LECTURES ON NAKAJIMA'S QUIVER VARIETIES

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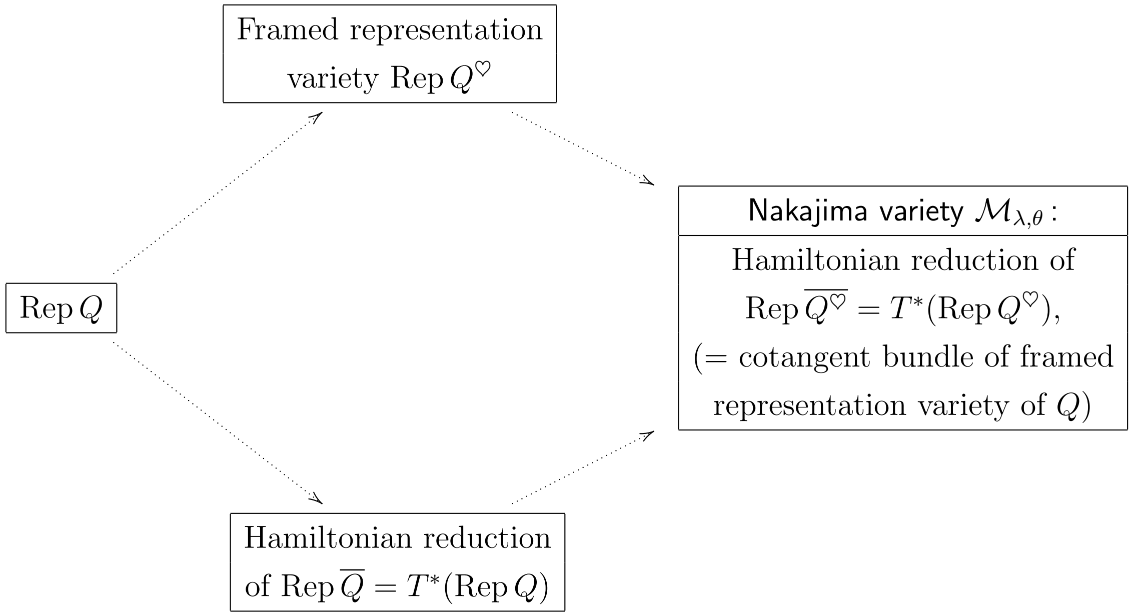
1. OUTLINE

1.1. **Introduction.** Nakajima's quiver varieties are certain smooth (non-affine) complex algebraic varieties associated with quivers. These varieties have been used by Nakajima to give a geometric construction of universal enveloping algebras of Kac-Moody Lie algebras (as well as of the corresponding quantized enveloping algebras) and of all irreducible integrable (e.g., finite dimensional) representations of those algebras. Nakajima's varieties provide also a large class of examples of algebraic symplectic manifolds whith extremely nice properties and rich structures, interesting in their own right.

A connection between quiver representations and Kac-Moody Lie algebras has been first discovered by C. Ringel around 1990. Ringel produced a construction of $U_q\mathfrak{n}$, the *positive part* of the quantized enveloping algebra $U_q\mathfrak{g}$ of a Kac-Moody Lie algebra $\mathfrak{g}$, in terms of a *Hall algebra* associated with an appropriate quiver. Shortly afterwards, G. Lusztig combined Ringel's ideas with the powerful technique of perverse sheaves to construct a *canonical basis* of $U_q\mathfrak{n}$.

The main advantage of Nakajima's approach (as opposed to the earlier one by Ringel and Lusztig) is that it yields a geometric construction of the whole algebra $U_q\mathfrak{g}$ rather than its positive part. In addition, it provides a geometric construction of simple integrable $U_q\mathfrak{g}$-modules.

There are several steps involved in the definition of Nakajima's quiver varieties. Given a quiver Q, we associate to it three other quivers, $Q^\heartsuit$, $\overline{Q}$, and $\overline{Q^\heartsuit}$, respectively. In terms of these quivers, various steps of the construction of Nakajima varieties may be illustrated schematically as follows



Our exposition will be close to the one given by Nakajima in [Na5].

1.2. **Reminder.** Throughout, the ground field is the field $\mathbb{C}$ of complex numbers.

We fix a quiver Q, i.e., a finite oriented graph, with vertex set I and edge set E. We write Q^{op} for the opposite quiver obtained from Q by reversing the orientation of edges.

For any pair $i, j \in I$, let a_{ij} denote the number of edges of Q going from j to i. The matrix $A_Q := \|a_{ij}\|$ is called the *adjacency matrix* of Q.

On $\mathbb{C}^I$, one has the standard euclidean inner product $\alpha \cdot \beta := \sum_{i\in I} \alpha_i\beta_i$. Thus, the (non-symmetric) bilinear form associated with the adjacency matrix reads

$$A_Q\alpha{\cdot}\beta = \sum_{x\in E} \alpha_{\mathrm{tail}(x)}\,\beta_{\mathrm{head}(x)}, \qquad \alpha, \beta \in \mathbb{C}^I.$$

One also defines an *Euler form* by the formula

$$\langle \alpha, \beta\rangle_Q := \alpha \cdot \beta - A_Q\alpha{\cdot}\beta, \tag{1.2.1}$$

Let $\mathbb{C}Q$ be the *algebra of* Q, and let $\mathbb{C}I \subset \mathbb{C}Q$ be the subalgebra spanned by trivial paths. Thus $\mathbb{C}I$ is a semisimple finite dimensional algebra isomorphic to the algebra of $\mathbb{C}$-valued functions on the set I, equipped with pointwise multiplication. We write 1_i for the trivial path at a vertex $i \in I$, equivalently, for the characteristic function of the one point set $\{i\} \subset I$.

Let B be a $\mathbb{C}$-algebra equipped with an algebra map $\mathbb{C}I \to B$, eg. B is a quotient of the algebra of Q. Abusing the notation, we also write 1_i for the image of the element $1_i \in \mathbb{C}I$ in B. Associated with any finite dimensional representation M, of B, is its *dimension vector* $\dim M \in \mathbb{Z}^I_{\geq 0}$, such that the ith coordianate of $\dim M$ equals $(\dim M)_i := \dim(1_i \cdot M)$.

Given an I-tuple $\mathbf{v} = (v_i)_{i\in I} \in \mathbb{Z}^I$, let $G_{\mathbf{v}} := \prod_{i\in I} GL(v_i)$. Further, write $\operatorname{Rep}(B, \mathbf{v})$ for the space of $\mathbf{v}$-dimensional representations of B. In the special case $B = \mathbb{C}Q$, we use simplified notation $\mathrm{Rep}(Q,\mathbf{v}) = \mathrm{Rep}(\mathbb{C}Q,\mathbf{v})$ for the space $\mathbf{v}$-dimensional representations of Q.

Thus, $G_{\mathbf{v}}$ is a reductive group, and $\mathrm{Rep}(Q,\mathbf{v})$ is a vector space that comes equipped with a linear $G_{\mathbf{v}}$-action, by 'base change' automorphisms. We have

$$\dim \mathrm{Rep}(Q,\mathbf{v}) = A_Q\mathbf{v}\cdot\mathbf{v}, \qquad \dim G_{\mathbf{v}} = \mathbf{v}\cdot\mathbf{v}. \tag{1.2.2}$$

Note the the subgroup $\mathbb{C}^\times \subset G_{\mathbf{v}}$, of *diagonnaly imbedded* invertible scalar matrices acts trivially on $\mathrm{Rep}(Q,\mathbf{v})$.

We will very often use the following elementary result

Lemma 1.2.3. *Let B be an algebra equipped with an algebra map $\mathbb{C}I \to B$. Then, the isotropy group of any point of* $\mathrm{Rep}(B,\mathbf{v})$ *is a* connected *subgroup of* $G_{\mathbf{v}}$.

Proof. Let M be a representation of B, and write $\mathrm{End}_B M$ for the algebra of B-module endomorphisms of M. It is known that the isotropy group G^M of the point $M \in \mathrm{Rep}(B,\mathbf{v})$ may be identified with the group of invertible elements of the algebra $\mathrm{End}_B M$.

We claim that, more generally, the set $A^\times$ of of invertible elements of any finite dimensional $\mathbb{C}$-algebra A is connected. To see this, we observe that the set A^{sing}, of noninvertible elements of A, is a hypersurface in A given by the equation $\det m_a = 0$, where m_a denotes the operator of left multiplication by an element $a \in A$.

Such a hypersurface has real codimension ≥ 2 in A, hence cannot disconnect A, a real vector space. Therefore, the set $A^\times = A \smallsetminus A^{\mathrm{sing}}$, of invertible elements, must be connected. ☐

2. Moduli of representations of quivers

2.1. **Categorical quotients.** Naively, one would like to consider a space of isomorphism classes of representations of Q of some fixed dimension $\mathbf{v}$. Geometrically, this amounts to considering the orbit space $\mathrm{Rep}(Q,\mathbf{v})/G_{\mathbf{v}}$. Such an orbit space is, however, rather 'badly behaved' in most cases. Usually, it does not have a reasonable Hausdorff topology, for instance.

One way to define a reasonable orbit space is to use a *categorical quotient*

$$\mathrm{Rep}(Q,\mathbf{v})//G_{\mathbf{v}} := \mathrm{Spec}\,\mathbb{C}[\mathrm{Rep}(Q,\mathbf{v})]^{G_{\mathbf{v}}},$$

the spectrum of the algebra of $G_{\mathbf{v}}$-invariant polynomials on the vector space $\mathrm{Rep}(Q,\mathbf{v})$. By definition, $\mathrm{Rep}(Q,\mathbf{v})//G_{\mathbf{v}}$ is an affine algebraic variety.

To understand the categorical quotient, we recall the following result of Le Bruyn and Procesi, [LBP],

Proposition 2.1.1. *The algebra $\mathbb{C}[\mathrm{Rep}(Q,\mathbf{v})]^{G_{\mathbf{v}}}$ is generated by the functions $\mathrm{Tr}_p:\ M \mapsto \mathrm{Tr}_M(p)$, where p runs over the set of oriented cycles in Q of the form $p = p_{i_1,i_2}\cdot p_{i_2,i_3}\cdot\ldots\cdot p_{i_{s-1},i_s}\cdot p_{i_s,i_1}$, $(p_{ij}\in E)$, and where $\mathrm{Tr}_M(p)$ denotes the trace of the operator corresponding to such a cycle in the representation $M \in \mathrm{Rep}(Q,\mathbf{v})$.*

Corollary 2.1.2. *For a quiver Q without oriented cycles, one has $\mathbb{C}[\mathrm{Rep}(Q,\mathbf{v})]^{G_{\mathbf{v}}} = \mathbb{C}$, hence, $\mathrm{Rep}(Q,\mathbf{v})//G_{\mathbf{v}} = pt$.* ☐

Theorem 2.1.3. *Geometric (= closed) points of the scheme* $\mathrm{Spec}\,\mathbb{C}[\mathrm{Rep}(Q,\mathbf{v})]^{G_{\mathbf{v}}}$ *are in a natural bijection with $G_{\mathbf{v}}$-orbits of semisimple representations of Q.* ☐

Corrolary 2.1.2 shows that the categorical quotient may often reduce to a point, so a lot of geometric information may be lost.

A better approach to the moduli problem is to use a stability condition and to replace the orbit space $\operatorname{Rep}(Q, \mathbf{v})/G_{\mathbf{v}}$, or the categorical quotient $\operatorname{Rep}(Q, \mathbf{v})//G_{\mathbf{v}}$, by an appropriate moduli space of (semi)-stable representations. There is a price to pay: moduli spaces arising in this way do depend on the choice of a stability condition, in general.

2.2. **Reminder on GIT.** The general theory of quotients by a reductive group action via stability conditions has been developed by D. Mumford, and is called Geometric Invariant Theory, cf. [GIT].

To fix ideas, let X be a not necessarily irreducible, affine algebraic G-variety, where G is a reductive algebraic group. Given a rational character (= algebraic group homomorphism) $\chi : G \to \mathbb{C}^{\times}$, Mumford defines a scheme $X//_{\chi}G$ in the following way. Let G act on the cartesian product $X \times \mathbb{C}$ by the formula $g : \ (x, z) \mapsto (gx, \chi(g)^{-1} \cdot z)$ (more generally, the cartesian product $X \times \mathbb{C}$ may be replaced here by the total space of any G-equivariant line bundle on X). The coordinate ring of $X \times \mathbb{C}$ is the algebra $\mathbb{C}[X \times \mathbb{C}] = \mathbb{C}[X][z]$, of polynomials in the variable z with coefficients in the coordinate ring of X. This algebra has an obvious grading by degree of the polynomial.

Let $A_{\chi} := \mathbb{C}[X \times \mathbb{C}]^G$ be the subalgebra G-invariants. Clearly, this is a graded subalgebra which is, moreover, a finitely generated algebra by Hilbert's theorem on finite generation of algebras of invariants. Explicitly, a polynomial $f(z) = \sum_{n=0}^{N} f_n \cdot z^n \in \mathbb{C}[X][z]$ is G-invariant if and only if, for each $n = 0, \ldots, n$, the function f_n is a χ^n-*semi-invariant*, i.e. if and only if one has

$$f_n(g^{-1}(x)) = \chi(g)^n \cdot f_n(x), \quad \forall g \in G,\ x \in X.$$

Write $\chi^n : \ g \mapsto \chi(g)^n$ for the n-th power of the character χ and let $\mathbb{C}[X]^{\chi^n} \subset \mathbb{C}[X]$ be the vector space of χ^n-semi-invariant functions. It is clear that we have

$$A_{\chi} := \mathbb{C}[X \times \mathbb{C}]^G = \bigoplus\nolimits_{n \geq 0} \mathbb{C}[X]^{\chi^n},$$

and the direct sum decomposition on the right corresponds to the grading on the algebra A_{χ}.

Let $X//_{\chi}G := \mathsf{Proj}\, A_{\chi}$ be the projective spectrum of that graded algebra. This is a quasi-projective scheme, called a *GIT quotient* of X by the G-action; the scheme $X//_{\chi}G$ is reduced whenever so is X (since A_{χ} has no nilpotents provided there are no nilpotents in $\mathbb{C}[X]$). The geometric points of the scheme $X//_{\chi}G$ correspond to graded maximal ideals of the algebra A_{χ} different from the augmentation ideal.

Remark 2.2.1. In the special case of the polynomial algebra $A = \mathbb{C}[u_1, \ldots, u_m]$, we have $\mathsf{Proj}\, A = \mathbb{P}^m = (\mathbb{C}^{m+1} \smallsetminus \{0\})/\mathbb{C}^{\times}$. Excluding the augmentation ideal of A corresponds in this case to removing the origin $0 \in \mathbb{C}^{m+1}$ from considerations. ◊

In general, for $n = 0$, we have $\mathbb{C}[X]^{\chi^n} = \mathbb{C}[X]^G$, is the algebra of G-invariants. Thus, we have a canonical algebra imbedding $\mathbb{C}[X]^G \hookrightarrow A_{\chi}$ as the degree zero subalgebra. Put another way, the algebra imbedding $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X \times \mathbb{C}]^G$ is induced by the first projection $X \times \mathbb{C} \to X$.

Standard results of algebraic geometry imply that the algebra imbedding $\mathbb{C}[X]^G \hookrightarrow A_{\chi}$ induces a *projective* morphism of schemes $\pi : \ \mathsf{Proj}\, A_{\chi} \to \operatorname{Spec} \mathbb{C}[X]^G = X//G$.

To get a better understanding of the GIT quotient $X//_{\chi}G$, one introduces the following definition, see [GIT].

Definition 2.2.2. (i) *A point* $x \in X$ *is called* χ*-semistable if there exists* $n \geq 0$ *and a* χ^n*-semi-invariant* $f \in \mathbb{C}[X]^{\chi^n}$ *such that* $f(x) \neq 0$.

(ii) *A point* $x \in X$ *is called* χ*-stable if there exists* $n \geq 1$ *and a* χ^n*-semi-invariant* $f \in \mathbb{C}[X]^{\chi^n}$ *such that* $f(x) \neq 0$*, in addition, the* G*-action on the set* $X \smallsetminus f^{-1}(0)$ *is required to be closed, and the isotropy group of the point* x *is required to be finite.*

Write X^{ss}_χ*, resp.* X^s_χ*, for the set of semistable, resp. stable, points. Thus, we have* $X^s_\chi \subset X^{ss}_\chi \subset X$.

(iii) *Two* χ*-semistable points* x, x' *are called* S-equivalent *if and only if the orbit closures* $\overline{G \cdot x}$ *and* $\overline{G \cdot x'}$ *meet in* X^{ss}_χ.

Note that the G-orbit of a stable point is an orbit of maximal dimension, equal to $\dim G$, moreover, this orbit is closed in X^{ss}_χ. Hence, two stable points are S-equivalent if and only if they belong to the same orbit.

Now, for any G-orbit $\mathcal{O} \subset X$, the polynomials $f \in A_\chi$ such that $f(\mathcal{O}) = 0$ form a graded ideal $\mathfrak{I}_\mathcal{O} \subset A_\chi$. It is easy to see that $\mathfrak{I}_\mathcal{O}$ is a *maximal* graded ideal in A_χ and that, for any G-orbit $\mathcal{O} \subset X^{ss}$, we have $\mathfrak{I}_\mathcal{O} \neq A_\chi$.

