\epsilon,d)$. Then there exists a unique sequence of Lipschitz maps $g_m : [-\Delta,\Delta] \to \mathbb{R}$ satisfying:

(i) $|g_m(0)| \leq \delta$;
(ii) $\text{lip}(g_m) \leq \gamma$;

(iii) $f_m(\text{graph}(g_m)) \supset \text{graph}(g_{m+1})$.
The underlying map $f \in \mathcal{F}(\lambda,\mu,\epsilon,d) \mapsto g_0 \in C^0([-\Delta,\Delta])$ *is continuous.*

We use the notations $v_n = C^{\pm 1}(\lambda \pm \epsilon)^n$ for:

$$C^{-1}(\lambda-\epsilon)^n \leq |v_n| \leq C(\lambda+\epsilon)^n$$

and $f_m^n = f_m \circ f_{m+1} \circ \cdots \circ f_{n-1}$ if $m \leq n$ and $f_m^n = (f_n^m)^{-1}$ otherwise.

Theorem 10. — *For all* $0 < \mu < 1 < \lambda < \infty$, $0 < \epsilon < \epsilon_1(\lambda,\mu)$, *there exists* $R_0 := R_0(\lambda,\mu,\epsilon) < \infty$ *with the following property.*

For any $d > 0$ *and* $f \in \mathcal{F}(\lambda,\mu,\epsilon,d)$, *there exists a unique sequence* $(z_m)_{m\in\mathbb{Z}}$ *of points in* $\mathbb{R}^2$ *satisfying:*

(i) $\sup_{m\in\mathbb{Z}} \|z_m\| < \infty$;
(ii) $f_m(z_m) = z_{m+1}$;

Additionally:

(iii) $\sup_{m\in\mathbb{Z}} \|z_m\| < R_0 d$
(iv) *z is uniformly hyperbolic: there exist* $C > 1$ *and a sequence of independent unit vectors* $v_m^s, v_m^u \in \mathbb{R}^2$ *such that* $f_m'(z_m).v_m^\sigma = v_{m+1}^\sigma$ *for* $\sigma = u, s$ *and:*

$$\|(f_m^n)'(z_m).v_m^s\| = C^{\pm 1}(\mu \pm 5\epsilon)^{n-m},\ \|(f_m^n)'(z_m).v_m^u\| = C^{\pm 1}(\lambda \pm 5\epsilon)^{n-m}$$

Moreover, if $f_m' = f_m$ *for* $m = 0,\dots,M$ *define two sequences of points* z_m *and* z_m' *respectively, then*

$$\|z_m - z_m'\| \leq R_0 d \left((\lambda-5\epsilon)^{-(M-m)} + (\mu+\epsilon)^m \right)$$

We call this exponential shadowing.

Remark 13. — *The following is a variant of standard material. We have freely used the textbook* [**9**] *(chapter 6).*

3.2. Invariant cone. —

Definition 14. — *The* γ**-horizontal cone** *is:*

$$H_\gamma := \{(u,v) \in \mathbb{R}^2 : |v| \leq \gamma |u|\}.$$

Lemma 15. — *There exists* $0 < \gamma' < \gamma$ *such that, for all* $m \in \mathbb{Z}$, $(x,y) \in \mathbb{R}^2$,

$$f_m'(x,y).H_\gamma \subset H_{\gamma'} \subset H_\gamma.$$

Also for all $w \in H_\gamma$,

$$(\lambda - 5\epsilon)\|w\| \leq \|f_m'(x,y).w\| \leq (\lambda+\epsilon)\|w\|$$

Proof: Let $(u',v') := f_m'(x,y).(u,v)$. Compute:

$$|u'| = |\lambda u + \alpha_x'.u + \alpha_y'.v| \geq (\lambda - \epsilon - \epsilon\gamma)|u|.$$

Also:

$$|v'| = |\mu v + \beta_x'.u + \beta_y'.v| \leq (\mu+\epsilon)|v| + \epsilon|u| \leq ((\mu+\epsilon)\gamma + \epsilon)|u| \leq \frac{\mu\gamma + 2\epsilon}{\lambda - 2\epsilon}|u'|$$

But $\gamma \geq \gamma_0$ implies:

$$\gamma' := \frac{\mu\gamma + 2\epsilon}{\lambda - 2\epsilon} < \gamma.$$

We turn to the growth of the norm of $w = (u, v) \in H_\gamma$. Obviously, writing $w' = f'_m(x, y).w$,

$$\|w'\|^2 = (u')^2 + (v')^2 \leq (\lambda + \epsilon)^2 u^2 + (\mu + \epsilon)^2 v^2 \leq (\lambda + \epsilon)^2 \|w\|^2$$

proving the upper bound. On the other hand,

$$\|w\| \leq \sqrt{1 + \gamma^2}|u| \leq (1 + \gamma^2)|u|$$

Also,

$$\|f'_m(x, y).w\| \geq |u'| \geq (\lambda - 2\epsilon)|u| \geq \frac{\lambda - 2\epsilon}{1 + \gamma^2}\|w\| \geq (\lambda - 5\epsilon)\|w\|$$

where the last inequality holds because $\gamma \leq \gamma_1$. □

3.3. The Graph Transform is Well-Defined. — Let

$$L(\gamma, \delta) := \{g \in \mathrm{lip}([-\Delta, \Delta]) : \mathrm{lip}(g) \leq \gamma \text{ and } \|g(0)\| \leq \delta\}.$$

