Threshold Saturation for Nonbinary Spatially-Coupled LDPC Codes on the Binary Erasure Channel

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Background

- Spatially-Coupled LDPC (SC-LDPC) codes are codes with a low-density parity-check matrix, the non-zero elements in which are located in the diagonal band [FZ99].
- (Iterative) Belief Propagation (BP) algorithm is used for decoding.
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- SC-LDPC codes show outstanding asymptotic performance for lots of channels and communication problems.
- For the binary erasure channel (BEC), the BP threshold of a binary SC-LDPC ensemble achieves the optimal MAP threshold of the underlying LDPC ensemble (*threshold saturation*) [KRU11]. Extended to BMS channels [KMRU10].

[FZ99] J. Felstrom and K. Zigangirov, "Time-varying periodic convolutional codes with low-density parity-check matrix," IEEE Trans. Inform. Theory, vol.45, no.6, pp.2181,2191, Sep 1999.
[KRU11] S. Kudekar, T. J. Richardson, and R. L. Urbanke, "Threshold saturation via spatial coupling: Why convolutional LDPC ensembles perform so well over the BEC," IEEE Trans. Inform. Theory, vol. 57, pp. 803–834, Feb. 2011.
[KMRU10] S. Kudekar, C. Méasson, T. J. Richardson, and R. L. Urbanke, "Threshold saturation on BMS channels via spatial coupling," 6th Int. Symp. Turbo Codes & Iterative Inform. Process., 2010.

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- An alternative proof technique based on potential functions for coupled scalar and vector recursions [YJNP12].

[YJNP12a] A. Yedla, Y.-Y. Jian, P. Nguyen, and H. Pfister, "A simple proof of threshold saturation for coupled scalar recursions," 7th Int. Symp. Turbo Codes & Iterative Inform. Process., 2012. Extended version: http://arxiv.org/abs/1309.7910

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- DE (BP decoding, vector recursion, BEC(ε)):

$$\boldsymbol{x}^{(\ell+1)} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}^{(\ell)}); \varepsilon),$$

where $\boldsymbol{f}(\boldsymbol{y})$ and $\boldsymbol{g}(\boldsymbol{x})$ are the variable and check node updates.

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Vector admissible system

- The functions $\boldsymbol{f}(\boldsymbol{y}; \varepsilon)$ and $\boldsymbol{g}(\boldsymbol{x})$ non-decreasing in $\boldsymbol{y}$ and $\boldsymbol{x}$;
- $\boldsymbol{f}(\boldsymbol{y}; \varepsilon)$ is differentiable in $\boldsymbol{y}$ and $\boldsymbol{g}(\boldsymbol{x})$ is twice differentiable in $\boldsymbol{x}$;
- $\boldsymbol{f}(\boldsymbol{0}; \varepsilon) = \boldsymbol{f}(\boldsymbol{y}; 0) = \boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0}$;
- $\boldsymbol{f}(\boldsymbol{y}; \varepsilon)$ is strictly increasing with ε.

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- Potential function [YJNP12b]:

$$U(\boldsymbol{x};\varepsilon) = \boldsymbol{g}(\boldsymbol{x})\boldsymbol{D}\boldsymbol{x}^{\mathsf{T}} - G(\boldsymbol{x}) - F(\boldsymbol{g}(\boldsymbol{x});\varepsilon),$$

  where F and G are scalar functions that satisfy $F'(\boldsymbol{y};\varepsilon) = \boldsymbol{f}(\boldsymbol{y};\varepsilon)\boldsymbol{D}$, and $G'(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})\boldsymbol{D}$, with $\boldsymbol{D}$ a positive diagonal matrix.

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A fixed point of the DE ($\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x});\varepsilon)$) corresponds to a stationary point of the corresponding potential function ($U(\boldsymbol{x};\varepsilon)$).

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- The potential threshold: $\varepsilon^{*} = \sup_{\varepsilon}\left(\varepsilon \in (\varepsilon^{\mathrm{BP}}, 1] \mid \Delta E(\varepsilon) \geq 0,\ \forall \boldsymbol{x} \in \mathcal{X}\right),$

  where $\Delta E(\varepsilon) = \inf_{\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{U}_{\boldsymbol{0}}(\boldsymbol{x})} U(\boldsymbol{x};\varepsilon)$ for some ε, $\varepsilon^{\mathrm{BP}} \leq \varepsilon \leq \varepsilon^{*}$, and $\mathcal{U}_{\boldsymbol{0}}(\varepsilon) = \{\boldsymbol{x} \in \mathcal{X} | \boldsymbol{x}^{\infty} = \boldsymbol{0}\}$ is the basin of attraction for $\boldsymbol{x}^{\infty} = \boldsymbol{0}$.