One of the main basic results of GIT reads

Theorem 2.2.3. (i) *The assignment* $\mathcal{O} \mapsto \mathfrak{I}_\mathcal{O}$ *induces a natural bijection between the set of* S*-equivalence classes of* G*-orbits in* X^{ss}_χ *and the set of geometric points of the scheme* $X/\!/_\chi G$.

(ii) *The bijection from (i) makes the orbit set* X^s/G *a Zariski open (possibly empty) subset in* $X/\!/_\chi G$.

Example 2.2.4. For the trivial character $\chi = 1$, a point $x \in X$ is χ-semistable if and only if the G-orbit of x is closed in X. Furthermore, we have $A_\chi = \mathbb{C}[X]^G[z]$. Therefore, we get

$$X/\!/_\chi G \;=\; \mathsf{Proj}\, A_\chi \;=\; \mathsf{Proj}\left(\mathbb{C}[X]^G[z]\right) \;=\; \operatorname{Spec} \mathbb{C}[X]^G \;=\; X/\!/G, \quad \text{for } \chi = 1.$$

In this case, the map π becomes an isomorphism $X/\!/_\chi G \xrightarrow{\sim} X/\!/G$.

We will frequently use the following result which is, essentially, a consequence of definitions.

Corollary 2.2.5. *Let* X *be a smooth* G*-variety such that the isotropy group of any point of* X *is connected. Then the set* X^s/G *is contained in the smooth locus of the scheme* $X/\!/_\chi G$.

Assume, in addition, that X *is affine and that the* G*-action on* X^{ss} *is free. Then any semistable point is stable and the scheme* $X/\!/_\chi G$ *is smooth.* □

2.3. **Stability conditions for quivers.** A. King introduced a totally different, purely algebraic, notion of stability for representations of algebras. He then showed that, in the case of quiver representations, his definition of stability is actually equivalent to Mumford's Definition 2.2.2.

To explain King's approach, fix a quiver Q and fix $\theta \in \mathbb{R}^I$. It will become clear shortly that the parameter θ is an analogue of the group character $\chi : G \to \mathbb{C}^\times$, in Mumford's theory.

Definition 2.3.1. *A representation* M *of* Q *is said to be* θ*-semistable (resp.* θ*-stable), if* $\theta \cdot \dim M = 0$ *and, for any subrepresentation* $N \subset M$*, we have* $\theta \cdot \dim N \leq 0$*, (resp. for any subrepresentation* $N \subset M$*, such that* $N \neq 0, M$*, we have* $\theta \cdot \dim N < 0$*).*

Clearly, if the vector $\theta \in \mathbb{R}^I_{<0}$ has negative coordinates then there are no θ-semistable representations.

At the opposite extreme, let the vector $\theta \in \mathbb{R}^I_{\geq 0}$ have non-negative coordinates. Then *any* representation of Q is θ-semistable. If, furthermore, $\theta \in \mathbb{R}^I_{>0}$, then any representation of Q is even θ-stable. We will often use the following vector

$$\theta^+ := (1, 1, \ldots, 1) \in \mathbb{Z}^I_{>0}. \tag{2.3.2}$$

In general, it may be rather hard to describe explicitely the class of (semi)stable representations.

Remark 2.3.3. Let $\dim M = \mathbf{v}$ and fix $\theta \in \mathbb{Z}^I$. Then, it is clear that $\theta \cdot \dim M = 0$ if and only if the character χ_θ vanishes on the subgroup $\mathbb{C}^\times \subset G_\mathbf{v}$.

For this reason, we will be mostly interested below in (semi)stable representations of dimension $\mathbf{v}$ such that $\theta \cdot \mathbf{v} = 0$. Clearly, this may only happen when not all the coordinates of the vector θ are of the same sign, in particular, we have $\theta^+ \cdot \mathbf{v} \neq 0$. ♢

The definition of (semi)stability given above is a special case of a more general definition due to A. King [Ki], who considers the case of an arbitrary associative $\mathbb{C}$-algebra A.

Given such an algebra A, let $K_{\mathrm{fin}}(A)$ denote the Grothendieck group of all *finite dimensional* A-modules. This is a free abelian group with the basis formed by the classes of simple finite dimensional A-modules. Fix an additive group homomorphism $\theta : K_{\mathrm{fin}}(A) \to \mathbb{R}$.

Following King, one says that a finite dimensional A-module M is θ-semistable, if $\theta([M]) = 0$ and, for any A-submodule $N \subset M$, we have $\theta([N]) \leq 0$. A semistable representation M is called θ-stable if the only subrepresentations $N \subset M$ with $\theta([N]) = 0$ are M and 0.

This definition specializes to Definition 2.3.1 as follows. One takes $A := \mathbb{C}Q$. Then, the assignment $[M] \mapsto \dim M$ yields a well defined group homomorphism $\dim : K_{\mathrm{fin}}(\mathbb{C}Q) \to \mathbb{Z}^I$. Now, for any $\theta \in \mathbb{R}^I$, define a group homomorphism $\phi_\theta : \mathbb{Z}^I \to \mathbb{R}$, $x \mapsto \sum_i \theta_i \cdot x_i$. This yields an obvious isomorphism $\mathbb{R}^I \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}^I, \mathbb{R})$, $\theta \mapsto \phi_\theta$. Thus, given $\theta \in \mathbb{R}^I$, one may form a composite homomorphism $K_{\mathrm{fin}}(\mathbb{C}Q) \to \mathbb{Z}^I \to \mathbb{R}$, $[M] \mapsto \theta \cdot \dim M$.

For this last homomorphism, King's general definition of stability for A-modules reduces to Definition 2.3.1.

Remark 2.3.4. Assume that the quiver Q has no oriented cycles. Then, it is easy to show that any simple representation M of Q is 1-dimensional, i.e., there exists a vertex $i \in I$ such that $M_i = \mathbb{C}$ and $M_j = 0$ for any $j \neq i$. It follows that the map $\dim : K_{\mathrm{fin}}(\mathbb{C}Q) \to \mathbb{Z}^I$, $[M] \mapsto \dim M$ is in this case a group isomorphism. ♢

Fix either a quiver or a $\mathbb{C}$-algebra. Let $f : M \to M'$ be a morphism of θ-semistable representations of (of possibly different dimensions). Then, it is immediate from definitions that $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are again semistable.

In this way, one obtains

Lemma 2.3.5. *For any $\theta \in \mathbb{R}^I$, the θ-semistable representations form a full abelian subcategory in the category of all finite dimensional representations. θ-stable representations are the simple objects of that category.* □

As a corollary, we deduce that any θ-semistable representation M has a Jordan-Hölder filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_m = M$, by subrepresentations, such that M_k/M_{k-1}

is a θ-stable representation for any $k = 1, \ldots, m$. The associated graded representation $\mathrm{gr}^s M := \oplus_k M_k/M_{k-1}$ does not depend, up to isomorphism, on the choice of such a filtration.

To relate Mumford's and King's notions of stability, we associate with an integral vector $\theta = (\theta_i)_{i\in I} \in \mathbb{Z}^I$, a rational character

$$\chi_\theta : \ G_{\mathbf{v}} \to \mathbb{C}^\times, \quad g = (g_i)_{i\in I} \ \mapsto \ \prod_{i\in I} \det(g_i)^{\theta_i}.$$

The main result of King relating the two notions of stability reads

Theorem 2.3.6. *For any dimension vector* $\mathbf{v}$ *and any* $\theta \in \mathbb{Z}^I$ *such that* $\theta \cdot \mathbf{v} = 0$, *we have*

(i) *A representation* $M \in \mathrm{Rep}(Q, \mathbf{v})$ *is* χ_θ*-semistable, resp.* χ_θ*-stable, in the sense of Definition 2.2.2 if and only if it is* θ*-semistable, resp.* θ*-stable, in the sense of Definition 2.3.1.*

(ii) *A pair* M, M', *of* χ_θ*-semistable representations, are* S-equivalent *in the sense of Definition 2.2.2 if and only if one has* $\mathrm{gr}^s M \cong \mathrm{gr}^s M'$. □

Let $\mathrm{Rep}^s_\theta(Q, \mathbf{v})$ denote the set of stable, resp. $\mathrm{Rep}^{ss}(Q, \mathbf{v})$ denote the set of semistable, representations of dimension $\mathbf{v}$. We write $\mathcal{R}_\theta(\mathbf{v}) = \mathcal{R}_\theta(Q, \mathbf{v}) := \mathrm{Rep}^{ss}_\theta(Q, \mathbf{v})//_{\chi_\theta} G_{\mathbf{v}}$. By Theorem 2.2.3, this is a quasi-projective variety.

Corollary 2.3.7 (A. King)**.** (i) *The group* $G_{\mathbf{v}}/\mathbb{C}^\times$ *acts freely on the set* $\mathrm{Rep}^s_\theta(Q, \mathbf{v})$, *of* θ-stable *representations. The orbit set* $\mathcal{R}^s_\theta(\mathbf{v}) := \mathrm{Rep}^s_\theta(Q, \mathbf{v})/G_{\mathbf{v}}$ *is contained in* $\mathcal{R}_\theta(\mathbf{v})$ *as a Zariski open (possibly empty) subset.*

(ii) *Assume that* Q *has no edge loops. Then, the vector* $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$ *is a* Schur vector *for* Q *if and only if there exists* $\theta \in \mathbb{Z}^I$ *such that*

$$\theta \cdot \mathbf{v} = 0 \quad \text{and} \quad \mathrm{Rep}^s_\theta(Q, \mathbf{v}) \neq \emptyset.$$

For such a θ, *we have* $\dim \mathcal{R}^s_\theta(\mathbf{v}) = 1 - \langle \mathbf{v}, \mathbf{v}\rangle_Q$.

Proof of (i). Let g be an element of the isotropy group of M, such that $g \notin \mathbb{C}^\times$, and let $c \in \mathbb{C}$ be an eigenvalue of g. Then $N := \mathrm{Ker}(g - c\mathrm{Id})$ is a nontrivial subrepresentation of M. Clearly, the group $\mathbb{C}^\times$ acts trivially on N. Hence, we have $\dim N \cdot \mathbf{v} = 0$, contradicting the definition of stability. It follows that the group $G_{\mathbf{v}}/\mathbb{C}^\times$ acts freely on $\mathrm{Rep}^s_\theta(Q, \mathbf{v})$. □

According to Example 2.2.4, we get

Corollary 2.3.8. *In the special case* $\theta = 0$, *one has*

$$\mathcal{R}_0(\mathbf{v}) \ = \ \mathrm{Rep}(Q, \mathbf{v})//G_{\mathbf{v}} \ = \ \mathrm{Spec}\, \mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{G_{\mathbf{v}}}.$$

For any $\theta \in \mathbb{Z}^I$, *there is a canonical projective morphism* $\pi : \ \mathcal{R}_\theta(Q, \mathbf{v}) \to \mathrm{Rep}(Q, \mathbf{v})//G_{\mathbf{v}}$. □

Remark 2.3.9. It may be expected that, for any $\theta \in \mathbb{R}^I$, the set $\mathcal{R}_\theta(\mathbf{v})$ has a natural structure of a complex analytic (not necessarily algebraic) variety. I am not aware of any serious work in this direction.

Remark 2.3.10. Note that a representation M of Q is θ-semistable if and only if M^*, the dual representation of Q^{op}, is $(-\theta)$-semistable. Thus, taking the dual representation yields canonical isomorphisms

$$\mathrm{Rep}^{ss}_\theta(Q, \mathbf{v}) \xrightarrow{\sim} \mathrm{Rep}^{ss}_{-\theta}(Q^{\mathrm{op}}, \mathbf{v}), \quad \text{resp.} \quad \mathcal{R}_\theta(Q, \mathbf{v}) \xrightarrow{\sim} \mathcal{R}_{-\theta}(Q^{\mathrm{op}}, \mathbf{v}).$$

3. Framings

3.1. The set $\mathcal{R}_\theta(Q, \mathbf{v})$ is often empty in various interesting cases of quiver Q and dimension vector $\mathbf{v}$. Introducing a framing is a way to remedy the situation.

To explain this, fix a quiver Q with vertex set I. We introduce another quiver $Q^\heartsuit$, called the *framing of*, Q as follows. The set of vertices of $Q^\heartsuit$ is defined to be $I \sqcup I'$, where I' is another copy of the set I, equipped with the bijection $I \xrightarrow{\sim} I', \ i \mapsto i'$. The set of edges of $Q^\heartsuit$ is, by definition, a disjoint union of the set of edges of Q and a set of additional edges $\mathbf{j}_i : i \to i'$, from the vertex i to the corresponding vertex i', one for each vertex $i \in I$.

Thus, giving a representation of $Q^\heartsuit$ amounts to giving a representation $\mathbf{x}$, of the original quiver Q, in a vector space $V = (V_i)_{i \in I}$ together with a collection of linear maps $V_i \to W_i, \ i \in I$, where $W = (W_i)_{i \in I}$ is an additional collection of finite dimensional vector W_i, placed at the vertex $i' \in I'$. We let $\mathbf{w} := (\dim W_i)_{i \in I}$ denote the corresponding dimension vector, and write $\mathbf{j} : V \to W$ to denote a collection of linear maps $\mathbf{j}_i : \ V_i \to W_i, \ i \in I$, as above

With this notation, a representation of $Q^\heartsuit$ is a pair $(\mathbf{x}, \mathbf{j})$, where $\mathbf{x}$ is a representation of Q in $V = (V_i)_{i \in I}$, and $\mathbf{j} : V \to W$ is arbitrary additional collection of linear maps. Accordingly, dimension vectors for the quiver $Q^\heartsuit$ are elements $\mathbf{v} \times \mathbf{w} \in \mathbb{Z}^I \times \mathbb{Z}^I = \mathbb{Z}^{I \sqcup I'}$. We write $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w}) := \mathrm{Rep}(Q^\heartsuit, \mathbf{v} \times \mathbf{w})$ for the space of representations $(\mathbf{x}, \mathbf{j})$, of $Q^\heartsuit$, of dimension $\dim V = \mathbf{v}, \ \dim W = \mathbf{w}$.

We define a $G_\mathbf{v}$-action on $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ by $g : \ (\mathbf{x}, \mathbf{j}) \mapsto (g^{-1}\mathbf{x}g, \mathbf{j} \circ g)$, where we write $\mathbf{j} \circ g$ for the collection of maps $V_i \xrightarrow{g_i} V_i \xrightarrow{\mathbf{j}_i} W_i$.

Remark 3.1.1. The group $G_\mathbf{v} = \prod_{i \in I} GL(v_i)$ may be viewed as a subgroup in $G_\mathbf{v} \times G_\mathbf{w} = \prod_{i \in I} GL(v_i) \times \prod_{i \in I} GL(w_i)$. The later group acts on $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ according to the general rule of §1.2 applied in the case of the quiver $Q^\heartsuit$. The $G_\mathbf{v}$-action defined above is nothing but the restriction of the $G_\mathbf{v} \times G_\mathbf{w}$-action to the subgroup $G_\mathbf{v}$.

From now on, we will view $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ as a $G_\mathbf{v}$-variety, and will ignore the action of the other factor, the group $G_\mathbf{w}$. $\Diamond$

3.2. We may apply the general notion of stability, cf. Definition 2.2.2, in the special case of the quiver $Q^\heartsuit$ and a character $\chi_\theta : \ G_\mathbf{v} \to \mathbb{C}^\times$. In this subsection, we restrict ourselves to the case where $\theta = \theta^+$, cf. (2.3.2); we write 'semistable' for 'θ^+-semistable', and let $\mathrm{Rep}^{ss}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ denote the set of semistable representations of $Q^\heartsuit$ of dimension $(\mathbf{v}, \mathbf{w})$. Further let $\mathcal{R}(\mathbf{v}, \mathbf{w}) := \mathrm{Rep}^{ss}(Q^\heartsuit, \mathbf{v}, \mathbf{w})//_{\chi_{\theta^+}} G_\mathbf{v}$ be the corresponding GIT quotient.

Imitating King's arguments, one proves

Lemma 3.2.1. (i) *A representation* $(\mathbf{x}, \mathbf{j}) \in \mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$, *in vector spaces* (V, W), *is semistable if and only if there is no nontrivial subrepresentation* $V' \subset V$, *of the quiver* Q, *contained in* $\mathrm{Ker}\, \mathbf{j}$.

