Exotic rotations

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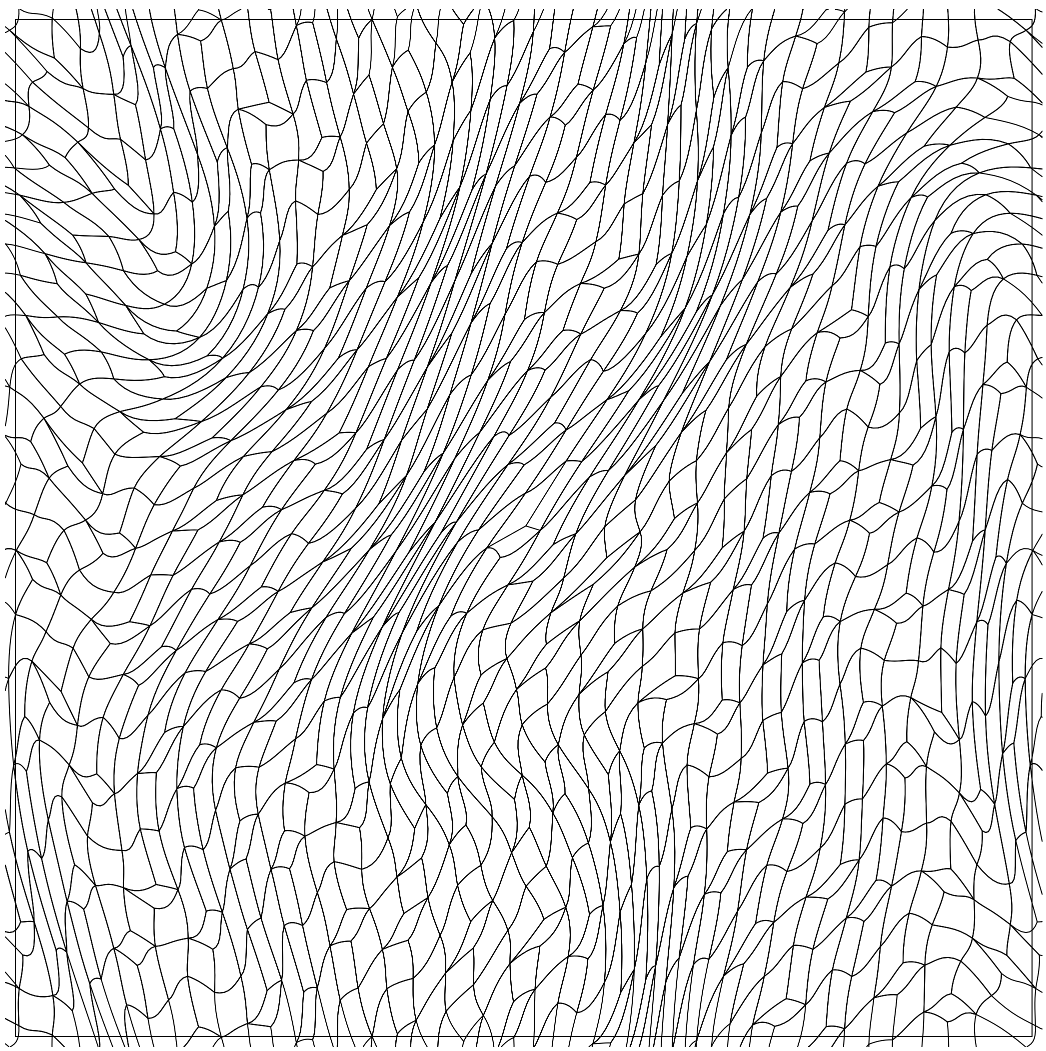


Dynamical partition of an irrational pseudo-rotation pictured by Jaroslaw Kwapisz.

Contents

In his course *Nombre de rotation en dimension* 1 *et* 2, François Béguin has introduced the rotation set for annulus or torus homeomorphisms that are isotopic to the identity. Roughly speaking, it is the set of all the rotations whose dynamics can be compared to the dynamics of the homeomorphism. In particular, it was shown that a torus homeomorphism whose rotation set is large has a rich dynamics (infinitely many periodic orbits, positive topological entropy...) In these notes, we take the opposite viewpoint and consider homeomorphisms (later called *irrational pseudo-rotations*) whose rotation set is very small (it is a singleton $\{\alpha\}$) and we try to answer the following questions: *To what extend is the dynamics of an irrational pseudo-rotation the same as the dynamics of the corresponding rotation R_α? Does an irrational pseudo-rotation have a poor dynamics?*

We will give some general answers (in parts 1 and 2) and also build some exotic examples (in parts 3, 4 and 5), which will in general have one of the following type:

- either the homeomorphism h is a C^∞-diffeomorphism but the construction only holds for irrational rotation numbers α satisfying a Diophantine condition,

- or the construction holds for any irrational number α but the dynamics h is only continuous.

Other interesting examples will be constructed in a more systematic way in the courses of Bassam Fayad and Tobias Jäger.

1 Towards a dynamical characterization of the rotations

In dynamical systems, we usually do not distinguish the dynamics of a homeomorphism R on a compact space X from its conjugates $h = g \circ R \circ g^{-1}$ by the homeomorphisms g of X. On some spaces (for instance on the circle $\mathbb{T}^1$) one can consider the rotations R: these are the isometries, their dynamics is quite simple, well understood and have an interesting combinatorics. It is natural to use them as models for the dynamics of the other homeomorphisms. Is it possible to decide if a homeomorphism h which is isotopic to the identity is conjugate to a rotation?

1.1 The dynamics on the circle

The case of the circle is classical: to any orientation preserving homeomorphism h of $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, H. Poincaré has associated [47] a *rotation number* $\rho(h) \in \mathbb{T}^1$ which has the following properties:

- If $\rho(h) \in \mathbb{Q}/\mathbb{Z}$, then h has a periodic orbit. Moreover, h is conjugate to a rotation if and only if $h^q = \mathrm{Id}$ for some integer $q \geq 1$. This is the periodic or the *rational* case.

- If $\rho(h) \notin \mathbb{Q}/\mathbb{Z}$ then h is semi-conjugate to the irrational rotation R of angle $\rho(h)$: there exists a surjective and increasing continuous map $g \colon \mathbb{T}^1 \to \mathbb{T}^1$ such that $R \circ g = g \circ h$. In particular, the cyclic order of the orbits of h is the same as the cyclic order of the orbits of R.

  If h is not smooth, it can happen that g is not injective and there exist some *wandering dynamics*: there is a non-trivial interval $I \subset \mathbb{T}^1$ which is disjoint from all its iterates $h^k(I)$, for $k \neq 0$. One says that h is a *Denjoy counterexample.*

- When $\rho(h) \notin \mathbb{Q}/\mathbb{Z}$ and h is a C^2-diffeomorphism, A. Denjoy has shown [11] that h is always conjugate to the rotation.

- When $\rho(h) \notin \mathbb{Q}/\mathbb{Z}$ satisfies an arithmetic (Diophantine) condition and h is a C^∞-diffeomorphism, M. Herman has shown [28] that the conjugacy g is also C^∞.

The following remark is less known:

- If $\rho(h) \notin \mathbb{Q}/\mathbb{Z}$, then h and R are "almost conjugate": there exist conjugates of R that are arbitrarily close to h and conjugates of h that are arbitrarily close to R, for the usual C^0 topology on the set of homeomorphisms.
  In other terms:
  The closure of the conjugacy classes of homeomorphisms having an irrational rotation number is classified by the rotation number.

Exercise. Prove that any homeomorphism h whose rotation number α is irrational may be conjugate to a homeomorphism close to the rotation R_α.
Indication: choose two long segments of orbit $(x, h(x), \dots, h^n(x))$ and $(y, R(y), \dots, R^n(y))$ for h and for the rotation $R = R_\alpha$ and consider a homeomorphism which satisfies $g(h^k(x)) = R^k(y)$ for each $0 \leq k \leq n$.

1.2 The dynamics on the sphere

Let us now consider a homeomorphism h of the two-sphere $\mathbb{S}^2$ that is isotopic to the identity (equivalently, h preserves the orientations) and let us wonder if h is conjugate to a rotation: there is no more any natural metric on the sphere and we will not try to characterize rotations as isometries but by some dynamical properties.

We will first describe several cases where this is not true.

a) First obstruction: the dissipation

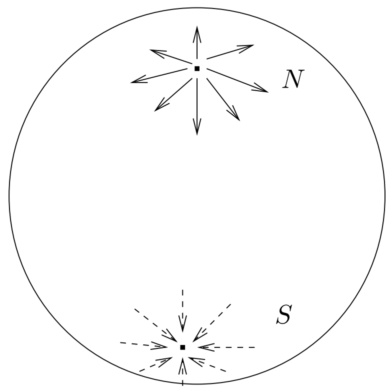


Figure 1: The North-South dynamics on the sphere.

> Some easy counterexamples to the conjugacy problem occur if some regions of the plane are attracting. For example, one can consider the North-South dynamics where two points are fixed, one is repelling the other one is attracting and all the other points are wandering (see figure 1).

In order to forbid the existence of attracting regions, we will need a hypothesis.

Conservation. *There exists a probability measure μ that is invariant by h and whose support is the whole sphere $\mathbb{S}^2$.*

Note that this hypothesis is preserved by conjugacy. For the rotations it is satisfied since the Lebesgue measure is invariant.

One gets the following consequence (see also the courses by Marc Bonino and Frédéric Le Roux).

Proposition 1.1. *A conservative homeomorphism of the sphere $\mathbb{S}^2$ which is isotopic to the identity has at least two fixed points.*

Proof. Any homeomorphism of the sphere $\mathbb{S}^2$ and isotopic to the identity has at least one fixed point (this is the Poincaré-Lefschetz formula). If h has only one fixed point P, the dynamics of h on $\mathbb{S}^2 \setminus \{P\}$ is a Brouwer homeomorphism of the plane. In particular, any point is wandering: it has a neighborhood U which is disjoint from its iterates $h^k(U)$, $k \neq 0$. This contradicts the Conservation Hypothesis. □

There are other possibles assumptions for the conservation:

- One could require that h preserves a smooth volume form. For this case, it is more natural to only consider smooth dynamics and conjugacies.
- A weaker and more topological property is to assume that h is *non-wandering*: any non-empty open set intersects one of its forward iterates. Proposition 1.1 still holds.
- If one considers a homeomorphism that fixes at least two points, N and S, one can consider the *Intersection Property* (relative to N and S): any simple curve γ contained in $\mathbb{S}^2 \setminus \{N, S\}$ intersects its image $h(\gamma)$. Many properties are still satisfied in this setting.

Exercise. Prove that the Conservation Hypothesis implies that h is non-wandering.

b) Second obstruction: the rational case

Let us first assume that h has at least three periodic points. Note that if h is a rotation, it is periodic. This case is thus similar to the rational case on the circle: the dynamics may "degenerate" and be far from the rotations.

- B. Kerékjártó has proven [35] that h is conjugate to a rational rotation if and only if $h^q = \mathrm{Id}$ for some $q \geq 1$. (In fact, we don't need to assume that h preserves the orientation nor that it is conservative, see [10].)
- There are many examples that are not conjugate to a rotation: for instance, one can start with the identity on $\mathbb{S}^2$ and modify the homeomorphism in a small disc. The obtained dynamics can be very rich and very different from the initial rotation.

Exercise. Build a (smooth) example having periodic orbits with arbitrarily large period, with positive topological entropy.

J. Franks [21] and P. Le Calvez [39] have shown that such a homeomorphism has always an infinite number of periodic points. (See the course by Patrice Le Calvez.)

If one fixes two fixed points N, S of h, one can study how the dynamics rotates around these points. The obtained rotation set can be large in this case.

In these notes we will always assume that we are not in the rational case.

Definition 1.2. An *irrational pseudo-rotation* of $\mathbb{S}^2$ is a conservative homeomorphism which is isotopic to the identity and has at most two periodic points.

As we explain above, it has exactly two fixed points, that will be marked and denoted by N and S.

Since an irrational pseudo-rotation has exactly two periodic points, the measure μ has at most two atoms. By removing them, one can assume that μ is non-atomic and gives a positive measure to all the open sets. A result of J. Oxtoby and S. Ulam [46] thus shows that the following property is equivalent to the Conservation hypothesis: *h can be C^0 conjugate to a homeomorphism that preserves the Lebesgue measure.*

Theorem 1.3 (Oxtoby-Ulam). *Let μ and ν be two probability measures on a closed manifold M, that have no atom and give positive measure to any non-empty open set. Then, there exists a homeomorphism g of M such that $g_*\mu = \nu$.*

By removing the fixed points of an irrational pseudo-rotation, one gets a homeomorphism of the (open) annulus. It is then possible to build a rotation set theory (see [38]), generalizing the case of the compact annulus, described in the course by François Béguin. Since there are no periodic orbit in the annulus, one obtains the following result (see [3]):

Proposition 1.4. *Let h be an irrational pseudo-rotation of the sphere $\mathbb{S}^2$ with two marked fixed points N, S. Then, h has a unique rotation number $\alpha \in \mathbb{T}^1$, which is irrational.*

Proof in the C^1 case. This case is much more simpler since one can easily reduce to a dynamics on the closed annulus. Then, one applies several variations on the Poincaré-Birkhoff theorem.

Blowing up the fixed points. Since h is C^1, one can remove the fixed point N and replace it by its unitary tangent space U_N: if one considers polar coordinates $(\theta, r) \in \mathbb{T}^1 \times [0, \varepsilon)$ in a neighborhood of N in $\mathbb{S}^2$, the point N is represented by all the coordinates $(\theta, 0)$. After blowing up the point N, the coordinates $(\theta, 0)$ parameterize the unitary tangent circle U_N that has replaced N.

The dynamics of h on $\mathbb{S}^2 \setminus \{N\}$ extends continuously to U_N by the action of the differential $D_N h$: a point in U_N is a unitary vector u at N. Its image will be the unitary vector $D_N h.u/\|D_N h.u\|$.

By blowing up both fixed points N and S one gets a homeomorphism $\bar{h}$ of the compact annulus $\mathbb{A} = \mathbb{T}^1 \times [0, 1]$. Since h preserved a probability measure with full support, $\bar{h}$ also preserves a probability measure with full support, which can be assumed to be the Lebesgue measure after topological conjugacy.

This Blowing up trick also appears in the course of Frédéric Le Roux.

The proposition 1.4 is now a consequence of the following lemma. □

Lemma 1.5. *Let h be a conservative homeomorphism of the compact annulus $\mathbb{A}$ whose rotation interval is not reduced to an irrational number α. Then h has a periodic point in the interior of the annulus.*

The proof uses the following generalization of Poincaré-Birkhoff theorem (see the course of Francois Béguin):

Franks generalization of Poincaré-Birkhoff theorem. Let us consider

- the open annulus $\mathbb{T}^1 \times (0, 1)$, which is the quotient of the plane by a translation T,
- a conservative homeomorphism h of the open annulus that is isotopic to the identity and a lift $\tilde{h}$ of h to the plane.

Let D be an open disk in the open annulus and $\tilde{D}$ a disk in the plane that lifts D. We say that $\tilde{D}$ is *free* if it is disjoint from its image by $\tilde{h}$. It is *positively (resp. negatively) returning for* $\tilde{h}$ if there exist some integers $n \geq 1$ and $p \geq 0$ (resp. $p \leq 0$) such that $\tilde{h}^n(\tilde{D})$ intersects $T^p(\tilde{D})$.

Then, Franks has proven [20]:
It there exist a free disk that is positively returning for $\tilde{h}$ and a free disk that is negatively returning for $\tilde{h}$, then h has a fixed point.

The proof of lemma 1.5 has two steps.

Reduction: the rotation set is the singleton $\{0\}$. This is a corollary of Franks argument (see the course of Francois Béguin): for any conservative homeomorphism h of the compact annulus $\mathbb{A}$, the rational numbers contained in the interior of the rotation set (which is an interval) are realized by periodic orbits contained in the interior of $\mathbb{A}$. It thus remains to consider the case where the rotation interval of h is a singleton $\{\alpha\}$ and α is a rational number. For simplification, one will assume that $\frac{p}{q} = 0$.

Conclusion: h has a periodic point in the interior of $\mathbb{A}$. We will work in the universal covering of the open annulus $\mathbb{A} \setminus \partial\mathbb{A}$. One can take the lift $\tilde{h}$ of h to the plane, which has rotation number 0. By Franks argument, if there both exist positively and negatively returning free disks for $\tilde{h}$, we are done. Hence, we will consider the case there are no negatively returning disks for $\tilde{h}$. We will argue by contradiction and assume that there are no periodic point in the open annulus.

Claim: there exist some positive integers k, n and a disk $\tilde{D}$ in the plane which lifts a disk in $\mathbb{A} \setminus \partial\mathbb{A}$ such that

- *$\tilde{D}$ is free by $T^{-1} \circ \tilde{h}^n$,*
- *$\tilde{h}^n(\tilde{D})$ meets $\tilde{T}^{k+1}(D)$.*

Proof. Let us consider any small disk D_0 in $\mathbb{A} \setminus \partial\mathbb{A}$ and a lift $\tilde{D}_0$ in the plane. The disk D_0 is *free* by h. But since the dynamics is non-wandering, there exists an integer $m \geq 1$ such that $h^m(D_0) \cap D_0 \neq \emptyset$. On the universal covering, this means that there exists an integer $p \in \mathbb{Z}$ such that $\tilde{h}^m(\tilde{D}_0)$ meets $T^p(\tilde{D}_0)$. We choose the smallest p with this property. Since there are no negatively returning disks for $\tilde{h}$, we have $p \geq 1$. Note that if $p > 1$, the claim follows with $D = D_0$, $n = m$ and $k = p - 1$. So we assume in the following that $p = 1$.

Let us choose a smaller disk $D \subset D_0$ such that $\tilde{h}^m(\tilde{D}) \subset T(\tilde{D}_0)$ and such that $h^m(D)$ is disjoint from D (this is possible since there are no periodic orbit of period m in the open annulus). Using again that the dynamics is non-wandering, we get another integer $n > m$ such that $h^n(D)$ meets D. Hence, there exists an integer $k \in \mathbb{Z}$ such that $\tilde{h}^n(\tilde{D})$ meets $T^{k+1}(\tilde{D})$. We consider the smallest one.

By construction, the image of $T(\tilde{D}_0)$ by $\tilde{h}^{n-m}$ meets $T^{k+1}(D_0)$. Since there are no negatively returning disk, this implies that $k \geq 1$. The disk $\tilde{D}$ is free for $\tilde{h}^n \circ T^{-1}$: otherwise k is not the smaller integer such that $\tilde{h}^n(\tilde{D})$ meets $T^{k+1}(\tilde{D})$. The proof is now complete. □

We thus have shown that the free disk $\tilde{D}$ is positively returning for $\tilde{g} = T^{-1} \circ \tilde{h}^n$. The rotation number on the boundary of $\mathbb{A}$ is 0, hence there exists some fixed point x_0 in $\partial\mathbb{A}$. Let us consider a disk D' which is a small neighborhood of the fixed point x_0 in $\mathbb{A} \setminus \partial\mathbb{A}$: $\tilde{D}'$ meets his image by $\tilde{h}^n$ but not by $T^{-p} \circ \tilde{h}^n$. Hence, $\tilde{D}'$ is a free disk by $\tilde{g}$ which is negatively returning. Franks argument implies that h^n has a fixed point in $\mathbb{A} \setminus \partial\mathbb{A}$, finishing the proof of the lemma.

Exercise. Using the results in the course by Frédéric Le Roux, show that the Lefschetz index of the fixed points of an irrational pseudo-rotation is 1.

c) Third obstruction: the non-smooth dynamics

It is possible to build an irrational pseudo-rotation which is not conjugate to a rotation as a variation of the Denjoy counterexamples: using the ideas of section 1.1 (see exercise 1.1), one can get a continuous family $(f_t)_{t\in[-1,1]}$ of homeomorphisms of the circle with a same irrational rotation number α such that for $t \neq 0$ the map f_t is conjugate to the rotation R_α and f_0 is a Denjoy counterexample.

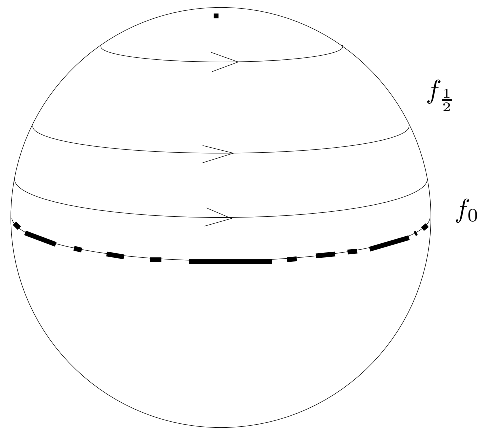


Figure 2: A Denjoy irrational pseudo-rotation.

We now consider the homeomorphism of the sphere $x^2 + y^2 + z^2 = 1$ that preserves each circle $z = t$ for each $t \in [-1, 1]$ and induces the dynamics of f_t on it (see figure 2). One easily check that this is a non-smooth irrational pseudo-rotation with rotation number α. It is not conjugate to the rotation since it contains a Denjoy counterexample.

Exercise. Fill the details in the construction of this example.