We are going to define a sequence of graph transforms $\Gamma_m : L(\gamma, \delta) \to L(\gamma, \delta)$ such that:

$$\mathrm{graph}(\Gamma_m g) = f_m(\mathrm{graph}(g)) \cap [-\Delta, \Delta] \times \mathbb{R}.$$

f_m and $g \in L(\gamma, \delta)$ induce the following mapping $G_m : [-\Delta, \Delta] \to \mathbb{R}$:

$$G_m(x) := \pi_1 \circ f_m(x, g(x))$$

where $\pi_1(x, y) = x$. Thus, where it makes sense:

$$(\Gamma_m g)(G_m(x)) = \pi_2 \circ f_m(x, g(x)).$$

Claim 16. — *(here as in other places we pretend that g is C^1 and not only Lipschitz for simplicity)*

$$|G_m(0)| \leq d + \epsilon\delta \tag{3.1}$$

$$G'_m(x) \geq \lambda - 2\epsilon \textit{ for all } x \in [-\Delta, \Delta]. \tag{3.2}$$

As $\Delta \geq (d + \epsilon\delta)/(\lambda - 1 - 2\epsilon)$,

$$(\lambda - 2\epsilon)\Delta - (d + \epsilon\delta) \geq \Delta$$

so the claim will imply that $G_m^{-1} : [-\Delta, \Delta] \to [-\Delta, \Delta]$ is well-defined. The proofs are simple computations:

$$|G_m(0)| = |\pi_1 \circ f_m(0, g(0))| \leq |x_m^0| + |\alpha'_y| \cdot |g(0)| \leq d + \epsilon\delta$$

and:

$$|G'_m(x)| = |\lambda + \alpha'_x + \alpha'_y.g'(x)| \geq \lambda - \epsilon - \epsilon\gamma \geq \lambda - 2\epsilon.$$

Thus, $\Gamma_m : L(\gamma, \delta) \to \mathrm{lip}([-\Delta, \Delta])$ is well-defined with:

$$\Gamma_m g(x') = \pi_2 \circ f_m(G_m^{-1}x', g(G_m^{-1}x')). \tag{3.3}$$

To see that $\Gamma_m : L(\gamma,\delta) \to L(\gamma,\delta)$ is well-defined we have to check the following. First, as f_m preserves the γ-horizontal cone,

$$\mathrm{lip}(\Gamma_m g) \le \gamma$$

Second, $|g(x_*)| \le \delta + \gamma|x_*|$ and

$$G_m(x_*) = x_m^0 + \lambda x_* + \alpha(x_*, g(x_*)) = 0$$
$$\implies \lambda|x_*| \le |x_m^0| + \epsilon|x_*| + \epsilon(\delta + \gamma|x_*|) \implies |x_*| \le \frac{d+\epsilon\delta}{\lambda - 2\epsilon}$$

therefore (using $\mu \le \lambda - 4\epsilon$)

$$|(\Gamma_m g)(0)| \le |y_m^0| + |\pi_2 \circ f_m(x_*, g(x_*)) - \pi_2(f_m(0,0))|$$
$$\le |y_m^0| + (\mu+\epsilon)|g(x_*)| + \epsilon|x_*| \le |y_m^0| + (\mu+\epsilon)(\delta + \gamma|x_*|) + \epsilon|x_*|$$
$$\le d + (\mu+\epsilon)\delta + ((\mu+\epsilon)\gamma+\epsilon)\frac{d+\epsilon\delta}{\lambda-2\epsilon} \le (\mu+2\epsilon)\delta + \left(1 + \frac{\mu\gamma+2\epsilon}{\lambda-2\epsilon}\right) d \le (\mu+2\epsilon)\delta + 2d \le \delta$$

by the assumption $\delta \ge \delta_0 d$. This completes the proof that $\Gamma_m g \in L(\gamma,\delta)$.

3.4. The Graph Transform is a Contraction.

— The previous results show that there are orbits remaining close to the origin for any finite time segment $[n,m]$: the pre-images by $f_n \circ \ldots f_{m-1}$ of $\Gamma_{m-1} \circ \cdots \circ \Gamma_n(0)$ where $0 \in L(\gamma,\delta)$. To extend this to infinite time segments we shall find a sequence of $g_m \in L(\gamma,\delta)$ with $\Gamma_m g_m = g_{m+1}$ by establishing contraction properties of the Γ_m's.