(ii) *The group* $G_\mathbf{v}$ *acts freely on the set* $\mathrm{Rep}^{ss}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$, *moreover, any semistable representation is automatically stable.*

(iii) $\mathcal{R}(\mathbf{v}, \mathbf{w})$ *is a smooth quasi-projective variety and the canonical map* $\mathrm{Rep}^{ss}(Q^\heartsuit, \mathbf{v}, \mathbf{w})/G_\mathbf{v} \to \mathcal{R}(\mathbf{v}, \mathbf{w})$ *is a bijection of sets.*

Proof. Part (i) is proved by mimicing King's proof of Theorem 2.3.6. To prove (ii), let $g \neq \mathrm{Id}$ be an element of the isotropy group of a representation $V \in \mathrm{Rep}^{ss}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$. Then,

$V' := \mathrm{Ker}(g - \mathrm{Id})$ is a subrepresentation of V that violates the condition of part (i). Part (ii) follows from this. Part (iii) follows from (ii) by Corollary 2.2.5. $\square$

Proposition 3.2.2 (King). (i) *Assuming Q has no edge loops and the set of θ-stable $\mathbf{v}$-dimensional representations of Q is nonempty, we have*

$$\dim \mathcal{R}_\theta(\mathbf{v}, \mathbf{w}) = \mathbf{v}\cdot\mathbf{w} - \langle \mathbf{v}, \mathbf{v}\rangle_Q, \tag{3.2.3}$$

(ii) *If Q has no oriented cycles then the scheme $\mathcal{R}_\theta(\mathbf{v}, \mathbf{w})$ is a (smooth) projective variety.*

Sketch of proof of formula (3.2.3). Observe first that we have

$$\dim \mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w}) = \mathbf{w} \cdot \mathbf{v} + A_Q\mathbf{v} \cdot \mathbf{v}.$$

Furthermore, one shows that, for θ as in the statement of the theorem, the set $\mathrm{Rep}^s_\theta(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ is Zariski open in $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$. The $G_\mathbf{v}$-action on $\mathrm{Rep}^s_\theta(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ being free, we compute

$$\begin{aligned} \dim \mathcal{R}_\theta(\mathbf{v}, \mathbf{w}) &= \dim \big(\mathrm{Rep}^s(Q^\heartsuit, \mathbf{v}, \mathbf{w})/G_\mathbf{v}\big) \\ &= \dim \mathrm{Rep}^s(Q^\heartsuit, \mathbf{v}, \mathbf{w}) - \dim G_\mathbf{v} = \dim \mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w}) - \dim G_\mathbf{v} \\ &= \mathbf{w} \cdot \mathbf{v} + A_Q\mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v} - \langle \mathbf{v}, \mathbf{v}\rangle_Q. \end{aligned}$$ $\square$

Remark 3.2.4. Note that the $G_\mathbf{v}$-action on $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ does *not* factor through the quotient $G_\mathbf{v}/\mathbb{C}^\times$. Therefore, the condition $\theta \cdot \dim V = 0$, on the vector $\theta \in \mathbb{R}^I$, is no longer so natural. Indeed, the vector $\theta^+ = (1, 1, \ldots, 1)$ used in the definition of $\mathcal{R}(\mathbf{v}, \mathbf{w})$ does not satisfy this condition. $\Diamond$

Example 3.2.5 (Jordan quiver). Let Q be a quiver with a single vertex and a single edge-loop at this vertex. For any positive integers $n, m \in \mathbb{Z}^I = \mathbb{Z}$, we have $\mathrm{Rep}(Q, n) = \mathrm{End}\,\mathbb{C}^n$. Further, we have

$$Q^\heartsuit : \quad \circlearrowleft \bullet \xrightarrow{\ \mathbf{j}\ } \bullet.$$

Hence, we get $\mathrm{Rep}(Q^\heartsuit, n, m) = \mathrm{End}\,\mathbb{C}^n \times \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$.

It is clear that, for $\theta = \theta^+ = 1$, any n-dimensional representation of Q is θ-stable. The map sending an $n \times n$-matrix to the (unordered) n-tuple of its eigenvalues yields an isomorphism $\mathcal{R}^s(n) = \mathrm{Rep}(Q, n)/\!/GL_n \xrightarrow{\sim} \mathbb{C}^n/\mathbb{S}_n$.

Now, take $m = 1$, so we get $\mathrm{Rep}(Q^\heartsuit, n, m) = \mathrm{End}\,\mathbb{C}^n \times \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$. A pair $(\mathbf{x}, \mathbf{j}) \in \mathrm{End}\,\mathbb{C}^n \times (\mathbb{C}^n)^*$ is semistable if and only if the linear function $\mathbf{j} : \mathbb{C}^n \to \mathbb{C}$ is a *cyclic vector* for $\mathbf{x}^* : (\mathbb{C}^n)^* \to (\mathbb{C}^n)^*$, the dual operator.

It is known that the GL_n-action on the set $\mathrm{Rep}^s(Q^\heartsuit, n, 1)$, of such pairs $(\mathbf{x}, \mathbf{j})$, is free. Moreover, sending $(\mathbf{x}, \mathbf{j})$ to the unordered n-tuple of the eigenvalues of $\mathbf{x}$ yields a bijection between the set of GL_n-orbits in $\mathrm{Rep}^s(Q^\heartsuit, n, 1)$ and $\mathbb{C}^n/\mathbb{S}_n$. Thus, in this case, we have isomorphisms $\mathcal{R}(Q^\heartsuit, n, 1) \cong \mathcal{R}^s(n) \cong \mathbb{C}^n/\mathbb{S}_n$.

Example 3.2.6 (Type **A** Dynkin quiver).

$$Q : \quad \overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \cdots \longleftarrow \overset{n-2}{\bullet} \longleftarrow \overset{n-1}{\bullet} \longleftarrow \overset{n}{\bullet}$$

In this case, we have $I = \{1, 2, \ldots, n\}$ and $\mathrm{Rep}(Q, \mathbf{v})/\!/GL_\mathbf{v} = pt$, since Q has no oriented cycles.

Now, let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (r, 0, 0, \ldots, 0)$, where $r > v_1 > v_2 > \ldots > v_n > 0$, is a strictly decreasing sequence of positive integers. An element of $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ has the form $(\mathbf{x}, \mathbf{j})$, where $\mathbf{x} = (x_{i-1,i} : \mathbb{C}^{v_i} \to \mathbb{C}^{v_{i-1}})_{i=2,\ldots,n}$, and the only nontrivial component of $\mathbf{j}$ is a linear map $j := \mathbf{j}_1 : \mathbb{C}^{v_1} \to \mathbb{C}^r$.

Observe that the collection of vector spaces

$$F_i := \operatorname{Image}(j \circ \mathbf{x}_{21} \circ \ldots \circ \mathbf{x}_{i-1,i}) \subset \mathbb{C}^r, \qquad i = 1, \ldots, n,$$

form an n-step partial flag, $F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^r$, in $\mathbb{C}^r$. Now, the stability condition amounts, in this case, to the *injectivity* of each of the maps $j,\ \mathbf{x}_{12}, \ldots,\ \mathbf{x}_{n-1,n}$. It follows that we have

$$\dim F_i = \dim \operatorname{Image}(j \circ \mathbf{x}_{21} \circ \ldots \circ \mathbf{x}_{i-1,i}) = \dim \mathbb{C}^{v_i} = v_i.$$

In this way, one shows that the corresponding moduli space $\mathcal{R}(\mathbf{v}, \mathbf{w})$ is isomorphic to a partial flag manifold. In particular, it is a smooth projective variety, in accordance with Proposition 3.2.2(ii).

4. HAMILTONIAN REDUCTION FOR REPRESENTATIONS OF QUIVERS

4.1. **Symplectic geometry.** To motivate later constructions, we first remind a few basic definitions.

Let X be a smooth manifold, write $T^*X \to X$ for the the cotangent bundle on X. The total space T^*X, of the cotangent bundle, comes equipped with a canonical symplectic structure, i.e. there is a canonically defined nondegenerated closed 2-form ω on T^*X.

In the case where X is a vector space, the only case we will use below, we have $T^*X = X \times X^*$, where X^* denotes the vector space dual to X. The canonical symplectic structure on $X \times X^*$ is given, in this special case, by the formula

$$\omega(x \times x^*,\ y \times y^*) := y^*(x) - x^*(y), \quad \forall x, y \in X,\ x^*, y^* \in X^*.$$

Now, let a Lie group G act on an arbitrary smooth manifold X. Let $\mathfrak{g}$ be the Lie algebra of G. Given $u \in \mathfrak{g}$, write $\vec{u}$ for the vector field on X corresponding to the 'infinitesimal u-action' on X, and let $\vec{u}_x$ be the value of that vector field at a point $x \in X$.

Associated with the G-action on X, there is a natural G-action on T^*X and a canonical *moment map*

$$\mu :\ T^*X \to \mathfrak{g}^*, \quad \alpha_x \mapsto \mu(\alpha), \quad \text{where } \mathfrak{g}^* \ni \mu(\alpha_x) : u \mapsto \langle \alpha, \vec{u}_x \rangle, \tag{4.1.1}$$

where $\alpha_x \in T^*_x X$ stands for a covector at a point $x \in X$.

The following properties of the map (4.1.1) are straightforward consequences of the definitions.

Proposition 4.1.2. (i) *If the group G is* connected *then the moment map is G-equiva-riant, i.e. it intertwines the G-action on T^*X and the coadjoint G-action on $\mathfrak{g}^*$.*

(ii) *Writing $T^*_Y X$ for the conormal bundle of a submanifold $Y \subset X$, one has*

$$\mu^{-1}(0) = \bigcup_{Y \in X/G} T^*_Y(X). \tag{4.1.3}$$

Here, X/G stands for the set of G-orbits on X.

From the last formula one easily derives the following result

Corollary 4.1.4. *Assume that the group G acts freely on X, and that the orbit space X/G is a well defined smooth manifold. Then,*

- *The G-action on T^*X is free, and the moment map* (4.1.1) *is a submersion.*

- *For any coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^*$, the orbit space $\mu^{-1}(\mathbb{O})/G$ has a natural structure of smooth symplectic manifold.*
- *For $\mathbb{O} = \{0\}$, there is, in addition, a canonical symplectomorphism*

$$T^*(X/G) \cong \mu^{-1}(0)/G. \tag{4.1.5}$$

Formula (4.1.5) explains the importance of the zero fiber of the moment map. Later on, we will consider quotients of $\mu^{-1}(0)$ by the group action in situations where the group action on X is no longer free, so the orbit set X/G can not be equipped with a reasonable structure of a manifold.

For any group G, the differential of a rational group homomorphism $G \to \mathbb{C}^\times$ gives a linear function $\mathfrak{g} \to \mathbb{C}$, i.e. a point $\lambda \in \mathfrak{g}^*$. The points of $\mathfrak{g}^*$ arising in this way are automatically fixed by the coadjoint action of G on $\mathfrak{g}^*$. If the group G is connected, then the corresponding fiber $\mu^{-1}(\lambda)$ is necessarily a G-stable subvariety, by Proposition 4.1.2(i). The varieties of that form play the role of 'twisted cotangent bundles' on X/G. These varieties share many features of the zero fiber of the moment map.

The following elementary result will be quite useful in applications to quiver varieties.

Lemma 4.1.6. Let $\lambda \in \mathfrak{g}^*$ be a fixed point of the coadjoint action of a connected group G, and let G act on a manifold X with an associated moment map μ as in (4.1.1). Then, the following holds:

A geometric point $\alpha \in \mu^{-1}(\lambda)$ is a smooth point of the scheme theoretic fiber $\mu^{-1}(\lambda)$ if and only if α has finite isotropy in G. In such a case, the symplectic form on T^*X induces a nondegenerate bilinear form on the vector space $T_\alpha(T^*X)/\operatorname{Lie} G^\alpha$.

Proof. Put $M := T^*X$, for short, let $\alpha \in M$, and write $G^\alpha \subset G$ for the isotropy group of the point α. One has the following sequence (not exact sequence) of linear maps

$$\mathfrak{g} \xrightarrow{\mathsf{act}} T_\alpha(M) \xrightarrow{\widetilde{\omega}:\ \xi \mapsto \omega(\xi, -)} T^*_\alpha(M) \xrightarrow{\mathsf{act}^*} \mathfrak{g}^*. \tag{4.1.7}$$

Here, the first map is induced by the infinitesimal $\mathfrak{g}$-action on M, the second map is an isomorphism induced by the symplectic 2-form ω on $M = T^*X$, and the last map is the transpose of the first one. Clearly, one has $\operatorname{Lie} G^\alpha = \operatorname{Ker}(\mathsf{act})$.

Now, the crucial observation, that follows directly from the definition of the moment map, cf. (4.1.1), is that the differential of μ at the point α is given by the formula $d\mu|_\alpha = \mathsf{act}^* \circ \widetilde{\omega}$.

Since the map $\widetilde{\omega}$ in (4.1.7) is a bijection, we deduce

$$\begin{aligned} G^\alpha \text{ is finite} \quad &\Longleftrightarrow \quad \operatorname{Lie} G^\alpha = 0 \\ &\Longleftrightarrow \quad \mathsf{act} \text{ is injective} \quad \Longleftrightarrow \quad \mathsf{act}^* \text{ is surjective} \\ &\Longleftrightarrow \quad d\mu|_\alpha \text{ is surjective} \quad \Longleftrightarrow \quad \alpha \text{ is a smooth point.} \end{aligned}$$

This proves the first statement of the lemma.

To prove the second statement, note that we have $T_\alpha(\mu^{-1}(\lambda)) = \operatorname{Ker}(d\mu|_\alpha)$. From (4.1.7) we see that the map $\widetilde{\omega}$ provides an isomorphism $\operatorname{Ker}(d\mu|_\alpha) \xrightarrow{\sim} \operatorname{Ker}(\mathsf{act}^*)$.

One has the following elementary result of linear algebra. Let $f : E \to F$ be a linear map of finite dimensional vector spaces. Then, any nondegenerate bilinear form β, on E, induces a well-defined nondegenerate bilinear form on $\operatorname{Ker}(f)/\operatorname{Image}(\beta^{-1} \circ f^*)$, where we write $\beta^{-1} : E^* \to E$ for the isomorphism induced by β.

The proof is completed by applying this general result of linear algebra in the case where $E := T^*_\alpha(M),\ F = \mathfrak{g}^*$, and $f := \mathsf{act}^*$. $\square$

4.2. Fix a finite set I and a dimension vector $\mathbf{v} \in \mathbb{Z}^I$.

From now on, we specialize to the case where the group G is a product of general linear groups of the form $G_\mathbf{v} = \prod_{i\in I} GL(v_i)$. Thus, we have $\mathfrak{g}_\mathbf{v} := \operatorname{Lie} G_\mathbf{v} = \oplus_{i\in I}\, \mathfrak{gl}(v_i)$. The center of each summand $\mathfrak{gl}(v_i)$ is a 1-dimensional Lie algebra of scalar matrices. Therefore, the center of $\mathfrak{g}_\mathbf{v}$ may be identified with the vector space $\mathbb{C}^I$.

Observe further that any Lie algebra homomorphism $\mathfrak{g}_\mathbf{v} \to \mathbb{C}$ has the form $\mathbf{x} = (x_i)_{i\in I} \mapsto \sum_{i\in I} \lambda_i \cdot \operatorname{Tr} x_i$. We deduce that the fixed point set of the coadjoint $G_\mathbf{v}$-action on $\mathfrak{g}^*_\mathbf{v}$ is a vector space $\mathbb{C}^I \subset \mathfrak{g}^*_\mathbf{v}$. Explicitly, an element $\lambda = (\lambda_i)_{i\in I} \in \mathbb{C}^I$ corresponds to the linear function $\mathbf{x} \mapsto \lambda \cdot \mathbf{x} = \sum_{i\in I} \lambda_i \cdot \operatorname{Tr} x_i$.

4.3. **The double $\overline{Q}$.** Given a quiver Q, let $\overline{Q} = Q \sqcup Q^{\mathrm{op}}$ be the *double* of Q, the quiver that has the same vertex set as Q and whose set of edges is a disjoint union of the sets of edges of Q and of Q^{op}, an opposite quiver. Thus, for any edge $x \in Q$, there is a reverse edge $x^* \in Q^{\mathrm{op}} \subset \overline{Q}$.

Assume the quiver Q has no edge-loops. Then, from formula (1.2.2) applied to the quiver $\overline{Q}$, we find

$$2 \dim G_\mathbf{v} - \dim \operatorname{Rep}(\overline{Q}, \mathbf{v}) = -C_Q \mathbf{v} \cdot \mathbf{v}, \quad \text{where} \quad C_Q := 2\mathrm{Id} - A_{\overline{Q}}, \tag{4.3.1}$$

is the *Cartan matrix* of the underlying graph of Q.

Definition 4.3.2. *For any $\lambda = (\lambda_i)_{i\in I} \in \mathbb{C}^I$, let $\Pi_\lambda = \Pi_\lambda(Q)$ be a quotient of the quiver algebra $\mathbb{C}\overline{Q}$, of the double quiver $\overline{Q}$, by the two-sided ideal generated by the following element*

$$\sum_{x\in Q} (xx^* - x^*x) - \sum_{i\in I} \lambda_i{\cdot}1_i.$$

Thus, Π_λ is an associative algebra called preprojective algebra *of Q with parameter λ.*

The defining relation for the preprojective algebra may be rewritten more explicitly as a collection of relations, one for each vertex $i \in I$, as follows:

$$\sum_{\{x\in Q:\ \mathrm{head}(x)=i\}} xx^* \quad - \sum_{\{x\in Q:\ \mathrm{tail}(x)=i\}} x^*x \;=\; \lambda_i{\cdot}1_i, \qquad i \in I.$$

Clearly, one has $\operatorname{Rep}(\overline{Q}, \mathbf{v}) \cong \operatorname{Rep}(Q, \mathbf{v}) \times \operatorname{Rep}(Q^{\mathrm{op}}, \mathbf{v})$. We will write a point of $\operatorname{Rep}(\overline{Q}, \mathbf{v})$ as a pair $(\mathbf{x}, \mathbf{y}) \in \operatorname{Rep}(Q, \mathbf{v}) \times \operatorname{Rep}(Q^{\mathrm{op}}, \mathbf{v})$.