From the dynamical point of view, there exists different kinds of irrational numbers. Some are not too strongly approximated by the rational numbers: one says that they satisfy a *Diophantine* condition. Let us recall the classical Diophantine condition: we say that α is Diophantine if there exist some constants $C > 0$ and $\tau \geq 0$ such that for any rational number p/q we have

$$|q\alpha - p| > \frac{C}{q^{1+\tau}}.$$

The other irrational numbers are said to be *Liouvillean.*

The next example shows that the irrational pseudo-rotations whose rotation number is Liouvillean may be reminiscent from the degenerated dynamics of the rational case, even if the diffeomorphism is smooth.

d) Fourth obstruction: the Liouvillean dynamics

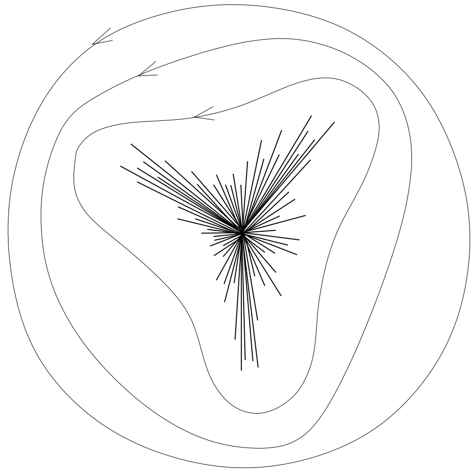


Figure 3: A hairy germ of diffeomorphism.

For some Liouvillean rotation numbers α, there exists a C^∞ irrational pseudo-rotation whose dynamics at the fixed point N is "hairy" (see figure 3): there exists a curve γ attached at N and a sequence of iterates $h^{n_k}(\gamma)$ that converges to $\{N\}$. Hence, the dynamics is not conjugate to the rotation. See [19]. The construction uses the so-called "Anosov-Katok technique" that will be described in part 3.

1.3 The Birkhoff conjecture

G. Birkhoff proposed [9] a conjecture that characterizes dynamically the rotations. The previous discussion motivates the following variation on his conjecture (see [32, Problem 3]).

Conjecture (Birkhoff[1]). Any Lebesgue measure preserving C^∞ irrational pseudo-rotation on $\mathbb{S}^2$ whose rotation number satisfies a Diophantine condition is conjugate to a rotation.

The same question can be asked on other manifolds:
In the closed annulus $\mathbb{A} = \mathbb{T}^1 \times [0,1]$, an irrational pseudo-rotation is a conservative homeomorphism that is isotopic to the identity and that has no fixed point. As it was explained in the proof of proposition 1.4, the conjecture on the annulus would imply the conjecture on the sphere.

Exercise. Show that on the closed disc $\mathbb{D} = \{(x,y) \in \mathbb{R}^2, x^2+y^2 \leq 1\}$, any conservative homeomorphism that is isotopic to the identity has a fixed point in the interior of $\mathbb{D}$.
Define the irrational pseudo-rotations of $\mathbb{D}$ and propose a conjecture in this case.

Some local versions of the conjecture are known and were shown by Herman (see [15]), using Kolmogorov-Arnold-Moser theory: the conjecture holds locally around the fixed points and globally for diffeomorphisms that are close to the rotations. This will be detailed in the course by Bassam Fayad.

[1] There was no Diophantine condition in the initial conjecture given by Birkhoff, but it was stated in the analytic category and his question is still open:
Does there exists a $\mathbb{R}$-analytic irrational pseudo-rotation of $\mathbb{S}^2$ which has a dense orbit?

Theorem 1.6 (Herman). *Any C^∞ irrational pseudo-rotation h of the sphere $\mathbb{S}^2$ whose rotation number α is Diophantine is C^∞-conjugated to the rotation R_α in a neighborhood of its fixed points N, S.*
If moreover h is C^∞ close to the rotation R_α, it is globally C^∞-conjugated to the rotation R_α.

Remark 1.7. The Conservation Hypothesis can be replaced here by the Intersection Property.

1.4 Dynamics on the torus

Let us now discuss the case of the torus: an irrational pseudo-rotation[2] is a conservative homeomorphism that is isotopic to the identity and whose rotation set is reduced to a unique rotation vector (α, α') which is not rational: for any triple $(p, p', q) \neq 0$ of integers we have $q(\alpha, \alpha') + (p, p') \neq 0$.

We can consider several cases:

- *The semi-irrational case.* there exits a triple $(p, q, q') \neq 0$ of integers such that

$$q\alpha + q'\alpha' + p = 0.$$

  The closure of the orbits of the rotation of angle (α, α') are finite unions of circles: by changing the base of $H_1(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z}^2$, the rotation vector of the irrational pseudo-rotation becomes of the form $(\frac{p}{q}, \omega)$ with $\frac{p}{q} \in \mathbb{Q}$ and $\omega \notin \mathbb{Q}$. Moreover, the dynamics has invariant circles if and only if there exists a pair of integers $(q, q') \neq 0$ such that $q\alpha + q'\alpha' = 0$ (or equivalently the rotation vector after changing the basis is $(0, \omega)$).

- *The totally irrational case.* All the orbits for the rotation of angle (α, α') are dense in $\mathbb{T}^2$. In this case, we say that (α, α') is Diophantine if there exist some constants $C > 0$ and $\tau \geq 0$ such that for each triple $(p, q, q') \neq 0$ of integers we have

$$|q\alpha + q'\alpha + p| > \frac{C}{(q + q')^{2+\tau}}.$$

A local version of the conjecture has been proven by A. Kolmogorov, V. Arnold and J. Moser (see [2, 44] and [28])

Theorem 1.8 (Kolmogorov-Arnold-Moser). *Let us consider a C^∞-diffeomorphism h of the torus $\mathbb{T}^2$ which preserves the Lebesgue measure and whose rotation set is reduced to a Diophantine vector (α, α'). If h is C^∞-close to the rotation $R = R_{(\alpha,\alpha')}$, then h is C^∞-conjugate to R.*

Remark 1.9. In this case, the proof uses the assumption that the Lebesgue measure is preserved. One could ask if this hypothesis can be relaxed.

One could propose some versions of the Birkhoff conjecture on $\mathbb{T}^2$ but few results are known.

[2]We would prefer to consider the a priori more general class of conservative homeomorphisms that are isotopic to the identity and have no periodic orbit. With this definition the rotation set is either a segment with rational slope that does not contain any rational point or a segment with irrational slope and whose endpoints are not rational. M. Misiurewicz and K. Ziemian have conjectured [43] that the rotation set of such a torus homeomorphism is reduced to a unique rotation vector, justifying our definition of irrational pseudo-rotation.

2 Topological dynamics of the irrational pseudo-rotations

We now study the dynamics of the irrational rotations of the annulus or of the torus, introduced in the last part.

From the topological viewpoints, one may ask the following questions:

- *Is the dynamics semi-conjugate to a rotation?*
  This is false, even if the dynamics is smooth. We will describe counter-examples in parts 3 and 4.

- *What are the invariant compact sets?*
  For the rotation, the minimal invariant compact sets are either finite sets, circle or the torus. Many examples will be given in the following parts.

- *Is the topological entropy positive?*
  The topological entropy already appeared in the course by François Béguin. In the homeomorphism is a C^2 diffeomorphism, the theorem of A. Katok [34], which will be stated in the course of Jérome Buzzi, implies that an irrational pseudo-rotation has always zero entropy. This theorem also implies another feature of the elliptic dynamics: the norm of the derivative $\|Dh^n\|$ of the iterates does not increase exponentially.

  The question in the continuous case will be addressed in part 5.

One may also be interested by the probability measures that are invariant by the dynamics (see section 4.1 for the basic definitions in ergodic theory). By Oxtoby-Ulam theorem given at section 1.2, one can assume (by conjugating the dynamics) that the standard Lebesgue measure is invariant. The following questions will be illustrated by many examples in the next parts.

- *Is the Lebesgue measure ergodic?*

- *How many ergodic measures has an irrational pseudo-rotation?*

- *What are the ergodic properties satisfied by the invariant measures? What are the measurable dynamics that can be realized by an irrational pseudo-rotation?*

In this part we will not discuss the asymptotic properties of the dynamics: we will show that the combinatorics of the irrational pseudo-rotations and the conjugates of the irrational rotations can not be distinguished by only a finite number of their iterates.

2.1 The arc translation theorem

2.1.1 Statement

The following theorem shows that the orbits of an irrational pseudo-rotation rotate uniformly in the annulus. An *essential simple arc* in the compact annulus $\mathbb{A} = \mathbb{T}^1 \times [0, 1]$ is a simple arc joining one of the boundary component to the other one.

Theorem 2.1 (Kwapisz)**.** *Let h be an irrational pseudo-rotation of the compact annulus of angle α. Then, for any $n \geq 1$, there exists an essential simple arc $\gamma = \gamma_n$ such that the arcs $\gamma, \dots, h^n(\gamma)$ are disjoint .*

Moreover, these arcs are cyclically ordered as the orbits of the rotation of angle α on the circle.

The property on cyclic order may be restated in the universal cover: the compact annulus is the quotient of $\tilde{\mathbb{A}} = \mathbb{R} \times [0,1]$ by the translation $T\colon (x,y) \mapsto (x+1,y)$.
First, one defines the essential simple arcs in $\mathbb{R} \times [0,1]$ as the simple arcs connecting the two lines $\mathbb{R} \times 0$ and $\mathbb{R} \times 1$. Two disjoint essential simple arcs $\tilde{\gamma}, \tilde{\gamma}'$ are ordered by the order on $\mathbb{R}$: we will denote $\tilde{\gamma} < \tilde{\gamma}'$ if $\tilde{\gamma}$ is "on the left" of $\tilde{\gamma}'$.
Let us now consider a lift $\tilde{h}$ of h to the universal cover of the annulus and denote by $\tilde{\alpha}$ its translation number. We have the following property: if one fixes any curve $\tilde{\gamma}$ that lifts γ, then, all the curves $\tilde{h}^k \circ T^\ell(\tilde{\gamma})$ with $k \in \{0, \dots, n\}$ and $\ell \in \mathbb{Z}$ are pairwise disjoint and satisfies

$$\tilde{h}^k \circ T^\ell(\gamma) < \tilde{h}^{k'} \circ T^{\ell'}(\gamma) \quad \Leftrightarrow \quad (k-k')\tilde{\alpha} + (\ell - \ell') < 0. \tag{2.1}$$

This result was proved by J. Kwapisz [36] in the torus $\mathbb{T}^2$: an essential simple curve in $\mathbb{T}^2$ is a simple closed path in $\mathbb{T}^2$ that is homotopic to the circle $\{0\} \times \mathbb{T}^1$. In the closed annulus, we refer to [24, 6]. The same theorem also holds in the sphere but the proof is much more difficult [26, 3] due to the lack of compactness: an *essential simple arc* in $\mathbb{S}^2 \setminus \{N, S\}$ for an irrational pseudo-rotation of the sphere is a simple arc joining the two fixed points N and S.

In general, there is no essential simple arc γ which is disjoint from all its iterates (we will give examples at section 3.3 remark 3.5.1) and at section 4.2.c)).

2.1.2 Proof of the arc translation theorem

Suppose we are given a family of k pairwise commuting maps of $\widetilde{\mathbb{A}}$, and consider sequences obtained by starting with any point in the closed band $\widetilde{\mathbb{A}}$ and iterating each time by one of the maps of the family (that is, we are considering a positive orbit of the $\mathbb{Z}^k$-action generated by the family). We prove that if the rotation sets of the k maps are all positive, then all the sequences obtained this way have a universally bounded leftward displacement. Moreover, by continuity, this remains true if we consider pseudo-orbits instead of orbits, *i.e.* if we allow a little "jump" (or "error") takes place at each step. Then we construct the essential simple arc $\tilde{\gamma}$ using a *brick decomposition.* Brick decompositions were already considered in the course by Patrice Le Calvez. In this text, we only need the easy version, without the maximality property.

The arc translation theorem will be a consequence of the following proposition.

Proposition 2.2. *Let Φ_1, $\Phi_2, \dots$, Φ_s be a family of homeomorphisms of $\widetilde{\mathbb{A}}$, isotopic to the identity, which commute and commute with the translation T, and whose rotation sets are included in $(0, +\infty)$. Then there exists an essential simple arc $\tilde{\gamma}$ such that $\tilde{\gamma} < \Phi_k(\tilde{\gamma})$ for each $1 \leq k \leq s$.*

Proof of the Arc Translation theorem. For each $k \in \{-n, \dots, n\}$, there exists a unique integer ℓ_k such that the rotation number of $\tilde{h}^k \circ T^{\ell_k}$ belongs to $(0,1]$. If one applies proposition 2.2 to the family of homeomorphisms $\tilde{h}^k \circ T^{\ell_k}$, one gets a simple essential arc $\tilde{\gamma}$ in $\tilde{\mathbb{A}}$ that satisfies $\tilde{\gamma} < \tilde{h}^k \circ T^{\ell_k}(\tilde{\gamma})$ for each k. Let γ be its image in $\mathbb{A}$. One gets the following properties:

- $\tilde{\gamma} < T(\tilde{\gamma})$. Hence, γ is a simple essential arc in $\mathbb{A}$.
- $\tilde{\gamma} < \tilde{h}^k \circ T^{\ell_k}(\tilde{\gamma}) < T(\tilde{\gamma})$. Hence, γ is disjoint from its image in $\mathbb{A}$. This also gives the property (2.1).

This concludes the proof of the theorem. □

The two next sections are devoted to the proof of proposition 2.2. In order to simplify the proof we will only consider two commuting maps Φ_1 and Φ_2. We denote by p_1 the projection $\tilde{\mathbb{A}} \to \mathbb{R}$ on the first coordinate.

2.1.3 Pseudo-orbits for commuting homeomorphisms with positive rotation sets

A sequence $(x_n)_{n\geq 0}$ of points in $\widetilde{\mathbb{A}}$ is called a (Φ_1, Φ_2)-*orbit* if for all n, we have $x_{n+1} = \Phi_1(x_n)$ or $\Phi_2(x_n)$. Let d denote the Euclidean distance on $\widetilde{\mathbb{A}} = \mathbb{R} \times [0, 1]$ and ε a positive real number. An ϵ-(Φ_1, Φ_2)-*pseudo-orbit* is a sequence $(x_n)_{n\geq 0}$ of points in $\widetilde{\mathbb{A}}$ such that for all n, we have $d(\Phi_1(x_n), x_{n+1}) < \varepsilon$ or $d(\Phi_2(x_n), x_{n+1}) < \varepsilon$. The main result that makes this definition useful is that we can choose $\varepsilon > 0$ such that the leftward displacement of any ε-(Φ_1, Φ_2)-pseudo-orbit is universally bounded.

Proposition 2.3. *There exists $\varepsilon > 0$ and $M > 0$ such that for any ε-(Φ_1, Φ_2)-pseudo-orbit $(x_n)_{n\geq 0}$, for any $n \geq 0$,*

$$p_1(x_n) \geq p_1(x_0) - M.$$

Proof. This is done in three steps.

a) Long segments of orbits: There exists an integer $N > 0$ with the following property. For every (Φ_1, Φ_2)-orbit $(x_0, \ldots, x_N)$ of length N, we have $p_1(x_N) - p_1(x_0) \geq 2$.

We first note that $p_1(x_N) - p_1(x_0)$ can be written as $p_1(\Phi_1^{N_1}\Phi_2^{N_2}(x_0)) - p_1(\Phi_2^{N_2}(x_0)) + p_1(\Phi_2^{N_2}(x_0)) - p_1(x_0)$ for some $N_1, N_2 \geq 0$ satisfying $N_1 + N_2 = N$. If our claim does not hold, then there exists a constant $C > 0$ and for arbitrarily large integer n, there exists a point x such that $p_1(\Phi_1^n(x)) - p_1(x) \leq C$ or $p_1(\Phi_2^n(x)) - p_1(x) \leq C$. This implies that the rotation set of Φ_1 or Φ_2 is not contained in $(0, +\infty)$ and contradicts the assumptions of the proposition.

b) Long segments of pseudo-orbits: There exists a constant $\varepsilon > 0$ with the following property. For every ε-(Φ_1, Φ_2)-pseudo-orbit $(x_0, \ldots, x_N)$ of length N, we have $p_1(x_N) - p_1(x_0) \geq 2$.

For $\varepsilon > 0$ small, any ε-(Φ_1, Φ_2)-pseudo-orbit of length N stays close to a (Φ_1, Φ_2)-orbit of length N (just by continuity). Hence, step a) implies step b).

c) Any segment of pseudo-orbit: Let us divide the ε-(Φ_1, Φ_2)-pseudo-orbit $(x_0, \ldots, x_n)$ into blocks $x_{kN}, \ldots, x_{(k+1)N}$ of size N. Each of these block (but maybe the last one which could be smaller) has a leftward displacement which is positive. The last block has length smaller or equal to N and has displacement bounded from below by some uniform constant M. This proves proposition 2.3. □

2.2 Brick decomposition

We consider a *brick decomposition* of $\mathbb{A}$, as shown on figure 4. Essentially, this amounts to taking an embedded triadic graph F in $\widetilde{\mathbb{A}}$ (*triadic* meaning that each vertex belongs to exactly three edges). We demand that F contains the boundary of $\widetilde{\mathbb{A}}$. A *brick* is defined to be the closure of a complementary domain of F in $\tilde{\mathbb{A}}$; it is a topological closed disk. The last requirement in the definition of F is the following key feature: every brick is of diameter less than the number ε given by proposition 2.3 (for the Euclidean metric on $\mathbb{A} = \mathbb{S}^1 \times [0, 1]$).

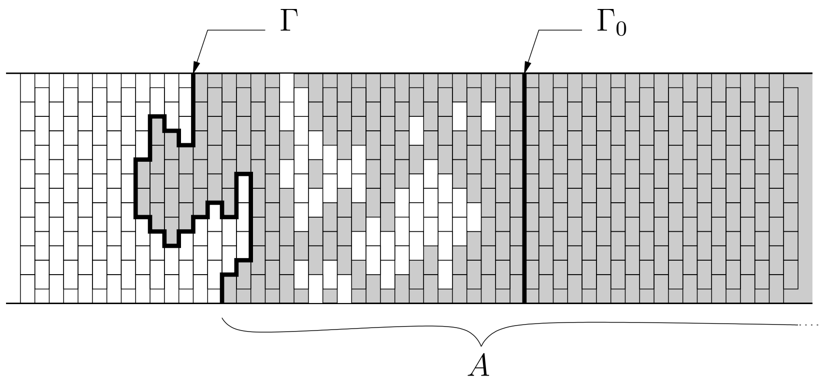


Figure 4: A brick decomposition

Remark 2.4. 1. Since F is triadic, the topological boundary of the union of any family of bricks is a 1-submanifold in $\widetilde{\mathbb{A}}$, with boundary included in the boundary of $\widetilde{\mathbb{A}}$.

2. We deduce from the definitions the following key property:
Any subset of $\widetilde{\mathbb{A}}$ *is included in the* interior *of the union of the bricks that it meets.*
Note that it is crucial that the bricks are defined to be topological *closed* disks.

A *brick chain (from the brick D to the brick D')* is a sequence $(D_0 = D, \dots D_n = D')$ of bricks in $\widetilde{\mathbb{A}}$ such that $\Phi_1(D_i) \cup \Phi_2(D_i)$ meets D_{i+1} for every $0 \leq i < n$.

Take $\gamma_0 = \{0\} \times [0, 1]$; we can suppose that γ_0 is included in F (as on figure 4). We define a subset A of $\widetilde{\mathbb{A}}$ in the following way:

- to any brick D, we associate the union $\mathcal{D}(D)$ of all the bricks D' of the decomposition such that there exists a brick chain from D to D';
- the set A is the union of all the sets $\mathcal{D}(D)$, where D ranges over the set of all the bricks lying on the right of the arc γ_0 (the brick D may meet γ_0).

Lemma 2.5. *The set A contains all the bricks on the right of Γ_0 and is bounded on the left: there exists a constant M such that A is included in $[-M, +\infty[\times[0, 1]$.*

Proof. Indeed, if $(D_0, \dots D_n)$ is a brick chain, and x is any point in D_n, then there exists an ε-(Φ_1,Φ_2)- pseudo-orbit $(x_0, \dots, x_n)$ such that x_0 is in D_0 and $x_n = x$. Remember that ε is given by proposition 2.3; let M be the other constant given by this proposition. Then we have $p_1(x_n) \geq p_1(x_0) - M$, so that if D_0 is on the right of γ_0, then $p_1(x_0) \geq 0$, and consequently D_n is included in $[-M, +\infty[\times[0, 1]$. We conclude that A is included in $[-M, +\infty[\times[0, 1]$. The fact that A contains all the bricks on the right of γ_0 follows from the definition of A (considering chains made of only one brick). □

Lemma 2.6. *The set A is a strict attractor for Φ_1 and Φ_2, i.e.*

$$\Phi_1(A) \subset \mathrm{Int}(A) \text{ and } \Phi_2(A) \subset \mathrm{Int}(A).$$

Proof. Indeed, let D be included in A. By definition, there exists a brick chain $(D_0, \dots, D)$ with D_0 on the right of γ_0. Then for any brick D' meeting $\Phi_1(D)$, the sequence $(D_0, \dots, D, D')$ is again a brick chain, so D' is also included in A. Then the lemma follows from remark 2.4.2): the set $\Phi_1(D)$ is included in the *interior* of the union of the bricks that it meets. Of course, the same argument can be applied to the homeomorphism Φ_2. □

We now finish the proof of proposition 2.2.