For $n \le m$, let $\Gamma_n^m = \Gamma_n \circ \Gamma_{n+1} \circ \cdots \circ \Gamma_{m-1} : L(\gamma,\delta) \to L(\gamma,\delta)$.

Proposition 17. — *Let $m \le n$ be two integers. Then, for any $g, h \in L(\gamma,\delta)$,*

$$\|\Gamma_m^n(g) - \Gamma_m^n(h)\|_0 \le (\mu+2\epsilon)^{n-m}\|g-h\|_0 \tag{3.4}$$

Proof: It enough to show (3.4) for $n = m+1$. Let G_m be defined by f_m and g as above and let H_m be defined likewise by f_m and h. Let $x \in G_m^{-1}([-\Delta,\Delta]) \subset [-\Delta,\Delta]$. Observe that:

$$|G_m(x) - H_m(x)| = |\pi_1(f_m(x,g(x)) - f_m(x,h(x)))| \le \epsilon|g(x) - h(x)|$$

Hence, using that $\mathrm{lip}(\Gamma h) \le 1$:

$$|\Gamma_m g(G_m(x)) - \Gamma_m h(G_m(x))| \le |\Gamma_m g(G_m(x)) - \Gamma_m h(H_m(x))| + |\Gamma_m h(H_m(x)) - \Gamma_m h(G_m(x))|$$
$$\le |\pi_2(f_m(x,g(x)) - f_m(x,h(x)))| + |H_m(x) - G_m(x)|$$
$$\le (\mu+\epsilon)|g(x)-h(x)| + \epsilon|g(x)-h(x)| \le (\mu+2\epsilon)\|g-h\|_0$$

□

3.5. Proof of the Theorem. — Observe that the properties (i)-(iii) of the g_m's announced in the statement of the Theorem are equivalent to $g_m \in L(\gamma, \delta)$ and $\Gamma_m g_m = g_{m+1}$.

Remark that, in the norm $\|\cdot\|_0$,

$$\text{diam}L(\gamma, \delta) \leq 2\delta + 2\gamma\Delta.$$

Existence of the sequence $(g_m)_{m\in\mathbb{Z}}$. — For $n \in \mathbb{Z}$, let $g_n^{(n)}$ be an arbitrary element of $L(\gamma, \delta)$ and for all $k > n$, let:

$$g_k^{(n)} = \Gamma_n^k(g_n^{(n)})$$

Observe that, for each $m \in \mathbb{Z}$:

$$g_m := \lim_{n\to-\infty} g_m^{(n)} \text{ exists in } L(\gamma, \delta).$$

Indeed, for any $p \leq q \leq m$:

$$\|g_m^{(p)} - g_m^{(q)}\|_0 = \|\Gamma_q^m(g_q^{(p)}) - \Gamma_q^m(g_q^{(q)})\|_0 \leq (\mu + 2\epsilon)^{q-m}\text{diam}L(\gamma, \delta)$$

which goes to zero as $q = \max(p, q) \to -\infty$. Thus we have a Cauchy sequence in the Banach space $(\text{lip}([-\Delta, \Delta]), \|\cdot\|_0)$. $L(\gamma, \delta)$ is a closed subset of this space, hence g_m belongs to it.

As Γ_m is continuous, $g_{m+1}^{(p)} = \Gamma_m g_m^{(p)}$ goes to the limit: $\Gamma_m g_m = g_{m+1}$.

Uniqueness of the sequence. — If $(g'_m)_{m\in\mathbb{Z}}$ is another such sequence, then, for any $m \in \mathbb{Z}$, for all $n \geq 0$,

$$\|g'_m - g_m\|_0 \leq (\mu + 2\epsilon)^n \|g'_{m-n} - g_{m-n}\|_0 \leq (\mu + 2\epsilon)^n \text{diam}L(\gamma, \delta)$$

so that $g'_m = g_m$.

Continuous dependence. — Let $a > 0$ and $f \in \mathcal{F}(\lambda, \mu, \epsilon, d)$. Let $n \geq 0$ be an integer large enough so that $(\mu + 2\epsilon)^n \text{diam}(L(\gamma, \delta)) < a/2$. Let $b > 0$ be so small that

$$\forall k \in [-m, n] \|f_k - f'_k\|_0 < b \tag{3.5}$$

imply:

$$\|\Gamma^0_{-n} g_{-n} - \Gamma'^0_{-n} g_{-n}\|_0 < a/2$$

where Γ and Γ' are defined by f and f' respectively. Let $\mathcal{U}$ be a neighborhood of f in $\mathcal{F}(\lambda, \mu, \epsilon, d)$ such that every $f' \in \mathcal{U}$ satisfies (3.5). Thus, for any such $f' \in \mathcal{U}$,

$$\begin{aligned}
\|g_0 - g'_0\|_0 &= \|\Gamma^0_{-n}(g_{-n}) - \Gamma'^0_{-n}(g'_{-n})\|_0 \\
&\leq \|\Gamma^0_{-n}(g_{-n}) - \Gamma'^0_{-n}(g_{-n})\|_0 + \|\Gamma'^0_{-n}(g_{-n}) - \Gamma'^0_{-n}(g'_{-n})\|_0 \\
&< \frac{a}{2} + (\mu + 2\epsilon)^n \|g_{-n} - g'_{-n}\|_0 < a
\end{aligned}$$

This concludes the proof of Theorem 9.