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- Proved threshold saturation of irregular LDPC codes for a Slepian-Wolf problem with erasures, joint decoding of irregular LDPC codes on an erasure multiple-access channel, and protograph codes on the BEC.

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Motivation

- Threshold saturation for binary SC-LDPC codes so far.
- Nonbinary SC-LDPC codes:
  - Construction method for nonbinary SC-LDPC codes [UKS11].
  - The threshold saturation also occurs for nonbinary SC-LDPC codes. Contrary to uncoupled ensembles, the BP threshold of nonbinary SC-LDPC codes improves with field size and tends to the Shannon limit [PGiA13].

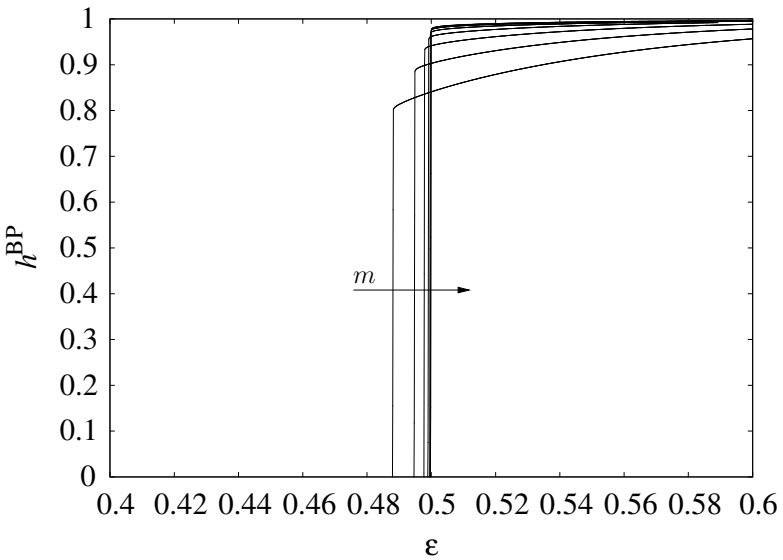

Figure: BP EXIT functions for nonbinary $(3, 6)$ SC-LDPC code ensembles over GF_2^m.

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[PGiA13] A. Piemontese, A. Graell i Amat, and G. Colavolpe, "Nonbinary spatially-coupled LDPC codes on the binary erasure channel," *IEEE Int. Conf. Commun.*, ICC'2013.

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- A symmetric $\boldsymbol{D}$ is sufficient so that F and G exist.
- Proof of threshold saturation.

DE for $(d_{\mathrm{v}}, d_{\mathrm{c}}, m)$ LDPC code ensembles on the BEC

- Code symbols $\alpha_0, \ldots, \alpha_{2^m-1} \in \mathcal{S} = \mathsf{GF}_2^m$

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- For the BEC the non-zero entries of a message are all equal, and the message is equivalent to a subspace of GF_2^m of dimension k.
- DE simplifies to exchange of probability vectors of length $m+1$, $(\tilde{p}_0, \ldots, \tilde{p}_m)$, where $\tilde{p}_i$ is the probability that the message has dimension i.

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- $\boldsymbol{x}_{\circ}^{\ell} = (x_{\circ 0}^{\ell}, \ldots, x_{\circ m}^{\ell})$; $x_{\circ i}^{\ell}$ is the probability that a message from variable nodes at iteration ℓ has dimension i.
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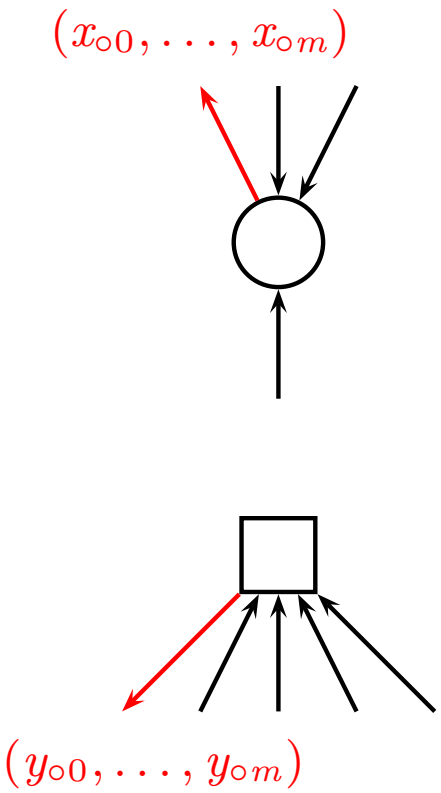

