VANISHING THEOREMS FOR TENSOR POWERS OF A POSITIVE VECTOR BUNDLE

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Abstract. — Let E be a holomorphic vector bundle of rank r over a compact complex manifold X of dimension n . It is shown that the Dolbeault cohomology groups $H^{p,q}(X, E^{\otimes k} \otimes (\det E)^l)$ vanish if E is positive in the sense of Griffiths and $p + q \geq n + 1$, $l \geq r + C(n, p, q)$. The proof rests on the well-known fact that every tensor power $E^{\otimes k}$ splits into irreducible representations of $\mathrm{Gl}(E)$, each component being canonically isomorphic to the direct image on X of a positive homogeneous line bundle over a flag manifold of E . The vanishing property is then obtained by a suitable generalization of Le Potier's isomorphism theorem, combined with a new curvature estimate for the bundle of X–relative differential forms on the flag manifold of E .

0. Statement of results.

The aim of this work is to prove a rather general vanishing theorem for cohomology groups of tensor powers of a positive vector bundle.

Let X be a complex compact n–dimensional manifold and E a hermitian vector bundle of rank r over X . We denote by $C^\infty_{p,q}(X, E)$ the space of smooth E–valued differential forms of type (p, q) on X and by

$$D_E = D'_E + D''_E \;:\; C^\infty_{p,q}(X, E) \longrightarrow C^\infty_{p+1,q}(X, E) \oplus C^\infty_{p,q+1}(X, E)$$

(or simply $D = D' + D''$) the Chern connection of E . Let $(x_1, \ldots, x_n)$ be holomorphic coordinates on X and $(e_1, \ldots, e_n)$ a local C^∞ orthonormal frame of E . The Chern curvature tensor $c(E)$ is defined by $D^2 = c(E) \wedge \bullet$ and may be written

$$c(E) = \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu}\, dx_i \wedge d\overline{x}_j \otimes e^\star_\lambda \otimes e_\mu \quad , \quad 1 \leq i, j \leq n \ , \ 1 \leq \lambda, \mu \leq r \ .$$

The curvature tensor $ic(E)$ is in fact a (1,1)–form with values in the bundle $\mathrm{Herm}(E, E)$ of hermitian endomorphisms of E , *i.e.* $c_{ij\lambda\mu} = \overline{c}_{ji\mu\lambda}$; thus $ic(E)$ can be identified with a hermitian form on $TX \otimes E$.

Let us recall that the bundle E is said to be *positive*, resp. *semi-positive* (in the sense of Griffiths [8]) if E can be endowed with a hermitian metric such that at every point $x \in X$ one has

$$ic(E)_x(\zeta \otimes v, \zeta \otimes v) = \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu}(x)\, \zeta_i \overline{\zeta}_j v_\lambda \overline{v}_\mu > 0 \quad , \quad \text{resp.} \quad \geq 0$$

for all non zero vectors $\zeta = \sum \zeta_i \, \partial/\partial z_i \in T_x X$, $v = \sum v_\lambda e_\lambda \in E_x$. Every vector bundle E generated by sections is ≥ 0 .

Another important concept is that of ampleness, for which we refer to Hartshorne [9]; a vector bundle E is ample if and only if the line bundle $O_E(1)$ over $P(E^\star)$ is ample (or equivalently > 0) . It is also well-known that $E > 0$ implies E ample, but the converse is unknown.

In the case of a positive or ample vector bundle E of rank $r > 1$, only very few general and optimal vanishing results are available for the Dolbeault cohomology groups $H^{p,q}$ of tensor powers of E . For example, the famous Le Potier vanishing theorem [13] :

$$E \text{ ample} \implies H^{p,q}(X, E) = 0 \text{ for } p + q \geq n + r$$

does not extend to symmetric powers $S^k E$, even when $p = n$ and $q = n - 2$ (*cf.* [11]) . Nevertheless, the following result shows that the vanishing property is true for tensor powers involving a sufficiently large power of $\det E$.