3.6. Proof of Theorem 10. —

Existence. — Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ be the symmetry wrt $x = y$. Under the assumptions of the theorem, one can apply Theorem 9 both to $f := (f_m)_{m\in\mathbb{Z}}$ and $f^{-1} := (\tau \circ f_{-m}^{-1} \circ \tau)_{m\in\mathbb{Z}}$. Indeed, f^{-1} is a $(\mu^{-1}, \hat{\epsilon}, \hat{d})$-hyperbolic sequence for $\hat{d} := 2\mu^{-1}d$, $\hat{\epsilon} := 2\lambda\mu^{-1}\epsilon$ (assuming ϵ small enough).

Thus, for $0 < \epsilon < \epsilon_1(\lambda, \mu)$ small enough, we can define, with obvious notations:

$$0 < \hat{\gamma}_0 := \max(\gamma_0(f), \gamma_0(f^{-1})) < \hat{\gamma}_1 := \min(\gamma_1(f), \gamma_1(f^{-1})) < 1$$

Also let $\hat{\lambda} := \min(\lambda, \mu^{-1})$, $\hat{\mu} := \max(\mu, \lambda^{-1})$, $\hat{\delta}_0 := \frac{2}{1-\hat{\mu}-\hat{\epsilon}}$ and

$$\hat{\Delta} := \frac{1 + \hat{\epsilon}\hat{\delta}_0}{\hat{\lambda} - 1 - 2\hat{\epsilon}} \hat{d}.$$

From now on, we drop the hats.

In this way we obtain maps g_m and $\tilde{g}_m$ in $L(\gamma, \delta)$. We claim that (1) graph$(g_m) \cap \tau$(graph$(\tilde{g}_{-m}$)) is a single point z_m and that (2) this sequence z_m satisfies properties (i)-(iv).

To see (1), observe that $\|g_m\|_0 \leq \delta + \gamma\Delta \leq \Delta$, hence graph$(g_m)$ is a line contained in the square $[-\Delta, \Delta]^2$ and joining its left side to its right side. Similarly, τ(graph(g_m)) is a line contained in the same square joining its top side to its bottom side. The two lines must obviously intersect (apply the Brouwer fixed point theorem –or just elementary continuity arguments– to $g_m \circ \tilde{g}_{-m} : [-\Delta, \Delta]$).

(2) easily follows from (1) upon remarking that τgraph(g_m) is a stable manifold.

Uniqueness. — Let $(z'_m)_{m\in\mathbb{Z}}$ be another solution satisfying $\|z'_m\| \leq \delta$ for all $m \in \mathbb{Z}$. We prove that $z'_m \in$ graph(g_m) in the above notation. To see this, repeat the construction of the g_m's starting with $g_n^{(n)} := y'_n$, the y-coordinate of z'_n, for each $n \in \mathbb{Z}$. It follows that $z'_n \in$ graph$(g_n^{(n)})$ hence $z'_m \in$ graph$(g_m^{(n)})$ for all $m \geq n$. Letting $n \to -\infty$ we obtain the claim.

Additional properties. — Clearly the above proof implies $\sup \|z_m\| \leq 2\sqrt{2}\Delta$, hence we can take

$$R_0 := 8 \frac{1 + \hat{\epsilon}\hat{\delta}_0}{\hat{\lambda} - 1 - 2\hat{\epsilon}} \mu^{-1}.$$

yielding property (iii).

To establish property (iv) we sketch the proof of the existence of an unstable direction with the stated properties.

First, the fact that f maps the cone H_γ strictly into itself (Lemma 15) implies that the projective version of the differentials $f'(z_m)$ are uniformly contracting in the Birkhoff metric of the projectivization of H_γ (see [**12**]) and this allows one to easily construct the invariant direction $v_m^u \in H_\gamma$ (in fact this argument shows that the functions g_m are differentiable). The second part of the same Lemma gives the statement about the rate of expansion.

Exponential shadowing. — Let $f, f' \in \mathcal{F}(\lambda, \mu, \epsilon, d)$ and $M \geq 1$ be as in the statement of the theorem. Let $g_m, \tilde{g}_m, g'_m, \tilde{g}'_m$ be the maps in $L(\gamma, \Delta)$ associated respectively to f, f^{-1}, f', f'^{-1} defined as above. Define $W_m^u :=$ graph(g_m), $W_m^s := \tau$(graph$(\tilde{g}_m$))

and define W'^u, W'^s similarly. Let $z_m = W^u_m \cap W^s_m$ and $z'_m = W^{u'}_m \cap W^{s'}_m$ be the associated points.