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- At check nodes, the BP decoder computes the check node updates (sum of the subspaces corresponding to the incoming messages),

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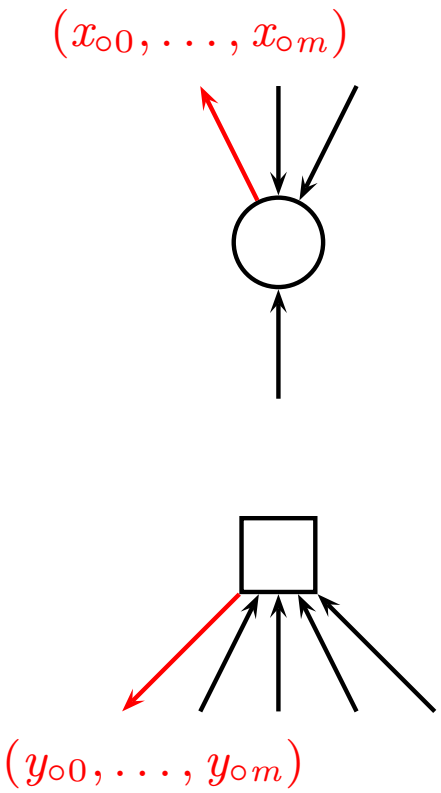

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- At variable nodes, the decoder computes the variable node updates (intersection of the subspaces of the incoming messages), corresponding to the incoming messages,

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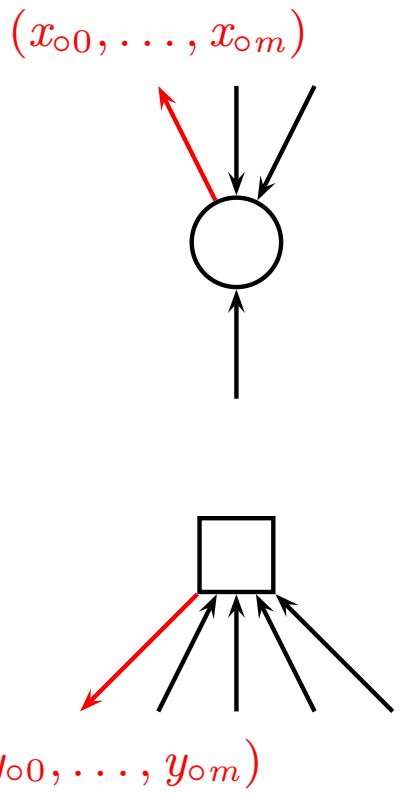

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- Does DE for nonbinary LDPC codes converge to a fixed point?

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- $\boldsymbol{f}_\circ$ and $\boldsymbol{g}_\circ$ are not monotone! $\Longrightarrow$ Not an admissible system.
- Does DE for nonbinary LDPC codes converge to a fixed point?

Idea

Rewrite the DE equation using complementary cumulative distribution function (CCDF) vectors.

DE for $(d_{\mathrm{v}}, d_{\mathrm{c}}, m)$ LDPC code ensembles on the BEC

Definition

Given a probability vector $\boldsymbol{x}_\circ$, define the CCDF vector $\boldsymbol{x} = (x_1, \ldots, x_m)$, where $x_i = \sum_{k=i}^{m} x_{\circ k}$. We also define $x_{m+1} = 0$. Then, it follows that $x_{\circ i} = x_i - x_{i+1}$. Note also that $x_0 = 1$.

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- We define new vector functions $\boldsymbol{f} = (f_0, \ldots, f_m)$ and $\boldsymbol{g} = (g_0, \ldots, g_m)$,

$$f_i = \sum_{k=i}^{m} f_{\circ k}(\boldsymbol{y}_\circ; \varepsilon) = \sum_{k=i}^{m} f_{\circ k}(\boldsymbol{y}^{-1} - \boldsymbol{y}; \varepsilon),$$

$$g_i = \sum_{k=i}^{m} g_{\circ k}(\boldsymbol{x}_\circ) = \sum_{k=i}^{m} g_{\circ k}(\boldsymbol{x}^{-1} - \boldsymbol{x}),$$

where $\boldsymbol{x}^{-1} = (1, x_1, \ldots, x_{m-1})$.