Recall that, for any pair, E, F, of finite dimensional vector spaces, there is a canonical perfect pairing

$$\operatorname{Hom}(E, F) \times \operatorname{Hom}(F, E) \to \mathbb{C}, \quad f \times f' \mapsto \operatorname{Tr}(f \circ f') = \operatorname{Tr}(f' \circ f).$$

Using this pairing, one obtains canonical isomorphisms of vector spaces

$$\operatorname{Rep}(Q^{\mathrm{op}}, \mathbf{v}) \cong \operatorname{Rep}(Q, \mathbf{v})^*, \quad \text{resp.} \quad \mathfrak{g}_\mathbf{v} \cong \mathfrak{g}^*_\mathbf{v}. \tag{4.3.3}$$

We deduce the following isomorphisms

$$\operatorname{Rep}(\overline{Q}, \mathbf{v}) \cong \operatorname{Rep}(Q, \mathbf{v}) \times \operatorname{Rep}(Q, \mathbf{v})^* \cong T^*\big(\operatorname{Rep}(Q, \mathbf{v})\big). \tag{4.3.4}$$

The natural $G_{\mathbf{v}}$-action on $\operatorname{Rep}(\overline{Q},\mathbf{v})$ corresponds, via the isomorphisms above, to the $G_{\mathbf{v}}$-action on the cotangent bundle induced by the $G_{\mathbf{v}}$-action on $\operatorname{Rep}(Q,\mathbf{v})$. Associated with the latter action, there is a moment map μ. It is given by the following explicit formula

$$\mu:\ \operatorname{Rep}(\overline{Q},\mathbf{v}) = T^*\big(\operatorname{Rep}(Q,\mathbf{v})\big) \longrightarrow \mathfrak{g}_{\mathbf{v}}^* = \mathfrak{g}_{\mathbf{v}}, \quad (\mathbf{x},\mathbf{y}) \mapsto \sum\,(x\circ y - y\circ x) \ \in\ \mathfrak{g}_{\mathbf{v}}.$$

We explain this general formula in the simplest case of the Jordan quiver.

Example 4.3.5. Let Q be a quiver with one vertex and one edge-loop. Then, $\overline{Q}$ is a quiver with a single vertex and two edge-loops at that vertex. Thus, given a positive integer $\mathbf{v}\in\mathbb{Z}^I=\mathbb{Z}$, we have $\operatorname{Rep}(\overline{Q},\mathbf{v})=\mathfrak{gl}_{\mathbf{v}}\times\mathfrak{gl}_{\mathbf{v}}$. The action of the group $G_{\mathbf{v}}$ on the space $\operatorname{Rep}(\overline{Q},\mathbf{v})$ becomes, in this case, the $\operatorname{Ad}GL_{\mathbf{v}}$-diagonal action on pairs of $(\mathbf{v}\times\mathbf{v})$-matrices.

Further, the isomorphism $\mathfrak{g}_{\mathbf{v}} \xrightarrow{\sim} \mathfrak{g}_{\mathbf{v}}^*$, resp. $\operatorname{Rep}(Q^{\mathrm{op}},\mathbf{v}) \xrightarrow{\sim} \operatorname{Rep}(Q,\mathbf{v})^*$, sends a matrix $\mathbf{x}\in\mathfrak{g}_{\mathbf{v}}$ to a linear function $\mathbf{y}\mapsto\operatorname{Tr}(\mathbf{x}\cdot\mathbf{y})$. Hence, in the notation of §4, for any $u\in\mathfrak{gl}_{\mathbf{v}}$, we have $\vec{u}=\operatorname{ad}u$.

Now, according to definitions, see formula (4.1.1), the moment map sends a point $(\mathbf{x},\mathbf{y})\in T^*(\mathfrak{gl}_{\mathbf{v}})=\mathfrak{gl}_{\mathbf{v}}\times\mathfrak{gl}_{\mathbf{v}}$ to a linear function

$$\mu(\mathbf{x},\mathbf{y}):\ \mathfrak{gl}_{\mathbf{v}}\to\mathbb{C}, \quad u\mapsto\langle\mathbf{y},\ \vec{u}_{\mathbf{x}}\rangle=\langle\mathbf{y},\ \operatorname{ad}u(\mathbf{x})\rangle=\operatorname{Tr}\big(\mathbf{y}\cdot[u,\mathbf{x}]\big)=\operatorname{Tr}\big([\mathbf{x},\mathbf{y}]\cdot u\big).$$

We see that, the linear function $\mu(\mathbf{x},\mathbf{y})\in\mathfrak{gl}_{\mathbf{v}}^*$ corresponds, under the isomorphism $\mathfrak{gl}_{\mathbf{v}}^*\xrightarrow{\sim}\mathfrak{gl}_{\mathbf{v}}$, to the matrix $[\mathbf{x},\mathbf{y}]$. We conclude that the moment map for the $\operatorname{Ad}GL_{\mathbf{v}}$-diagonal action on $T^*(\mathfrak{gl}_{\mathbf{v}})=\mathfrak{gl}_{\mathbf{v}}\times\mathfrak{gl}_{\mathbf{v}}$ has the following final form

$$\mu:\ \mathfrak{gl}_{\mathbf{v}}\times\mathfrak{gl}_{\mathbf{v}}\longrightarrow\mathfrak{gl}_{\mathbf{v}}, \quad \mathbf{x}\times\mathbf{y}\mapsto[\mathbf{x},\mathbf{y}].$$

This is nothing but the general formula (4.3.4) in a special case of the Jordan quiver Q. $\Diamond$

It is clear from Definition 4.3.2 that, inside $\operatorname{Rep}(\mathbf{v},\overline{Q})$, one has an equality:

$$\operatorname{Rep}(\Pi_\lambda,\mathbf{v})=\mu^{-1}(\lambda):=\{(\mathbf{x},\mathbf{y})\in\operatorname{Rep}(\overline{Q},\mathbf{v})\mid[\mathbf{x},\mathbf{y}]=\lambda\}, \qquad \lambda\in\mathbb{C}^I. \tag{4.3.6}$$

This is, in fact, an isomorphism of schemes.

It is important to emphasize that, up to a relabelling $\lambda\mapsto\lambda'$ of parameters, **both the scheme $\mu^{-1}(\lambda)$ and the algebra $\Pi_\lambda(Q)$ depend only on the underlying graph of Q, and do not depend on the orientation of the quiver.**

Remark 4.3.7. Observe that, for any Π_λ-representation V of dimension $\mathbf{v}$, in view of the defining relation for the preprojective algebra, one must have

$$\lambda\cdot\mathbf{v}=\sum_{i\in I}\ \lambda_i\cdot\operatorname{Tr}_V(1_i)=\operatorname{Tr}_V\left(\sum_{i\in I}\ \lambda_i 1_i\right)=\operatorname{Tr}_V\left(\sum_{x\in Q}\ (xx^*-x^*x)\right)=0,$$

where in the last equation we have used that the trace of any commutator vanishes. We deduce that the algebra Π_λ has no $\mathbf{v}$-dimensional representations unless $\lambda\cdot\mathbf{v}=0$.

This is consistent with (4.3.6). Indeed, the group $\mathbb{C}^\times\subset G_{\mathbf{v}}$ acts trivially on $\operatorname{Rep}(\overline{Q},\mathbf{v})$, hence the image of the moment map μ is contained in the hyperplane $(\operatorname{Lie}\mathbb{C}^\times)^\perp\subset\mathfrak{g}_{\mathbf{v}}^*$. Therefore, the fiber $\mu^{-1}(\lambda)$ over a point $\lambda\in\mathbb{C}^I\subset\mathfrak{g}_{\mathbf{v}}^*$ is empty unless we have $\lambda\cdot\mathbf{v}=0$. Conversely, it is easy to see that the condition $\lambda\cdot\mathbf{v}=0$ insures that $\mu^{-1}(\lambda)\neq\emptyset$. $\Diamond$

Given a dimension vector $\mathbf{v}$, we introduce the following notation

$$\mathbb{C}^I_{\mathbf{v}} := \{\lambda \in \mathbb{C}^I \mid \lambda \cdot \mathbf{v} = 0\}, \quad \text{resp.} \quad \mathbb{Z}^I_{\mathbf{v}} := \mathbb{Z}^I \cap \mathbb{C}^I_{\mathbf{v}}.$$

The cotangent bundle projection $p : T^*\big(\operatorname{Rep}(Q, \mathbf{v})\big) \to \operatorname{Rep}(Q, \mathbf{v})$, may be clearly identified with the natural projection to $\operatorname{Rep}(Q, \mathbf{v}) \to \operatorname{Rep}(Q, \mathbf{v})$, $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}$, cf. (4.3.4). Restricting the projection p to a fiber of the moment map one obtains a map $p_\lambda : \operatorname{Rep}(\Pi_\lambda, \mathbf{v}) = \mu^{-1}(\lambda) \to \operatorname{Rep}(Q, \mathbf{v})$.

Observe further that the composite $\mathbb{C}Q \hookrightarrow \mathbb{C}\overline{Q} \twoheadrightarrow \Pi_\lambda$ yields an algebra imbedding $\mathbb{C}Q \hookrightarrow \Pi_\lambda$. In terms of the latter imbedding, the map p_λ amounts to restricting representations of the algebra Π_λ to the subalgebra $\mathbb{C}Q$. Thus, we obtain, cf. [**?**],

Proposition 4.3.8. *For any $\mathbf{x} \in \operatorname{Rep}(Q, \mathbf{v})$, the set $p_\lambda^{-1}(X)$ is canonically identified with the set of extensions of $\mathbf{x}$ to a Π_λ-module $(\mathbf{x}, \mathbf{y}) \in \operatorname{Rep}(\Pi_\lambda, \mathbf{v})$.* □

4.4. **Hamiltonian reduction.** For any $\lambda \in \mathbb{C}^I_{\mathbf{v}}$, the fiber $\mu^{-1}(\lambda)$ is a nonempty closed $G_{\mathbf{v}}$-stable subscheme of $\operatorname{Rep}(\overline{Q}, \mathbf{v})$. We consider the following GIT quotient

$$\mathcal{M}_{\lambda,\theta}(\mathbf{v}) := \mu^{-1}(\lambda)/\!/_{\chi_\theta} G_{\mathbf{v}} = \operatorname{Rep}^{ss}(\Pi_\lambda, \mathbf{v})/S\text{-equivalence}, \qquad \lambda \in \mathbb{C}^I_o,\ \theta \in \mathbb{Z}^I_{\mathbf{v}}. \tag{4.4.1}$$

Remark 4.4.2. One may identify $\mathbb{C}^I \times \mathbb{R}^I = \mathbb{R}^3 \otimes \mathbb{R}^I$ and view a pair $(\lambda, \theta) \in \mathbb{C}^I \times \mathbb{R}^I$ as a point in $\mathbb{R}^3 \otimes \mathbb{R}^I$. In this way, the Hamiltonian reduction $\mu^{-1}(\lambda)/\!/_{\chi_\theta} G_{\mathbf{v}}$ may be identified with a *hyper-Kähler reduction* of the vector space $\operatorname{Rep}(\overline{Q^\heartsuit}, \mathbf{v})$, equipped with a Hermitian metric, with respect to the maximal compact subgroup of the complex algebraic group $G_{\mathbf{v}}$ formed by the elements which preserve the metric.

Corollary 4.4.3. (i) *Any simple Π_λ-module of dimension $\mathbf{v}$ corresponds to a point in $\mu^{-1}(\lambda)^{\mathrm{reg}}$, the smooth locus of the scheme* (4.3.6)

(ii) *The group $G_{\mathbf{v}}/\mathbb{C}^\times$ acts freely on $\mu^{-1}(\lambda)^{\mathrm{reg}}$.*

(iii) *Let $T_{G_{\mathbf{v}}\alpha}(\mu^{-1}(\lambda))$ be the normal space, at $\alpha \in \mu^{-1}(\lambda)^{\mathrm{reg}}$, to the orbit $G_{\mathbf{v}}\alpha \subset \mu^{-1}(\lambda)^{\mathrm{reg}}$. Then, the vector space $T_{G_{\mathbf{v}}\alpha}(\mu^{-1}(\lambda))$ has a canonical symplectic structure, and we have*

$$\dim T_{G_{\mathbf{v}}\alpha}(\mu^{-1}(\lambda)) = 2 - C\mathbf{v} \cdot \mathbf{v}.$$

Proof. Part (i) follows, thanks to Schur's lemma, from Lemma 1.2.3 and Lemma 4.1.6. The last lemma also yields part (ii).

To prove (iii), put $U := \mu^{-1}(\lambda)^{\mathrm{reg}}$. For any $\alpha \in U$, the tangent space to U/G at the point corresponding to the image of α equals $T_\alpha(M)/\operatorname{Lie} G^\alpha$. The symplectic structure on this space is provided by the last statement of Lemma 4.1.6.

Further, the (proof of) Lemma 4.1.6 implies that the map $d\mu|_\alpha$ is surjective, and the map act in diagram (4.1.7) is injective. Thus, we compute

$$\begin{aligned}\dim U/G = \dim\big(T_\alpha(M)/\operatorname{Lie} G^\alpha\big) &= \dim \operatorname{Ker}(d\mu|_\alpha) - \dim \operatorname{Image}(\mathsf{act})\\ &= \big[\dim(T^*X) - \dim \mathfrak{g}^*\big] - \dim \mathfrak{g} = 2(\dim X - \dim G). \quad \square\end{aligned}$$

Many of the results concerning stability of quiver representations carry over in a straightforward way to Π_λ-modules. In particular, we have

Theorem 4.4.4. (i) *For $\theta = 0$, the scheme $\mathcal{M}_{\lambda,0}(\mathbf{v}) = \operatorname{Spec} \mathbb{C}[\mu^{-1}(\lambda)]^{G_{\mathbf{v}}}$, is affine; geometric points of this scheme correspond to semisimple Π_λ-modules.*

(ii) *Geometric points of the scheme* $\mathcal{M}_{\lambda,\theta}(\mathbf{v})$ *correspond to* S*-equivalence classes of* θ*-semistable* Π_λ*-modules.*

(iii) *The group* $G_\mathbf{v}$ *acts freely on the set* $\mu^{-1}(\lambda)^s$*, of* θ*-stable points; isomorphism classes of* θ*-stable* Π_λ*-modules form a Zariski open subset* $\mathcal{M}^s_{\lambda,\theta}(\mathbf{v}) \subset \mathcal{M}_{\lambda,\theta}(\mathbf{v})$*, of dimension* $2-(\mathbf{v}, C\mathbf{v})$*.*

(iv) *The canonical map* $\pi : \mathcal{M}_{\lambda,\theta}(\mathbf{v}) \to \mathcal{M}_{\lambda,0}(\mathbf{v})$ *is a projective morphism that takes a* Π_λ*-module* V *to its semi-simplification.*

Sketch of Proof. Part (i) of the theorem is a consequence of Corollary 2.3.8.

To prove (iii), let $V \in \mu^{-1}(\lambda)^s$ be a stable $\Pi_\lambda(Q)$-module. A version of Corollary 2.3.7(ii) implies that the isotropy group of V is equal to $\mathbb{C}^\times$. It follows that V gives a smooth point of the fiber $\mu^{-1}(\lambda)$, by Lemma 4.1.6. Furthermore, Corollary 4.4.3 applies and we find

$$\dim \mathcal{M}^s_{\lambda,\theta}(\mathbf{v}) = 2(\dim \mathrm{Rep}(Q,\mathbf{v}) - \dim(G_\mathbf{v}/\mathbb{C}^\times)) = 2-(\mathbf{v}, C\mathbf{v}).$$

Other statements of the theorem are obtained by applying Theorem 2.3.6 to the quiver $\overline{Q}$. □

In the special case $\lambda = 0$, using isomorphism (4.1.3), we deduce

Proposition 4.4.5. *The variety* $\mathcal{M}_{0,\theta}(\mathbf{v})$ *contains* $T^*\mathcal{R}^s_\theta(\mathbf{v})$*, the cotangent space to the moduli space* $\mathcal{R}^s_\theta(\mathbf{v})$*, as an open (possibly empty) subset of the smooth locus of* $\mathcal{M}_{0,\theta}(\mathbf{v})$*.* □

Example 4.4.6 (Dynkin quivers). Let Q be an ADE quiver, and fix $\mathbf{v}$ and λ such that $\lambda\cdot\mathbf{v} = 0$.

The number of $G_\mathbf{v}$-orbits in $\mathrm{Rep}(Q,\mathbf{v})$ is finite by the Gabriel theorem. Thus, we see from (4.1.3) that $\mu^{-1}(0)$ is in this case a finite union of conormal bundles, a Lagrangian subvariety in $T^*\,\mathrm{Rep}(Q,\mathbf{v})$.

Further, for any $\mathbf{x} \in \mathrm{Rep}(Q,\mathbf{v})$ and any $t \in \mathbb{C}^\times$, let $t\cdot\mathbf{x}$ be the representation obtained by rescaling all maps in $\mathbf{x}$ by t. It is easy to see that $\mathbf{x}$ and $t\cdot\mathbf{x}$ belong to the same $G_\mathbf{v}$-orbit. Hence, each $G_\mathbf{v}$-orbit in $\mathrm{Rep}(Q,\mathbf{v})$ is a $\mathbb{C}^\times$-stable subvariety. It follows that the conormal bundle to such an orbit contains the origin $0 \in \mathrm{Rep}(\overline{Q},\mathbf{v})$ in its closure. We deduce that any homogeneous $G_\mathbf{v}$-invariant polynomial on $\mu^{-1}(0)$ of positive degree vanishes.

One proves that any simple $\Pi_\lambda(Q)$-module is 1-dimensional, and one has

$$\mathcal{M}_0(\mathbf{v}) = pt.$$

4.5. **The McKay correspondence.** Fix a 2-dimensional symplectic vector space (E,ω), and a finite subgroup $\Gamma \subset Sp(E,\omega) = SL_2(\mathbb{C})$. Let $Q = Q_\Gamma$ be the McKay graph of Γ.