THEOREM. — *Let L be a holomorphic line bundle over X . Assume that $E > 0$ and $L \geq 0$, or $E \geq 0$ and $L > 0$. For all integers p, q such that $p + q \geq n$, set*

$$A(n,p,q) = \frac{n(n+1)(p+1)(q+1)}{4(p+q-n+1)} \quad \text{if } p < n$$

and $A(n,p,q) = 0$ if $p = n$. Let $h \in \{1, \ldots, r-1\}$ and let $\Gamma^a E$ be the irreducible tensor power representation of $\mathrm{Gl}(E)$ of highest weight $a \in \mathbb{Z}^r$, with

$$a_1 \geq a_2 \geq \ldots \geq a_h > a_{h+1} = \ldots = a_r = 0 \ .$$

If $p + q \geq n + 1$ then

$$H^{p,q}(X, \Gamma^a E \otimes (\det E)^l \otimes L) = 0 \quad \text{for } l \geq h + A(n,p,q) \ .$$

The proof of this theorem is based on analysis and differential geometry, but an analogous result can be obtained in a purely algebraic way (*cf.* [5]); in that case the positivity hypothesis can be replaced by ampleness, the semi-positivity hypothesis by the fact that the bundle is generated by its global sections; then, the condition required on l is $l \geq n - p + r - 1$. Both results overlap in most cases, but the above analytic result can be better if $r - h$ is very large.

Observing that $S^k E$ is the irreducible representation of highest weight $(k, 0, \ldots, 0)$ and that $E^{\otimes k}$ splits into irreducible representations of the type $\Gamma^a E \otimes (\det E)^l$ with $h \leq \min\{k, r-1\}$ (*cf.* formula (2.17)), we obtain :

COROLLARY. — *Under the positivity hypotheses of the theorem, then for all p, q such that $p + q \geq n + 1$ one has*

(0.1) $H^{p,q}(X, S^k E \otimes (\det E)^l \otimes L) = 0$ *if* $l \geq 1 + A(n,p,q)$;

(0.2) $H^{p,q}(X, E^{\otimes k} \otimes (\det E)^l \otimes L) = 0$ *if* $l \geq \min\{k, r-1\} + A(n,p,q)$.

The special case $p = n$ of (0.1) is due to P. Griffiths [8] . For $p = n$ and arbitrary r , $k_0 \geq 2$, Peternell-Le Potier and Schneider [11] have constructed an example of a vector bundle $E > 0$ of rank r over a manifold X of dimension $n = 2r$ such that

$$H^{n,n-2}(X, S^k E) \neq 0 \quad , \quad 2 \leq k \leq k_0 \quad . \tag{0.3}$$

This result shows that the lower bound $l \geq 1$ in (0.1) cannot be improved. More generally, the following example (for which we refer to [5]) shows that our condition $l \geq h$ in the theorem is optimal. This example gives a negative answer to a question of Sommese [15].

EXAMPLE. — *Let $X = G_r(V)$ be the Grassmannian of subspaces of codimension r of a vector space V of dimension d , and E the tautological quotient vector bundle of rank r over X (then $E \geq 0$ and $L = \det E$ is ample) . Let $h \in \{1, \ldots, r-1\}$ and $a \in \mathbb{Z}^r$, $\beta \in \mathbb{Z}^d$ be such that*

$$a_1 \geq \ldots \geq a_h \geq d - r \quad , \quad a_{h+1} = \ldots = a_r = 0 \quad ,$$
$$\beta = (a_1 - d + r, \ldots, a_h - d + r, 0, \ldots, 0) \quad .$$

Set $n = \dim X = r(d-r)$, $q = (r-h)(d-r)$. Then

$$H^{n,q}\big(X, \Gamma^a E \otimes (\det E)^h\big) = \Gamma^\beta V \otimes (\det V)^h \neq 0 \quad . \tag{0.4}$$

Our approach is based on three well-known facts. First, every tensor power of E splits into irreducible representations of the linear group $\mathrm{Gl}(E)$. It is thus sufficient to consider "irreducible" tensor powers of E . Secondly, every irreducible tensor power of E appears in a natural way as the direct image on X of a *positive line bundle* over a suitable flag manifold of E . This follows from Bott's theory of homogeneous vector bundles [3]. The third fact is the isomorphism theorem of Le Potier [13], which relates the cohomology groups of E over X to those of the line bundle $\mathrm{O}_E(1)$ over $P(E^\star)$. We generalize here this isomorphism to the case of arbitrary flag bundles associated to E .