Proof of proposition 2.2. Consider the essential arc γ "bounding A on the left" (see figure 4); more precisely, using lemma 2.5 and remark 2.4.1), this can be defined as the boundary of the connected component of $\widetilde{\mathbb{A}} \setminus A$ containing $]-\infty, -M[\times[0,1]$. From lemma 2.6 it follows that γ is disjoint from its images $\Phi_1(\gamma)$ and $\Phi_2(\gamma)$. This ends the proof of proposition 2.2. □

2.2.1 Combinatorics of the rotation of the circle

The order of the iterates of the curve γ in the arc translation theorem is described by the following classical result about the irrational rotations on the circle (see also figure 5):

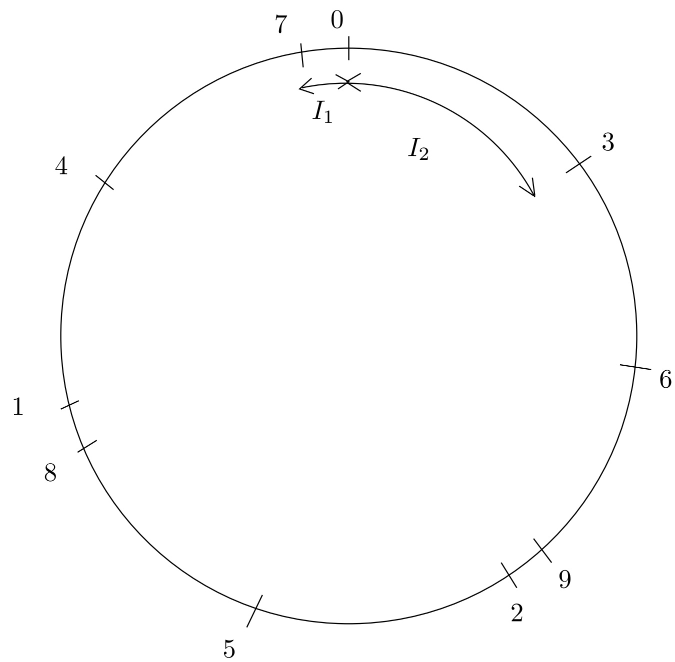


Figure 5: The two intervals I_1 and I_2 for a rotation whose angle $\tilde{\alpha}$ belongs to $(\frac{2}{3}, \frac{5}{7})$: we have $q_1 = 3$ and $q_2 = 7$.

Proposition 2.7. *Let r be an irrational rotation of the circle. Then, there exists arbitrarily large integers $q_1, q_2 \geq 1$ and two closed intervals I_1, I_2 in $\mathbb{T}^1$ such that:*

- *The family $\mathcal{I} = \{I_1, r(I_1), \dots, r^{q_1-1}(I_1), I_2, r(I_2), \dots, r^{q_2-1}(I_2)\}$ covers the circle and its elements have pairwise disjoint interiors.*
- *The union $I_1 \cup I_2$ is an interval.*
- *The intervals $r^{q_1}(I_1)$ and $r^{q_2}(I_2)$ have disjoint interior and their union coincides with $I_1 \cup I_2$.*

Remark 2.8. The integers q_1, q_2 and the intervals I_1, I_2 are chosen in the following way:

- First, one fixes a large integer $Q > 0$ and a point x_0 in the circle.
- Among the iterates $r^k(x_0)$ with $0 < k \leq Q$, one is the closest to x_0 and will be denoted by $r^{q_1}(x_0)$. One will assume that for instance it is on the right to x_0 and define $I_2 = [x_0, r^{q_1}(x_0)]$.
  By definition, the Q first iterates of x_0 do not intersect the interior of the interval $J = [r^{-q_1}(x_0), r^{q_1}(x_0]$.
- Let us consider the first iterate $r^{q_2}(x_0)$ that belongs to the interior of J. It can not intersect I_2 since this would imply that $r^{q_2-q_1}(x_0)$ belongs to $r^{-q_1}(I_2) \subset J$ and contradict the minimality of q_2. We now define $I_1 = [r^{q_2}(x_0), x_0]$.
  Note that if one replaces the point x_0 by another point on $\mathbb{T}^1$, the integers q_1, q_2 remain the same.

This combinatorics allows to define a *renormalization* of the dynamics: one can identify the two endpoints of the interval $I_1 \cup I_2$ and obtain a circle $\mathcal{C}_{I_1,I_2}$. The return map from $I_1 \cup I_2$ into itself (which is r^{q_1} on I_1 and r^{q_2} on I_2) induces a homeomorphism of $\mathcal{C}_{I_1,I_2}$ which is a new rotation.

From proposition 2.7 and theorem 2.1, one immediately deduces a same result for the irrational pseudo-rotations of the compact annulus, the sphere or the torus.

Corollary 2.9. *The irrational pseudo-rotations of the compact annulus, the sphere or the torus can be renormalized in the same way as the irrational rotations of the circle.*

Proof of the proposition. Let us introduce two integers $q_1 < q_2$ and the intervals $I_1 = [r^{q_2}(x_0), x_0]$ and $I_2 = [x_0, r^{q_1}(x_0)]$.

Fact 1. The intervals $r^k(I_1)$ and $r^\ell(I_2)$ with $0 \leq k < q_1$ and $0 \leq \ell < q_2$ cover the circle $\mathbb{T}^1$.
Proof. It suffices to check that each iterate $r^i(x_0)$ with $0 \leq i < q_1 + q_2$ is the left point and the right point of some of these intervals. There are three cases:

- If $0 \leq i < q_1$, then $r^i(x_0)$ belongs to $r^i(I_1) \cup r^i(I_2)$.
- If $q_1 \leq i < q_2$, then $r^i(x_0)$ belongs to $r^{i-q_1}(I_2)$ and $r^i(I_2)$.

- If $q_2 \leq i < q_1 + q_2$, then $r^i(x_0)$ belongs to $r^{i-q_1}(I_2)$ and $r^{i-q_2}(I_1)$.

Let us assume furthermore that the integers q_1, q_2 have been chosen as in remark 2.8.

Fact 2. The interior of the interval $I_1 \cup I_2$ *does not contain any iterate* $r^k(x_0)$ *with* $0 < k < q_1 + q_2$.
Proof. Let us assume that $r^k(x_0)$ belongs to $I_1 \cup I_2$. (Note that $k > q_2$ by the choice of q_2.)

- If it belongs to I_2, then the length of $[r^k(x_0), r^{q_1}(x_0)]$ is smaller than the distance from $r^{q_1}(x_0)$ to x_0. The same is true for the length of $[r^{k-q_1}(x_0), x_0]$ but since $0 < k - q_1 < q_2$, this contradicts the minimality of q_2.
- If $r^k(x_0)$ belongs to I_2 one also gets a contradiction by the same argument.

Fact 3 : The interior of the intervals $r^k(I_1)$ *and* $r^\ell(I_2)$ *with* $0 \leq k < q_1$ *and* $0 \leq \ell < q_2$ *are pairwise disjoint.*
Proof. Otherwise, the interior of some interval $r^j(I_1)$ or $r^j(I_2)$ contains some iterate $r^\ell(x_0)$ with $j < \ell < q_1 + q_2$. This implies that the point $r^{\ell-j}(x_0)$ belongs to the interior of I_1 or I_2 and this contradicts the Fact 3.

To finish, one notes that $r^{q_1}(I_1)$ and $r^{q_2}(I_2)$ are adjacent at the point $r^{q_1+q_2}(x_0)$. Hence, they have disjoint interior and $r^{q_1}(I_1) \cup r^{q_2}(I_2)$ is an interval whose endpoints are $r^{q_1}(x_0)$ and $r^{q_2}(x_0)$. One deduces that it is equal to $I_2 \cup I_1$.

□

One can characterize the integers q_1, q_2 that satisfy proposition 2.7.

Proposition 2.10. *Let* $\tilde{r}$ *be a lift of* r *to the universal cover* $\tilde{\mathbb{A}}$ *and let* $\tilde{\alpha}$ *be its rotation number. Then, the integers* $q_1, q_2 \geq 1$ *satisfy proposition 2.7 if and only if there exist two integers* $p_1, p_2 \in \mathbb{Z}$ *such that* $\frac{p_1}{q_1} < \tilde{\alpha} < \frac{p_2}{q_2}$ *and* $q_1 p_2 - p_1 q_2 = 1$.

A pair of rational numbers $(p_1/q_1, p_2/q_2)$ with $q_1, q_2 > 0$ is said to be a *Farey interval* if and only if $\begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix}$ belongs to $GL(2, \mathbb{Z})$.

Proof. Let us consider two integers $q_1, q_2 \geq 1$. Let us introduce the largest integer $p_1 \in \mathbb{Z}$ such that $q_1\tilde{\alpha} - p_1 > 0$ and the smallest integer $p_2 \in \mathbb{Z}$ such that $q_2\tilde{\alpha} - p_2 < 0$. The intervals $I_2 = [x_0, r^{q_1}(x_0)]$ and $I_1 = [r^{q_2}(x_0), x_0]$ have length $q_1\tilde{\alpha} - p_1$ and $p_2 - q_2\tilde{\alpha}$ respectively. The q_1 first iterates of I_1 and the q_2 first iterates of I_2 cover the circle $\mathbb{T}^1$ by the fact 1 of the proof of proposition 2.7. Hence, the properties of proposition 2.7 are satisfied if and only if

$$1 = q_2(q_1\tilde{\alpha} - p_1) + q_1(p_2 - q_2\tilde{\alpha}) = q_1 p_2 - p_1 q_2.$$

□

Remark 2.11. If $(p_1/q_1, p_2/q_2)$ is a Farey interval, the order of the iterates $r^k(x_0)$ with $0 \leq k < q_1 + q_2$ does not depend on the angle of the rotation $\tilde{\alpha} \in (p_1/q_1, p_2/q_2)$.

2.3 The tiling theorem

2.3.1 Statement

Kwapisz also gave a generalization [37] of the arc translation theorem 2.1 and of the tiling provided by proposition 2.7 to totally irrational pseudo-rotations of the torus $\mathbb{T}^2$.

A triple of rational vectors $\{(p_i/q_i, p'_i/q_i), i = 1, 2, 3\}$ with $q_1, q_2, q_3 > 0$ is a *Farey triple* if

$$\begin{bmatrix} p_1 & p'_1 & q_1 \\ p_2 & p'_2 & q_2 \\ p_3 & p'_3 & q_3 \end{bmatrix} \in GL(3, \mathbb{Z}).$$

Let us consider an irrational vector $(\tilde{\alpha}, \tilde{\alpha}')$ in $\mathbb{R}^2$. A Farey triple for $(\tilde{\alpha}, \tilde{\alpha}')$ is a Farey triple containing $(\tilde{\alpha}, \tilde{\alpha}')$ in the interior of its convex hull.

Remark 2.12. The vector $(\tilde{\alpha}, \tilde{\alpha}')$ belongs to the convex hull of the rational vectors $(p_i/q_i, p'_i/q_i)$, $i = 1, 2, 3$ if and only if 0 belongs to the interior of its convex hull of the vectors $q_i(\tilde{\alpha}, \tilde{\alpha}') - (p_i, p'_i)$.

In dimension 2, there is no canonical combinatorial way to find Farey triples for totally irrational vectors as in dimension 1 (see the remark 2.8). The next proposition shows however that it always exist, but the construction depends on the choice of a metric $\|.\|$ on $\mathbb{T}^2$ that is invariant by the rotations.

Proposition 2.13. *For any totally irrational vector $(\tilde{\alpha}, \tilde{\alpha}')$, there exist Farey triples $\{(p_i/q_i, p'_i/q_i), i = 1, 2, 3\}$ such that the following quantity is arbitrarily small.*

$$\max_{i \in \{1,2,3\}} \|q_i(\tilde{\alpha}, \tilde{\alpha}') - (p_i, p'_i)\|.$$

Proof. One first notice that there always exists a Farey triple for $(\tilde{\alpha}, \tilde{\alpha}')$: by translation by a vector in $\mathbb{Z}^2$, one can always assume that $0 < \alpha, \alpha' < 1$. Note that 0 belongs to the interior of the triangle whose vertices are (α, α'), $(\alpha, \alpha') - (1, 1)$ and either $(\alpha, \alpha') - (0, 1)$ or $(\alpha, \alpha') - (1, 0)$. Hence, one can set $q_1 = q_2 = q_3$, $p_1 = p'_1 = 1$, $p_3 = p'_3 = 0$ and $(p_2, p'_2) = (1, 0)$ or $(0, 1)$.

We now explain how to build a new Farey triple $(P_i/Q_i, P'_i/Q_i)$, $i = 1, 2, 3$ such that for some constant $\gamma \in (0, 1)$ we have

$$\sum_{i \in \{1,2,3\}} \|Q_i(\tilde{\alpha}, \tilde{\alpha}') - (P_i, P'_i)\|^2 \leq \gamma \sum_{i \in \{1,2,3\}} \|q_i(\tilde{\alpha}, \tilde{\alpha}') - (p_i, p'_i)\|^2.$$

Let us define the return vectors $v_i = q_i(\tilde{\alpha}, \tilde{\alpha}') - (p_i, p'_i)$. One chooses the vector whose length is maximal (we will assume that it is v_3) and one chooses two integers n, m such that $V_3 = nv_1 + mv_2 + v_3$ belongs to the domain

$$\{w = a.v_1 + b.v_2,\ |a| \leq \frac{1}{2},\ | < w, v_2 > | \leq \frac{1}{2}\|v_2\|^2\}$$

which is a fundamental domain of the plane for the translations by v_1 and v_2.

Note that $V_3 = a.v_1 + b.v_2$ can be decomposed as

$$V_3 = a < v_1, v_2^\perp > v_2^\perp + c.v_2,$$

where $v_2^\perp$ is a unitary vector that is orthogonal to v_2 and $|c| \leq \frac{1}{2}$. This gives

$$\|V_3\|^2 \leq \frac{1}{4}(\|v_1\|^2 + \|v_2\|^2).$$

Let us define $V_1 = \pm v_1$ and $V_2 = \pm v_2$ so that $V_3 = -A.V_1 - B.V_2$ with $A, B > 0$. One deduces that 0 belongs to the convex hull of V_1, V_2 and V_3 and that

$$\|V_1\|^2 + \|V_2\|^2 + \|V_3\|^2 \leq \frac{5}{4}(\|v_1\|^2 + \|v_2\|^2) \leq \frac{5}{6}(\|v_1\|^2 + \|v_2\|^2 + \|v_3\|^2).$$

Moreover, V_1, V_2, V_3 are the image of v_1, v_2, v_3 by some matrix

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ n & m & 1 \end{bmatrix} \in GL(3, \mathbb{Z})$$

Hence, they are the return vectors of some Farey triple $(P_i/Q_i, P_i'/Q_i)$, $i = 1, 2, 3$ for $(\tilde{\alpha}, \tilde{\alpha}')$.

□

Here is the tiling theorem (see also figure 6):

Theorem 2.14 (Kwapisz). *Let h be an irrational pseudo-rotation of the torus of angles (α, α') and a Farey triple $\{(p_i/q_i, p_i'/q_i), i = 1, 2, 3\}$ for some lift $(\tilde{\alpha}, \tilde{\alpha}')$ of (α, α'). Then, there exists three topological rectangles A_1, A_2, A_3 in $\mathbb{T}^2$ such that:*

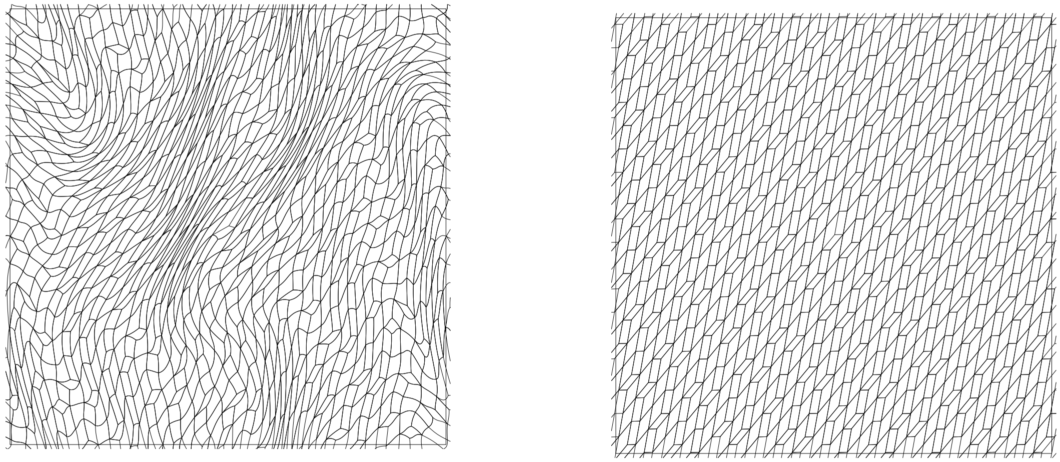
- *The family $\mathcal{I} = \{h^k(A_i), i = 1, 2, 3,\ 0 \leq k < q_i\}$ covers the torus and its elements have pairwise disjoint interiors.*
- *The union $A_1 \cup A_2 \cup A_3$ is a topological hexagon.*
- *The rectangles $h^{q_i}(A_i)$, with i=1,2,3 have disjoint interior and their union coincides with $A_1 \cup A_2 \cup A_3$.*

Similarly for the rotation R by (α, α') on the torus, one can choose three affine rectangles $\bar{A}_1, \bar{A}_2, \bar{A}_3$ with the same property and there exists a homeomorphism Φ of $\mathbb{T}^2$ that is homotopic to the identity and satisfies $\Phi(h^k(A_i)) = R^k(\bar{A}_i)$.

The vertices of the rectangles of the family $\mathcal{I}$ are the $q_1 + q_2 + q_3 - 1$ first iterates of the point x_0 which belongs to the boundary of the three rectangles A_1, A_2 and A_3. By identification, the hexagon $A_1 \cup A_2 \cup A_3$ is mapped on a torus $\mathcal{T}$. The return map from $A_1 \cup A_2 \cup A_3$ into itself (which is h^{q_i} on A_i for each $i = 1, 2, 3$) induces a homeomorphism of $\mathcal{T}$ which is a new irrational pseudo-rotation: this is a *renormalized dynamics of* h (see figure 7).

Remark 2.15. As for remark 2.11, if one chooses a Farey triple $\mathcal{F} = \{(p_i/q_i, p_i'/q_i), i = 1, 2, 3\}$, the tilings obtained for any two irrational pseudo-rotations $\tilde{h}$ and $\tilde{g}$ having rotations vectors $(\tilde{\alpha}, \tilde{\alpha}')$ and $(\tilde{\beta}, \tilde{\beta}')$ in the interior of the convex hull of $\mathcal{F}$.

Remark 2.16. The tiling for the irrational rotation can be found in the following way in $\mathbb{R}^3$: Let us denote by P the plane $\mathbb{R}^2 \times 0$ and by T, S, H the three translations $(x, y, z) \mapsto (x + 1, y, z)$ $(x, y, z) \mapsto (x, y + 1, z)$ and $(x, y, z) \mapsto (x, y, z + 1)$. One considers the lattice L of $\mathbb{R}^3$ generated by the three translations $F_i = H^i \circ S^{-p_i} \circ T^{-p_i'}$. The family of cubes of the lattice L that are



(a) (b)

Figure 6: Dynamical tilings: a) for a standard map h, b) for a rotation $R_{\alpha,\alpha'}$.
The standard map h is the composition $R_{\alpha,\alpha'} \circ V_b \circ H_a$ with

$$H_a(x,y) = (x + \tfrac{a}{2\pi}\sin(2\pi y), y),\ a = 0.45,$$
$$V_b(x,y) = (x, y + \tfrac{b}{2\pi}\sin(2\pi x)),\ b = 1.8.$$

The rotation vector (α, α') is the same for h and $R_{\alpha,\alpha'}$ and satisfies $\alpha^3 - \alpha^2 - \alpha - 1 = 0, \quad \alpha' = \alpha^{-1} - 1$.
The Farey triple is $(927, 778, 504), (504, 423, 274), \ (274, 230, 149)$.

The picture has been realized by J. Kwapisz (see [37]).

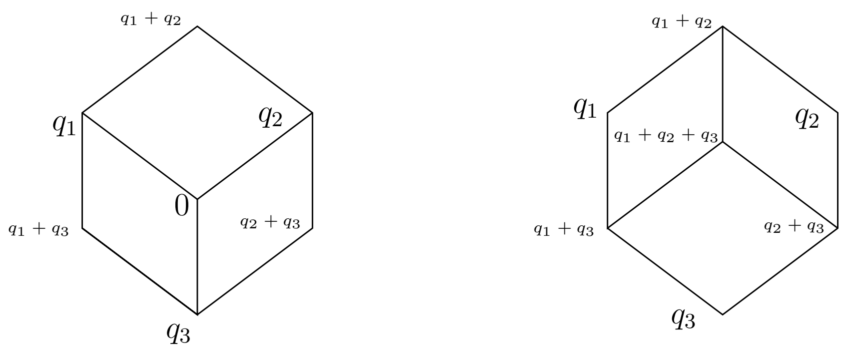


Figure 7: The tiles A_1, A_2, A_3 and their image by the return map.

contained in the closed upper half space $z \geq 0$ defines a region bounded by a surface S: this surface is an approximation of the plane P by squares of the lattice. One can projects S on P by a linear projection which maps the points $H^n(0) = (0,0,n)$ on the points $(n\tilde{\alpha}, n\tilde{\alpha}', 0)$. It can be shown that the decomposition of S into squares is mapped on a tiling of P, satisfying the theorem 2.14 for the irrational rotation R of angle (α, α'). (See figure 6 b).)

2.3.2 Proof for the irrational rotations of the torus

We will assume that $q_1 \leq q_2 \leq q_3$. Let us consider the lift $\tilde{h}$ of h to the universal cover $\mathbb{R}^2$ of the torus, having $(\tilde{\alpha}, \tilde{\alpha}')$ as a translation vector and the two standard translations $S\colon (x, y) \mapsto (x+1, y)$ and $T\colon (x, y) \mapsto (x, y+1)$. We then define the commuting maps $f_i = \tilde{h}^i \circ S^{-p_i} \circ T^{-p'_i}$.