It is well known that the assignment $\Gamma \mapsto Q_\Gamma$ gives a one-to-one correspondence between conjugacy classes of finite subgroups of the group $SL_2(\mathbb{C})$ and quivers of the form $\overline{Q}$, where Q is an extended Dynkin quiver of type $\widetilde{A}$, $\widetilde{D}$, $\widetilde{E}$.

The set I of vertices of Q is identified, via the McKay correspondence, with the set of equivalence classes of irreducible representations of Γ. We write L_i for the irreducible representation corresponding to the vertex $i \in I$ and put $\delta_i := \dim L_i$. In particular, there is a distinguished vertex $o \in I$ corresponding to the trivial representation.

Associated with the extended Dynkin diagram, is a root system $R \subset \mathbb{Z}^I$. The vector $\delta = (\delta_i)_{i\in I} \in \mathbb{Z}^I$ is the minimal imaginary root of that root system.

Let $\mathbb{C}\Gamma$ be the group algebra of Γ, and for each $i \in I$, choose a minimal idempotent $e_i \in \mathbb{C}\Gamma$ such that $\mathbb{C}\Gamma\cdot e_i \cong L_i$. Put $e = \sum_{i\in I} e_i$, an idempotent in $\mathbb{C}\Gamma$.

One way of stating the McKay correspondence is as follows

Theorem 4.5.1. (i) *There is a canonical algebra isomorphism* $e\cdot\Pi_0(Q)\cdot e \cong \mathrm{Sym}\,E\rtimes\Gamma$*; in particular, the algebras* $\Pi_0(Q)$ *and* $(\mathrm{Sym}\,E)\rtimes\Gamma$ *are Morita equivalent.*

(ii) *There is a canonical algebra isomorphism* $1_o\cdot\Pi_0(Q)\cdot 1_o\cong(\mathrm{Sym}\,E)^\Gamma$.

The orbit space $\mathbb{C}^2/\Gamma=\mathrm{Spec}\,\mathbb{C}[x,y]^\Gamma$ is an irreducible normal 2-dimensional variety with an isolated singularity at the origin. Such a variety is known to have a minimal resolution, unique up to isomorphism.

The following result is a reformulation of a result of P. Kronheimer in the language of quiver moduli.

Theorem 4.5.2. (i) *There is a natural isomorphism* $\mathcal{M}_0(\delta)\cong\mathbb{C}^2/\Gamma$, *of algebraic varieties.*

(ii) *Assume that* $\theta\in\mathbb{Z}^I$ *does not belong to root hyperplanes of the affine root system. Then, the variety* $\mathcal{M}_\theta(\delta)$ *is smooth and the canonical map* $\pi:\ \mathcal{M}_\theta(\delta)\to\mathcal{M}_0(\delta)=\mathbb{C}^2/\Gamma$ *is the minimal resolution of* $\mathbb{C}^2/\Gamma$.

5. NAKAJIMA VARIETIES

5.1. We now combine together all the previous constructions and consider the quiver $\overline{Q^\heartsuit}$. For any dimension vector $(\mathbf{v},\mathbf{w})\in\mathbb{Z}^I\times Z^I$, by definition, we have

$$\begin{aligned}\mathrm{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w}) &= T^*\,\mathrm{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})\\ &= \mathrm{Rep}(Q,\mathbf{v})\times\mathrm{Rep}(Q^{\mathrm{op}},\mathbf{v})\times\mathrm{Hom}_I(\mathbb{C}^{\mathbf{v}},\mathbb{C}^{\mathbf{w}})\times\mathrm{Hom}_I(\mathbb{C}^{\mathbf{w}},\mathbb{C}^{\mathbf{v}}).\end{aligned}$$

Thus, one may view an element of $\mathrm{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w})$ as a quadruple $(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j})$, where $\mathbf{x}\in\mathrm{Rep}(Q,\mathbf{v})$, $\mathbf{y}\in\mathrm{Rep}(Q^{\mathrm{op}},\mathbf{v})$, $\mathbf{i}\in\mathrm{Hom}_I(\mathbb{C}^{\mathbf{w}},\mathbb{C}^{\mathbf{v}})$, and $\mathbf{j}\in\mathrm{Hom}_I(\mathbb{C}^{\mathbf{v}},\mathbb{C}^{\mathbf{w}})$.

The vector space has the symplectic structure of a cotangent bundle and the group $G_{\mathbf{v}}$ acts on $\mathrm{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w})$ by symplectic automorphisms. The associated moment map is given by

$$\mu:\ \mathrm{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w})\to\mathfrak{g}_{\mathbf{v}}^*=\mathfrak{g}_{\mathbf{v}},\quad(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j})\mapsto\sum\,[x,y]+\mathbf{i}\otimes\mathbf{j}\ \in\ \mathfrak{g}_{\mathbf{v}},\tag{5.1.1}$$

where $\mathbf{i}_i\otimes\mathbf{j}_i:\ V_i\to V_i$ is a rank one operator, and we use the notation $\mathbf{i}\otimes\mathbf{j}:=\sum_{i\in I}\ \mathbf{i}_i\otimes\mathbf{j}_i$.

For any $\lambda\in\mathbb{C}^I$ we have

$$\mu^{-1}(\lambda)=\{(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j})\in\mathrm{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w})\mid[\mathbf{x},\mathbf{y}]+\mathbf{i}\otimes\mathbf{j}=\lambda\}.$$

The equation on the right is often called the *moment map equation*, or the *ADHM-equation*, since an equation of this form was first considered by Atiyah, Hitchin, Drinfeld, and Manin in their work on instantons on $\mathbb{P}^2$.

From that point of view, it is natural to view the equation above as part of a larger system of hyper-Kähler moment map equations, cf. Remark 4.4.2. Accordingly, we will refer to an element of the real vector space $\mathbb{R}^3\otimes\mathbb{R}^{\mathbf{v}}$ as a ‘hyper-Kähler parameter’.

Now, given $\theta\in\mathbb{Z}^I$, we may apply general Definition 2.3.1 to the variety $\mu^{-1}(\lambda)$ and the character χ_θ of the group $G_{\mathbf{v}}$. This way, one proves

Proposition 5.1.2. A quadruple $(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j})\in\mu^{-1}(\lambda)$ is θ-stable if and only if the following holds:

For any collection of vector subspaces $S = (S_i)_{i\in I} \subset V = (V_i)_{i\in I}$ which is stable under the maps $\mathbf{x}$, by we have

$$
\begin{aligned}
S_i \subset \operatorname{Ker} \mathbf{j}_i,\ \ \forall i \in I \quad &\Longrightarrow \quad \theta(\dim S) \leq 0; && (5.1.3)\\
S_i \supset \operatorname{Image} \mathbf{i}_i,\ \ \forall i \in I \quad &\Longrightarrow \quad \theta(\dim S) \leq \theta(\dim V). && (5.1.4)
\end{aligned}
$$

In the special case, cf. (2.3.2)*, where $\theta = \theta^+$, resp. $\theta = -\theta^+$, condition* (5.1.4)*, resp.* (5.1.3)*, is superfluous.* ☐

Definition 5.1.5. *The variety $\mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w}) := \mu^{-1}(\lambda)^{ss}_{\theta}/\chi_\theta G_{\mathbf{v}}$ is called the* Nakajima variety *with parameters (λ,θ). Let $\mathcal{M}^s_{\lambda,\theta}(\mathbf{v},\mathbf{w}) \subset \mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w})$ denote the subset corresponding to stable points.*

To formulate the main properties of Nakajima varieties, recall first that, in the case where the quiver Q has no edge loops, there is a root system $R \subset \mathbb{Z}^I$ associated with the Cartan matrix C_Q. Let R_+ be the set of roots $\alpha \in R$ with nonnegative coordinates. For any root $\alpha \in R$, we put $\alpha^\perp := \{\lambda \in \mathbb{R}^I \mid \lambda \cdot \alpha = 0\}$.

Given a dimension vector $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$, the parameter $(\lambda,\theta) \in \mathbb{C}^I \times \mathbb{Z}^I$ will be called $\mathbf{v}$-*regular* if, viewed as a hyper-Kähler parameter $(\lambda,\theta) \in \mathbb{R}^3 \otimes \mathbb{R}^I$, it satisfies

$$(\lambda,\theta) \in \mathbb{R}^3 \otimes \mathbb{R}^I \smallsetminus \bigcup_{\{\alpha \in R_+ \mid \mathbf{v}-\alpha \in \mathbb{Z}^I_{\geq 0}\}} \mathbb{R}^3 \otimes \alpha^\perp. \tag{5.1.6}$$

We note that $(\lambda,\theta) := (0,\theta^+)$ is a regular parameter.

Theorem 5.1.7. (i) *We have $\mathcal{M}_{\lambda,0}(\mathbf{v},\mathbf{w}) = \mu^{-1}(\lambda)//G_{\mathbf{v}}$ is an affine algebraic variety for any $\lambda \in \mathbb{C}^I$; given $\theta \in \mathbb{Z}^I$, there is a natural projective morphism $\pi:\ \mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w}) \to \mathcal{M}_{\lambda,0}(\mathbf{v},\mathbf{w})$.*

(ii) *If Q has no edge loops and the parameter (λ,θ) is $\mathbf{v}$-regular, then $\mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w}) = \mathcal{M}^s_{\lambda,\theta}(\mathbf{v},\mathbf{w})$ is a smooth and connected algebraic symplectic manifold of dimension*

$$\dim \mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w}) \ =\ 2\mathbf{w}\cdot\mathbf{v} - C_Q\mathbf{v}\cdot\mathbf{v}.$$

(ii) *The variety $\mathcal{M}_{0,\theta^+}(\mathbf{v},\mathbf{w})$ contains $T^*\mathcal{R}^+_\theta(\mathbf{v},\mathbf{w})$ as a Zariski open subset.*

Sketch of Proof. Part (i) is clear. To prove (ii), one shows that the isotropy group of any point $(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j}) \in \mu^{-1}(\lambda)$ that satisfies the stability conditions (5.1.3)-(5.1.4) is trivial, provided the parameter (λ,θ) is $\mathbf{v}$-regular. It follows, in particular, that the $G_{\mathbf{v}}$-orbit of a semistable point $(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j}) \in \mu^{-1}(\lambda)$ must be an orbit of maximal dimension equal to $\dim G_{\mathbf{v}}$. We conclude that one semistable orbit can not be contained in the closure of another semistable orbit. Thus, all semistable orbits are closed in $\mu^{-1}(\lambda)^{ss}$, hence any semistable point is actually stable.

Further, by Corollary 4.1.6, the triviality of stabilizers implies that the set $\mu^{-1}(\lambda)^{ss}$ of θ-stable points is smooth and $\mu^{-1}(\lambda)^{ss}/G_{\mathbf{v}}$ is a symplectic manifold. Further, we have $\dim \operatorname{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w}) = 2\mathbf{w}\cdot\mathbf{v} + (2\mathrm{Id} - C_Q)\mathbf{v}\cdot\mathbf{v}$. Therefore, we find

$$\dim\big(\mu^{-1}(\lambda)^{ss}/G_{\mathbf{v}}\big) = 2\mathbf{w}\cdot\mathbf{v} + (2\mathrm{Id} - C_Q)\mathbf{v}\cdot\mathbf{v} - 2\dim G_{\mathbf{v}} = 2\mathbf{w}\cdot\mathbf{v} - C_Q\mathbf{v}\cdot\mathbf{v}.$$

The connectedness of the varieties $\mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w})$ is a much more difficult result proved by Crawley-Boevey [CB1]. ☐

Let $\mu^{-1}(\lambda)^\circ \subset \mu^{-1}(\lambda)$ be the subset of points with trivial isotropy group. We let $\mathcal{M}^\circ_{\lambda,0}(\mathbf{v},\mathbf{w}) \subset \mathcal{M}_{\lambda,0}(\mathbf{v},\mathbf{w})$ be the image of this set in $\mu^{-1}(\lambda)//G_{\mathbf{v}}$. We have

Proposition 5.1.8. *Assume the quiver Q has no edge loops. Then,*

(i) *$\mathcal{M}^\circ_{\lambda,0}(\mathbf{v},\mathbf{w})$ is a Zariski open (possibly empty) subset of $\mathcal{M}_{\lambda,0}(\mathbf{v},\mathbf{w})$ and the map π restrics to an isomorphism*

$$\pi: \ \pi^{-1}\big(\mathcal{M}^\circ_{\lambda,0}(\mathbf{v},\mathbf{w})\big) \xrightarrow{\sim} \mathcal{M}^\circ_{\lambda,0}(\mathbf{v},\mathbf{w}). \qquad \square$$

(ii) *Assume that (λ,θ) is a $\mathbf{v}$-regular parameter and the set $\mathcal{M}^\circ_{\lambda,0}(\mathbf{v},\mathbf{w})$ is nonempty. Then the set $\pi^{-1}\big(\mathcal{M}^\circ_{\lambda,0}(\mathbf{v},\mathbf{w})\big)$ is dense in $\mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w})$ and map $\pi: \ \mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w}) \to \mathcal{M}_{\lambda,0}(\mathbf{v},\mathbf{w})$ is a symplectic resolution of singularities.* $\square$

5.2. **A Lagrangian subvariety.**

In this subsection, we let $\lambda = 0$. Note the the affine scheme $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$ contains a distinguished point 0, the image of the zero representation $0 \in \operatorname{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w})$ in the categorical quotient $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w}) = \mu^{-1}(0)//G_{\mathbf{v}}$.

For any $\theta \in \mathbb{Z}^I$, we define a closed subscheme $\Lambda_\theta(\mathbf{v},\mathbf{w}) \subset \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$ to be $\Lambda_\theta(\mathbf{v},\mathbf{w}) := [\pi^{-1}(0)]_{\mathrm{red}}$, the zero fiber of the canonical morphism $\pi: \ \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w}) \to \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$ equipped with reduced scheme structure.

Definition 5.2.1. *A subvariety Λ of a symplectic manifold (M,ω) is called* Lagrangian *if the tangent space to Λ at any smooth point of Λ is a maximal isotropic subspace of the symplectic 2-form ω.*

Theorem 5.2.2. *Assume Q has no edge-loops. Then, we have*

(i) *If the parameter $(0,\theta)$ is $\mathbf{v}$-regular then, each irreducible component of the scheme $\Lambda_\theta(\mathbf{v},\mathbf{w})$ is a Lagrangian subvariety of $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$, a symplectic manifold.*

(ii) *If $\theta = \theta^+$, then the $G_{\mathbf{v}}$-orbit of a quadruple $(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j}) \in \mu^{-1}(0)^{ss}$ represents a point of $\Lambda_\theta(\mathbf{v},\mathbf{w})$ if and only if we have $\mathbf{i} = 0$ and the $G_{\mathbf{v}}$-orbit of the pair $(\mathbf{x},\mathbf{y}) \in \operatorname{Rep}(\overline{Q},\mathbf{v})$ contains the pair $(0,0) \in \operatorname{Rep}(\overline{Q},\mathbf{v})$ in its closure.*

Remark 5.2.3. Part (ii) of the theorem motivates the name 'nilpotent variety' for the variety $\Lambda_\theta(\mathbf{v},\mathbf{w})$. $\Diamond$

We will prove part (i) of Theorem 5.2.2. To this end, we introduce a $\mathbb{C}^\times$-action on various varieties in question. First of all, let $\mathbb{C}^\times$ act naturally on $T^*\operatorname{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})$, by dilations along the fibers of the contangent bundle. This $\mathbb{C}^\times$-action corresponds, via the identification $\operatorname{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w}) = T^*\operatorname{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})$, to the action $\mathbb{C}^\times \ni t: \ (\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j}) \mapsto (\mathbf{x}, t\cdot\mathbf{y}, \mathbf{i}, t\cdot\mathbf{j})$. The latter action keeps the subvariety $\mu^{-1}(0)$ stable and commutes with the $G_{\mathbf{v}}$-action on $\operatorname{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w})$. Therefore, for any θ, there is an induced $G_{\mathbf{v}}$-action $\mathbb{C}^\times \ni t: \ z \mapsto t(z)$, on $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$. Furthermore, the map π becomes a $G_{\mathbf{v}}$-equivariant morphism of $G_{\mathbf{v}}$-varieties, and the fiber $\pi^{-1}(0) \subset \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$ becomes a $G_{\mathbf{v}}$-stable subvariety.

Observe next that the symplectic form ω on $T^*\operatorname{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})$ gets rescaled under the $\mathbb{C}^\times$-action as follows $\mathbb{C}^\times \ni t: \ \omega \mapsto t\cdot\omega$. Hence, the induced symplectic form on $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$, to be denoted by ω again, transforms in a similar way.

Lemma 5.2.4. *If the quiver Q has no oriented cycles, then the $\mathbb{C}^\times$-action on $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$ is a contraction to $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})^{\mathbb{C}^\times} = \{0\}$, the only fixed point.*

Proof. A fixed point in $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w}) = \mu^{-1}(0)/G_{\mathbf{v}}$ is represented, by definition, by a $\mathbb{C}^\times$-stable and closed $G_{\mathbf{v}}$-orbit $\mathcal{O} \subset \mu^{-1}(0)$. For any point $(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j}) \in \mathcal{O}$, in this closed orbit, we must have $\lim_{t\to\infty}(\mathbf{x}, t\cdot\mathbf{y}, \mathbf{i}, t\cdot\mathbf{j}) \in \mathcal{O}$. Thus, we get $(\mathbf{x},0,\mathbf{i},0) \in \mathcal{O}$. We conclude that the orbit $\mathcal{O}$ is contained in $\operatorname{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})$, the zero section of the cotangent bundle $T^*\operatorname{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w}) = \operatorname{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w})$.