Let w_m be the unique intersection point of W'^u_m and W^s_m for $m = 0, \dots, M$ (this is well-defined by the same arguments as those given above for z_m).

w_m and z'_m are both on W'^u_m whose tangent vectors are expanded by a factor at least $\lambda - 5\epsilon$ under f_m. Thus,

$$d(w_M, z'_M) \geq \frac{1}{1+\gamma}(\lambda - 5\epsilon)^{M-m} d(w_m, z'_m)$$

that is, as both w_M and z'_m belong to $[-\Delta, \Delta]^2$ and $\gamma < 1$:

$$d(w_m, z'_m) \leq 4\sqrt{2}\Delta(\lambda - 5\epsilon)^{-(M-m)}$$

Using that w_m and z_m are both on W^s we obtain:

$$d(w_m, z_m) \leq 4\sqrt{2}\Delta(\mu^{-1} - 5\epsilon)^{-m}$$

The triangular inequality yields the claim about $d(z_m, z'_m)$ for $m = 0, \dots, M$.

This concludes the proof of Theorem 10.

4. Katok Non-Uniform Shadowing Lemma

Theorem 11. — *Let $f : M \to M$ be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of a compact manifold possessing a hyperbolic measure*[(3)] *μ. There exists a compact set Λ such that $\mu(\Lambda) > 0$ and for all $r_1 > 0$, there exist a number $r_2 > 0$ such that if $x \in \Lambda \cap f^{-n}\Lambda$ and $d(x, f^n x) < r_2$, then there exists a periodic point $z \in M$ such that:*

- *$f^n(z) = z$;*
- *$d(f^k z, f^k x) < r_1$ for all $k = 0, \dots, n-1$.*

Exercise 7. — *Show that if μ above is not reduced to a periodic orbit then f has infinitely many periodic orbits.*

We give as usual the proof for the two dimensional case. We further assume that $M = \mathbb{T}^d$ for simplicity.

Let μ be some hyperbolic measure of f. Let $\lambda > 0 > \mu$ be the exponents of μ. Let $L := e^\lambda > 1 > K := e^\mu$ and $0 < \epsilon < \min(\lambda/100, -\mu/100, \epsilon_2(L, K))$ where $\epsilon_1 > 0$ is defined in Theorem 10.

Recall that Pesin's theory defines linear charts $\chi_x(z) = x + C_\epsilon(x).z$ on $B(0, \rho(x))$ — see Theorem 8. Now, let's use some of measure theory's magic (see [**17**], Theorems 2.18 and 2.23):

***Theorem 12* (Lusin's Theorem)**. — *Let X be a locally compact Hausdorff space such that every open set is σ-compact and μ be a Borel positive measure which is finite on compact subsets. Then μ is regular and for every measurable function $f : X \to \mathbb{R}$ such that $\mu(\{x : f(x) \neq 0\}) < \infty$ and every $\epsilon > 0$, there exists a continuous fonction $g : X \to \mathbb{R}$ with bounded support satisfying:*

$$\mu(\{x : g(x) \neq f(x)\}) < \epsilon$$

(3) This means ergodic, invariant probability measure with no zero Lyapunov exponent.

In particular, there is a compact subset of measure arbitrarily close to 1 *on which the restriction of f is continuous.*

Applying this to the functions ρ and C_ϵ (eg, the values of the coefficients), we obtain a compact subset $\Lambda_1 \subset M$ such that:

– $\mu(\Lambda_1) > 0$;
– $C_\epsilon(x)$ and $\rho(x)$ are continuous on Λ_1.

Recall that Theorem 10 has defined a number $R_0 > 0$. Let

$$d := \frac{\min(r_1, \min_{\Lambda_1} \rho)}{R_0}.$$

Let $r_2 > 0$ be small enough so that for arbitrary $x \in \Lambda_1$, $y \in f^{-1}\Lambda_1$, we have:

$$d(fx, y) < r_2 \implies \text{Global}(f, x, fy, \rho(x)) \text{ is } (L, K, \epsilon, d)\text{-hyperbolic}.$$

where Global$(\cdot)$ has been defined in the proof of Theorem 8, eq. (2.2).

We now define a hyperbolic sequence $f = (f_m)_{m\in\mathbb{Z}} \in \mathcal{F}(L, K, \epsilon, d)$ as:

– $f_m = \text{Global}(f, f^m x, f^{m+1}x, \rho(f^m x))$ for $0 \le m < n - 1$;
– $f_{n-1} = \text{Global}(f, f^{n-1}x, x, \rho(f^{n-1}x))$;
– $f_{k+qn} = f_k$ for $k \in [0, n-1]$ and $q \in \mathbb{Z}$.