Tilings of $\mathbb{R}^2$. We introduce the vertex set

$$\mathcal{X} = \{\tilde{h}^k(0,0) + (n,m),\ (n,m) \in \mathbb{Z}^2,\ 0 \leq k < q_1 + q_2 + q_3\}.$$

An *edge* (of type $i \in \{1,2,3\}$) is a segment whose vertices are in $\mathcal{X}$ and can be written as $\{P, f_i(P)\}$. A *tile* (of *type* (i,j) with $i \neq j$ in $\{1,2,3\}$) is a parallelogram whose vertices are in $\mathcal{X}$ and can be written as $\{P, f_i(P), f_j(P), f_i \circ f_j(P)\}$. For instance, the point $O = (0,0)$ is the vertex of three tiles, A_1, A_2, A_3, one of each type, given by:

$$A_k = \{O, f_i(O), f_j(O), f_i \circ f_j(O)\}, \quad \text{with } i \neq j \neq k.$$

A tiling is a covering of $\mathbb{R}^2$ by tiles whose interior are pairwise disjoint.

Lemma 2.17. *The tiles A_1, A_2, A_3 cover a neighborhood of O and have disjoint interior.*

Proof. This is equivalent to the assumption that O belongs to the convex hull of the vectors $q_i(\tilde{\alpha}, \tilde{\alpha}') - (p_i, p'_i)$. ☐

Lemma 2.18. *For any point $x \in \mathcal{X}$, all the tiles whose vertex set contains x have disjoint interior and cover a neighborhood of x. For any edge e, there are exactly two tiles whose edge set contains e and they have disjoint interior*

Proof. Each point $x \in \mathcal{X}$ can be written as $\tilde{h}^k(O) + (n,m)$. The proof is done by induction on k. The case $k = 0$ is a consequence of the previous lemma.

Let us assume that it has been proven for some integer $k < q_1 + q_2 + q_3 - 1$ and consider a point $x = \tilde{h}^{k+1}(O) + (n,m)$. By induction, a neighborhood of $\tilde{h}^{-1}(x)$ is covered by tiles $T_1, T_2, \ldots, T_s$ whose interior are disjoint. Moreover, any edge or tile whose vertex set contains $\tilde{h}^{-1}(x)$ is one of these tiles T_i. Taking the image by $\tilde{h}$, one deduces the same property at x, unless $k+1$ is equal to one of the following values: q_1, q_2, q_3, q_1+q_2, q_1+q_3, q_2+q_3. In these last case, the edges at x are not all images of edges at $\tilde{h}^{-1}(x)$ and these situations should be considered more carefully. The local picture at any of these points is represented on figure 8 (these are the vertices of the central hexagon).

If for instance $k+1 = q_1$, the point $\tilde{h}^k(O)$ is surrounded by the tiles $\tilde{h}^{q_1-1}(A_1)$, $\tilde{h}^{q_1-1}(A_2)$ and $\tilde{h}^{q_1-1}(A_3)$. The image of the tile A_1 by $\tilde{h}^{q_1}$ is no more a tile, but if $q_1 < q_2$, one can replace it by the union $T^{p_1} \circ S^{p'_1}(A_2 \cup A_3)$ and with the tiles $\tilde{h}^{q_1}(A_2)$ and $\tilde{h}^{q_1}(A_3)$, a neighborhood of O is still covered. Moreover, the point x does not belong to any other edge since the points $f_2^{-1}(x)$ and $f_3^{-1}(x)$ do not belong to $\mathcal{X}$. (*N.b.:* The situation is different in the three degenerate cases $0 < q_1 = q_2 < q_3$, $0 = q_1 = q_2 < q_3$ and $0 < q_1 = q_2 = q_3$.)

The other cases can be described in a similar way. ☐

Corollary 2.19. *Any point $x \in \mathcal{X}$ belongs to a unique tiling of $\mathbb{R}^2$. Any edge belongs to a unique tiling of $\mathbb{R}^2$.*

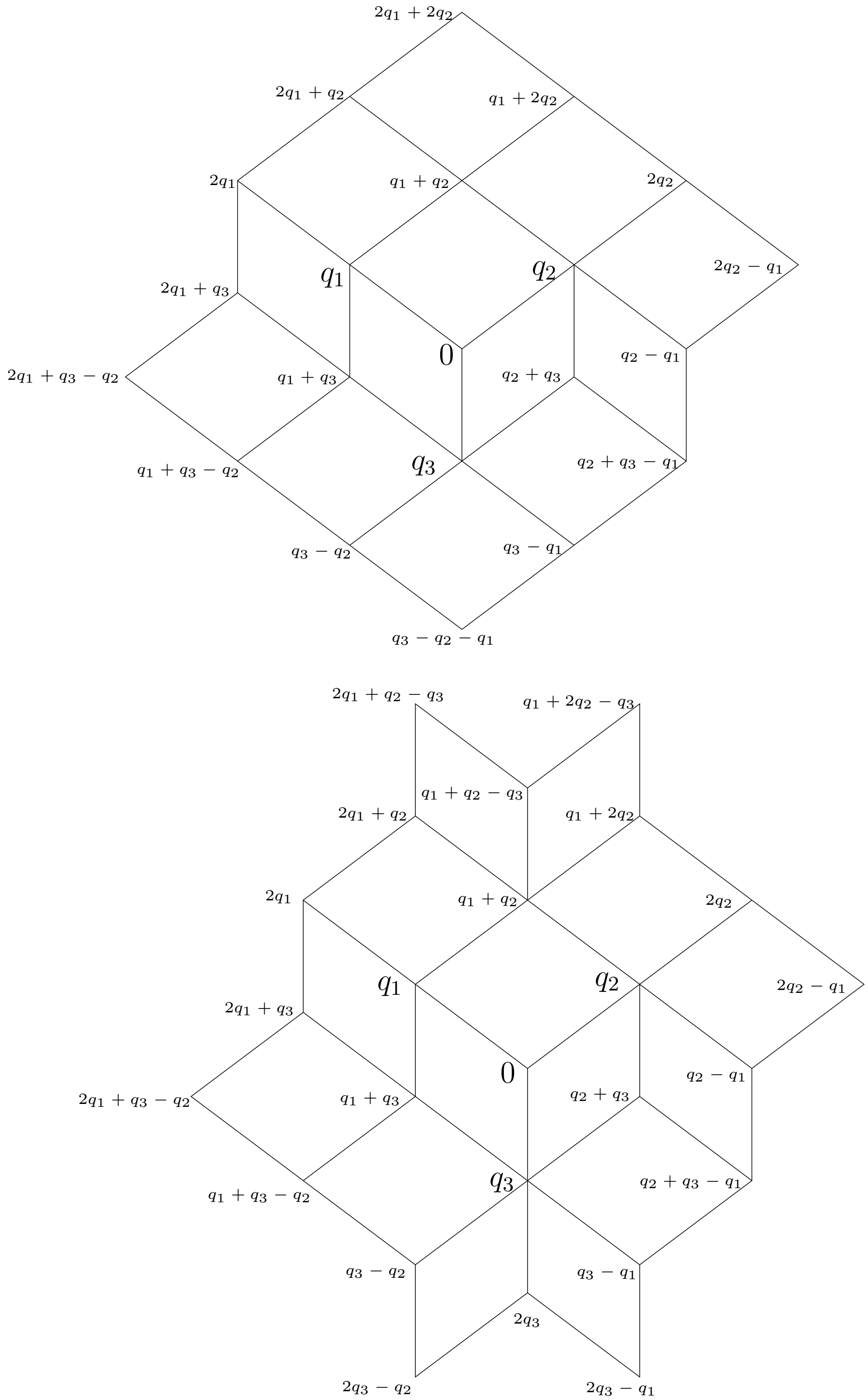


Figure 8: The tiling at the point $(0, 0)$: at each vertex, we give the number of iterations. We assumed $q_1 < q_2 < q_3$. The two pictures represent the cases $q_1 + q_2$ is less (first picture) or larger or equal (second picture) than q_3.

Let us denote by $\mathcal{T}$ the partition of $\mathcal{X}$ by points having the same tiling.

Action of $\mathbb{Z}^3$ *on the tilings.* The maps S and T act on $\mathcal{X}$ and preserve the edges, the tiles and the tilings. Hence they also act on $\mathcal{T}$. Note that $\tilde{h}$ does not preserve the set $\mathcal{X}$: the points $\tilde{h}^k(0,0)+(n,m)$ have no image in $\mathcal{X}$ by $\tilde{h}$ if $k=q_1+q_2+q_3-1$ or by $\tilde{h}^{-1}$ if $k=0$. One thus define a bijection $H\colon \mathcal{X}\to\mathcal{X}$ such that

- $H(\tilde{h}^k(0,0)+(n,m))=\tilde{h}^{k+1}(0,0)+(n,m)$ if $0\le k<q_1+q_2+q_3-1$
- $H(\tilde{h}^{q_1+q_2+q_3-1}(0,0)+(n,m))=(p_1,p_1')+(p_2,p_2')+(p_3,p_3')+(n,m)$.

Note that the tile are preserved unless they contain one of the points $\tilde{h}^{q_1+q_2+q_3-1}(0,0)+(n,m)$ as a vertex. This point is surrounded by three tiles $f_1^{q_1-1}(A_1)$, $f_2^{q_2-1}(A_2)$ and $f_1^{q_3-1}(A_3)$ that define an hexagon. This hexagon is mapped by $\tilde{h}$ on the hexagon which is the union of the three tiles A_1, A_2, A_3 translated by $(p_1,p_1')+(p_2,p_2')+(p_3,p_3')$. This shows that the map H preserve however the tilings. One gets three commuting maps S, T, H, hence an action of $\mathbb{Z}^3$ which is transitive on the sets $\mathcal{X}$ and $\mathcal{T}$.

There is only one tiling. Let us denote by $\mathcal{P}$ the tiling that contains the point $(0,0)$ as a vertex and define the maps $F_i=H^{q_i}\circ S^{-p_i}\circ T^{-p_i'}$ for $i\in\{1,2,3\}$. These maps send $(0,0)$ on other vertices of $\mathcal{P}$. This implies that the tiling $\mathcal{P}$ is invariant by the maps F_1,F_2,F_3. Since the matrix $\begin{bmatrix} p_1 & p_1' & q_1\\ p_2 & p_2' & q_2\\ p_3 & p_3' & q_3\end{bmatrix}$ belongs to $GL(3\mathbb{Z})$, one deduces that the groups $<S,T,H>$ and $<F_1,F_2,F_3>$ coincide. This implies that the group $<S,T,H>$ fix the tiling $\mathcal{P}$. It also acts transitively on the tilings, hence, $\mathcal{P}=\mathcal{X}$ and there is only one tiling.

Conclusion. By iterating the point $(0,0)$ under H, one checks that for any point P in $\mathcal{X}$, the tiles that contain P as a vertex are the image by some map $H^k\circ S^{-n}\circ T^{-m}$ with $0\le k<q_1,q_2+q_3$, $(n,m)\in\mathbb{Z}^2$ of the tiles A_1,A_2, A_3. Since $\mathcal{P}$ is invariant by S and T, it projects on a tiling of $\mathbb{T}^2$ by parallelograms.

2.3.3 Restatement

Let $\tilde{h}$ be the lift of h to the plane whose rotation vector is $(\tilde{\alpha},\tilde{\alpha}')$, let S and T be the translations $S\colon(x,y)\mapsto(x+1,y)$ and $T\colon(x,y)\mapsto(x,y+1)$ and let us define the plane homeomorphisms $\tilde{h}_i=\tilde{h}^{q_i}\circ S^{-p_i}\circ T^{-p_i'}$. Note that they have no fixed point: they are Brouwer homeomorphisms, see the course by Marc Bonino for a presentation of Brouwer's theory.

Then, the tiling theorem is equivalent to the following property:

Under these notations, there exist three topological lines $\Delta_1,\Delta_2,\Delta_3\subset\mathbb{R}^2$ that are properly embedded in the plane and that satisfy the following properties:

- *There exists a point x_0 such that the intersection of any two distinct lines Δ_i and Δ_j is reduced to x_0.*
- *The line Δ_i is invariant by $\tilde{h}_i$ for each i.*

- *For each $i \neq j \neq k$, one component of $\mathbb{R}^2 \setminus \Delta_i$ contains $\tilde{h}_j(\Delta_i)$ and $\tilde{h}_k^{-1}(\Delta_i)$, the other one contains $\tilde{h}_j^{-1}(\Delta_i)$ and $\tilde{h}_k(\Delta_i)$.*

The third property says that Δ_i is a Brouwer line for $\tilde{h}_j$ and $\tilde{h}_k$ which is pushed by these two maps on different sides of Δ_i. The tile A_1 (for instance) may be obtained as the disk bounded by Δ_2, Δ_3, $\tilde{h}_3(\Delta_2)$ and $\tilde{h}_2(\Delta_3)$.

It would be interesting to prove the tiling theorem by constructing the lines $\Delta_1, \Delta_2, \Delta_3$ (the proof by J. Kwapisz uses flows in $\mathbb{T}^3$).

2.4 An application to the closure of the conjugacy classes

The arc translation theorem and the tiling theorem have the following consequences that generalize a property we already gave in section 1.1 for circle homeomorphisms.

Corollary 2.20. *Let h be an irrational pseudo-rotation of M which is the sphere $\mathbb{S}^2$ or the compact annulus $\mathbb{A}$ and denote by α its rotation number. Then, there exist conjugates of h that are arbitrarily close to the rotation R_α in the space of homeomorphisms of M.*

Corollary 2.21. *Let h be a totally irrational pseudo-rotation of the torus $\mathbb{T}^2$ and denote by (α, α') its rotation vector. Then, there exist conjugates of h that are arbitrarily close to the rotation $R_{\alpha,\alpha'}$ in the space of homeomorphisms of $\mathbb{T}^2$.*

This shows that the closure of the conjugacy class of a pseudo-rotation contains the closure of the conjugacy class of the corresponding rotation. We do not know if the converse is true. However Patrice Le Calvez has shown [40] that if h is a diffeomorphism of $\mathbb{T}^2$ whose orbits are all dense in $\mathbb{T}^2$ (i.e. if h is a minimal torus diffeomorphism), then it is the limit (for the C^0-topology) of homeomorphisms that are conjugate to the rotation.
N.b.: Because of the semi-continuity properties on the rotation set (see the course of François Béguin), if a sequence of pseudo-rotations h_n converges towards a pseudo-rotation h,then the sequence of the rotation numbers of h_n converges towards the rotation number of h.

We give the proof of the second corollary (which is simpler than the proof of the first one).

Proof of the corollary 2.21. Let h be a totally irrational pseudo-rotation of the torus and fix a Farey triple for its rotation vector $(\tilde{\alpha}, \tilde{\alpha}')$.

Consider a dynamical tiling $\mathcal{T}$ provided by theorem 2.14 (it is generated by three tiles A_1, A_2, A_3) and a dynamical tiling $\mathcal{T}_0$ for the rotation $R = R_{\alpha,\alpha'}$ (it is generated by three tiles B_1, B_2, B_3). If the Farey triple has small returns (as in proposition 2.13), then the tiles of $\mathcal{T}_0$ have arbitrarily small diameters.

We will define an "almost" conjugacy Θ, that is a homeomorphism of $\mathbb{T}^2$ that sends the tiles A_i of $\mathcal{T}$ on the tiles B_i of $\mathcal{T}_0$ and that satisfies $\Theta \circ h = R \circ \Theta$ everywhere but maybe on the union $h^{-1}(A_1 \cup A_2 \cup A_3)$. This implies that R and the conjugate $g = \Theta \circ h \circ \Theta^{-1}$ coincide outside $R^{-1}(B_1 \cup B_2 \cup B_3)$. Since the diameter of $B_1 \cup B_2 \cup B_3$ can be chosen small, this implies that R and g are arbitrarily close for the C^0-distance on the space of homeomorphisms of $\mathbb{T}^2$.

Let us define x_0 as the point which is the intersection of the tiles A_i and y_0 as the point which is the intersection of the tiles B_i. The map Θ should send the point $h^i(x_0)$ on the point $h^i(y_0)$ for each $0 \leq i < q_1 + q_2 + q_3$. The map Θ is then defined on the 1-skeleton of $\mathcal{T}$: one chooses any homeomorphism that sends the edge of $\mathcal{T}$ between the points x_0 and $f^{q_i}(x_0)$ on the corresponding edge of $\mathcal{T}_0$ between the points y_0 and $f^{q_i}(y_0)$. Any edge e of $\mathcal{T}$ is the image by some iterate h^k of some edge e_0 between the point x_0 and $f^{q_i}(x_0)$. The map Θ on e is defined by the formula $R^k \circ \Theta \circ h^{-k}$.

Now the homeomorphism Θ has been defined on the boundary of each tile. One extends it arbitrarily on the interior of the tiles A_1, A_2, A_3 (by using Schoenfliess theorem, see the courses by Marc Bonino and Frédéric Le Roux). On any tile $A = h^k(A_i)$, the map Θ is now defined by the formula $R^k \circ \Theta \circ h^{-k}$. $\square$

3 Smooth Liouvillean examples

This part is devoted to the construction of smooth irrational pseudo-rotations having pathological dynamical properties. Its ancestors are examples by L. Shnirelman [50] and A. Besicovitch [7] of dynamics on the sphere having a dense orbit. The method was developed later on by D. Anosov and Katok in [1] and many generalizations were given recently. A modern exposition of the subject can be found in [14].

In this introduction, we will only consider dynamics on the sphere $\mathbb{S}^2$ and build two different examples (that are obviously not conjugate to a rotation):

- A C^∞ irrational pseudo-rotation of $\mathbb{S}^2$ having a dense orbit. (The dynamics is then said to be *topologically transitive.*)
- A C^∞ irrational pseudo-rotation of $\mathbb{S}^2$ having a minimal invariant compact set with positive Lebesgue measure and which is connected but not locally connected at any point. (This set will be a *pseudo-circle.*)

Other examples will be presented in part 4. The control of finer ergodic properties will be presented by Bassam Fayad in his course (see also [16]).

One feature of this construction is that it produces irrational pseudo-rotations h with Liouvillean rotation number. The reason is that it uses the action of a whole one-parameter group of rotations $(R_t)_{t\in\mathbb{T}^1}$. The dynamics h is obtained as the limit of "trivial dynamics" h_n that are conjugates $\varphi_n \circ R_{\frac{p_n}{q_n}} \circ \varphi_n^{-1}$ to rational rotations. As n goes to infinity, the rational rotation numbers $\frac{p_n}{q_n}$ converge toward the rotation number α of h but the conjugacies φ_n become more and more intricate and prevent h from beeing conjugate to the rotation R_α. One can however ensure that h is also the limit of the conjugates $\varphi_n \circ R_\alpha \circ \varphi_n^{-1}$ of the rotation R_α: this provides us with an example of rotation R_α whose (smooth) conjugacy class contains in its closure some wild dynamics h.

3.1 General presentation of the method

We introduce the standard sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$, the points $N = (0, 0, 1)$ and $S = (0, 0, -1)$ and the equator $\mathcal{C} = \mathbb{S}^2 \cap \left(\mathbb{R}^2 \times \{0\}\right)$.

Let us consider the standard action $R\colon \mathbb{T}^1 \to \mathrm{Diff}^\infty(\mathbb{S}^2)$ by the group of the rotations of the sphere $\mathbb{S}^2$ that fixes each point N and S. The method consists in building inductively

- a sequence of rational numbers $\left(\frac{p_n}{q_n}\right)_{n\geq 0}$,
- a sequence of C^∞ diffeomorphisms $(\varphi_n)_{n\geq 0}$ that are isotopic to the identity, preserve the Lebesgue measure and coincide with the identity in (non-uniform) neighborhoods of the fixed points N, S,

so that the sequence of diffeomorphisms

$$h_n = \varphi_n \circ R_{\frac{p_n}{q_n}} \circ \varphi_n^{-1}$$

converges towards a diffeomorphism h for the C^∞-topology. Clearly, h preserves the points N, S, the Lebesgue measure and is isotopic to the identity.

Each conjugacy φ_n is obtained from the previous one φ_{n-1} by pre-composition with a diffeomorphism M_n that is isotopic to the identity, preserves the Lebesgue measure and coincide with the identity in neighborhoods of the fixed points N, S, so that we have

$$\varphi_0 = M_0 = \mathrm{Id}_{\mathbb{S}^2} \quad \text{and} \quad \varphi_n = M_1 \circ \cdots \circ M_n.$$

A key property is the following.

Commutation. *The diffeomorphism M_{n+1} commutes with the rotation $R_{\frac{p_n}{q_n}}$ for each $n \geq 0$.*

As a consequence h_n also coincides with $\varphi_{n+1} \circ R_{\frac{p_n}{q_n}} \circ \varphi_{n+1}^{-1}$.

The convergence is ensured by the following assumption.

Convergence. *The angle $\frac{p_{n+1}}{q_{n+1}}$ is chosen after φ_{n+1} and arbitrarily close to $\frac{p_n}{q_n}$.*

In this way, one immediately deduces that $h_{n+1} = \varphi_{n+1} \circ R_{\frac{p_{n+1}}{q_{n+1}}} \circ \varphi_{n+1}^{-1}$ can be chosen arbitrarily close to the previous diffeomorphism $h_n = \varphi_{n+1} \circ R_{\frac{p_n}{q_n}} \circ \varphi_{n+1}^{-1}$ so that the sequence (h_n) converges towards a diffeomorphism h.

Note that we did not assume that the sequence of the conjugacies (φ_n) converges: usually this is not the case. However, the map h_{n+1} is close to the map h_n and during many iterates their orbits look like very similar. However, for big iterates of the form $k.q_n$, precisethe rotations $R_{\frac{p_n}{q_n}}^{k.q_n} = \mathrm{Id}$ and $R_{\frac{p_{n+1}}{q_{n+1}}}^{k.q_n}$ can be very different and the huge distortion of φ_n appears.