When $p = n$ the above-mentioned algebraic facts suffice to prove the theorem. However, when $p < n$, the generalized Borel-Le Potier spectral sequence does not degenerate at the E_1 level (*cf.* [12]). A possible way in order to overcome this difficulty is to establish a curvature estimate for the bundle of X–relative differential forms on the flag manifold of E , using the standard Kodaira-Akizuki-Nakano inequality [1] . Our estimate (*cf.* §4) measures in some sense how far is the spectral sequence from being E_1–degenerate. The following related problem is interesting, but its complete solution certainly requires a better understanding of the Borel-Le Potier spectral sequence for flag bundles.

PROBLEM. — *Given a dominant weight $a \in \mathbb{Z}^r$ with $a_r = 0$, determine the smallest admissible constant $A(n, p, q)$ in the theorem.*

It is shown in [5] that if the Borel-Le Potier spectral sequence degenerates in E_2 , then it is always sufficient to take $l \geq r - 1 + \min\{n-p, n-q\}$. In view of the above main theorem, one may suspect that the correct answer could be $l \geq h + \min\{n-p, n-q\}$.

The above results have been annouced in the note [4]. The author wishes to thank warmly Prof. Michel Brion, Friedrich Knopp, Thomas Peternell and Michael Schneider for valuable remarks which led to substantial improvements of this work.

1. Kodaira-Akizuki-Nakano vanishing theorem.

We recall here the basic Akizuki-Nakano inequality [1] which will be used several times in the sequel. Assume that X carries a Kähler metric ω , and let L be a hermitian line bundle over X . At each point $x \in X$, one can write

$$ic(L) = i \sum_{1 \le j \le n} \gamma_j \, dz_j \wedge d\overline{z}_j$$

where $(dz_1, \ldots, dz_n)$ is an ω–orthonormal basis of $T^\star X$ and where $\gamma_1 \le \ldots \le \gamma_n$ are the curvature eigenvalues of L . For every $v \in C^\infty_{p,q}(X, L)$ we have

$$(1.1) \qquad ||D''_L v||^2 + ||D''^{\star}_L v||^2 \ge \langle \Theta_L v \, , \, v \rangle \quad ,$$

where Θ_L is the hermitian endomorphism defined by

$$(1.2) \qquad \langle \Theta_L v \, , \, v \rangle = \sum_{|I|=p, |J|=q} \big(\gamma_I + \gamma_J - \sum_{1 \le j \le n} \gamma_j \big) |v_{I,J}|^2 \quad ,$$

with $\gamma_I = \sum_{m \in I} \gamma_m$. When L is > 0 , one can choose $\omega = ic(L)$ as the Kähler metric on X; in that case $\gamma_1 = \ldots = \gamma_n = 1$ and therefore

$$(1.3) \qquad ||D''_L v||^2 + ||D''^{\star}_L v||^2 \ge (p + q - n) ||v||^2 \quad .$$

Corollary (Kodaira-Akizuki-Nakano). — *One has*

$$(1.4) \qquad L > 0 \implies H^{p,q}(X, L) = 0 \quad \textit{for} \quad p + q \ge n + 1 \quad ,$$

$$(1.5) \qquad L < 0 \implies H^{p,q}(X, L) = 0 \quad \textit{for} \quad p + q \le n - 1 \quad .$$

2. Homogeneous line bundles over flag manifolds and irreducible representations of the linear group.

The aim of this section is to settle notations and to recall a few basic results on homogeneous line bundles over flag manifolds. The classical foundation works on this subject are Borel-Weil [2] and R. Bott [3] , which contain all the required material (*cf.* also Demazure [6] for a very simple proof of Bott's formula). We will give however an independent self-contained exposition in order to prepare the tools needed in the differential geometric approach of §4 .