Observe next that, for any homogeneous polynomial $f \in \mathbb{C}[\operatorname{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})]^{G_{\mathbf{v}}}$, of positive degree, we have $f|_{\operatorname{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})} = 0$, since Q has no oriented cycles, see Proposition 2.1.1. Also, the restriction map $\mathbb{C}[\operatorname{Rep}(Q^\heartsuit,\mathbf{v},\mathbf{w})]^{G_{\mathbf{v}}} \to \mathbb{C}[\mu^{-1}(0)]^{G_{\mathbf{v}}}$ is a surjection, since $\mu^{-1}(0)$ is a closed subvariety and the group $G_{\mathbf{v}}$ is reductive. It follows from this that any homogeneous invariant polynomial $f \in \mathbb{C}[\mu^{-1}(0)]^{G_{\mathbf{v}}}$, of positive degree, vanishes on the orbit $\mathcal{O}$. But $G_{\mathbf{v}}$-invariant polynomials are known to separate closed $G_{\mathbf{v}}$-orbits. Thus, $\mathcal{O} = \{0\}$. $\square$

We note that the $\mathbb{C}^\times$-action on $\operatorname{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w})$ depends on the orientation of Q (that is, on the position of $Q^\heartsuit$ inside $\overline{Q^\heartsuit}$) while the scheme $\mu^{-1}(0)$ is independent of such an orientation. From now on, we choose an orientation such that Q has no oriented cycles, so that Lemma 5.2.4 holds.

The fixed point set of the $\mathbb{C}^\times$-action in the smooth variety $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$ is a (necessarily smooth) subvariety $F := \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})^{\mathbb{C}^\times} \subset \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$. We write $F_1,\ldots,F_r$ for the connected components of F, and introduce the following sets

$$\Lambda_s := \{z \in \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w}) \mid \lim_{t\to\infty} t(z) \text{ exists, and we have } \lim_{t\to\infty} t(z) \in F_s\}, \quad s = 1,\ldots,r. \tag{5.2.5}$$

Lemma 5.2.6. *The set F is contained in $\pi^{-1}(0)$, and there is a decomposition $\Lambda_{0,\theta}(\mathbf{v},\mathbf{w}) = \bigsqcup_{1\le s\le r} \Lambda_s$.*

Proof. Since $\pi\big(\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})^{\mathbb{C}^\times}\big) \subset \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})^{\mathbb{C}^\times} = \{0\}$, it follows that $F \subset \pi^{-1}(0)$.

Fix $z \in \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$, and consider the map $\mathbb{C}^\times \to \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$, $t \mapsto t(z)$. The $\mathbb{C}^\times$-action being a contraction, this map extends to the point $t = 0$. It is clear that if the map $t \mapsto t(z)$ also has a limit as $t \to \infty$ then it extends to a regular map $\mathbb{P}^1 \to \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$. Such a map must be a constant map, since $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$ is affine. Thus, we have shown that, for any $z \in \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w}) \smallsetminus \{0\}$, the map $t \mapsto t(z)$ has no limit as $t \to \infty$. Therefore, for any $z \in \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w}) \smallsetminus \pi^{-1}(0)$, the map $t \mapsto t(z)$ can not have a limit as $t \to \infty$.

We conclude that the limit $\lim_{t\to\infty} t(z)$ exists if and only if we have $z \in \pi^{-1}(0)$. Moreover, since $\pi^{-1}(0)$ is proper, in such a case one has $\lim_{t\to\infty} t(z) \in \pi^{-1}(0)$. Finally, it is clear that the limit point must be a $\mathbb{C}^\times$-fixed point. The result follows. $\square$

Remark 5.2.7. The $\mathbb{C}^\times$-action on $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$ is a contraction to F.

Theorem 5.2.2(i) is clearly a consequence of the following more precise result

Proposition 5.2.8. *Each piece Λ_s is a smooth, connected, locally closed Lagrangian subvariety of $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$.*

Furthermore, the closures $\overline{\Lambda}_s$, $s = 1,\ldots,r$, are precisely the irreducible components of Λ.

Proof. The pieces defined by equation (5.2.5) are known as the Bialinicki-Birula pieces. They are shown by Bialinicki-Birula to be smooth, connected, and locally closed subsets, in a much more general setup. The first statement of the proposition follows.

Next, we fix a connected component F_s and a point $\phi \in F_s$. The tangent space to $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$ at ϕ has a weight decomposition with respect to the $\mathbb{C}^\times$-action

$$T_\phi\big(\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})\big) = \bigoplus\nolimits_{m\in\mathbb{Z}} H_m, \tag{5.2.9}$$

such that $t \in \mathbb{C}^\times$ acts on the direct summand H_m via multiplication by t^m. In particular, we see that $H_0 = T_\phi F$, is the tangent space to the fixed point set F.

Recall that the symplectic 2-form ω on $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$ has weight $+1$ with respect to the $\mathbb{C}^\times$-action. Hence, a pair of direct summands H_k and H_l are ω-orthogonal unless $k+l=1$; furthermore, the 2-form gives a perfect pairing $\omega:\ H_m \times H_{1-m} \to \mathbb{C}$, for any $m \in \mathbb{Z}$. We see, in particular, that $\bigoplus_{m\leq 0} H_m$ is a Lagrangian subspace in $\bigoplus_{m\in\mathbb{Z}} H_m$.

To complete the proof, pick $z \in \Lambda_s$ such that $\lim\limits_{t\to\infty} t(z) = \phi$. It is clear that, for the curve $t \mapsto t(z)$ to have a limit as $t \to \infty$, the tangent vector to the curve at $t=\infty$ must belong to the span of nonpositive weight subspaces. In other words, we must have

$$\frac{d(t(z))}{dt}\Big|_{t=\infty} \ \in\ \bigoplus\nolimits_{m<0} H_m.$$

Since Λ_s is smooth at ϕ, we deduce the equation $T_\phi(\Lambda_s) = \bigoplus_{m\leq 0} H_m$. It follows, by the above, that $T_\phi(\Lambda_s)$ is a Lagrangian subspace in $T_\phi\big(\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})\big)$, and the first statement of the proposition is proved.

Now, the decomposition of Lemma 5.2.6 presents Λ as a union of irreducible varieties of equal dimensions, and the second statement of the proposition follows. ☐

5.3. **Hilbert scheme of points.** Let Q be the Jordan quiver, and let $\mathbf{v} \in \mathbb{Z}$ be a positive integer.

In the setting of Example 4.3.5, the fiber of the moment map over a central element $\lambda \cdot \mathrm{Id} \in \mathfrak{gl}_{\mathbf{v}}$ equals

$$\mu^{-1}(\lambda\cdot\mathrm{Id}) = \{(\mathbf{x},\mathbf{y}) \in \mathfrak{gl}_{\mathbf{v}} \times \mathfrak{gl}_{\mathbf{v}} \mid [\mathbf{x},\mathbf{y}] = \lambda\cdot\mathrm{Id}\}.$$

This variety is empty for $\lambda \neq 0$ since we have $\mathrm{Tr}([\mathbf{x},\mathbf{y}]) = 0$. For $\lambda = 0$, we get $\mu^{-1}(0) = \mathscr{Z}$, the *commuting variety* of the Lie algebra $\mathfrak{gl}_{\mathbf{v}}$.

Let $\imath : \mathbb{C}^{\mathbf{v}} \hookrightarrow \mathfrak{gl}_{\mathbf{v}}$ be the imbedding of diagonal matrices. Since any two diagonal matrices commute, we get a closed imbedding $\imath\times\imath:\ \mathbb{C}^{\mathbf{v}} \times \mathbb{C}^{\mathbf{v}} \hookrightarrow \mathscr{Z}$. The group $\mathbb{S}_{\mathbf{v}} \subset GL_{\mathbf{v}}$, of permutation matrices, acts diagonally on $\mathscr{Z} \subset \mathfrak{gl}_{\mathbf{v}} \times \mathfrak{gl}_{\mathbf{v}}$, and clearly preserves the image of the map $\imath\times\imath$. Therefore, restriction of $\mathrm{Ad}\,GL_{\mathbf{v}}$-invariant functions induces an algebra map $(\imath\times\imath)^*:\ \mathbb{C}[\mathscr{Z}]^{\mathrm{Ad}\,GL_{\mathbf{v}}} \to \mathbb{C}[\mathbb{C}^{\mathbf{v}}\times\mathbb{C}^{\mathbf{v}}]^{\mathbb{S}_{\mathbf{v}}}$. The latter map can be shown to be an algebra isomorphism.

Thus, we deduce

$$\mathcal{M}_{0,0}(\mathbf{v}) = \mu^{-1}(0)/\!/GL_{\mathbf{v}} = \operatorname{Spec}\mathbb{C}[\mathscr{Z}]^{\mathrm{Ad}\,GL_{\mathbf{v}}} = \operatorname{Spec}\mathbb{C}[\mathbb{C}^{\mathbf{v}}\times\mathbb{C}^{\mathbf{v}}]^{\mathbb{S}_{\mathbf{v}}} = (\mathbb{C}^{\mathbf{v}}\times\mathbb{C}^{\mathbf{v}})/\mathbb{S}_n. \tag{5.3.1}$$

Next, we study Nakajima varieties $\mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w})$ for the Jordan quiver Q. We have

$$\overline{Q^\heartsuit} \quad = \quad \begin{tikzcd} \mathbb{C}^{\mathbf{v}} \arrow[loop above, "\mathbf{x}"] \arrow[loop below, "\mathbf{y}"] \arrow[r, bend left, "\mathbf{j}"] & \mathbb{C}^{\mathbf{w}} \arrow[l, bend left, "\mathbf{i}"] \end{tikzcd}$$

Therefore, writing $M_\lambda(\mathbf{v},\mathbf{w}) := \mu^{-1}(\lambda\cdot \mathrm{Id})$ for the corresponding fiber of the moment map, we get

$$M_\lambda(\mathbf{v},\mathbf{w}) = \{(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j}) \in \mathfrak{gl}_\mathbf{v}\times\mathfrak{gl}_\mathbf{v}\times \mathrm{Hom}(\mathbb{C}^\mathbf{w},\mathbb{C}^\mathbf{v})\times\mathrm{Hom}(\mathbb{C}^\mathbf{v},\mathbb{C}^\mathbf{w}) \mid [\mathbf{x},\mathbf{y}]+\mathbf{i}\otimes\mathbf{j} = \lambda\cdot\mathrm{Id}\}.$$

Here, $\mathbf{i}\otimes\mathbf{j}$ denotes a rank one linear operator $\mathbb{C}^\mathbf{v}\to\mathbb{C}^\mathbf{v},\ u\mapsto\langle\mathbf{j},u\rangle\cdot\mathbf{i}$.

The above variety $M_\lambda(\mathbf{v},\mathbf{w})$ is nonempty for any $\lambda\in\mathbb{C}$. Below, we restrict ourselves to the special case $\mathbf{w}=1$. In this case, we may view $\mathbf{i}$ as a vector in $\mathbb{C}^\mathbf{v}=\mathrm{Hom}(\mathbb{C},\mathbb{C}^\mathbf{v})$, resp. $\mathbf{j}$ as a covector in $(\mathbb{C}^\mathbf{v})^*=\mathrm{Hom}(\mathbb{C}^\mathbf{v},\mathbb{C})$.

Assume first that $\lambda\neq 0$. Then, one proves that there is no proper subspace $0\neq S\subsetneq\mathbb{C}^\mathbf{v}$ such that $\mathbf{i}\in S$ and such that S is stable under the maps $\mathbf{x},\mathbf{y}$. It follows, by Corollary 4.4.3(i) that $M_\lambda(\mathbf{v},1)$ is a smooth affine variety and that the $GL_\mathbf{v}$-action on this variety is free. Therefore, each $GL_\mathbf{v}$-orbit in $M_\lambda(\mathbf{v},1)$ is closed. We conclude

$$\mathcal{M}_{\lambda,0}(\mathbf{v},1) = M_\lambda(\mathbf{v},1)//GL_\mathbf{v} = M_\lambda(\mathbf{v},1)/GL_\mathbf{v} =: \text{Calogero-Moser space.}$$

Also, we compute

$$\dim\mathcal{M}_{\lambda,0}(\mathbf{v},1) = \dim M_\lambda(\mathbf{v},1)-\dim GL_\mathbf{v} = (2\mathbf{v}^2+2\mathbf{v}-\mathbf{v}^2)-\mathbf{v}^2 = 2\mathbf{v}.$$

Next, let $\lambda=0$. Then, from the equation $[\mathbf{x},\mathbf{y}]+\mathbf{i}\otimes\mathbf{j}=0$ we deduce

$$\langle\mathbf{j},\mathbf{i}\rangle = \mathrm{Tr}(\mathbf{i}\otimes\mathbf{j}) = -\mathrm{Tr}([\mathbf{x},\mathbf{y}]) = 0.$$

It follows that $\mathbf{i}\otimes\mathbf{j}$ is a nilpotent rank one linear operator in $\mathbb{C}^\mathbf{v}$.

The following result of linear algebra will play an important role in our analysis.

Lemma 5.3.2. *Let $\mathbf{x},\mathbf{y}\in\mathfrak{gl}_\mathbf{v}$ be a pair of linear operators such that $[\mathbf{x},\mathbf{y}]$ is a nilpotent rank one operator. Then there exists a basis of $\mathbb{C}^\mathbf{v}$ such the matrices of $\mathbf{x}$ and $\mathbf{y}$ in that basis are both upper-triangular.* □

In the case $\lambda=0$, the variety $M_0(\mathbf{v},1)$ is neither smooth nor irreducible. Thus, to get a good quotient one has to impose a stability condition. First, let $\theta=0$, so $\mathcal{M}_{0,0}(\mathbf{v},1)$ is an affine algebraic variety, by Theorem 5.1.7.

To describe this variety explicitly, one shows using the above lemma, that the assignment sending a quadruple $(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j})\in M_0(\mathbf{v},1)$ to the joint spectrum $(\mathrm{Spec}\,\mathbf{x},\mathrm{Spec}\,\mathbf{y})\in\mathbb{C}^\mathbf{v}\times\mathbb{C}^\mathbf{v}$, of the operators $\mathbf{x}$ and $\mathbf{y}$, written in an upper-triangular form provided by lemma 5.3.2, gives a well-defined morphism $M_0(\mathbf{v},1)\to(\mathbb{C}^\mathbf{v}\times\mathbb{C}^\mathbf{v})/\mathbb{S}_\mathbf{v}$, of algebraic varieties. Moreover, this morphism turns out to induce an algebra isomorphism $\mathbb{C}[\mathbb{C}^\mathbf{v}\times\mathbb{C}^\mathbf{v}]^{\mathbb{S}_\mathbf{v}}\xrightarrow{\sim}\mathbb{C}[M_0(\mathbf{v},1)]^{G_\mathbf{v}}$.

We conclude that the Nakajima variety with parameters $(\lambda,\theta)=(0,0)$ is an affine variety

$$\mathcal{M}_{0,0}(\mathbf{v},1)\cong(\mathbb{C}^\mathbf{v}\times\mathbb{C}^\mathbf{v})/\mathbb{S}_\mathbf{v}. \tag{5.3.3}$$

Remark 5.3.4. The $\mathbb{C}^\times$-action on $\mathcal{M}_{0,0}(\mathbf{v},1)$ that has been defined in the previous subsection goes under the above isomorphsm to a $\mathbb{C}^\times$-action on $(\mathbb{C}^\mathbf{v}\times\mathbb{C}^\mathbf{v})/\mathbb{S}_\mathbf{v}$. The latter action is given by $\mathbb{C}^\times\ni t:\ (u,v)\mapsto(u,t\cdot v)$. The fixed points of that action form a subset $(\mathbb{C}^\mathbf{v}/\mathbb{S}_\mathbf{v})\times\{0\}\subset(\mathbb{C}^\mathbf{v}\times\mathbb{C}^\mathbf{v})/\mathbb{S}_\mathbf{v}$. Note the subset in question does not reduce to a single point. Indeed, the Jordan quiver has an edge loop and, therefore, Lemma 5.2.4 does not apply in our present situation. ◊

Next, we take $\theta := -\theta^+ = -1\in\mathbb{Z}$. With this choice of θ, a point $(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j})\in M_0(\mathbf{v},1)$ is stable if and only if condition (5.1.4) holds, and we have

Proposition 5.3.5. *The set of θ-semistable points equals*

$$M_0(\mathbf{v},1)^s_{-\theta^+} = \{(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j}) \mid [\mathbf{x},\mathbf{y}]=0,\ \mathbf{j}=0,\ \mathbf{i} \textit{ is a cyclic vector for } (\mathbf{x},\mathbf{y})\}. \tag{5.3.6}$$

Proof. According to Theorem 5.1.2, a point $(\mathbf{x},\mathbf{y},\mathbf{i},\mathbf{j}) \in M_0(\mathbf{v},1)$ is stable if and only if condition (5.1.4) holds. The condition means that $\mathbf{i}$ is a cyclic vector for $(\mathbf{x},\mathbf{y})$, i.e., we have $\mathbb{C}\langle\mathbf{x},\mathbf{y}\rangle\mathbf{i} = \mathbb{C}^{\mathbf{v}}$.