Also, it is difficult to control how close are $\frac{p_{n+1}}{q_{n+1}}$ and $\frac{p_n}{q_n}$: in general, the convergence is rather strong. In particular, the sequence $\left(\frac{p_n}{q_n}\right)$ converges toward an angle α which is well approximated by the rationals; hence α is irrational and Liouvillean. We would however need estimations if one wanted to give an explicit class of Liouvillean numbers that can be obtained by this method.

For each construction that follows the Anosov-Katok technique, two points should be explained:

- How close should be $\frac{p_{n+1}}{q_{n+1}}$ from $\frac{p_n}{q_n}$?
- What are the properties satisfied by M_{n+1} and φ_{n+1}?

Choice of the angle $\frac{p_{n+1}}{q_{n+1}}$. Let us illustrate some properties that can be obtained from the Convergence Hypothesis by requiring a strong enough convergence of the sequence (h_n).

Properties. - *The diffeomorphism h is arbitrarily close to the identity.*

- *The diffeomorphism h is contained in the closure of the C^∞-conjugacy class of R_α.*
- *The sequence of iterates (h^{q_n}) goes to the identity. (One sometimes says that the dynamics is* quasiperiodic.*)*

- *The diffeomorphism h is an irrational pseudo-rotation.*

Proof. One can choose $h_0 = \mathrm{Id}_{\mathbb{S}^2}$ and h arbitrarily close to h_0, giving the first property. One can require that h_n is ε_n-close to $\varphi_n \circ R_\alpha \circ \varphi_n^{-1}$ for some sequence (ε_n) that goes to zero, proving the second property. Similarly, the diffeomorphism h^{q_n} can be chosen δ_n-close to the diffeomorphism $h_n^{q_n} = \mathrm{Id}_{\mathbb{S}^2}$ for some sequence (δ_n) that goes to zero, showing the third property.

There exists some constants $C_n, \eta_n > 0$ such that the distance between $h_n^k(x)$ and x is larger than η_n for each point x outside the C_n-neighborhood of N, P and any integer $|k| < q_n$. This property is still satisfied by the diffeomorphisms close, hence by h. One deduces that N and S are the only periodic points of h. This implies that h is an irrational pseudo-rotation

□

Construction of the diffeomorphism M_{n+1}. Let us finish this general description of the Anosov-Katok technique by discussing the choice of the diffeomorphism M_{n+1}. Note that since M_{n+1} commutes with h_n and since h_n is conjugate to a periodic rotation, it is easier to first define a map $\overline{M}_{n+1}$ on the quotient space $\left(\mathbb{S}^2 \setminus \{N, S\}\right)/h_n$ and then to consider the lift M_{n+1} on $\mathbb{S}^2 \setminus \{N, S\}$ which coincides with the identity in a neighborhood of N and S.

The quotient $\left(\mathbb{S}^2 \setminus \{N, S\}\right)/h_n$ is an open annulus which is diffeomorphic to the annulus $A_n = \left(\mathbb{S}^2 \setminus \{N, S\}\right)/R_{\frac{p_n}{q_n}}$, through the quotient map φ_n. It is usually on the annulus A_n that we will define $\overline{M}_{n+1}$.

The diffeomorphism $\overline{M}_{n+1}$ will be chosen accordingly to the property of h one wants to obtain. One delicate point is to ensure that the Lebesgue measure is preserved. The following consequence of Moser's argument [45] is a "smooth version" of Oxtoby-Ulam theorem.

Theorem 3.1 (Moser)**.** *Let us consider a compact manifold $\mathcal{M}$ (with boundary) and two smooth volume forms ω and ω' with the same total volume (i.e. $\int_M \omega = \int_M \omega'$) and which coincide in a neighborhood of $\partial\mathcal{M}$.*

Then, there exists a smooth diffeomorphism ψ of M which is equal to the identity in a neighborhood of $\partial\mathcal{M}$ and which sends ω on ω': we have $\psi_*\omega = \omega'$.

3.2 Transitivity

Up to here we did not prove that the limit diffeomorphism may not be conjugate to a rotation. A first example is ensured by the following property: a homeomorphism h of a topological space X is *transitive* if it has a dense orbit.

Theorem 3.2. *There exists a (Lebesgue measure preserving) C^∞ irrational pseudo-rotation of $\mathbb{S}^2$ which is transitive.*

One may wonders what is the size of the set $\mathcal{D}$ of points whose orbit is dense in $\mathbb{S}^2$: we will see that it is a dense G_δ subset of $\mathbb{S}^2$. Anosov and Katok have shown that $\mathcal{D}$ may be chosen with full Lebesgue measure (the Lebesgue measure can be chosen ergodic). On the other hand, Le Calvez and J.-C. Yoccoz [41] have shown that $\mathcal{D} \subsetneq \mathbb{S}^2 \setminus \{N, S\}$ (the dynamics is never minimal).

Let us begin with a criterion for the transitivity:

Proposition 3.3. *Let h be a homeomorphism of a compact space X. There exists a positive orbit $\{h^n(x),\ n \geq 0\}$ which is dense in X if and only if for any non-empty open sets $U, V \subset X$,*

there exists an integer $n \geq 1$ such that $h^n(U) \cap V$ is non-empty.
In this case, the set of points whose orbit is dense in X is a dense G_δ subset of X.

Exercise. The direct implication in the proposition is immediate, prove the reverse one.
Indication: Let (U_i) be a countable basis of open sets in X. Prove that for each i, the set of points $x \in X$ having a positive iterate in U_i is a dense open set of X.

The former criterion motivates the following assumption:

Transitivity. *The image of the equator $\mathcal{C}$ by φ_n is $\frac{1}{n}$ dense in $\mathbb{S}^2$.*

We now explain the construction.

a) Construction of the angle $\frac{p_{n+1}}{q_{n+1}}$. Let us see how the Transitivity Hypothesis implies the transitivity provided that the convergence of the sequence (h_n) is chosen strong enough. We will assume that in the induction the angle $\frac{p_n}{q_n}$ and the map φ_{n+1} have already been build. Note that for any choice of $\frac{p_{n+1}}{q_{n+1}}$, the circle $\varphi_{n+1}(\mathcal{C})$ will be preserved by h_{n+1}.

By choosing the angle $\frac{p_{n+1}}{q_{n+1}}$ close enough to $\frac{p_n}{q_n}$, one ensures that q_{n+1} is arbitrarily large and that the orbits contained in the circle $\varphi_{n+1}(\mathcal{C})$ are $\frac{1}{n+1}$-dense in the circle. One deduce that there exists a point $x \in \mathbb{S}^2$ whose q_{n+1} first iterates by h_{n+1} are $\frac{2}{n+1}$-dense in the sphere. If the convergence of (h_n) is strong enough, this is also true for h. As a consequence, the diffeomorphism h satisfies the criterion stated in proposition 3.3 and h is transitive.

b) Construction of the diffeomorphism M_{n+1}. The Transitivity Hypothesis will obtained by a suitable choice of the diffeomorphism M_{n+1}: it has to send the circle $\varphi_n(\mathcal{C})$ on a closed curve which is $\frac{1}{n}$-dense in $\mathbb{S}^2$. As we explained in section 3.1, we define the quotient diffeomorphism $\overline{M}_{n+1}$ on the quotient space $(\mathbb{S}^2 \setminus \{N, s\})/h_n$ so that the circle $C_n = \varphi_n(\mathcal{C})/h_n$ is mapped on a curve which is ε_n-dense for a constant $\varepsilon_n > 0$ arbitrarily small. Such a conservative diffeomorphism $\overline{M}_{n+1}$ is easy to define since one needs to control only a finite number of points in C_n (of course one can apply Moser's argument).

3.3 Pathological minimal sets

We now describe a second example which is not transitive but which shows that for irrational pseudo-rotations, the minimal separating sets are not necessary invariant closed curves. M. Handel was the first to observe [27] how pathological invariant sets can appear in dynamics and Herman adapted [31] his construction to the Anosov-Katok technique. The positive Lebesgue measure of the minimal set is written in [14].

An invariant non-empty compact set K of a homeomorphism h on X is *minimal* if it does not contain any proper invariant compact set: equivalently, all the orbits of points in K are dense in K. Zorn lemma implies that if X is compact, it always contains a minimal set (consider the family of non-empty invariant compact sets, ordered by the inclusion).

Let us also recall that a set K is *locally connected at some point $x \in K$* if there exists a basis of neighborhood of x in K which are connected.

Theorem 3.4. *There exists a (Lebesgue measure preserving) C^∞ irrational pseudo-rotation h of $\mathbb{S}^2$ which has a minimal invariant set Λ which is connected, not locally connected at any point and has positive Lebesgue measure.*

We consider a belt obtained by thickening the equator $\mathcal{C}$: this is a diffeomorphism between the annulus $C_0 = \mathbb{T}^1 \times [-1, 1]$ and a neighborhood of $\mathcal{C}$ which preserves the Lebesgue measure (up to a multiplicative constant for the normalization) so that the action of the rotations R_t on the sphere is identified with the action of the rotations $(x, y) \mapsto (x + t, y)$ on C_0. All the construction will take place in the annulus C_0.

We fix a constant $s \in (0, 1)$ and consider a nested sequence of annuli

$$C_n = \mathbb{T}^1 \times \left[-(1 + n^{-1}) \cdot \frac{s}{2}, (1 + n^{-1}) \cdot \frac{s}{2}\right].$$

The definition the compact set Λ uses the following assumption.

Support of M_n. *The map M_n coincides with the identity outside C_{n-1}.*

This assumption implies that φ_n coincides with all the maps φ_{n+k}, $k \geq 0$ outside C_n. In particular, the diffeomorphism h coincides with $\varphi_n \circ R_\alpha \circ \varphi_n^{-1}$ outside $\varphi_n(C_n)$. We thus obtain a decreasing sequence $(\varphi_n(C_n))$ of annuli and the intersection is a connected compact set Λ which is invariant by h. One deduces the following property:

Property. *The set Λ is a connected invariant compact set whose Lebesgue measure is s. The open set $\mathbb{S}^2 \setminus \Lambda$ has two connected components, invariant by h. The induced dynamics is conjugate to the rotation R_α.*

We now come to the minimality property.

Minimality. *The image of each circle $C_y = \mathbb{T}^1 \times \{y\} \subset C_{n+1}$ by φ_{n+1} is $\frac{1}{n}$-dense in $\varphi_{n+1}(C_{n+1})$.*

Let us explain the construction.

a) Construction of the angle $\frac{p_{n+1}}{q_{n+1}}$. Let us see how the Minimality Hypothesis implies the minimality of the dynamics of h on Λ, provided that the convergence of the sequence (h_n) is chosen strong enough. We will assume that in the induction the angle $\frac{p_n}{q_n}$ and the map φ_{n+1} have been build. Note that for any choice of $\frac{p_{n+1}}{q_{n+1}}$, the circle $\varphi_{n+1}(C_y)$ will be preserved by h_{n+1}.

By choosing the angle $\frac{p_{n+1}}{q_{n+1}}$ close enough to $\frac{p_n}{q_n}$, one ensures that q_{n+1} is arbitrarily large and that all the orbits contained in the circle $\varphi_{n+1}(C_t)$ are $\frac{1}{n}$-dense in the circle. One deduces that there exists for any point $x \in \varphi_{n+1}(C_{n+1})$, the q_{n+1} first iterates are $\frac{2}{n}$-dense in $\varphi_{n+1}(C_{n+1})$. If the convergence of (h_n) is strong enough, this is also true for h. As a consequence, all the orbit of points of $\Lambda \subset \varphi_{n+1}(C_{n+1})$ are $\frac{2}{n}$-dense in Λ giving the minimality.

b) Construction of the diffeomorphism M_{n+1}.
In order to prove the Minimality Hypothesis, one needs to show that the annulus $\varphi_{n+1}(C_{n+1})$ is thin; since its measure is larger than s, it should also be very long and twisted in the sphere.

Hence, one should carefully choose the embedding $\varphi_{n+1}(C_{n+1})$ inside $\varphi_n(C_n)$. Let us consider the partition of C_{n+1} into q_{n+1} rectangles $C_{n+1,i}$ of the form

$$C_{n+1,i} = \left[\frac{i}{q_{n+1}}, \frac{i+1}{q_{n+1}}\right] \times \left[-(1+(n+1)^{-1}) \cdot \frac{s}{2}, (1+(n+1)^{-1}) \cdot \frac{s}{2}\right].$$

Each rectangle $C_{n+1,i}$ has a diameter bounded from below but has a small area. One can thus expect that the diameters of the images $\varphi_{n+1}(C_{n+1,i})$ are small, implying the Minimality Hypothesis.

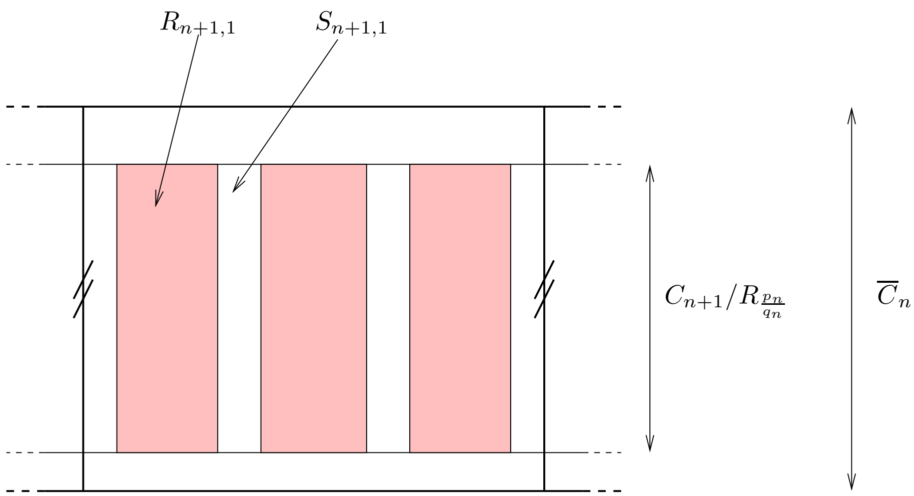


Figure 9: The covering of $C_{n+1}/R_{\frac{p_n}{q_n}}$ by vertical rectangles and separating strips.

We will now work inside the quotient annulus $\overline{C}_n = C_n/R_{\frac{p_n}{q_n}}$. Let us consider a large odd integer $2\ell_n+1$, a small positive constant $\varepsilon_n \ll (2\ell_n+1)^{-1}$ and introduce a covering (see figure 9) of the quotient annulus $C_{n+1}/R_{\frac{p_n}{q_n}} \subset \overline{C}_n$ by

- $2\ell_n + 1$ disjoint isometric vertical rectangles $R_{n,k}$ of size $L_n \times (1+(n+1)^{-1}).s$,
- $2\ell_n + 1$ separating strips $S_{n,k}$ of of size $\varepsilon_n^2 \times (1+(n+1)^{-1}).s$.

The map $\overline{M}_{n+1}$ defined on $\overline{C}_n$ sends (see figure 10)

- *the vertical rectangles $R_{n,k}$ on disjoint horizontal rectangles $R'_{n,k}$* of size $(q_n^{-1} - 2\varepsilon_n) \times L'_n$ having the same area as the rectangles $R_{n,k}$. One can assume that the interior of the image rectangles are disjoint from a vertical gap Δ_n which is a rectangle of size $2\varepsilon_n \times (1+(n+1)^{-1})$. Also the vertical distance between two consecutive rectangles is ε_n. Moreover, the map $\overline{M}_{n+1} \colon R_{n,k} \to R'_{n,k}$ can be chosen affine.
- *the separating strips $S_{n,k}$ on loops $S'_{n,k}$* having the same area and contained in the gap Δ_n.

The vertical rectangles $R_{n,k}$ are labeled by $k \in \mathbb{Z}/(2\ell_n+1)\mathbb{Z}$ according to their circular ordering in the annulus but we do not assume that this ordering matches with the vertical ordering of the rectangles $R'_{k,n}$: for each integer k, the rectangle $R'_{n,k+1}$ could be any of the three rectangles

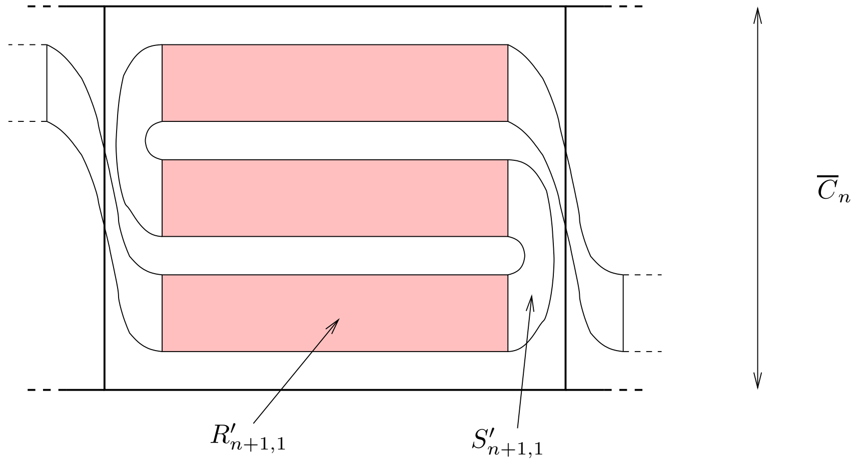


Figure 10: The image of $C_{n+1}/R_{\frac{p_n}{q_n}}$ by $\overline{M}_{n+1}$ contains horizontal rectangles and loops.

above or below $R'_{n,k}$. The way one chooses this ordering will be detailed later. However, this shows that one can choose each loop S'_{n+k} (whose area is bounded by ε_n^2) in a rectangle of size $2\varepsilon_n \times (3L_n + 2\varepsilon_n)$.

To complete the construction, on has to extend the map $\overline{M}_{n+1}$ to the whole annulus $\overline{C}_n$ by using Moser's theorem 3.1. This is possible if the areas of the two components of $C_n \setminus C_{n+1}$ coincide with the area of the components of $\overline{C}_n \setminus M_{n+1}(C_{n+1})$. Note that $M_{n+1}(C_{n+1})$ is contained in an annulus of height close to $(1 + (n+1)^{-1})$ whereas the height of C_n is $(1 + n^{-1})$. Hence, it is possible to compose by a vertical construction in order to adjust the areas. One then lifts the map $\overline{M}_n$ as a diffeomorphism M_n.

Note that by this construction, if the integer q_{n+1} is chosen large enough, then the image $M_{n+1}(C_{n+1,i})$ of each rectangle $C_{n+1,i}$ of the partition of C_{n+1} will have a diameter bounded by $4L_n$ which can be assumed arbitrarily small if the integer ℓ_n has been chosen large enough. In this way, one gets the following properties:

Property. - *The rectangles $\varphi_{n+1}(C_{n+1,i})$ have arbitrarily small diameter.*

- *The Minimality Hypothesis is satisfied.*
- *Each rectangle $\varphi_{n+1}(C_{n+1,i})$ meets at most two rectangles $\varphi_n(C_{n,j})$.*

c) Combinatorics of the loops $S'_{n,i}$. Let us add one more hypothesis:

Winding. *For any rectangle C_{n,i_0}, there exists two rectangles $C_{n+1,j}$ and $C_{n+1,k}$ such that*

- *both rectangles $\varphi_{n+1}(C_{n+1,j})$ and $\varphi_{n+1}(C_{n+1,k})$ are contained in $\varphi_n(C_{n,i_0})$;*
- *the image by φ_{n+1} of each component of $C_{n+1} \setminus (C_{n+1,j} \cup C_{n+1,k})$ meets all the rectangles $\varphi_n(C_{n,i})$.*

This assumption implies that the minimal set Λ is wild:

Property. *The minimal set Λ is not locally connected at any point.*

Proof. Let us consider any point $x \in \Lambda$ and an arbitrarily small neighborhood K of x in Λ which is connected. It contains the rectangle $\varphi_n(C_{n,i_0})$ of some partition. By the Winding Hypothesis, there should exist two rectangles $C_{n+1,j}$ and $C_{n+1,k}$ such that $\varphi_{n+1}(C_{n+1,j})$ and $\varphi_{n+1}(C_{n+1,k})$ are contained in K and such that the image by φ_{n+1} of each component of $C_{n+1} \setminus (C_{n+1,j} \cup C_{n+1,k})$ meets all the rectangles $\varphi_n C_{n,i}$. Since K is connected and contained in the annulus $\varphi_{n+1}(C_{n+1})$, this implies that K meets the image of all the rectangles $C_{n+1,\ell}$ that belongs to one of the components of $C_{n+1} \setminus (C_{n+1,j} \cup C_{n+1,k})$. Since the rectangles $\varphi_{n+1}(C_{n+1,\ell})$ have a diameter bounded by some small constant δ_{n+1}, this shows that K is δ_{n+1}-dense in $\varphi_{n+1}(C_{n+1})$. In particular, $K = \Lambda$ which is a contradiction.

□

Exercise. Prove that Λ is *indecomposable*: it is not the union of two proper connected compact subsets.

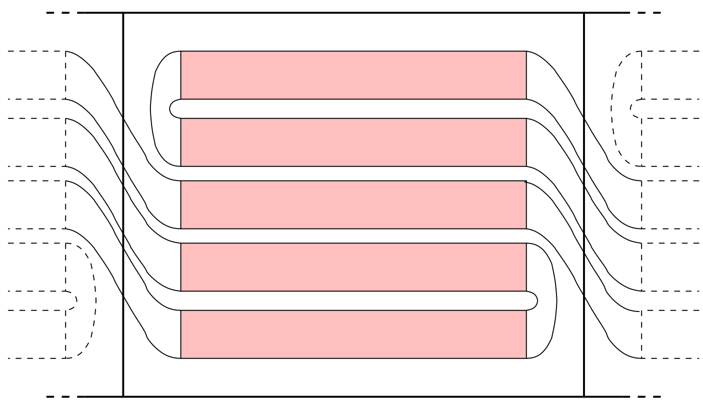


Figure 11: The combinatorics of the loops $S'_{n+1,i}$.

