Corrigé CC2B Mat 244 2014-15

Question de cours

1) $(e_1 - e_n)$, famille orthogonale, est une base orthonormée si $n = \dim V$ et $\forall i$, $\phi(e_i, e_i) = 1$

2) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\4\end{pmatrix}$

$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$.

Exercise 2.

1) $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 \iff \begin{cases} 2\lambda_1 + \lambda_3 = 0 \\ \lambda_1 + \lambda_2 = 0 \\ \lambda_3 = 0 \\ 2\lambda_1 + \lambda_2 - \lambda_3 = 0 \end{cases}$

$\iff \begin{cases} 2\lambda \quad + \lambda_3 = 0 \\ \lambda_2 - \lambda_3/2 = 0 \\ \lambda_3 = 0 \\ \lambda_2 - 2\lambda_3 = 0 \end{cases} \iff \lambda = \lambda_2 = \lambda_3 = 0$

Cette famille est donc libre

2) $f_1 = v_1 / \|v_1\| = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \\ 2/3 \end{pmatrix}$

$f_2' = v_2 - \langle v_2, f_1 \rangle f_1 = v_2 - f_1 = \begin{pmatrix} -2/3 \\ 2/3 \\ 0 \\ 1/3 \end{pmatrix}$

$f_2 = f_2' / \| f_2' \| = \begin{pmatrix} -2/3 \\ 2/3 \\ 0 \\ 1/3 \end{pmatrix}$

$f_3' = v_3 - \langle v_3, f_1 \rangle f_1 - \langle v_3, f_2 \rangle f_2 = v_3 - 0 f_1 + f_2$

$= \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \\ -2/3 \end{pmatrix}$

$f_3 = f_3' / \| f_3' \| = \begin{pmatrix} \frac{1}{3\sqrt{2}} \\ \sqrt{2}/3 \\ 1/\sqrt{2} \\ -\sqrt{2}/3 \end{pmatrix}$

La BON recherchée est $\left(\begin{pmatrix} 2/3 \\ 1/3 \\ 0 \\ 2/3 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 2/3 \\ 0 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1/3\sqrt{2} \\ \sqrt{2}/3 \\ 1/\sqrt{2} \\ -\sqrt{2}/3 \end{pmatrix} \right)$

Exercise 3

1) $q(x,y) = 2x^2 + 2\sqrt{2}xy + 3y^2 = (\sqrt{2}x + y)^2 + 2y^2$

Rang $q = 2$, signature $q = (2,0)$

$\rightsquigarrow$ q et un produit scalaire.

2) Posons $(\sqrt{2}x + y)v_1 + y v_2 = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\Rightarrow \sqrt{2}v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad v_1 + v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}; \quad v_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1 \end{pmatrix}.$$

base q-orthogonale.

3) Matrice de q : $\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}$

$$P_m(X) = \begin{vmatrix} 2-X & \sqrt{2} \\ \sqrt{2} & 3-X \end{vmatrix} = (X-3)(X-2) - 2$$
$$= X^2 - 5X + 4$$

$P_m(X) = 0 \Leftrightarrow X^2 - 5X + 4 = 0 \Leftrightarrow X = 4, 1.$

4) $E_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{array}{l} 2x + \sqrt{2}y = x \\ \sqrt{2}x + 3y = y \end{array} \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x + \sqrt{2}y = 0 \right\}$

Base de E_1 : $\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ BON de E_1 : $\begin{pmatrix} -\sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix}$

$E_4 : \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{matrix} 2x + \sqrt{2}y = 4x \\ \sqrt{2}x + 3y = 4y \end{matrix} \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \sqrt{2}x - y = 0 \right\}$

Base de E_4 : $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$. BON de E_4 : $\begin{pmatrix} \sqrt{1/3} \\ \sqrt{2/3} \end{pmatrix}$.

Base recherchée : $\begin{pmatrix} -\sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix}, \begin{pmatrix} \sqrt{1/3} \\ \sqrt{2/3} \end{pmatrix}$.

5) Matrice de q : $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

<u>Exercice 4</u>

1) $\langle 1, \cos \rangle = \int_{-\pi}^{\pi} \cos x \, dx = \left[\sin x \right]_{-\pi}^{\pi} = 0$

$\langle 1, \sin \rangle = \int_{-\pi}^{\pi} \sin x \, dx = \left[-\cos x \right]_{-\pi}^{\pi} = 0$

$\langle \cos, \sin \rangle = \int_{-\pi}^{\pi} \cos x \sin x \, dx = \left[\frac{\sin^2 x}{2} \right]_{-\pi}^{\pi} = 0$

$\Rightarrow$ la famille est orthogonale.

2) Normalisons cette famille

$\| 1 \| = \sqrt{\int_{-\pi}^{\pi} 1^2 \, dx} = \sqrt{2\pi}$; la normalisation est $\frac{1}{\sqrt{2\pi}}$

$$\|\cos x\| = \sqrt{\int_{-\pi}^{\pi} \cos^2 x \, dx} = \sqrt{\pi}$$; la normalisation est $\dfrac{\cos x}{\sqrt{\pi}}$

$$\|\sin x\| = \sqrt{\int_{-\pi}^{\pi} \sin^2 x \, dx} = \sqrt{\pi}$$; la normalisation est $\dfrac{\sin x}{\sqrt{\pi}}$.

$$\text{Projection} = \int_{-\pi}^{\pi} \frac{t^2}{\sqrt{2\pi}} \, dt \times \frac{1}{\sqrt{2\pi}} + \int_{-\pi}^{\pi} \frac{t^2 \cos t}{\sqrt{\pi}} \, dt \, \frac{\cos t}{\sqrt{\pi}}$$

$$+ \underbrace{\int_{-\pi}^{\pi} \frac{t^2}{\sqrt{\pi}} \sin t \, dt \times \frac{\sin t}{\sqrt{\pi}}}_{0 \text{ par parité}}$$

$$= \frac{\pi^2}{3} + \cancel{} - 4 \cdot \cos t = \frac{1}{\sqrt{2\pi}} \times \left(\frac{\sqrt{2\pi}\,\pi^2}{3} \right) - 4\sqrt{\pi} \left(\frac{\cos t}{\sqrt{\pi}} \right)$$

3) Borne inférieur $= \|t^2\|^2 - \left(\dfrac{\sqrt{2\pi} \cdot \pi^2}{3} \right)^2 - 16\pi$

$$= \frac{2\pi^5}{5} - \frac{2\pi^5}{9} - 16\pi$$

Réalisé pour $a = \dfrac{\pi^2}{3}$, $b = -4$, $c = 0$