We claim that the last equation implies $\mathbf{j}=0$. To see this, we observe that for any $a \in \mathfrak{gl}_{\mathbf{v}}$, we have

$$\langle \mathbf{j},\ a\mathbf{i}\rangle = \operatorname{Tr}\big(a\circ(\mathbf{i}\otimes\mathbf{j})\big) = -\operatorname{Tr}\big(a\circ[\mathbf{x},\mathbf{y}]\big), \qquad \forall a \in \mathfrak{gl}_{\mathbf{v}}. \tag{5.3.7}$$

Assume now that $a \in \mathbb{C}\langle\mathbf{x},\mathbf{y}\rangle$, is a noncommutative polynomial in $\mathbf{x}$ and $\mathbf{y}$. Then, we may write the matrices $\mathbf{x},\mathbf{y}$, and a in an upper-triangular form, by Lemma 5.3.2. In this form, the principal diagonal of the matrix $[\mathbf{x},\mathbf{y}]$ vanishes, and we get $\operatorname{Tr}\big(a\circ[\mathbf{x},\mathbf{y}]\big)=0$. Thus, (5.3.7) implies that the linear function $\mathbf{j}$ vanishes on the vector space $\mathbb{C}\langle\mathbf{x},\mathbf{y}\rangle\mathbf{i} = \mathbb{C}^{\mathbf{v}}$, and the proposition follows. ☐

For any commuting pair $(\mathbf{x},\mathbf{y}) \in \mathfrak{gl}_{\mathbf{v}}$ and any vector $\mathbf{i} \in \mathbb{C}^{\mathbf{v}}$, we introduce a set of polynomials in two indeterminates, x and y, as follows

$$J_{\mathbf{x},\mathbf{y},\mathbf{i}} := \{f \in \mathbb{C}[x,y] \mid f(\mathbf{x},\mathbf{y})\mathbf{i}=0\}.$$

It is clear that $J_{\mathbf{x},\mathbf{y},\mathbf{i}}$ is an ideal of the algebra $\mathbb{C}[x,y]$. Furthermore, this ideal has codimension $\mathbf{v}$ in $\mathbb{C}[x,y]$ if and only if the map $\mathbb{C}[x,y]/J_{\mathbf{x},\mathbf{y},\mathbf{i}} \to \mathbb{C}^{\mathbf{v}},\ f \mapsto f(\mathbf{x},\mathbf{y})\mathbf{i}$, is surjective. The latter holds if and only if $\mathbf{i}$ is a cyclic vector for the pair $(\mathbf{x},\mathbf{y})$. In fact, one proves

Corollary 5.3.8. *The assigment $(\mathbf{x},\mathbf{y},\mathbf{i}) \mapsto J_{\mathbf{x},\mathbf{y},\mathbf{i}}$ establishes a bijection between the orbit set $M_0(\mathbf{v},1)^s_{-\theta^+}/GL_{\mathbf{v}}$ and the set of ideals $J \subset \mathbb{C}[x,y]$ such that* $\dim\mathbb{C}[x,y]/J = \mathbf{v}$. ☐

The set of codimension $\mathbf{v}$ ideals in the algebra $\mathbb{C}[x,y]$ has a natural scheme structure. The resulting scheme $\operatorname{Hilb}^n(\mathbb{C}^2)$ turns out to be a smooth connected variety of dimension $2n$, called the *Hilbert scheme* of n points in the plane. Thus, we see that, for $\mathbf{w}=1$ and $\lambda=0,\ \theta=-\theta^+$, one has a natural isomorphism

$$\mathcal{M}_{0,-\theta^+}(\mathbf{v},1) \cong \operatorname{Hilb}^n(\mathbb{C}^2).$$

In this case, the canonical projective morphism π, cf. (5.3.3),

$$\pi:\ \mathcal{M}_{0,-\theta^+}(\mathbf{v},1) = \operatorname{Hilb}^n(\mathbb{C}^2) \quad\longrightarrow\quad \mathcal{M}_{0,0}(\mathbf{v},1) = (\mathbb{C}^{\mathbf{v}}\times\mathbb{C}^{\mathbf{v}})/\mathbb{S}_{\mathbf{v}},$$

turns out to be a resolution of singularities, called the *Hilbert-Chow morphism.*

Remark 5.3.9. One can show that changing our choice of stability condition from $\theta=-\theta^+$ to $\theta=\theta^+$ leads to isomorphic quiver varieties, because of the isomorphisms of Remark 2.3.10.

5.4. **A Steinberg type variety.** Throughout this subsection, we let $(\lambda,\theta) := (0,\theta^+)$, and we use simplified notation $\mathcal{M}(\mathbf{v},\mathbf{w}) := \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$, resp. $\mathcal{M}_0(\mathbf{v},\mathbf{w}) = \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$. Also, we write $\mathbf{v} \leq \mathbf{v}'$ whenever $\mathbf{v}'-\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$.

The variety $\mathcal{M}_0(\mathbf{v},\mathbf{w})$ has a distinguished origin $0 \in \mathcal{M}_0(\mathbf{v},\mathbf{w})$ that corresponds to the zero representation of $\overline{Q^\heartsuit}$.

For any pair $0 \leq \mathbf{v} \leq \mathbf{v}'$, of dimension vectors, we have a natural vector space imbedding

$$\operatorname{Rep}(\overline{Q^\heartsuit},\mathbf{v},\mathbf{w}) \hookrightarrow \operatorname{Rep}(\overline{Q^\heartsuit},\mathbf{v}',\mathbf{w}),\ M \mapsto M\oplus 0_{\mathbf{v}'-\mathbf{v}}, \quad\text{where}\ \ 0_{\mathbf{v}'-\mathbf{v}} \in \operatorname{Rep}(\overline{Q^\heartsuit},\mathbf{v}'-\mathbf{v},\mathbf{w}),$$

denotes the zero representation. The imbedding above clearly induces a closed imbedding $\mathcal{M}_0(\mathbf{v},\mathbf{w}) \hookrightarrow \mathcal{M}_0(\mathbf{v}',\mathbf{w})$, of categorical quotients. In this way, we may (and will) view $\mathcal{M}_0(\mathbf{v},\mathbf{w})$ as a closed subscheme of $\mathcal{M}_0(\mathbf{v}',\mathbf{w})$.

Given $\theta \in \mathbb{Z}^I$ and any pair $\mathbf{v},\mathbf{v}' \in \mathbb{Z}^I$, of dimension vectors, we define an associated *Steinberg variety*

$$Z(\mathbf{v},\mathbf{v}',\mathbf{w}) \;:=\; \mathcal{M}(\mathbf{v},\mathbf{w}) \times_{\mathcal{M}_0(\mathbf{v}+\mathbf{v}',\mathbf{w})} \mathcal{M}(\mathbf{v}',\mathbf{w}) \;\subset\; \mathcal{M}(\mathbf{v},\mathbf{w}) \times \mathcal{M}(\mathbf{v}',\mathbf{w}), \tag{5.4.1}$$

as a fiber product in the following diagram

$$\begin{tikzcd}
 & Z(\mathbf{v},\mathbf{v}',\mathbf{w}) \arrow[dl] \arrow[dr] & \\
\mathcal{M}(\mathbf{v},\mathbf{w}) \arrow[d, "\pi"] & & \mathcal{M}(\mathbf{v}',\mathbf{w}) \arrow[d, "\pi"] \\
\mathcal{M}_0(\mathbf{v},\mathbf{w}) \arrow[r, hook] & \mathcal{M}_0(\mathbf{v}+\mathbf{v}',\mathbf{w}) & \mathcal{M}_0(\mathbf{v}',\mathbf{w}) \arrow[l, hook']
\end{tikzcd} \tag{5.4.2}$$

Thus, one has a canonical projective morphism $\pi_Z:\ Z(\mathbf{v},\mathbf{v}',\mathbf{w}) \to \mathcal{M}_0(\mathbf{v}+\mathbf{v}',\mathbf{w})$.

Recall next that $\mathcal{M}(\mathbf{v},\mathbf{w})$, resp. $\mathcal{M}(\mathbf{v}',\mathbf{w})$, is a symplectic manifold with symplectic 2-form ω, resp. ω'. We equip the cartesian product $\mathcal{M}(\mathbf{v},\mathbf{w}) \times \mathcal{M}(\mathbf{v}',\mathbf{w})$ with the symplectic 2-form $\omega + (-\omega')$.

Theorem 5.4.3. *$Z(\mathbf{v},\mathbf{v}',\mathbf{w})$ is a Lagrangian subvariety of $\mathcal{M}(\mathbf{v},\mathbf{w}) \times \mathcal{M}(\mathbf{v}',\mathbf{w})$.*

The Steinberg variety is typically quite singular and has many irreducible components. In the special case where $\mathbf{v} = \mathbf{v}'$, the diagonal $\mathcal{M}(\mathbf{v},\mathbf{w}) \subset \mathcal{M}(\mathbf{v},\mathbf{w}) \times \mathcal{M}(\mathbf{v},\mathbf{w})$ is one such component, which is smooth.

6. Lie algebras and Quiver varieties

6.1. **Convolution.** Let $\mathbb{C}[X]$ denote the vector space of $\mathbb{C}$-valued functions on a finite set X. Characteristic functions of one element subsets form a $\mathbb{C}$-base of $\mathbb{C}[X]$.

Let $X_r,\ i = 1,2$, be a pair of finite sets. A linear operator $K : \mathbb{C}[X_1] \to \mathbb{C}[X_2]$ is given, in the bases of characteristic functions, by a rectangular $|X_1| \times |X_2|$-matrix $|K(x_2,x_1)|_{x_i \in X_i}$. We may view this matrix as a $\mathbb{C}$-valued function $(x_1,x_2) \mapsto K(x_1,x_2)$, on $X_1 \times X_2$, called the *kernel of the operator K*.

The action of K is then given, in terms of that kernel, by the formula

$$K:\ f \mapsto K * f, \quad \text{where} \quad (K * f)(x_2) := \sum_{x_1 \in X_1} K(x_2,x_1)\cdot f(x_1). \tag{6.1.1}$$

Now, let $X_i,\ i = 1,2,3$, be a triple of finite sets, and let $K : \mathbb{C}[X_1] \to \mathbb{C}[X_2]$ and $K' : \mathbb{C}[X_2] \to \mathbb{C}[X_3]$ be a pair of operators, with kernels $K_{32} \in \mathbb{C}[X_3 \times X_2]$ and $K_{21} \in \mathbb{C}[X_2 \times X_1]$, respectively. One may form the composite operator $K \circ K' :\ \mathbb{C}[X_1] \to \mathbb{C}[X_3]$, $f \mapsto K(K'(f))$.

Explicitly, in terms of the kernels, for any $f \in \mathbb{C}[X_1]$, the function $K(K'(f))$ is given by

$$\begin{aligned} x_3 \ \mapsto\ K(K'(f))(x_3) &= \sum_{x_2 \in X_2} K_{32}(x_3, x_2) \cdot \left(\sum_{x_1 \in X_1} K_{21}(x_2, x_1) \cdot f(x_1) \right) \\ &= \sum_{x_1 \in X_1} \left(\sum_{x_2 \in X_2} K_{32}(x_3, x_2) \cdot K_{21}(x_2, x_1) \right) \cdot f(x_1). \end{aligned}$$

Thus, the kernel of the composite operator $K \circ K'$ is a function $K_{32} * K_{21}$, on $X_3 \times X_1$, given by the formula

$$(x_3, x_1) \ \mapsto\ (K_{32} * K_{21})(x_3, x_1) \ :=\ \sum_{x_2 \in X_2} K_{32}(x_3, x_2) \cdot K_{21}(x_2, x_1). \tag{6.1.2}$$

The operation

$$* : \ \mathbb{C}[X_3 \times X_2] \times \mathbb{C}[X_2 \times X_1] \longrightarrow \mathbb{C}[X_3 \times X_1], \quad K_{32} \times K_{21} \mapsto K_{32} * K_{21} \tag{6.1.3}$$

is called *convolution* of kernels. Thinking of kernels as of rectangular matrices, the convolution becomes nothing but matrix multiplication. Thus, formula (6.1.2) corresponds to the standard matrix multiplication for $|X_3| \times |X_2|$-matrix by a $|X_2| \times |X_1|$-matrices. So, all we have done so far was a reinterpretation of the fact that composition of linear operators corresponds to a product of corresponding matrices.

Remark 6.1.4. A There is an equivalent, but slightly more elegant, way to write formula (6.1.2) as follows.

For any map $p : \ X \to Y$, of finite sets, one has a pull-back map $p^* : \ \mathbb{C}[Y] \to \mathbb{C}[X]$, of functions given by $(p^*f)(x) := p(f(x)),\ \forall x \in X$. We also define a *push-forward* linear map on functions by

$$p_* : \ \mathbb{C}[X] \to \mathbb{C}[Y], \quad f \mapsto p_* f, \quad \text{where} \quad (p_*f)(y) := \sum_{\{x \in p^{-1}(y)\}} f(x). \tag{6.1.5}$$

For any pair $i, j \in \{1, 2, 3\}$, let $p_{ij} : \ X_3 \times X_2 \times X_1 \to X_i \times X_j$ be the projection along the factor not named. It is clear that, with the above notation, formula (6.1.2) may be rewritten as follows

$$K_{32} * K_{21} \ :=\ (p_{31})_* \big((p_{32}^* K_{32}) \cdot (p_{21}^* K_{21}) \big). \tag{6.1.6}$$

We will be especially interested in a special case of convolution (6.1.6) where $X_1 = X_2 = X_3 = X$ is a set with n elements. Then, the convolution product (6.1.6) makes $\mathbb{C}[X \times X]$ an associative algebra. According to the preceeding discussion, this algebra is isomorphic to the algebra of $n \times n$-matrices.

One may get more interesting examples of convolution algebras by considering an equivariant version of the above construction, where there is a group G acting on a finite set X. We let G act diagonally on $X \times X$ and let $\mathbb{C}[X \times X]^G \subset \mathbb{C}[X \times X]$ be the subspace of G-invariant functions. This space is clearly isomorphic to $\mathbb{C}[(X \times X)/G]$, the space of functions on the set of G-diagonal orbits in $X \times X$.

It is immediate to check that the convolution product (6.1.2)-(6.1.3) is G-equivariant, hence it makes $\mathbb{C}[X \times X]^G$ a subalgebra of $\mathbb{C}[X \times X]$. The resulting algebra $\big(\mathbb{C}[X \times X]^G,\ *\big)$ may be shown to be always semisimple. Such an algebra need not be simple, so it is not necessarily isomorphic to a matrix algebra, in general.

Example 6.1.7 (Group algebra). Given a finite group G, we take $X = G$. We let G act on X by left translations, and act diagonally on $G \times G$, as before. Observe that the map $G \times G \to G$, $(g_1, g_2) \mapsto g_1^{-1} \cdot g_2$ descends to a well defined map $(G \times G)/G \to G$. Moreover, the latter map is easily seen to be a bijection.

We deduce the following chain of vector space isomorphisms

$$\mathbb{C}[G \times G]^G \xrightarrow{\sim} \mathbb{C}[(G \times G)/G] \xrightarrow{\sim} \mathbb{C}[G]. \tag{6.1.8}$$

It is straightforward to check that the restriction of convolution (6.1.2)-(6.1.3) to $\mathbb{C}[G \times G]^G$ goes, under the composite isomorphism in (6.1.8), to the standard convolution on a group. The latter is given by

$$(f * f')(g) = \sum_{h \in G} f(gh^{-1}) \cdot f'(h), \qquad \forall f, f' \in \mathbb{C}[G].$$

We conclude that the algebra $\big(\mathbb{C}[G \times G]^G,\ *\big)$, with convolution product (6.1.6), is isomorphic to the *group algebra* of G.

Example 6.1.9 (Hecke algebra). Let $G = G(\mathbb{F})$ be a split reductive group over a finite field $\mathbb{F} = \mathbb{F}_q$. Let $B \subset G$ be a Borel subgroup of G. We put $X := G/B$, and let G act on X by left translations. It is known, thanks to the Bruhat decomposition, that G-diagonal orbits in $G/B \times G/B$ are labelled by the elements of W, the Weyl group of G.

The resulting convolution algebra $H_q(G) := \big(\mathbb{C}[G/B \times G/B]^G,\ *\big)$ is called the *Hecke algebra* of G.

6.2. **Borel-Moore homology.** We are going to extend the constructions of the previous subsection to the case where finite sets are replaced by smooth C^∞-manifolds.

Thus, we let $X_i,\ i = 1, 2, 3$, be a triple of smooth manifolds. One might try to replace the summation in formula (6.1.2) by integration to get a convolution product of the form $* :\ C^\infty(X_3 \times X_2) \times C^\infty(X_2 \times X_1) \to C^\infty(X_3 \times X_1)$, cf (6.1.3).

To make this work, one still needs additional ingredients. One such ingredient is a *measure* on X_2 that is necessary in order to define the integral that replaces summation in formula (6.1.2).

An alternate approach, that does not require introducing a measure, is to replace functions by differential forms. In this way, one defines a convolution product

$$\Omega^p(X_3 \times X_2) \times \Omega^q(X_2 \times X_1) \to \Omega^{p+q-\dim X_2}(X_3 \times X_1),\ \ K_{32} \times K_{21} \mapsto \int_{X_2} (p_{32}^* K_{32}) \wedge (p_{21}^* K_{21}). \tag{6.2.1}$$

To insure the convergence of the integral in (6.2.1) one may assume, for instance, that the manifold X_2 is compact. A slightly weaker assumption, that is sufficient for (6.2.1) to make sence, is to restrict considerations to differential forms with certain support condition that would insure, in particular, that the set

$$p_{32}^{-1}(\operatorname{supp} K_{32})\ \cap\ p_{21}^{-1}(\operatorname{supp} K_{21})\ \text{ be compact.} \tag{6.2.2}$$

Unfortunately, none of the above works in the examples arising from quiver varieties that we would like to consider below. In those examples, the manifolds $X_i,\ i = 1, 2, 3$, are the quiver varieties, which are *noncompact* complex algebraic varieties. It turns out that the only natural support condition one could make in those cases in order for (6.2.2) to hold, is to require supports of K_{32} and K_{21}, in (6.2.1), be contained in appropriate *closed* algebraic subvarieties.