It remains to explain how to connect the horizontal rectangles $R'_{n+1,k}$ by small loops in order to satisfy the Winding Hypothesis. Let us order the horizontal rectangles accordingly to their vertical position: $R'_{n+1}(1)$ will be the rectangle at the top and $R'_{n+1}(2\ell_n + 1)$ the rectangle at the bottom. Let us connect

- the left side of $R'_{n+1}(1)$ to the left side of $R'_{n+1}(2)$ and the right side of $R'_{n+1}(1)$ to the left side of $R'_{n+1}(3)$,
- the right sides of $R'_{n+1}(2i)$ and $R'_{n+1}(2i{+}1)$ to the left sides of $R'_{n+1}(2i{+}2)$ and $R'_{n+1}(2i{+}3)$, for $1 \leq i < \ell_n$,
- the right sides of $R'_{n+1}(2\ell_n)$ and $R'_{n+1}(2\ell_n + 1)$ together.

In this way (see figure 11), the union of the rectangles $R'_{n+1,k}$ with the loops $S'_{n+1,k}$ is an annulus essentially embedded in $\overline{C}_n$.

Moreover, each rectangle C_{n,i_0} is a fundamental domain of $\overline{C}_n$. Hence, one can choose two rectangles $C_{n+1,j}$ and $C_{n+1,k}$ whose images by φ_{n+1} are contained in $\varphi_n(C_n)$ and whose images by $\overline{M}_n$ are contained in $R'_{n+1}(1)$ and in $R'_{n+1}(2\ell_n+1)$ respectively. Since each component of $\overline{M}_n(C_{n+1}) \setminus \left(R'_{n+1}(1) \cup R'_{n+1}(2\ell_n+1)\right)$ spirals ℓ_n times around $\overline{C}_n$, the images of the two components of $C_{n+1} \setminus (C_{n+1,j} \cup C_{n+1,k})$ by φ_{n+1} intersect ℓ_n successive rectangles $C_{n,i}$. If ℓ_n is larger than q_n, one gets the Winding Hypothesis.

Remark 3.5.

1. For the irrational pseudo-rotation h we have built, *there does not exist any curve γ which satisfies the arc translation theorem 2.1 for any integer $n \in \mathbb{N}$.*

   Proof. Let us consider the components of $\mathbb{S}^2 \setminus \Lambda$: these are two open disks D_N and D_S. The map h is conjugate to R_α on each of these disks. Thus, each disk is a union of invariant circles C that are the closure of each orbit by h.

   If there exists a curve γ that is disjoint from all its iterates, it intersects each circle C in exactly one point (otherwise the iterates of γ would not be disjoint and cyclically ordered as for the rotation R_α. Consequently, γ can be decomposed into three connected curves $\gamma_N \subset D_N$, $\gamma_\Lambda \subset \Lambda$ and $\gamma_S \subset D_S$.

   Let γ' be an iterate of γ. For n large enough, the rectangles $C_{n,j}$, $j \in \mathbb{Z}/q_n\mathbb{Z}$, that intersect γ (resp. γ') are consecutive rectangles $C_{n,k}, C_{n,k+1}, \dots, C_{n,\ell}$ (resp. $C_{n,r}, C_{n,r+1}, \dots, C_{n,s}$). The intersection of the rectangles $C_{n,k} \cup C_{n,k+1} \cup \dots C_{n,s}$ (resp. $C_{n,s} \cup C_{n,s+1} \cup \dots C_{n,\ell}$) as n goes to $+\infty$ define a connected compact set Λ_1 (resp. Λ_2) and by construction $\Lambda = \Lambda_1 \cup \Lambda_2$.

   Since γ and γ' are disjoint, Jordan's theorem implies that $\mathbb{S}^2 \setminus (\gamma \cup \gamma')$ has two connected components. Both intersect Λ. Moreover, Λ_1 is disjoint from one of them and Λ_2 from the other. Hence, both sets Λ_i are proper subset of Λ. This shows that Λ is decomposable and contradicts the exercise above.

2. It is possible to strengthen the Winding Hypothesis so that Λ is *hereditarily indecomposable*: any connected compact subset $\Lambda' \subset \Lambda$ is indecomposable (see exercise 3.3). Under this additional property, all the limit sets Λ are homeomorphic to a universal set, called the *pseudo-circle*, see [8, 18].

3. One can replace the Winding Hypothesis by a very different one:
   For any rectangle C_{n,i_0}, and any two rectangles $\varphi_{n+1}(C_{n+1,j})$ and $\varphi_{n+1}(C_{n+1,k})$ that are contained in $\varphi_n(C_{n,i_0})$, the image by φ_{n+1} of one of the components of $C_{n+1} \setminus (C_{n+1,j} \cup C_{n+1,k})$ is contained in $\varphi_n(C_{n,i_0-1} \cup C_{n,i_0} \cup C_{n,i_0+1})$.
   Under this assumption, the set Λ is a simple closed curve with positive Lebesgue measure.

3.4 Hairy germs

Exercise. With the same technique, it is possible to build the example d) described at section 1.2: we identify a neighborhood of N is identified with the disk D_0 of radius 1 and we consider a sequence of disks D_n of radius $(1+n^{-1}).s$, for some constant $s > 0$. The map M_n will coincide with the identity outside D_{n-1}.

Imagine an hypothesis on φ_n which implies that the germ of the diffeomorphism h at N is hairy.

4 Fibered examples

In this part, we introduce another class of examples on the torus which use the product decomposition $\mathbb{T}^2 = \mathbb{T}^1 \times \mathbb{T}^1$. These are the *skew products over an irrational rotation of the circle*: one considers the maps of the form

$$h\colon (x, y) \mapsto (x + \alpha, h_2(x, y)).$$

These skew products have a more rigid dynamics than the general diffeomorphisms of $\mathbb{T}^2$ that are isotopic to the identity but exhibit interesting properties that will be discussed in the course by Tobias Jaeger (see also [33]). In particular, they have a unique rotation vector, of the form $(\tilde{\alpha}, \tilde{\beta})$ and $\tilde{\beta}$ is called the *fibrewise rotation number* (see [30]).

Two classes of examples will be defined:

- The Lebesgue-measure preserving homeomorphisms were first studied by H. Furstenberg [22]. They are or the form

  $$h\colon (x, y) \mapsto (x + \alpha, y + \beta + \varphi(x)),$$

  with $\int_{\mathbb{T}^1} \varphi(x)dm = 0$ for the Haar measure m on $\mathbb{T}^1$. The rotation vector in this case may be computed by integrating the displacement $\tilde{h}(x, y) - (x, y)$ and is equal to $(\tilde{\alpha}, \tilde{\beta})$.

- Another class of examples is induced by the projective action of cocycles in $SL(2, \mathbb{R})$.

This allows to produce irrational pseudo-rotations that are not conjugate to a rotation. Due to the fibered structure, we will not be able to build strongly exotic topological properties. What we will show however is that the measurable dynamics can be pathological. As explained in the introduction of these notes, many examples of irrational pseudo-rotations either are not differentiable or have a Liouvillean rotation vector.

4.1 Background in ergodic theory

We will study *measurable dynamical systems*, i.e. bijective bimeasurable maps T on a set X with a σ algebra $\mathcal{A}$. A *measured dynamical system* is a measurable dynamical system endowed with an invariant probability measure μ. If X is a compact set, $\mathcal{A}$ the Borel σ algebra and T is a homeomorphism, such a measure always exists (this is the Bogoliouboff-Kryloff theorem).

When there exists only one invariant measure, one says that the dynamics is *uniquely ergodic*. Otherwise, any invariant measure can be decomposed into "elementary" measures, called *ergodic measures*: these are the invariant measures μ such that for any invariant measurable set $A \subset X$ the measure $\mu(A)$ equals 0 or 1.

For example th eirrational rotation son the circle are uniquely ergodic.

Two measured dynamics (X, μ, S) and (Y, ν, T) are *metrically isomorphic* if there exist two measurables sets $X_0 \subset X$ and $Y_0 \subset Y$ such that $\mu(X_0) = \nu(Y_0) = 1$ and a bimeasurable bijection $\Theta\colon X_0 \to Y_0$ such that $\Theta \circ S = T \circ \Theta$. If Θ is only surjective, one says that (X, μ, S) is an *extension* of (Y, ν, T) (and that (Y, ν, T) is a *factor* of (X, μ, S)). A measured dynamics which does not have any factor which is a (non-trivial) rotation is said to be *weakly mixing*.

More generally, two measurable dynamical systems $(X, \mathcal{A}, S)$ and $(Y, \mathcal{B}, T)$ are *isomorphic* if there exist two measurables sets $X_0 \subset X$ and $Y_0 \subset Y$ such that $\mu(X_0) = \nu(Y_0) = 1$ for any S-invariant measure ν and any T-invariant measure S and a bimeasurable bijection $\Theta \colon X_0 \to Y_0$ such that $\Theta \circ S = T \circ \Theta$.

4.2 The Furstenberg examples

Let us consider a continuous map $\varphi \colon \mathbb{T}^1 \to \mathbb{R}^1$ such that $\int_{\mathbb{T}^1} \varphi(x) dm = 0$ (for the Haar measure m on $\mathbb{T}^1$) and define a homeomorphism of $\mathbb{T}^2$ isotopic to the identity by

$$h \colon (x, y) \mapsto (x + \alpha, y + \beta + \varphi(x)).$$

First remarks.

1. The map h preserves the Lebesgue measure and has a unique rotation vector, equal to (α, β). Hence it is an irrational pseudo-rotation.

2. h commutes with the *vertical translations*

$$\mathrm{Id} \times R_\omega \colon (x, y) \mapsto (x, y + \omega).$$

For this section, the reader should remember the definition of minimal sets defined in section 3.3.

a) Conjugacy to the rotation
One may want to conjugate h by a homeomorphism of the form $(x, y) \mapsto (x, y - \psi(x))$. One obtains the map: $(x, y) \mapsto (x + \alpha, y + \beta + \varphi(x) + \psi(x) - \psi(x + \alpha))$. Hence, in order to conjugate to the rotation $R_{\alpha,\beta}$, one has to solve the following (additive) *cohomological equation*:

$$\varphi(x) = \psi(x + \alpha) - \psi(x). \tag{4.1}$$

W. Gottschalk and G. Hedlund have shown [23] the following criterion. It uses the *Birkhoff sums* of φ which are the functions $S_n\varphi \colon \mathbb{T}^1 \to \mathbb{R}$, $n \geq 0$ defined by

$$S_n\varphi \colon x \mapsto \sum_{k=0}^{n-1} \varphi(x + k\alpha).$$

Theorem 4.1. *Let $\varphi \colon \mathbb{T}^1 \to \mathbb{R}$ be a continuous map.*
The cohomological equation (4.1) has a continuous solution $\psi \colon \mathbb{T}^1 \to \mathbb{R}$ if and only if the Birkhoff sums $S_n\varphi$ are uniformly bounded.

Proof. If ψ is a solution of (4.1), the Birkhoff sum $S_n\varphi(x)$ is equal to $\psi(x + n\alpha) - \psi(x)$. Hence the Birkhoff sums are uniformly bounded by $2\|\psi\|_\infty$.

For the other implication, one considers the lifted homeomorphism H on $\mathbb{T}^1 \times \mathbb{R}$ defined by $(x, y) \mapsto (x + \alpha, y + \varphi(x))$. Note that $H^n(x, y) = (x + n\alpha, y + S_n\varphi(x))$. Since the Birkhoff sums are uniformly bounded, each orbit by H is bounded. Hence there exists some non-empty invariant compact set K by H. By taking K smaller, one can assume that K is a minimal set.

Claim. *If (x_0, y_0) and $(x_0, y_0 + \omega)$ belong to K, then K is invariant by $T_\omega\colon (x, y) \mapsto (x, y + \omega)$.*
Proof. Let us consider the points (x, y) such that $T_\omega(x, y)$ also belongs to K. Since H and T_ω commute, this set is invariant by H. Moreover, it is non-empty and closed. Hence, this is K.

If K is invariant by some map $T_\omega\colon (x, y) \mapsto (x, y + \omega)$, with $\omega \neq 0$, then it is not bounded. So, the claim implies that each line $\{x\} \times \mathbb{R}$ meets K in at most one point. One the other hand, since the orbits of the rotation R_α are dense in $\mathbb{T}^1$, Each line $\{x\} \times \mathbb{R}$ meets K. Hence K is a graph of some map $\psi\colon \mathbb{T}^1 \to \mathbb{R}$. Since K is closed, ψ is continuous. □

Corollary 4.2. *The map h is conjugate to $R_{\alpha,\beta}$ if and only if the lifted dynamics satisfies*

$$\sup_{n\geq 0} \|\tilde{h}^n - n(\tilde{\alpha}, \tilde{\beta})\|_\infty < +\infty. \tag{4.2}$$

A (general) irrational pseudo-rotations h that satisfies (4.2) is said to be *regular.*

Proof. If $h = g \circ R_{\alpha,\beta} \circ g^{-1}$, we have $\|\tilde{h}^n - n(\tilde{\alpha}, \tilde{\beta})\| \leq \|\tilde{g}\|_\infty + \|\tilde{g}^{-1}\|_\infty < +\infty$.
In order to prove the other implication, one first notes that

$$\tilde{h}^n - n(\tilde{\alpha}, \tilde{\beta})(x, y) = (x, y + S_n\varphi(x)).$$

Hence (4.2) implies that the Birkhoff sums of φ are uniformly bounded. Theorem 4.1 now implies that h is conjugate to the rotation by a map of the form $(x, y) \mapsto (x, y + \psi(x))$. □

b) Minimality

Proposition 4.3. *If h is not conjugate to the rotation, it is minimal.*

Proof. Let K be a minimal invariant compact set of h. One introduces the closed subgroup $\Omega \subset \mathbb{T}^1$ of angles ω such that K is invariant by the vertical rotation $\mathrm{Id} \times R_\omega$. As for the claim in the proof of theorem 4.1, one can show that if K contains two points (x, y) and $(x, y + \omega)$, then ω belongs to Ω.

If $\Omega = \mathbb{T}^1$, we have $K = \mathbb{T}^2$ and the dynamics on $\mathbb{T}^2$ is minimal. If Ω is a finite group, each set $K \cap \left(x \times \mathbb{T}^1\right)$ has the same cardinal as Ω. Since K is closed, one deduces that K is a finite union of closed curves that are locally continuous graphs. One will prove that h is regular. By corollary 4.2 this will imply that h is conjugate to the rotation. Note that in order to show that h is regular, it is enough to replace h by any positive iterate. Hence, one can assume that h preserves a closed curve γ.

Let us consider a lift $\tilde{\gamma}$ of γ in the plane $\mathbb{R}^2$. It is the graph of a map $\theta\colon \mathbb{R} \to \mathbb{R}$ that satisfies $\theta(x + k) = \theta(x) + \ell$ for some integers k, ℓ with $k \geq 1$. One chooses a lift $\tilde{h}$ of h that fixes θ.

Let us consider a point (x_0, y_0) in the fundamental domain

$$\{(x, y),\ x \in [0, 1],\ \theta(x) \leq y \leq \theta(x) + 1\}.$$

The point $(x_n, y_n) = \tilde{h}^n(x_0, y_0)$ satisfies $\theta(x_0+n\alpha) \leq y_n \leq \theta(x_0+n\alpha)+1$. Let us decompose $n\alpha = Nk + \alpha_0$ with $|\alpha_0| < k$. One gets $|y_n - N\ell| \leq 2 \sup_{|x|<k} |\theta(x)|$. Hence, $y_n - n\alpha\ell/k$ is bounded. One deduces that $\beta = \alpha\ell/k$ (so we are in the semi-irrational case) and $\|(x_n, y_n) - n(\alpha, \beta)\|$ is uniformly bounded: the dynamics is regular, as claimed. □

c) Construction of examples

Proposition 4.4. *There exist an irrational angle α and an analytic map $\varphi\colon \mathbb{T}^1 \to \mathbb{R}$ such that the cohomological equation (4.1) has no continuous solution.*

Proof. Let us define
$$\alpha = \sum_{k\geq 0} 10^{-k^2},$$
$$\varphi(x) = \sum_{n\in\mathbb{Z}} a_n.e^{2i\pi n.x}, \quad \text{with} \quad a_n = 2^{-|n|}.$$
The map φ is analytic.

Let us assume that there exists a continuous solution ψ. One can write its Fourier expansion $\psi(x) = \sum_{n\in\mathbb{Z}} b_n.e^{2i\pi n.x}$ in $L^2(\mathbb{T}^2)$. The uniqueness of the Fourier coefficients implies that for each $n \in \mathbb{Z}\setminus\{0\}$ we have
$$b_n = \frac{a_n}{1-e^{2i\pi n\alpha}}.$$
We have to control the small denominators: $|1-e^{2i\pi n\alpha}|$ is smaller than the distance of $n\alpha$ to $\mathbb{Z}$. For $n = 10^{k_0}$ we have
$$n\alpha = \sum_{k\leq k_0} 10^{k_0^2-k^2} + \sum_{k>k_0} 10^{-(k^2-k_0^2)}$$
and $\sum_{k>k_0} 10^{-(k^2-k_0^2)} \leq \sum_{j>0} 10^{-(2jk_0)} \leq 10^{-k_0}$ so that $|b_n| \geq 5^{k_0}$ which is impossible if ψ is continuous. □

Corollary 4.5. *Let α be the angle given by proposition 4.4. Then, for any β, there exists an analytic irrational pseudo-rotation h which has rotation vector (α,β), is minimal, irregular, and hence, not conjugate to the rotation.*

Remarks.

1. We will see in exercise that the angle α has to be Liouvillean.
2. In section 4.3 b), we will give a method for constructing for each irrational α a continuous map φ such that the cohomological equation has no solution.
3. When β is irrational, for each integer n the arc translation theorem 2.1 provides us with a simple curve γ isotopic to $\mathbb{T}^1\times 0$ that is disjoint from its n first iterates. *The curve γ can not be disjoint from all its iterates:* otherwise, one can consider the covering $\mathbb{T}^1\times\mathbb{R}$ of the torus, a lift $\tilde{\gamma}$ of γ and any point (x,y) in the fundamental domain bounded by $\tilde{\gamma}$ and $\tilde{\gamma}+(0,1)$. Let $\tilde{h}$ be a lift of h. The iterate $\tilde{h}^n(x,y)$ of (x,y) belongs to the annulus bounded by $\tilde{h}^n(\tilde{\gamma})$ and $\tilde{h}^n(\tilde{\gamma}+(0,1))$. If γ is disjoint from all its iterates, this annulus is contained in the annulus bounded by two curves $\tilde{\gamma}+(0,k-1)$ and $\tilde{\gamma}+(0,k+2)$. This implies that the dynamics is regular, which is a contradiction.

Exercise. *(Baby KAM)*
a) Recall the characterization of C^∞ functions $\mathbb{T}^1\to\mathbb{R}$ by their Fourier coefficient.
b) Prove that if α is Diophantine, then, there exist constants $C>0$ and $\tau\geq 0$ such that
$$|1-e^{2i\pi n\alpha}| > \frac{C}{n^{1+\tau}}.$$

c) Prove that if α is Diophantine and if φ is a C^∞ function that satisfies $\int \varphi dm = 0$, then the cohomological equation (4.1) has a C^∞ solution.

Exercise. *(Alternative proof)* By using Anosov-Katok method, build a Liouvillean irrational angle α and a C^∞ map φ such that the cohomological equation 4.1 has no continuous solution.

d) Ergodic properties

It is interesting to study the invariant probability measures. For instance if there exists measurable function $\psi\colon \mathbb{T}^1 \to \mathbb{R}$ that satisfies the cohomological equation (4.1), then its graph carries an invariant measure which is ergodic and the measured dynamics is metrically isomorphic to $(\mathbb{T}^1, m, R_\alpha)$. In particular, Furstenberg gave an example such that the equation (4.1) has no continuous solution but has a measurable solution. This implies that the corresponding pseudo-rotation (with $\beta = 0$) is minimal (hence irreducible from the topological viewpoint) but not uniquely ergodic (it has many invariant measurable graphs, hence it is not irreducible from the measurable viewpoint).

Furstenberg has also shown that a dichotomy occurs:

- Either the dynamics preserves a subset that is locally a measurable graph over $\mathbb{T}^1$. In this case, the dynamics preserves many ergodic measures, supported by this set and its images by the vertical translations.
- Or the dynamics is uniquely ergodic (the Lebesgue measure is the unique invariant measure).

Since we are in $\mathbb{T}^2$, it is better to work with maps $\chi\colon \mathbb{T}^1 \to \mathbb{T}^1$ instead of the maps $\psi\colon \mathbb{T}^1 \to \mathbb{R}$. Let us identify the circle $\mathbb{T}^1$ with the unitary complex circle $\mathbb{U} \subset \mathbb{C}$. For any γ, one introduces the *multiplicative cohomological equation*

$$e^{2i\pi\varphi(x)-\gamma}\chi(x) = \chi(x+\alpha). \tag{4.3}$$

If this equation is satisfied, then the dynamics is isomorphic to the rotation $R_{\alpha,\beta+\gamma}$ by the map $\Theta\colon (x,y) \to (x, \chi(x).y)$.