Applying Theorem 10, we get a periodic sequence $(z_m)_{m\in\mathbb{Z}}$, $z_{m+n} = z_m$ which is close to the origin. Indeed, the last claim of Theorem 10 for the sequences f and f' defined by the Lyapunov charts around the true orbit of x gives, for $0 \le m < n$:

$$\|z_m - 0\| \le R_0 d((L-5\epsilon)^{(n-m)} + (K+\epsilon)^m) \le \max(\rho(x)e^{-\epsilon m}, \rho(f^n x)e^{-\epsilon(n-m)}) \le \rho(f^m x)$$

Hence, $(z_m)_{m\in\mathbb{Z}}$ stays in the domains of the charts $\chi_{f^{m \bmod n}x}$ and therefore gives a f-orbit on M,

$$z = \chi_x(z_0)$$

Finally, for $0 \le m < n$,

$$d(f^m z, f^m x) \le \sup_m \|z_m\| < R_0 d < r_1.$$

as claimed, finishing the proof of the Shadowing Theorem.

5. Existence of Hyperbolic Horseshoes

5.1. Statements. — Katok applied the previous idea to get not only a periodic orbit but a whole horseshoe.

Theorem 13. — *Let $f : M \to M$ be a $C^{1+\epsilon}$ diffeomorphism of a compact manifold. Let μ be a hyperbolic, ergodic invariant probability measure. For any $h < h(f, \mu)$, there exists a hyperbolic invariant compact set $\Lambda \subset M$ such that*

$$h(f|\Lambda) \ge h.$$

Moreover,

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\{x \in M : f^n x = x \text{ is hyperbolic}\} \ge \sup_{\mu\in\text{Prob}_{hyp}(f)} h(f, \mu)$$

5.2. Proof of the Theorem. — We proceed basically as in the proof of the Shadowing Theorem 11, but for many orbits at once.

Let $r_0 > 0$ small enough so that, according to the relevant variant of Katok's formula:

$$\liminf_{n\to\infty} \frac{1}{n} \log s(r_0, n, \mu) > h.$$

Let Λ and $r_2 > 0$ be the compact set and positive number defined in the proof of Theorem 11 for $r_1 = r_0/4$. Maybe after reducing it (but keeping positive measure), we can assume that $\Lambda \subset B(x_*, r_2)$, for some $x_* \in \Lambda$.

As $\mu(\Lambda) > 0$,

$$s(r_0, n, \Lambda) \geq s(r_0, n, \mu) = \inf_{\mu(Y)\geq\mu(\Lambda)} s(r_0, n, Y)$$

so,

$$\liminf_{n\to\infty} \frac{1}{n} \log s(r_0, n, \Lambda) > h.$$

For each n take an (r_0, n)-separated set $C_n \subset \Lambda$ with maximum cardinality. Observe that

$$\Lambda \subset \bigcup_{x\in C_n} B(x, r_0, n)$$

(otherwise the cardinality wouldn't be maximum).

A simple application of Birkhoff ergodic theorem yields:

Claim 18. — *Maybe after reducing Λ by an arbitrarily small measure, for any $\delta > 0$, there exists $n_1 < \infty$ such that for all $x \in \Lambda$ and $n \geq n_1$,*

$$\exists (1-\delta)n \leq k < n \ s.t. f^k x \in \Lambda.$$

Therefore one can find an integer $n' \in [(1-\delta)n, n]$ and a subset $C_n' \subset C_n$ with $\#C_n' \geq \#C_n/n \geq e^{(h+\epsilon)n}$ such that all $x \in C_n'$ satisfy $f^{n'}x \in \Lambda$.

We claim that one can find an (ϵ, n')-separated (and not just (ϵ, n)-separated) subset $C_n'' \subset C_n'$ with $\#C_n'' \geq e^{hn'}$. To this end, partition C_n' according to $(c(x,k))_{n'<k<n}$. This partition has at most $Ce^{(n-n')(h_{\mathrm{top}}(T)+\epsilon)}$ elements, hence contains a set C_n'' with cardinality at least $e^{n(h+\epsilon)-\delta n}/C > e^{nh}$, proving the claim.

For simplicity we pretend now that $n' = n$ and $C_n'' = C_n$.

Now define a basic sequence to be a finite sequence of maps $f^x := (f_0^x, \dots, f_{n-1}^x)$ for each $x \in C_n$ by setting

$$f_m^x := \mathrm{Global}(f, f^m x, f^{m+1}x, \rho(f^m(x))) \quad (0 \leq m < n-1)$$
$$f_{n-1}^x := \mathrm{Global}(f, f^{n-1}x, x_*, \rho(f^{n-1}x_*))$$

Claim 19. — *Each f_m^x is (L, K, ϵ, d)-hyperbolic.*

Therefore, given $(x_i)_{i\in\mathbb{Z}}$, an arbitrary sequence of points of C_n, the concatenation f of the basic sequences f^{x_i} belongs to $\mathcal{F}(L, K, \epsilon, d)$ and hence, as in the proof of the Shadowing Theorem 11, defines a sequence $(z_m)_{m\in\mathbb{Z}}$. These points are close enough to the origins of the charts so as to correspond to the f-orbit on M of the point $z = \chi_{x_0}(z_0)$.