Obviously, any C^∞-differential form on a manifold whose support is contained in a closed (proper) submanifold must vanish identically. There are, however, plenty of 'distribution-like' differential forms, called *currents*, which may be supported on closed submanifolds. Indeed, replacing differential forms by currents resolves the convergence problem for integration. Unfortunately, introducing currents creates another problem: the wedge-product operation, which is used in (6.2.1), is not well defined for currents.

All the above difficulties may be resolved by introducing homology. Recall that there is the de Rham differential acting on the (graded) vector space $\Omega^\bullet(X)$, of differential forms on a manifold X. The homology of the resulting de Rham complex $\big(\Omega^\bullet(X),\ d\big)$ is isomorphic to $H^\bullet(X,\mathbb{C})$, the singular cohomology of X with complex coefficients. Similarly, there is a natural de Rham differential on the (graded) vector space of currents on X, and the homology of the resulting complex is known to be isomorphic to $H^{BM}_\bullet(X,\mathbb{C})$, the *Borel-Moore homology* of X with complex coefficients. The latter is the homology theory that we are going to use.

For practical purposes, it is more convenient to use a different (*a posteriori* equivalent) definition of Borel-Moore homology based on Poincaré duality rather than on the de Rham complex of currents. We now recall this definition.

Let M be a smooth *oriented* C^∞-manifold of real dimension m. One defines Borel-Moore homology of a closed subset $X \subset M$ to be the following relative cohomology

$$H^{BM}_\bullet(X) := H^\bullet(M, M \smallsetminus X; \mathbb{C}). \tag{6.2.3}$$

It can be shown that the group on the right is, in fact, independent of the choice of a closed imbedding of X into a smooth manifold.

Notation 6.2.4. **From now on, we drop the superscript 'BM' and let $H_\bullet(X)$ stand for Borel-Moore homology (rather than ordinary homology) of X.**

A property that makes Borel-Moore homology so useful for our purposes is that, for any X, which is either a smooth connected, and *oriented* C^∞-manifold or an irreducible *complex* algebraic variety, the space $H_m(X)$, where $m := \dim_{\mathbb{R}} X$, is 1-dimensional; furthermore, there is a canonical base element $[X] \in H_m(X)$, called the *fundamental class* of X.

Remark 6.2.5. Note that, in the ordinary homology theory, fundamental classes only exist for *compact* manifolds, while such a compactness condition is not necessary for the fundamental class to exist in Borel-Moore homology. ◊

We record a few basic properties of the Borel-Moore homology theory. First, for any *proper* map $p : X \to Y$, there is a push-forward functor $p_* :\ H_\bullet(X) \to H_\bullet(Y)$.

Second, there is a cap-product on Borel-Moore homology. In more detail, given two closed subsets $X, Y \subset M$, where M is a smooth oriented manifold of real dimension m, one has a cup product

$$\cup :\ H^{m-i}(M, M \smallsetminus X;\ \mathbb{C}) \times H^{m-j}(M, M \smallsetminus Y;\ \mathbb{C}) \to H^{2m-i-j}(M, M \smallsetminus (X \cup Y);\ \mathbb{C}).$$

We define a cap-product on Borel-Moore homology by transporting the above cup product via formula (6.2.3); this way we obtain a cap-product pairing

$$\cap :\ H_i(X) \times H_j(Y) \to H_{i+j-m}(X), \qquad m = \dim_{\mathbb{R}} M. \tag{6.2.6}$$

It should be emphasized that the cap-product so defined *does* depend on the ambient smooth manifold M.

6.3. **Convolution in Borel-Moore homology.** There is a convolution product in Borel-Moore homology that provides an adequate generalization, from the case of finite sets to the case of manifolds, of the convolution product (6.1.6).

To define the convolution product , fix $M_i,\ i = 1, 2, 3$, a triple of smooth oriented manifolds, and let $p_{ij} :\ M_1 \times M_2 \times M_3 \to M_i \times M_j$ denote the projection along the factor not named, cf. (6.1.6).

Definition 6.3.1. *A pair of closed subsets $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ is said to be* composable *if the following map* (6.3.2) *is* proper

$$p_{13} :\ (p_{12}^{-1} Z_{12}) \cap (p_{23}^{-1} Z_{23}) \to M_1 \times M_3. \tag{6.3.2}$$

Given composable subsets as above, we define their composite *to be*

$$Z_{12} \circ Z_{23} := p_{13}\big[(p_{12}^{-1} Z_{12}) \cap (p_{23}^{-1} Z_{23})\big] \quad \subset \quad M_1 \times M_3.$$

Now, let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ be as above, and put $m_i := \dim M_i$.

We use $M := M_1 \times M_2 \times M_3$ as an ambient manifold and apply formula (6.2.6). In this way, we get a cap product map

$$\cap :\ H_{i+m_3}(p_{12}^{-1} Z_{12}) \times H_{j+m_1}(p_{23}^{-1} Z_{23}) \ \longrightarrow\ H_{i+j-m_2}((p_{12}^{-1} Z_{12}) \cap (p_{23}^{-1} Z_{23})).$$

Assume further that Z_{12} and Z_{23} are composable. Then, we have a push-forward morphism $(p_{13})_*$, on Borel-Moore homology, induced by the *proper* map (6.3.2).

One defines the convolution in Borel-Moore homology as the following map, cf. (6.1.6), (6.2.1),

$$\begin{aligned} * :\ H_i(Z_{12}) \times H_j(Z_{23}) \ &\longrightarrow\ H_{i+j-\dim M_2}(Z_{12} \circ Z_{23}), \\ c_{12} \times c_{23} \ &\mapsto\ c_{12} * c_{23} := (p_{13})_*\Big((c_{12} \boxtimes [M_3]) \ \cap\ ([M_1] \boxtimes c_{23})\Big). \end{aligned} \tag{6.3.3}$$

6.4. **Convolution algebra.** Fix M, a smooth complex algebraic variety, not necessarily connected, in general. Further, let Y be a (not necessarily smooth) algebraic variety and $\pi : M \to Y$, a *proper* morphism. Thus, we may form a fiber product $Z := M \times_Y M$, a closed subvariety of $M \times M$.

One may apply the convolution in Borel-Moore homology operation in a special case where $M_1 = M_2 = M_3 = M$, and $Z_{12} = Z_{23} = Z$. The assumption the morphism π be proper insures that the set Z is composable with itself in the sense of Definition 6.3.1. Furthermore, it is immediate to check that one has $Z \circ Z = Z$. Thus, the convolution product (6.3.3) gives $H_\bullet(Z)$, the total Borel-Moore homology group of Z, a structure of associative algebra. The fundamental class $[\Delta]$, of the diagonal $\Delta \subset M \times M$, is the unit of the algebra $\big(H_\bullet(Z),\ *\big)$.

Next, pick a point $y \in Y$ and put $M_y := \pi^{-1}(y)$. Consider the setting of section 6.3 in the special case where $M_1 = M_2 = M$, and where $M_3 = pt$ is a point. Thus, we have $M_2 \times M_3 = M_2 \times pt = M$, and put $Z_{12} := M \times_Y M = Z$, as before, and $Z_{23} := M_y = \pi^{-1}(y)$, viewed as a closed subset in $M_2 \times M_3 = M$.

It is immediate to check that the sets Z and M_y are composable and, moreover, one has $Z \circ M_y = M_y$. Therefore, convolution in BM homology gives the space $H_\bullet(M_y)$ an $H_\bullet(Z)$-module structure.

Let $\mathscr{V}$ denote a set that provides a labelling for connectected components of the manifold M. We write $M^{(r)}$ for the connectected component with label $r \in \mathscr{V}$. For any pair $M^{(r)}, M^{(s)}$, of connectected components, we put $Z^{(r,s)} := Z \cap (M^{(r)} \times M^{(s)})$. Similarly, we put $M_y^{(r)} := M_y \cap M^{(r)}$, for any $r \in \mathscr{V}$. Clearly, we have $H_\bullet(Z) = \bigoplus_{r,s\in\mathscr{V}} H(Z^{(r,s)})$, resp. $H_\bullet(M_y) = \bigoplus_{r\in\mathscr{V}} H(M_y^{(r)})$.

We introduce a special notation, $H_{\operatorname{top}}(Z^{(r,s)}) := H_d(Z^{(r,s)})$, $d := \dim_{\mathbb{R}} Z^{(r,s)}$, for the top Borel-Moore homology group of $Z^{(r,s)}$, resp. $H_{\operatorname{top}}(M_y^{(r)})$ for the top Borel-Moore homology group of $M_y^{(r)}$. These groups have natural bases formed by the fundamental classes of irreducible components of the variety $Z^{(r,s)}$, resp. of the variety $M_y^{(r)}$, of maximal dimension.

The following result is an immediate consequence of formula (6.3.3).

Lemma 6.4.1. (i) *The vector space* $H_{\operatorname{top}}(Z) := \bigoplus_{r,s\in\mathscr{V}} H_{\operatorname{top}}(Z^{(r,s)})$ *is a subalgebra of the convolution algebra* $\big(H_\bullet(Z),\ *\big)$.

(ii) *For any* $y \in Y$, *the vector space* $H_{\operatorname{top}}(M_y) := \bigoplus_{r\in\mathscr{V}} H_{\operatorname{top}}(M_y^{(r)})$ *is stable under the convolution-action of the subalgebra* $H_{\operatorname{top}}(Z) \subset H_\bullet(Z)$ *on* $H_\bullet(M_y)$. □

Remark 6.4.2. Note that, in the direct sum $\bigoplus_{r,s\in\mathscr{V}} H_{\operatorname{top}}(Z^{(r,s)})$, resp. $\bigoplus_{r\in\mathscr{V}} H_{\operatorname{top}}(M_y^{(r)})$, different direct summands may have different degrees, in general. ♢

6.5. **Geometric construction of $U(\mathfrak{g})$.** We return to the quiver setting. Thus, we fix a quiver Q, without edge loops, and a dimension vector $\mathbf{w} \in \mathbb{Z}^I$. We keep the assumptions of §5.4, in particular, we put $\theta := \theta^+$ and we let the parameter λ be zero.

For each dimension vector $\mathbf{v}$, we have defined a smooth Nakajima's quiver variety $\mathcal{M}(\mathbf{v}, \mathbf{w})$, an affine quiver variety $\mathcal{M}_0(\mathbf{v}, \mathbf{w})$, and a projective morphism $\pi : \mathcal{M}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w})$. Further, for each pair $\mathbf{v}, \mathbf{v}'$, we have defined a Steinberg variety $Z(\mathbf{w}, \mathbf{v}, \mathbf{v}') \subset \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v}', \mathbf{w})$, cf. (5.4.1).

We introduce the following disconnected varieties

$$M(\mathbf{w}) := \bigsqcup_{\mathbf{v}\in\mathbb{Z}^I} \mathcal{M}(\mathbf{v}, \mathbf{w}), \quad M_0(\mathbf{w}) := \bigsqcup_{\mathbf{v}\in\mathbb{Z}^I} \mathcal{M}_0(\mathbf{v}, \mathbf{w}), \quad Z(\mathbf{w}) := \bigsqcup_{(\mathbf{v},\mathbf{v}')\in\mathbb{Z}^I\times\mathbb{Z}^I} Z(\mathbf{w}, \mathbf{v}, \mathbf{v}').$$

Thus, the morphisms $\pi : \mathcal{M}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w})$ may be assembled together to give a morphism $M(\mathbf{w}) \to M_0(\mathbf{w})$, and we have $Z(\mathbf{w}) = M(\mathbf{w}) \times_{M_0(\mathbf{w})} M(\mathbf{w})$. Also, we define

$$H_{\mathbf{w}} := \bigoplus_{(\mathbf{v},\mathbf{v}')\in\mathbb{Z}^I\times\mathbb{Z}^I} H_{\operatorname{top}}\big(Z(\mathbf{w}, \mathbf{v}, \mathbf{v}')\big).$$

Thus, at a heuristic level, one has $H_{\mathbf{w}} = H_{\operatorname{top}}(Z(\mathbf{w}))$. At any rate, applying convolution in Borel-Moore homology for various direct summands, one by one, one makes $(H_{\mathbf{w}},\ *)$ a $\mathbb{Z}^I \times \mathbb{Z}^I$-graded $\mathbb{C}$-algebra.

We also let $\Lambda(\mathbf{v}, \mathbf{w}) = \pi^{-1}(0)$ be the zero fiber of the morphism π, cf. §5.2. We put

$$\Lambda_{\mathbf{w}} := \bigsqcup_{\mathbf{v}\in\mathbb{Z}^I} \Lambda(\mathbf{v}, \mathbf{w}), \text{ resp. } L_{\mathbf{w}} := \bigoplus_{\mathbf{v}\in\mathbb{Z}^I} H_{\operatorname{top}}\big(\Lambda(\mathbf{v}, \mathbf{w})\big).$$

Thus, heuristacally, one has $L_{\mathbf{w}} = H_{\mathrm{top}}(\Lambda_{\mathbf{w}})$.

Recall next that, associated with the Cartan matrix C_Q, of the quiver Q, there is a Kac-Moody Lie algebra $\mathfrak{g}_Q$. Let $U(\mathfrak{g}_Q)$ be its universal enveloping algebra.

One of the main results of Nakajima's theory reads

Theorem 6.5.1. (i) *There is a natural algebra homomorphism* $\Psi: U(\mathfrak{g}_Q) \to H_{\mathbf{w}}$.

(ii) *The* $U(\mathfrak{g}_Q)$*-action on the vector space* $L_{\mathbf{w}}$*, induced by the homomorphism* Ψ *via Lemma 6.4.1(ii), makes the latter a simple integrable* $\mathfrak{g}_Q$*-module with highest weight* $\mathbf{w}$.

Theorem 5.2.2 implies that $\Lambda_{\mathbf{w}}$ is a (disconnected) *Lagrangian* subvariety of $M(\mathbf{w})$, a disconnected symplectic manifold. It follows that the fundamental classes of *all* irreducible components of the variety $\Lambda_{\mathbf{w}}$ form a natural basis in the vector space $L_{\mathbf{w}} = H_{\mathrm{top}}(\Lambda_{\mathbf{w}})$. This basis goes, via the identification provided by Theorem 6.5.1(ii), to a so-called *semicanonical* basis in the corresponding simple $U(\mathfrak{g}_Q)$-module.

Recall that the Lie algebra $\mathfrak{g}_Q$ has a set of Chevalley generators $e_i,\ h_i, f_i,\ i \in I$. The homomorphism Ψ, of Theorem 6.5.1(i), is constructed by sending each Chevalley generator to an appropriate explicit linear combination of the fundamental classes of some carefully chosen *smooth* irreducible components of the Steinberg variety $Z(\mathbf{w})$.

Specifically, fix $i \in I$ and let $\mathbf{e}^i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^I$ denote the i-th coordinate vector. Then, the generator h_i is sent to a linear combination of the form $\sum_{\mathbf{v}} a_{\mathbf{v}} \cdot [\mathcal{M}(\mathbf{v}, \mathbf{w})]$, where $[\mathcal{M}(\mathbf{v}, \mathbf{w})]$ denotes the fundamental class of the diagonal $\mathcal{M}(\mathbf{v}, \mathbf{w}) \subset \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v}, \mathbf{w})$, and $a_{\mathbf{v}} \in \mathbb{C}$ are certain coefficients.

The generator e_i is sent to a linear combination of the form $\sum_{\mathbf{v}} b_{\mathbf{v}} \cdot [Z^i(\mathbf{v}, \mathbf{w})]$. Here $Z^i(\mathbf{v}, \mathbf{w}) \subset \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v} + \mathbf{e}^i, \mathbf{w})$, is a smooth irreducible component of the Steinberg varietiy $Z(\mathbf{v}, \mathbf{v} + \mathbf{e}^i, \mathbf{w})$, and $b_i \in \mathbb{C}$ are some coefficients. Similarly, the generator f_i is sent to a linear combination of the form $\sum_{\mathbf{v}} c_{\mathbf{v}} \cdot [Z^i(\mathbf{v} - \mathbf{e}^i, \mathbf{w})^{\mathrm{op}}]$. In the last formula, $Z^i(\mathbf{v} - \mathbf{e}^i, \mathbf{w})^{\mathrm{op}} \subset \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v} - \mathbf{e}^i, \mathbf{w})$ is a subvariety which is obtained from the variety $Z^i(\mathbf{v} - \mathbf{e}^i, \mathbf{v}) \subset \mathcal{M}(\mathbf{v} - \mathbf{e}^i, \mathbf{w}) \times \mathcal{M}(\mathbf{v}, \mathbf{w})$, involved in the formula for the generator e_i, by the flip-isomorphism $\mathcal{M}(\mathbf{v} - \mathbf{e}^i, \mathbf{w}) \times \mathcal{M}(\mathbf{v}, \mathbf{w}) \cong \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v} - \mathbf{e}^i, \mathbf{w})$.

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