It should be noticed that this equations could have solutions even for $\beta \neq 0$ (see [17, 25]). This shows that the corresponding irrational pseudo-rotation h can be isomorphic (from the measurable viewpoint) to a rotation whose angle is different from the rotation vector of h. For some function φ, this equation may have no solution for any β and the dynamics is not isomorphic to any rotation.

4.3 Linear cocycles

Let us consider an irrational angle α and a map $A\colon \mathbb{T}^1 \to SL(2,\mathbb{R})$ that is homotopic to the identity and define a homeomorphism $H\colon \mathbb{T}^1 \times \mathbb{R}^2$ by

$$H(x,t,s) = (x+\alpha, A(x).(t,s)).$$

By identification of non-zero collinear vectors in $\mathbb{R}^2$, one gets a homeomorphism h on $\mathbb{T}^1 \times P^1\mathbb{R} = \mathbb{T}^2$ that is a skew product isotopic to the identity over the rotation R_α. If for instance the values of A are in $SO(2,\mathbb{R})$, we just find the Furstenberg examples: we have $A(x) = R_{2\varphi(x)}$.

a) Non-standard realizations

The examples we are aimed to study here have the following features:

- They are not conjugate to the rotation: their topological dynamics is "wild".
- Their measurable dynamics is isomorphic to the rotation R_α. (The dynamics of the rotation on $\mathbb{T}^1$ has been realized on $\mathbb{T}^2$. For this reason, one says that h is a *non-standard realization of* R_α.)

In [5], we build the following example:

Proposition 4.6. *For any irrational α, there exists a continuous map $A\colon \mathbb{T}^1 \to SL(2,\mathbb{R})$ whose associated homeomorphism h has the following properties:*

- *The dynamics is minimal. Hence, h is an irrational pseudo-rotation. Its rotation vector is $(\alpha, 0)$.*
- *The dynamics is uniquely ergodic. Let μ be the (unique) invariant probability measure. The measurable dynamics $(\mathbb{T}^2, \mu, h)$ is isomorphic to the rotation $(\mathbb{T}^1, m, R_\alpha)$.*

Exercise. Using the Anosov-Katok technique, adapt the previous construction and build an example which is C^∞ for some (Liouvillean) α.

4.4 Reparametrization of irrational flows

Note that for a fibered dynamics h over a rotation R_α on $\mathbb{T}^1$, any measure dynamics $(\mathbb{T}^2, \mu, h)$ projects on a measured dynamics of $\mathbb{T}^1, R_\alpha$. Since the irrational rotations of $\mathbb{T}^1$ ar euniquely ergodic, this shows that the dynamics $(\mathbb{T}^1, m, R_\alpha)$ is a factor of $(\mathbb{T}^2, \mu, h)$. In particular, a weakly mixing measured dynamics will never appear in fibered systems.

We will give a slightly different construction which produces weakly mixing dynamics: let us consider on $\mathbb{T}^2$ the foliation by lines whose slope is irrational and equal to η and let us consider a continuous function $\phi\colon \mathbb{T}^2 \to (0, +\infty)$. One then defines the reparametrized irrational translation flow to be the flow along the irrational foliation which is associated to the vector field $\phi(x,y)^{-1}.(1,\alpha)$. The flow preserves a unique measure, which is $\phi.m$. One then introduces the time-one map h of this flow. One easily check that it is an irrational pseudo-rotation whose rotation vector is

$$\frac{1}{\int_{\mathbb{T}^2} \phi(x)dm(x)}.(1,\alpha).$$

Sklover has shown [51] that by this construction, it is possible to build analytic pseudo-rotations which are weakly mixing. We refer to the course by Bassam Fayad for a more detailed exposition of this subject.

5 Continuous Denjoy examples

There are not known obtruction to realize a measured dynamical system (X, μ, T) as a uniquely ergodic irrational pseudo-rotation. In this section we will show that it is indeed possible in general to build such a pseudo-rotation: we proved [4] this with F. Béguin and F. Le Roux by adapting a previous construction by Mary Rees of a minimal homeomorphism of the 2-torus with positive topological entropy. Her technique can be viewed as a generalization of Denjoy's counter examples on the circle and it produces homeomorphisms that are in general only continuous.

5.1 Denjoy-Rees technique

Twenty-five years ago, M. Rees has constructed a homeomorphism of the torus $\mathbb{T}^d$ ($d \geq 2$) which is minimal and has positive topological entropy (see [48]). The existence of such an example is surprising for several reasons:

– classical examples of minimal homeomorphisms (irrational rotations, time t maps of horocyclic flows, etc.) are also typical examples of zero entropy maps.

– a classical way for proving that a map f has positive topological entropy is to show that the number of periodic orbits of period $\leq n$ for f grows exponentially fast when $n \to \infty$. So, in many situations, "positive topological entropy" is synonymous of "many periodic orbits". But a minimal homeomorphism do not have any periodic orbit.

– a beautiful theorem of A. Katok states that, if f is a $C^{1+\alpha}$ diffeomorphism of a compact surface S with positive topological entropy, then there exists an f-invariant compact set $\Lambda \subset S$ such that some power of $f_{|\Lambda}$ is conjugate to a full shift (see [34, corollary 4.3] and the course by Jérome Buzzi). In particular, a $C^{1+\alpha}$ diffeomorphism of a compact surface with positive topological entropy cannot be minimal.

Beyond the mere existence of minimal homeomorphisms of $\mathbb{T}^d$ with positive topological entropy, the technique used by Rees to construct such a homeomorphism is very interesting. This technique can be seen as a very sophisticated generalization of the one used by A. Denjoy to construct his famous counter-example (a periodic orbit free homeomorphism of $\mathbb{S}^1$ which is not conjugate to a rotation, [11]). Indeed, the basic idea of Rees construction is to start with an irrational rotation of $\mathbb{T}^d$, and to "blow-up" some orbits, just as in Denjoy counter-example. Of course, the construction of Rees is much more complicated and delicate than the one of Denjoy; for example, to get a homeomorphism with positive topological entropy, one has to blow up a set of orbits of positive Lebesgue measure.

The aim of the present paper is to describe a general setting for what we call the *Denjoy-Rees technique*. This general setting includes as particular cases the construction of various "Denjoy counter-examples" in any dimension, and Rees construction of a minimal homeomorphism of $\mathbb{T}^d$ with positive topological entropy. Moreover, we will develop a new technique which allows to control that the homeomorphisms we obtain "do not contain too much dynamics". This yields new results such as the existence of minimal *uniquely ergodic* homeomorphisms with positive topological entropy, or the possibility to realize many measurable dynamical systems as minimal homeomorphisms on manifolds.

5.2 Strictly ergodic homeomorphisms with positive topological entropy

A homeomorphism is said to be *strictly ergodic* if it is minimal and uniquely ergodic. As an application of Denjoy-Rees technique, we will prove the following theorem.

Theorem 5.1. *Any compact manifold of dimension $d \geq 2$ which carries a strictly ergodic homeomorphism also carries a strictly ergodic homeomorphism with positive topological entropy.*

A. Fathi and M. Herman have proved that every compact manifold of dimension $d \geq 2$ admitting a locally free action of the circle[3] carries a strictly ergodic homeomorphism (see [13]). Putting theorem 5.1 together with Fathi-Herman result yields many examples. In particular, the torus $\mathbb{T}^d$ for $d \geq 2$, the sphere $\mathbb{S}^{2n+1}$ for $n \geq 1$, any Seifert manifold, any manifold obtained as a quotient of a compact connected Lie group, *etc.*, carry strictly ergodic homeomorphisms with positive topological entropy (see the discussion in [13]).

Theorem 5.1 (as well as Rees example and Katok theorem cited above) can be seen as a piece of answer to a general question of Herman asking "whether, for diffeomorphisms, positive topological entropy is compatible with minimality, or strict ergodicity" (see [34, page 141]). Katok answered negatively to Herman question in the case of $C^{1+\alpha}$ diffeomorphisms of surfaces. Then, Herman himself constructed an analytic minimal diffeomorphism with positive topological entropy on a 4-manifold (see [29]), and Rees constructed a minimal homeomorphism with positive topological entropy on $\mathbb{T}^d$. But neither Herman, nor Rees managed to make their examples strictly ergodic (see the introductions of [48] and [34]). Theorem 5.1 shows that positive topological entropy is compatible with strict ergodicity for homeomorphisms (in any dimension). To complete the answer to Herman question, it essentially remains to determine what is the best possible regularity for a minimal (resp. strictly ergodic) homeomorphism on $\mathbb{T}^2$ with positive entropy (Hölder? C^1?). We do not have any idea of the best regularity one can obtain for a homeomorphism constructed via Denjoy-Rees technique.

Note that, by Oxtoby-Ulam theorem [46], one can assume that the unique measure preserved by the homeomorphism provided by theorem 5.1 is a Lebesgue measure.

5.3 Realizing measurable dynamical systems as homeomorphisms on manifolds

The main difference between Rees result and our theorem 5.1 is the fact that the homeomorphisms we construct are uniquely ergodic. More generally, we develop a technique which allows us to control the number of invariant measures of the homeomorphisms obtained by constructions *à la Denjoy-Rees*. What is the point of controlling the invariant measures? In short:

– the Denjoy-Rees technique by itself is a way for constructing examples of "curious" minimal homeomorphisms,

– the Denjoy-Rees technique combined with the possibility of controlling the invariant measures is not only a way for constructing examples, but also a way for realizing measurable dynamical systems as homeomorphisms on manifolds.

Let us explain this. In her paper, Rees constructed a homeomorphism f on $\mathbb{T}^d$ which is minimal and possesses an invariant probability measure μ such that f has a rich dynamics

[3] An action of the circle is said to be *locally free* if no orbit of this action is reduced to a point.

from the point of view of the measure μ: in particular, the metric entropy $h_\mu(f)$ is positive. By the variational principle, this implies that the topological entropy $h_{\rm top}(f)$ is also positive. Nevertheless, f might possess some dynamics that is not detected by the measure μ (for example, $h_{\rm top}(f)$ might be much bigger than $h_\mu(f)$). So, roughly speaking, Rees constructed a minimal homeomorphism which has a rich dynamics, but without being able to control how rich this dynamics is. Now, if we can control what are the invariant measures of f, then we know exactly what f looks like from the measurable point of view.

An interesting general question is:

Question. *Given any measurable dynamical system $(X, \mathcal{A}, S)$, does there exist a homeomorphism on a manifold which is isomorphic to $(X, \mathcal{A}, S)$?*

In this direction, using Denjoy-Rees technique together with our technique for controlling the invariant measures, we obtained the following "realization theorem" (which implies theorem 5.1, see below).

Theorem 5.2. *Let R be a uniquely ergodic aperiodic homeomorphism of a compact manifold $\mathcal{M}$ of dimension $d \geq 2$. Let h_C be a homeomorphism on some Cantor space C. Then there exists a homeomorphism $f : \mathcal{M} \to \mathcal{M}$ isomorphic to $R \times h_C : \mathcal{M} \times C \to \mathcal{M} \times C$.*

Furthermore, the homeomorphism f is a topological extension of R: there exists a continuous map $\Phi : \mathcal{M} \to \mathcal{M}$ such that $\Phi \circ f = R \circ \Phi$. If R is minimal (resp. transitive), then f can be chosen minimal (resp. transitive).

Remark 5.3. In [48], Rees considered the case where R is an irrational rotation of $\mathbb{T}^d$ and h_C is a full shift. She constructed a minimal homeomorphism g which had a subsystem isomorphic to $R \times h_C$, but was not isomorphic to $R \times h_C$.

Let us recall that the problem of realizing measurable dynamical systems as topological dynamical systems admits many alternative versions. One possibility is to prescribe the invariant measure, that is, to deal with *measured* systems instead of *measurable* systems. In this context, the realizability problem consists in finding homeomorphisms f on a manifold $\mathcal{M}$ and an f-invariant measure ν such that $(\mathcal{M}, f, \nu)$ is metrically isomorphic to a given dynamical system (X, T, μ). In this direction, D. Lind and J.-P. Thouvenot proved that every finite entropy measured dynamical system is metrically conjugate to some shift map on a finite alphabet, and thus also to a Lebesgue measure-preserving homeomorphism of the two-torus (see [42]).

Then, one can consider the same problem but with the additional requirement that the realizing homeomorphism is uniquely ergodic. In this direction (but not on manifolds), one has the celebrated Jewett-Krieger theorem: any ergodic system is metrically conjugate to a uniquely ergodic homeomorphism on a Cantor space (see e.g. [12]).

Observe that, if one applies theorem 5.2 with the homeomorphism R being an irrational rotation of the torus $\mathbb{T}^2$ (and h_C being any homeomorphism of a Cantor set), then it is easy to see that the resulting homeomorphism f is also an irrational pseudo-rotation. Varying the homeomorphism h_C, one gets lots of examples of "exotic" irrational pseudo-rotations on the torus $\mathbb{T}^2$. We point out that with this method the angle of the rotation R may be any irrational point of $\mathbb{R}^2/\mathbb{Z}^2$ (but the pseudo-rotation we obtained are just homeomorphisms). If one seeks realizations of measurable (or measured) dynamical systems by smooth maps then the Denjoy-Rees technique seems to be useless.

Let us try to give a brief idea of what the homeomorphism f provided by theorem 5.2 looks like (see also section 5.5). On the one hand, from the topological point of view, f looks very much like the initial homeomorphism R (which typically can be a very simple homeomorphism, like an irrational rotation of the torus $\mathbb{T}^2$). Indeed, the continuous map $\Phi : \mathcal{M} \to \mathcal{M}$ realizing the topological semi-conjugacy between f and R is an "almost conjugacy": there exists an f-invariant G_δ-dense set X on which Φ is one-to-one. This implies that f is minimal. On the other hand, from the measurable point of view, f is isomorphic to the product $R \times h_C$ (which might exhibit a very rich dynamics since h_C is an arbitrary homeomorphism on a Cantor set). As often, the paradox comes from the fact that the set X is big from the topological viewpoint (it is a G_δ dense set), but small from the measurable viewpoint (it has zero measure for every f-invariant measure).

We end this section by explaining how theorem 5.2 implies theorem 5.1. Let $\mathcal{M}$ be a manifold of dimension $d \geq 2$, and assume that there exists a strictly ergodic homeomorphism R on $\mathcal{M}$. We have to construct a strictly ergodic homeomorphism with positive entropy on $\mathcal{M}$. For this purpose, we may assume that the topological entropy of R is equal to zero, otherwise there is nothing to do.

Let σ be the shift map on $\{0,1\}^{\mathbb{Z}}$, and μ be the usual Bernoulli measure on $\{0,1\}^{\mathbb{Z}}$. By Jewett-Krieger theorem (see e.g. [12]), there exists a uniquely ergodic homeomorphism h_C of a Cantor set C which is metrically conjugate to $(\{0,1\}^{\mathbb{Z}}, \sigma, \mu)$. Since the shift map is a K-system, and since R is uniquely ergodic and has zero topological entropy, the product map $R \times h_C$ is also uniquely ergodic (see [52, proposition 4.6.(1)]). Denote by ν the unique invariant measure of $R \times h_C$. Then the metric entropy $h_\nu(R \times h_C)$ is equal to $h_\mu(\sigma) = \log 2$. Now theorem 5.2 provides us with a minimal homeomorphism f on $\mathcal{M}$, which is isomorphic to $R \times h_C$. Denote by Θ the map realizing the isomorphism between $R \times h_C$ and f. Since $R \times h_C$ is uniquely ergodic, f is also uniquely ergodic: $\Theta_* \nu$ is the unique f-invariant measure). Moreover, the metric entropy $h_{\Theta_*\nu}(f)$ is equal to $h_\nu(R \times h_C)$, which is positive. The variational principle then implies that the topological entropy of f is positive. Hence, f is a strictly ergodic homeomorphism with positive topological entropy.

5.4 A more general statement

Theorem 5.2 is only a particular case of a more general statement: theorem 5.5 below will allow to consider a homeomorphism R that is not uniquely ergodic, and to replace the product map $R \times h_C$ by any map that fibers over R.

To be more precise, let R be a homeomorphism of a manifold $\mathcal{M}$ and A be any measurable subset of $\mathcal{M}$. Let C be a Cantor set. We consider a bijective bi-measurable map which is fibered over R:

$$\begin{array}{rcl} h : \bigcup_{i \in \mathbb{Z}} R^i(A) \times C & \longrightarrow & \bigcup_{i \in \mathbb{Z}} R^i(A) \times C \\ (x, c) & \longmapsto & (R(x), h_x(c)) \end{array}$$

where $(h_x)_{x \in \cup R^i(A)}$ is a family of homeomorphisms of C. We make the following continuity assumption: for every integer i, the map h^i is continuous on $A \times C$.

Remark 5.4. In the case $A = \mathcal{M}$, the continuity assumption implies that h is a homeomorphism of $\mathcal{M} \times C$; if $\mathcal{M}$ is connected, then h has to be a product as in theorem 5.2. Thus we do not restrict ourselves to the case where $A = \mathcal{M}$; we rather think of A as a Cantor set in $\mathcal{M}$. Also note that the continuity assumption amounts to requiring that h is continuous on $R^i(A) \times C$ for every i. Note that this does *not* imply that h is continuous on $\bigcup_{i\in\mathbb{Z}} R^i(A) \times C$.

We will prove the following general statement.

Theorem 5.5. *Let R be a homeomorphism on a compact manifold $\mathcal{M}$ of dimension $d \geq 2$. Let μ be an aperiodic*[4] *ergodic measure for R, and $A \subset \mathcal{M}$ be a set which has positive measure for μ and has zero-measure for every other ergodic R-invariant measure. Let h be a bijective map which is fibered over R and satisfies the continuity assumption as above. Then there exists a homeomorphism $f : \mathcal{M} \to \mathcal{M}$ such that $(\mathcal{M}, f)$ is isomorphic to the disjoint union*

$$\left(\bigcup_{i\in\mathbb{Z}} R^i(A) \times C, h\right) \bigsqcup \left(\mathcal{M} \setminus \bigcup_{i\in\mathbb{Z}} R^i(A), R\right).$$

Furthermore, f is a topological extension of R: there exists a continuous map $\Phi : \mathcal{M} \to \mathcal{M}$, which is one-to-one outside the set $\Phi^{-1}(\mathrm{Supp}\mu)$, such that $\Phi \circ f = R \circ \Phi$. If R is minimal (resp. transitive), then f can be chosen to be minimal (resp. transitive).

The case where R is not minimal on $\mathcal{M}$ is considered in the following addendum.

Addendum 5.6. *In any case, the dynamics f is transitive on $\Phi^{-1}(\mathrm{Supp}\mu)$. Moreover, if R is minimal on $\mathrm{Supp}\mu$, then f can be chosen to be minimal on $\Phi^{-1}(\mathrm{Supp}\mu)$.*

In the case where R is uniquely ergodic, theorem 5.5 asserts that there exists a homeomorphism $f : \mathcal{M} \to \mathcal{M}$ which is isomorphic to the fibered map h. In the general case, it roughly says that there exists a homeomorphism $f : \mathcal{M} \to \mathcal{M}$ which, from the measurable point of view, "looks like" R outside $\Phi^{-1}\left(\bigcup_{i\in\mathbb{Z}} R^i(A)\right)$ and like h on $\Phi^{-1}\left(\bigcup_{i\in\mathbb{Z}} R^i(A)\right)$. In other words, it allows to replace the dynamics of R on the iterates of A by the dynamics of h.

5.5 Outline of Denjoy-Rees technique

In this section, we give an idea of the Denjoy-Rees technique. For this purpose, we first recall one particular construction of the famous Denjoy homeomorphism on the circle. Many features of Rees construction already appear in this presentation of Denjoy construction, especially the use of microscopic perturbations (allowing the convergence of the construction) with macroscopic effects on the dynamics.

General method for constructing Denjoy counter-examples. To construct a Denjoy counter-example on the circle, one starts with an irrational rotation and blows up the orbit of some point to get an orbit of wandering intervals. There are several ways to carry out the construction, let us outline the one that suits our needs. We first choose an irrational rotation R. The homeomorphism f is obtained as a limit of conjugates of R,

$$f = \lim f_n \text{ with } f_n = \Phi_n^{-1} R \Phi_n$$

[4]An invariant measure μ is *aperiodic* if the set of periodic points of R has measure 0 for μ. When μ is ergodic, this just says that μ is not supported by a periodic orbit of R.

where the sequence of homeomorphisms (Φ_n) will converge towards a non-invertible continuous map Φ that will provide a semi-conjugacy between f and R. To construct this sequence, we pick some interval I_0 (that will become a wandering interval). The map Φ_n will map I_0 to some interval I_n which is getting smaller and smaller as n increases, so that I_0 is "more and more wandering"; more precisely, I_0 is disjoint from its n first iterates under f_n.

In order to make the sequence (f_n) converge, the construction is done recursively. The map Φ_{n+1} is obtained by post-composing Φ_n with some homeomorphism M_{n+1} that maps I_n on I_{n+1}, and whose support is the disjoint union of the n backward and forward iterates, under the rotation R, of an interval $\hat{I}_n$ which is slightly bigger than I_n. Thus the uniform distance from Φ_n to Φ_{n+1} is roughly equal to the size of I_n. This guarantees the convergence of the sequence (Φ_n) if the size of I_n tends to 0 quickly enough.