Let $\mathbf{S}_1$ be the union of points obtained in this way. Let $\mathbf{S} := \bigcup_{k=0}^{n} f^k\mathbf{S}_1$ be the union of the corresponding orbits. $\mathbf{S}_1$ is the image by a continuous map of a compact set, hence it is compact. Thus $\mathbf{S}$ is a compact and invariant set.

Note that, as a consequence of the Shadowing Theorem,

$$d(f^{m+in}z, f^m x_i) < r_1/2$$

so that $\mathbf{S}$ is $r_1/2$ separated: if z and z' are obtained from two distinct sequences then there exists some $k \in \mathbb{Z}$ such that $d(f^k z, f^k z') > r_0/2$. In particular, $f^n|\mathbf{S}_1$ is topologically conjugate to the full shift $\{1, \dots, \#C_n\}^{\mathbb{Z}}$.

Finally Theorem 10 (iv) also shows that $\mathbf{S}$ satisfies uniform hyperbolicity estimates which can be carried to the manifold.

This concludes the proof of the theorem.

5.3. Corollaries. —

Remark 20. — *It is not difficult to ensure the following additional property ("weak star closeness" of Λ to μ). Given any number of continuous functions $\phi_1, \dots, \phi_N$ and $\epsilon > 0$, one can choose Λ so that for some $n_0 < \infty$, for all $x \in \Lambda$ and all $n \geq n_0$, for all $i = 1, \dots, N$,*

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \phi_i(f^k x) - \mu(\phi_i) \right| < \epsilon.$$

Let us give three especially interesting corollaries.

Corollary 21. — *Let $f : M \to M$ be a $C^{1+\epsilon}$ diffeomorphism of a compact surface. Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\{x \in M : f^n x = x \text{ is hyperbolic}\} \geq h_{\text{top}}(f).$$

Remark 22. — *It has been shown by Chung and Hirayama* [**3**] *that the inequality above is an equality provided that one counts only "uniformly hyperbolic periodic points", i.e., $x \in M$ with $f^n x = x$ and*

$$\|T_x f^k(f^m x)\| \geq \gamma e^{\lambda k}$$

for arbitrary $\gamma > 0$ and $0 < \lambda < h_{\text{top}}(f)$.

It is on the other hand easy to construct examples such that f^n has infinitely many fixed points and V. Kaloshin [**6**] *has shown that there is an open subset of C^2 diffeomorphisms for which, generically, the number of hyperbolic periodic points grows arbitrarily fast (see also* [**7**] *for which an "prevalent" upper bound is proved by controlling the above number γ as $n \to \infty$).*

Corollary 23. — **(of the proof)** *Let $f : M \to M$ be a $C^{1+\epsilon}$ diffeomorphism of a compact manifold. If f has a hyperbolic measure which is not periodic then $h_{\text{top}}(f) > 0$.*

In particular, on surfaces, zero topological entropy implies that all ergodic invariant probability measure have a zero exponent, except possibly for the periodic ones.

Exercise 8. — *1. Find an example where the hyperbolic measure itself has zero entropy.*

2. Prove the above corollary. Hint: *Construct two* **distinct** *periodic points by applying the shadowing theorem 11 to two returns $x_i, f^{n_i}x_i$ $(i = 0, 1)$ to the compact set (use the exponential shadowing). Check that for $p \geq 1$ large enough, the following map is one-to-one: $\pi : \{0,1\}^{\mathbb{Z}} \to M$ defined by $\pi(\alpha)$ is the point shadowing:*

$$(\dots, x_{\alpha_n}, f(x_{\alpha_n}), \dots, f^{pn_1n_2-1}(x_{\alpha_n}, \dots).$$

Question 3. — *Find a $C^{1+\epsilon}$-surface diffeomorphism with zero entropy, infinitely many periodic points, all of them hyperbolic. Can you ensure that the Lyapunov exponents are bound away from zero?*

Corollary 24. — *For any compact surface M^2, the map*

$$h_{\text{top}} : \text{Diff}^{1+\epsilon}(M^2) \to [0, \infty)$$

is lower semi-continuous. Also, $h_{\text{top}} : \text{Diff}^{\infty}(M^2) \to [0, \infty)$ is continuous.

We now give the proofs of the Corollaries.