Convergence of the DE for nonbinary LDPC codes

- Using CCDF vectors, the DE equation can be written in an equivalent form as

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}); \varepsilon). \tag{1}$$

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The functions $\boldsymbol{f}(\boldsymbol{y}; \varepsilon)$ and $\boldsymbol{g}(\boldsymbol{x})$ are increasing in $\boldsymbol{y}$ and $\boldsymbol{x}$.

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Successful decoding

Successful decoding corresponds to convergence of the DE equation to the fixed point $\boldsymbol{x}^{\infty} = \mathbf{0} = (0, 0, \ldots, 0)$.

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However...
$U(\boldsymbol{x}; \varepsilon)$ with diagonal $\boldsymbol{D}$ does not exist! $\Longrightarrow$ Conditions on the existence of $U(\boldsymbol{x}; \varepsilon)$.

Properties of $\boldsymbol{D}$ and calculation of $U(\boldsymbol{x};\varepsilon)$

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Theorem 4 [$U(\boldsymbol{x};\varepsilon)$ for the $(d_{\mathrm{v}}, d_{\mathrm{c}}, m)$ ensemble]

$F(\boldsymbol{y};\varepsilon)$ and $G(\boldsymbol{x})$ exist (hence $U(\boldsymbol{x};\varepsilon)$ exists) if there exist sets of values $\{d_{js}\}$, $\{\varphi_{(i_1,\ldots,i_m)}\}$ and $\{\mu_{(k_1,\ldots k_m)}\}$ that satisfy the following equations,

$$\begin{cases} i_s\varphi_{(i_1,\ldots,i_s,\ldots,i_m)} = \sum_{j=1}^{m} d_{js}\varphi^{(j)}_{(i_1,\ldots i_s-1,\ldots,i_m)}(\varepsilon) \\ k_t\mu_{(k_1,\ldots,k_t,\ldots,k_m)} = \sum_{j=1}^{m} d_{jt}\gamma^{(j)}_{(k_1,\ldots,k_t-1,\ldots k_m)} \end{cases}, \tag{2}$$

for all possible m-tuples $(i_1,\ldots,i_m)$ and $(k_1,\ldots,k_m)$ and all i_s and k_t varying from 1 to m. The coefficients φ's and γ's in (2) are given by

$$\varphi^{(j)}_{(i_1,\ldots,i_m)}(\varepsilon) = \mathrm{coeff}(f_j(\boldsymbol{x};\varepsilon), x_1^{i_1}\cdots x_m^{i_m}),$$
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Similar result for the coupled case. General result for coupled vector systems.

Properties of $\boldsymbol{D}$ and calculation of $U(\boldsymbol{x};\varepsilon)$

Theorem 5 [Existence of $U(\boldsymbol{x};\varepsilon)$ with diagonal $\boldsymbol{D}$]

A solution to (2) exists if, for all i from 1 to m

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- Many vector systems satisfy (3)–(4): The examples in [YJNP12b], bilayer LDPC codes for the relay channel, ...
- Not satisfied for nonbinary LDPC codes! $\Longrightarrow$ The potential function does not exist for diagonal $\boldsymbol{D}$.

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Proposition 2 [Nonbinary LDPC codes]

A positive symmetric matrix $\boldsymbol{D}$ is sufficient for the existence of a solution of (2).

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Theorem 3 [Threshold saturation]

Given the spatially coupled $(d_{\mathrm{v}}, d_{\mathrm{c}}, m, L, \mathtt{w})$ LDPC code ensemble, for $\varepsilon < \varepsilon^*$ and $\mathtt{w} > \frac{mK}{2\Delta E(\varepsilon)}$, the only fixed point of the system is $\boldsymbol{x}^{\infty} = \boldsymbol{0}$.

Conclusion

- Threshold saturation occurs for nonbinary SC-LDPC codes on the BEC!
- Monotonicity of the variable node and check node updates for nonbinary LDPC codes, and the existence of a fixed point in the DE.
- Existence and computation of $U(\boldsymbol{x};\varepsilon)$: Solution of a system of linear equations.
- A necessary condition on the existence of F and G (thus $U(\boldsymbol{x};\varepsilon)$), assuming diagonal $\boldsymbol{D}$.
- A symmetric $\boldsymbol{D}$ is sufficient for $U(\boldsymbol{x};\varepsilon)$ exist for nonbinary codes.

[AGiA13] I. Andriyanova, A. Graell i Amat, "Threshold Saturation for Nonbinary SC-LDPC Codes on the Binary Erasure Channel," submitted to *IEEE Transactions on Information Theory*. [Online]. Available: http://arxiv.org/abs/1311.2003/