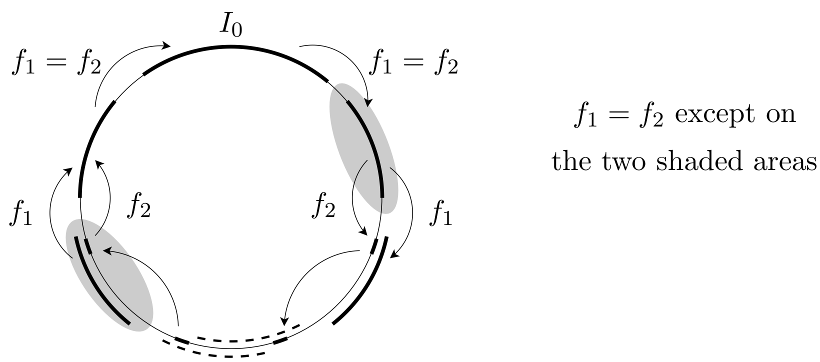


Figure 12: Denjoy construction on the circle: the interval I_0 has four consecutive disjoint iterates under f_1, and six consecutive disjoint iterates under f_2.

Clearly, this is not enough for the convergence of (f_n) (for example, if Φ_n was the identity outside a little neighborhood of I_0, then (f_n) could converge to a map that crashes I_0 onto a point). We also demand that M_{n+1} commutes with the rotation R except on the union of two small intervals I_n^{in} and I_n^{out} (namely, $I_n^{\text{in}} = R^{-(n+1)}(\hat{I}_n)$ and $I_n^{\text{out}} = R^n(\hat{I}_n)$). Then an immediate computation shows that the map f_{n+1} coincides with f_n except on the set $\Phi_n^{-1}(I_n^{\text{in}} \cup I_n^{\text{out}})$, which happens to be equal to $\Phi_{n-1}^{-1}(I_n^{\text{in}} \cup I_n^{\text{out}})$ (due to the condition on the support of M_n, see figure 12). Note that the interval $\hat{I}_n$ *is chosen after the map* Φ_{n-1} *has been designed*, and thus we see that this set $\Phi_{n-1}^{-1}(I_n^{\text{in}} \cup I_n^{\text{out}})$ can have been made arbitrarily small (by choosing $\hat{I}_n$ small enough), so that f_{n+1} is arbitrarily close to f_n.

The Denjoy-Rees technique. We now turn to the generalization of the Denjoy construction developed by Rees. We have in mind the easiest setting: the map R is an irrational rotation of the two-torus $\mathcal{M} = \mathbb{T}^2$, we are given some homeomorphism h_C on some abstract Cantor space C, and we want to construct a minimal homeomorphism f of $\mathbb{T}^2$ which is isomorphic to $R \times h_C$. In some sense, we aim to blow up the dynamics of R and to "embed the dynamics of h_C" into the blown-up homeomorphism f. The main difference with the Denjoy construction is that we have to blow up the orbits of all the points of a positive measure Cantor set K. Whereas for the previous construction the point to be blown up was disjoint from all its iterates under the rotation R, obviously K *will meet some of its iterates*: one has to deal with the recurrence of K, which adds considerable difficulty.

At step k, we will know an approximation E_k^0 of K by a union of small rectangles. The key property of these rectangles, that enables the construction in spite of the recurrence, is that they are *dynamically coherent*: if X_1, X_2 are two connected components of E_k^0, if $-k \leq l, l' \leq k$, then the rectangles $R^l(X_1)$ and $R^{l'}(X_2)$ are either disjoint or equal.

The final map f will again be a limit of conjugates of R,

$$f = \lim f_k \text{ with } f_k = \Phi_k^{-1} R \Phi_k \text{ and } \Phi = \lim \Phi_k.$$

We fix an embedding of the abstract product Cantor set $K \times C$ in the manifold $\mathcal{M}$ (see figure 13); in some sense, this set $K \times C$ will play for the Rees map the same role as the interval I_0 for the Denjoy map. Indeed, each point x of K will be blown-up by Φ^{-1}, so that the fiber $\Phi^{-1}(x)$ contains the "vertical" Cantor set $\{x\} \times C$ embedded in $\mathcal{M}$. Furthermore, the first return map of f in $\widetilde{K} := \Phi^{-1}(K)$ will leave the embedded product Cantor space $K \times C$ invariant: in other words, for each point $x \in K$ and each (return time) p such that $R^p(x)$ belongs to K, the homeomorphism f^p will map the vertical Cantor set $\{x\} \times C$ onto the vertical Cantor set $\{R^p(x)\} \times C$. When both vertical Cantor sets are identified to C by way of the second coordinate on $K \times C$, the map f^p induces a homeomorphism of C, which will be equal to h_C^p. This is the way one embeds the dynamics of h_C into the dynamics of f, and gets an isomorphism between the product $R \times h_C$ and the restriction of the map f to the set $\bigcup_i f^i(K \times C)$. Note that theorem 5.2 requires more, namely an isomorphism between $R \times h_C$ and the map f *on the whole manifold* $\mathcal{M}$. We will explain in the last paragraph how one can further obtain that f is isomorphic to its restriction to the set $\bigcup_i f^i(K \times C)$.

Once again the construction will be carried out recursively. At step k we will take care of all return times less than $2k+1$, that is, the map Φ_k will be constructed so that the approximation f_k of f will satisfy the description of figure 13 for $| p | \leq 2k+1$. This property will be transmitted to f_{k+1} (and so gradually to f) because for any point x of K whose return time in K is less than or equal to $2k+1$, $f_{k+1}^p(x) = f_k^p(x)$. Actually the equality $f_{k+1} = f_k$ will hold except on a very set which becomes smaller and smaller as k increases (just as in the Denjoy construction). On the other hand f_{k+1} will take care of return times equal to $2k+2$ and $2k+3$. The convergence of the sequences (Φ_k) and (f_k) will be obtained using essentially the same argument as in the Denjoy construction.

Another feature of the construction is that we want f to inherit from the minimality of R. This will be an easy consequence of the two following properties. Firstly the fiber $\Phi^{-1}(x)$ above a point x that does not belong to an iterate of K will be reduced to a point. Secondly the other fibers will have empty interior.

Control of invariant measures Until here, we have been dealing with the control of the dynamics on the iterates of the product Cantor set $K \times C$. This is enough for f to admit the product $R \times h_C$ as a subsystem, and to get an example with positive topological entropy (Rees initial result). If we want the much stronger property that f is uniquely ergodic (in theorem 5.1) or isomorphic to $R \times h_C$ (in theorem 5.2), we need to gain some control of the dynamics outside the iterates of $K \times C$, on the whole manifold $\mathcal{M}$. With this in view, we first note that, since K has positive measure, the (unique) invariant measure for R gives full measure to $\bigcup_i R^i(K)$. The automatic consequence for f is that any invariant measure for f gives full measure to $\bigcup_i f^i(\widetilde{K})$ (with $\widetilde{K} = \Phi^{-1}(K)$). It now remains to put further constraints on the construction to ensure that any invariant measure for f will give measure 0 to $\widetilde{K} \setminus K \times C$. This will be done by

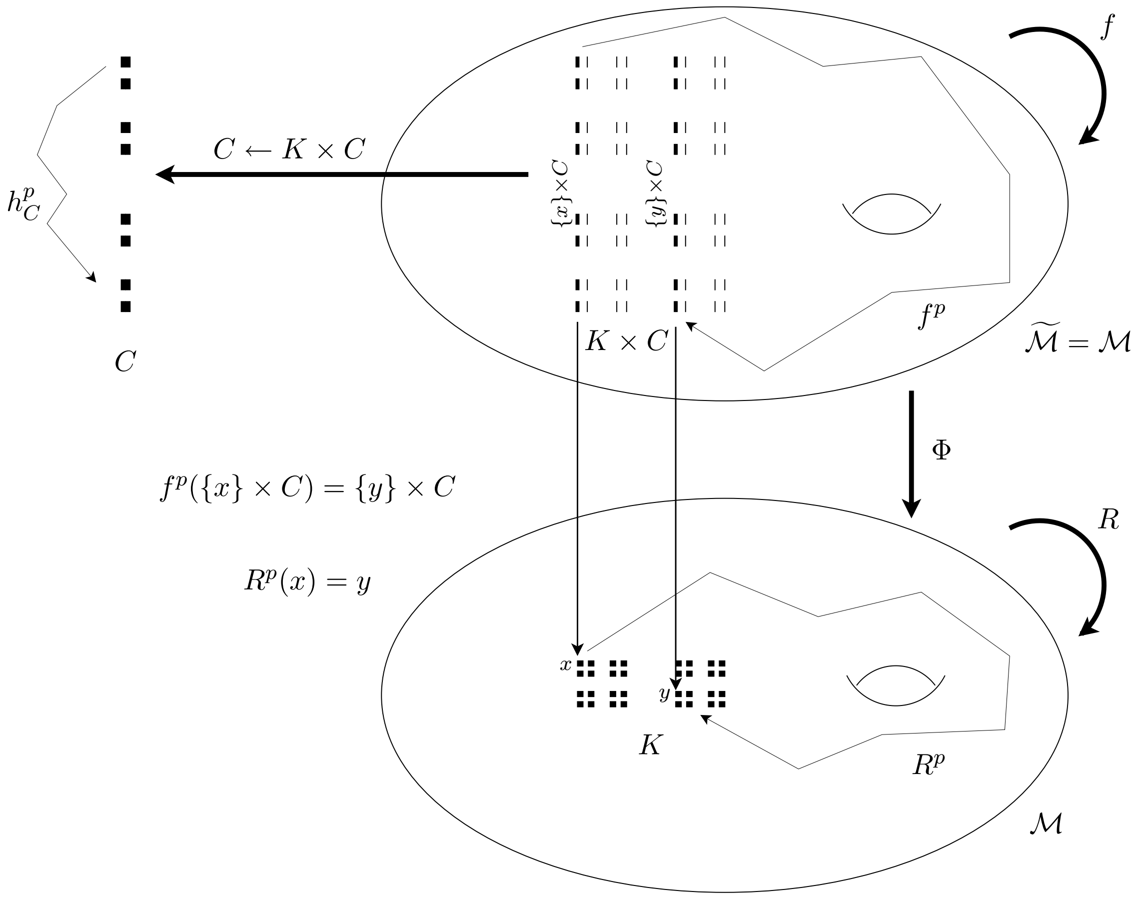


Figure 13: Rees construction on the torus: description of the isomorphism between f and $R \times h_C$. The topological flavor is given by the vertical map Φ, the measurable flavor is given by the horizontal map $K \times C \to C$.

considering the first return map of f in $\widetilde{K}$, and by forcing the ω-limit set of any point $x \in \widetilde{K}$, with respect to this first return map, to be included in $K \times C$.

5.6 Structure of the proof

Recall that our goal is to prove theorem 5.5 (which implies theorem 5.2 and theorem 5.1, see the end of section 5.3). So we are given, in particular, a homeomorphism R on a manifold $\mathcal{M}$ and a map h which fibers over R, and we aim to construct a homeomorphism f on $\mathcal{M}$ which is isomorphic to $\left(\bigcup_{i\in\mathbb{Z}} R^i(A) \times C, h\right) \bigsqcup \left(\mathcal{M} \setminus \bigcup_{i\in\mathbb{Z}} R^i(A), R\right)$.

The proof is divided into three steps.

- First, we construct a Cantor set K, obtained as a decreasing intersections of a sequence sets $(E_n^0)_{n\in\mathbb{N}}$, where E_n^0 is a finite collection of pairwise disjoint rectangles for every n.
- Then, we explain how to blow-up the orbits of the points of K: we construct a sequence of homeomorphisms (M_n) whose infinite composition is a map $\Psi : \mathcal{M} \to \mathcal{M}$ such that $\Psi^{-1}(K)$ contains a copy of the product Cantor set $K \times C$.
- Finally, we explain how to insert the dynamics of h in the blowing-up of the orbits of the points of K. In order to improve the convergence, one will define an extracted sequence $(\boldsymbol{M}_k)$ of (M_n). One also needs to "twist" the dynamics by constructing a sequence of homeomorphisms (H_k) and by replacing each $\boldsymbol{M}_k$ by the homeomorphism $H_k \circ \boldsymbol{M}_k$. The infinite composition of the homeomorphisms $H_k \circ \boldsymbol{M}_k$ is a map Φ and the desired homeomorphism $f : \mathcal{M} \to \mathcal{M}$ is a topological extension of R by Φ.

5.7 Some examples

Let us now illustrate some of these results by a few examples.

a) Denjoy counter-examples

The simplest setting is when the set K is a single point. This yields various generalizations of the classical Denjoy counter-examples on $\mathbb{S}^1$. These examples can have wandering domains and will not give irrational pseudo-rotations.

Proposition 5.7. *Let R be a homeomorphism on a compact manifold $\mathcal{M}$, and x a point of $\mathcal{M}$ which is not periodic under R. Consider a compact subset D of $\mathcal{M}$ which can be written as the intersection of a strictly decreasing sequence $(\widetilde{X}_n)_{n\geq 0}$ of topological closed balls. Then there exist a homeomorphism $f : \mathcal{M} \to \mathcal{M}$ and a continuous onto map $\Phi : \mathcal{M} \to \mathcal{M}$ such that $\Phi \circ f = R \circ \Phi$, and such that*

- *$\Phi^{-1}(x) = D$;*
- *$\Phi^{-1}(y)$ is a single point if y does not belong to the R-orbit of x.*

Remarks 5.8.

- The properties of Φ and f imply that, if R is minimal, then the set $\mathcal{M} \setminus \bigcup_{n\in\mathbb{Z}} f^n(\mathrm{Int}(D))$ is the only minimal closed invariant set for f.

– So, if R is an irrational rotation on $\mathcal{M} = \mathbb{S}^1$ and D is a non-trivial interval of $\mathbb{S}^1$, then f is a classical Denjoy counter-example.

– In any case, if the interior of D is non empty, then it is an open wandering set for f. In particular, if R is minimal and D has non-empty interior, then the dynamics of f is very similar to the dynamics of Denjoy counter-examples on the circle.

– If R is minimal and D has empty interior, then f is minimal. In this case, we obtain a kind of "Denjoy-counter-example" whose dynamical behavior is actually quite different from those of the classical Denjoy counter-examples on the circle.

b) Different ways of blowing-up an invariant circle

Now, we would like to build other examples of irrational pseudo-rotations of $\mathbb{S}^2$ having a pathological minimal set (see section 3.3).

For this purpose, we consider an irrational rigid rotation R of the sphere $\mathbb{S}^2$ (fixing the two poles N and S). We denote by Λ the equatorial circle of $\mathbb{S}^2$ (which is invariant under R), and we pick a point $x \in \Lambda$. Using proposition 5.7, we can construct a homeomorphism f and a map Φ such that $\Phi \circ f = R \circ \Phi$, such that $\Phi^{-1}(x)$ is a non-trivial "vertical" segment J and such that $\Phi^{-1}(y)$ is a single point if y does not belong to the R-orbit of x. It follows that $\widetilde{\Lambda} = \Phi^{-1}(\Lambda)$ is a one-dimensional (connected with empty interior) f-invariant compact set which separates $\mathbb{S}^2$ into two connected open sets.

Moreover, according to the way we choose the M_n's, we can get quite different topologies for the set $\widetilde{\Lambda}$ and quite different dynamics for the restriction of f to $\widetilde{\Lambda}$. Here are three possible types of behaviors:

– $\widetilde{\Lambda}$ is a non-arcwise connected set which is minimal for f (figure 14, I);

– $\widetilde{\Lambda}$ is a topological circle which is not minimal for f: the restriction of f to $\widetilde{\Lambda}$ is a Denjoy counter-example on the circle, the vertical segment J is wandering (figure 14, II);

– $\widetilde{\Lambda}$ contains a circle which is a minimal set for f, but is not equal to this circle (figure 14, III, where the minimal set is the equatorial circle).

Remark 5.9. The construction of the above examples can be made in such a way that Φ is C^∞ on $\mathbb{S}^2 \setminus \widetilde{\Lambda}$. Moreover, if we identify $\mathbb{S}^2 \setminus \{N, S\}$ to the annulus $\mathbb{S}^1 \times \mathbb{R}$ and see f as a homeomorphism of $\mathbb{S}^1 \times \mathbb{R}$, then all the constructions can be made in such a way that f is a fibered homeomorphism (i.e. is of the form $f(x, y) = (x + \alpha, f_x(y))$).

c) Pseudo-rotations with positive topological entropy

We would like to apply theorem 5.5 to obtain a more sophisticated example of pseudo-rotation on $\mathbb{S}^2$. The possibility of constructing such a homeomorphism was mentioned in [49]; more details are given in appendix 5.7. We choose any irrational angle $\alpha \in \mathbb{S}^1$ and denote by R_α the rigid rotation of angle α. We denote by Λ be the equatorial circle invariant by R_α.

Proposition 5.10. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exists an irrational pseudo-rotation f on $\mathbb{S}^2$ of angle α with positive topological entropy.*

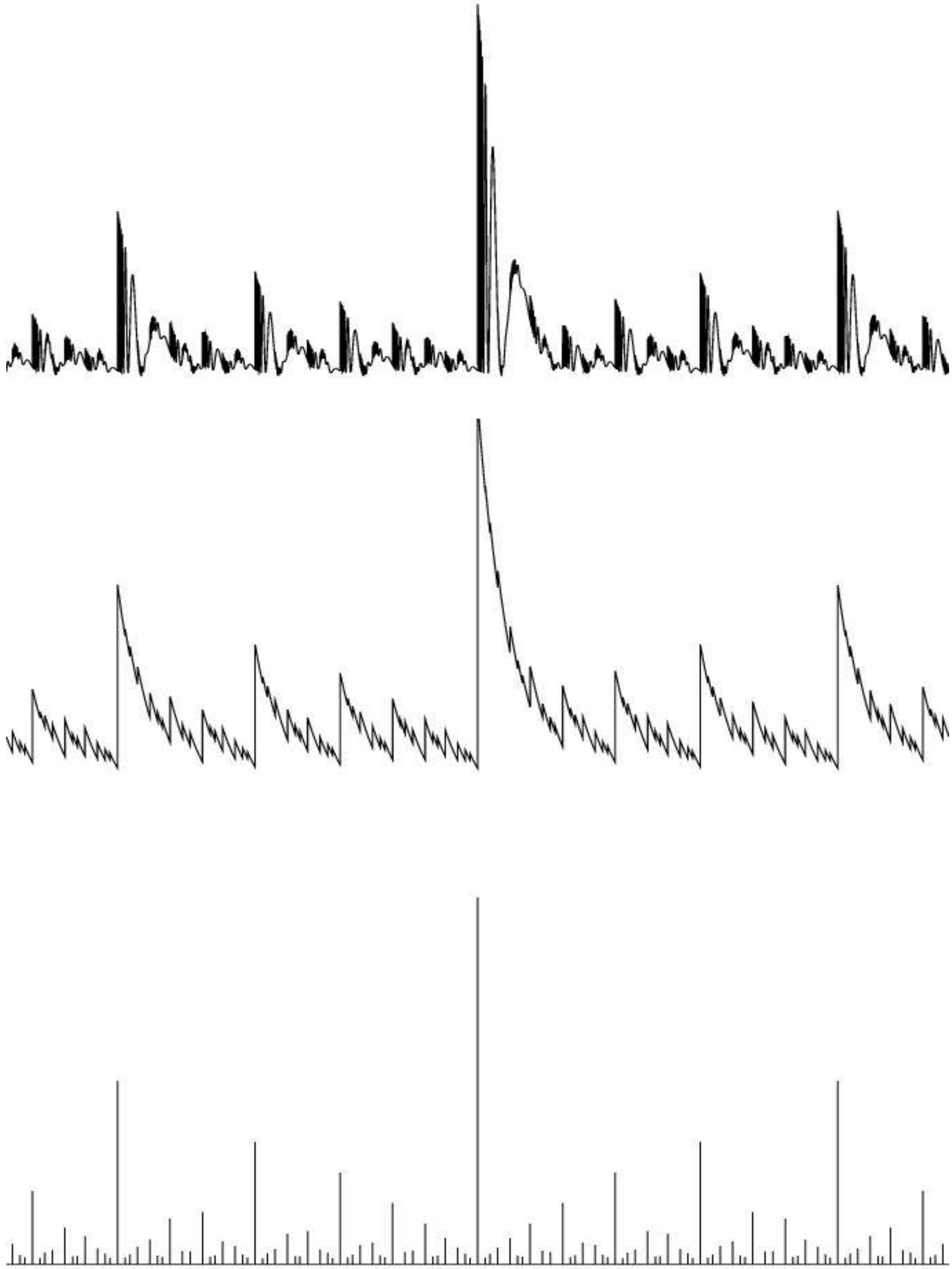


Figure 14: Different ways of blowing-up an invariant circle

Furthermore, there exists a continuous onto semi-conjugacy Φ *between* f *and the rigid rotation* R_α*. If* Λ *is the equatorial circle of* $\mathbb{S}^2$ *(invariant under* R_α*), then the set* $\widetilde{\Lambda} = \Phi^{-1}(\Lambda)$ *is a one-dimensional (connected with empty interior) minimal closed* f*-invariant set which carries all the entropy of* f*. It separates the sphere into two connected open sets. The map* Φ *is smooth on* $\mathbb{S}^2 \setminus \widetilde{\Lambda}$*; thus the restriction* f *to* $\mathbb{S}^2 \setminus \widetilde{\Lambda}$ *is* C^∞*-conjugate to the restriction of* R_α *to* $\mathbb{S}^2 \setminus \Lambda$*.*

Proof. The proposition is almost a corollary of theorem 5.5 applied in the case where the manifold $\mathcal{M}$ is the sphere $\mathbb{S}^2$, the homeomorphism R is the rigid rotation R_α, the measure μ is the unique R-invariant measure supported by the equatorial circle Λ, the set A is the equatorial circle Λ and the map h is the product of $R_{\alpha|\Lambda}$ by a Cantor homeomorphism with positive topological entropy. The only point which does not follow from theorem 5.5 is the fact that Φ is C^∞ on $\mathbb{S}^2 \setminus \widetilde{\Lambda}$. □

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