The first corollary is a consequence of Ruelle-Margulis Inequality (Theorem 6). Let μ be an ergodic and invariant probabiltiy measure with non-zero entropy and exponents $\mu^1 \geq \mu^2$. We must have $0 < h(f, \mu) \leq \max(\lambda^1, 0)$. Hence $\lambda^1 > 0$. On the other hand, μ is also invariant by f^{-1} with exponents $-\mu^2 \geq -\mu^1$ and the previous reasoning applied to it yields $-\mu^2 > 0$. Thus, μ is hyperbolic. Hence:

$$\sup_{\mu \in \text{Prob}_{hyp}(f)} h(f, \mu) = \sup_{\mu \in \text{Prob}_e(f)} h(f, \mu) = h_{\text{top}}(f)$$

by the Variational Principle Theorem 2.

The first assertion of the second corollary is a consequence of the structural stability of uniformly hyperbolic dynamics (see Theorem 18.2.1 of [**9**]). The second assertion follows now from Yomdin's theory [**19, 2**]:

$$f \in C^{\infty}(M) \mapsto h_{\text{top}}(f)$$

is upper semi-continuous.

6. Conclusion - Comments

6.1. Counterexamples. — *In higher dimensions:* it is not true that there are invariant hyperbolic sets with entropy approximating the entropy of the whole. Indeed, just consider the product of your favorite nonzero topological entropy system with the identity on the circle. To kill all periodic points without modifying the entropy, replace the identity with an irrational rotation.

For homeomorphisms: Rees [**16**] has built surface homeomorphisms with nonzero topological entropy which are minimal, hence without any periodic points. She obtains these dynamics as limits of zero entropy transformations. Hence the topological entropy is not lower semicontinuous on $Homeo(M^2)$.

It is a major open problem to decide if C^1 surface diffeomorphisms are like homeomorphisms or $C^{1+\alpha}$ diffeomorphisms in this respect (see [**10**] for a "combinatorial" distortion control for C^1 interval maps).

6.2. Global results. — Stronger results are known for interval maps, which go beyond approximating to give a global structure:

Theorem 14 **(B 1995, B 2000, B-Ruette 2001)**. — *Let $f : [0,1] \to [0,1]$ be a $C^{1+\epsilon}$ map. If $h_{\text{top}}(f) > 0$ then f is entropy-conjugate to a countable Markov shift.*

If, additionally, f has a measure of maximum entropy (which is the case if f is C^r with r large enough), then, for some positive integer p:

$$\liminf_{n\to\infty} \frac{\#\{x : f^{pn}x = x\}}{e^{pn\cdot h_{\text{top}}(f)}} \geq 1$$

On surfaces we have the same result for a class of models:

Theorem 15 **(B)**. — *Let f be a piecewise affine homeomorphism of a compact surface (say $\mathbb{T}^2$). If $h_{\text{top}}(f) > 0$, then f has finitely many ergodic invariant probability measures with maximum entropy (for short* **maximum measures**. *Moreover, we have a* **multiplicative lower bound for the periodic points***: for some positive integer p:*

$$\liminf_{n\to\infty} \frac{\#\{x : f^{pn}x = x\}}{e^{pn\cdot h_{\text{top}}(f)}} \geq 1$$

The above results motivate the:

Conjecture 1. — *Any $C^{1+\epsilon}$ diffeomorphism of a compact surface with nonzero topological entropy has an at most countable, possibly empty, collection of maximum measures.*

If there exists a maximum measure, then a multiplicative lower bound for the periodic points holds.

Conjecture 2. — *Any C^∞ diffeomorphism of a compact surface with nonzero topological entropy has a finite collection of maximum measures, non-empty by Newhouse's result. In particular, a multiplicative lower bound for the periodic points holds.*

Appendice A
On the Interval

Theorem 16 **(Misiurewicz (1979))**. — *If $f : [0,1] \to [0,1]$ is a continuous map then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\{x : f^n x = x\} \geq h_{top}(f).$$

Exercise 9. — *Let $f : [0,1] \to [0,1]$ be continuous. Let P be a finite partition of $[0,1]$. A P-graph is a finite graph which satisfies the following properties:*

- *its set of vertices includes P;*
- *if $U \to U_1 \to \cdots \to U_n \to V$ with $U, V \in P$ and $U_i \notin P$ for all $1 \leq i \leq n$, then $f^{n+1}(U) \supset V$ and $f^k(U) \not\supset W$ for all $1 \leq k \leq n$ and $W \in P$;*

– *if some vertex* U *has a successor not in* P *then it is its only successor;*
– *each vertex has some successor.*

Show that there exists $c > 0$ *such that if* P *is a fine enough partition of* $[0,1]$ *into subintervals and any* P*-graph has entropy, defined as* $\limsup_{n\to\infty} \frac{1}{n} \log N(n)$ *if* $N(n)$ *is the number of paths of length* n*, at least* $h_{top}(f^M) - c$.(4) *and compare with a graph with more arrows.*

Prove the above Misiurewicz theorem.

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(4) *Hint:* Take N so large that $h_{top}(f) \leq h_{top}(f) + 1$ and M so large that $f^M(I_i)$ contains at least one

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