Let B_r (resp. B^r) be the Borel subgroup of $\mathrm{Gl}_r = \mathrm{Gl}(\mathbb{C}^r)$ of lower (resp. upper) triangular matrices, $U_r \subset B_r$, $U^r \subset B^r$ the subgroups of unipotent matrices, and $T^r = B_r \cap B^r$ the complex torus $(\mathbb{C}^\star)^r$ of diagonal matrices. Let V be a complex vector space of dimension r . We denote by $M(V)$ the manifold of complete flags

$$V = V_0 \supset V_1 \supset \ldots \supset V_r = \{0\} \quad , \quad \mathrm{codim}_{\mathbb{C}} V_\lambda = \lambda \quad .$$

To every linear isomorphism $\zeta \in \mathrm{Isom}(\mathbb{C}^r, V) \ : \ (u_1, \ldots, u_r) \longmapsto \sum_{1 \le \lambda \le r} u_\lambda \zeta_\lambda$, one can associate the flag $[\zeta] \in M(V)$ defined by $V_\lambda = \mathrm{Vect}(\zeta_{\lambda+1}, \ldots, \zeta_r)$, $1 \le \lambda \le r$. This leads to the identification

$$M(V) = \mathrm{Isom}(\mathbb{C}^r, V)/B_r$$

where B_r acts on the right side. We denote simply by V_λ the tautological vector bundle of rank $r-\lambda$ on $M(V)$, and we consider the canonical quotient line bundles

$$(2.1)\qquad \begin{cases} Q_\lambda = V_{\lambda-1}/V_\lambda \ , & 1 \le \lambda \le r \ , \\ Q^a = Q_1^{a_1} \otimes \ldots \otimes Q_r^{a_r} \ , & a = (a_1, \ldots, a_r) \in \mathbb{Z}^r \ . \end{cases}$$

The linear group $\mathrm{Gl}(V)$ acts on $M(V)$ on the left, and there exist natural equivariant left actions of $\mathrm{Gl}(V)$ on all bundles V_λ , Q_λ , Q^a .

We compute now the tangent and cotangent vector bundles of $M(V)$. The action of $\mathrm{Gl}(V)$ on $M(V)$ yields

$$(2.2)\qquad TM(V) = \mathrm{Hom}(V,V)/W$$

where W is the subbundle of endomorphisms $g \in \mathrm{Hom}(V,V)$ such that $g(V_\lambda) \subset V_\lambda$, $1 \le \lambda \le r$. Using the self-duality of $\mathrm{Hom}(V,V)$ given by the Killing form $(g_1, g_2) \mapsto \mathrm{tr}(g_1 g_2)$, we find

$$(2.3)\qquad \begin{cases} T^\star M(V) = \big(\mathrm{Hom}(V,V)/W\big)^\star = W^\perp \\ W^\perp = \{g \in \mathrm{Hom}(V,V) \ ; \ g(V_{\lambda-1}) \subset V_\lambda \ , \ 1 \le \lambda \le r\} \ . \end{cases}$$

There exists a filtration of $T^\star M(V)$ by subbundles of the type

$$\{g \in \mathrm{Hom}(V,V) \ ; \ g(V_\lambda) \subset V_{\mu(\lambda)} \ , \ \lambda < \mu(\lambda) \ , \ 1 \le \lambda \le r\}$$

in such a way that the corresponding graded bundle is the direct sum of the line bundles $\mathrm{Hom}(Q_\lambda, Q_\mu) = Q_\lambda^{-1} \otimes Q_\mu$, $\lambda < \mu$; their tensor product is thus isomorphic to the canonical line bundle $K_{M(V)} = \det(T^\star M(V))$:

$$(2.4)\qquad K_{M(V)} = Q_1^{1-r} \otimes \ldots \otimes Q_\lambda^{2\lambda-r-1} \otimes \ldots \otimes Q_r^{r-1} = Q^c$$

where $c = (1-r, \ldots, r-1)$; c will be called the canonical weight of $M(V)$.

• *Case of incomplete flag manifolds.*

More generally, given any sequence of integers $s = (s_0, \ldots, s_m)$ such that $0 = s_0 < s_1 < \ldots < s_m = r$, we may consider the manifold $M_s(V)$ of incomplete flags

$$V = V_{s_0} \supset V_{s_1} \supset \ldots \supset V_{s_m} = \{0\} \quad , \quad \mathrm{codim}_{\mathbb{C}} V_{s_j} = s_j \ .$$

On $M_s(V)$ we still have tautological vector bundles $V_{s,j}$ of rank $r - s_j$ and line bundles

$$(2.5)\qquad Q_{s,j} = \det(V_{s,j-1}/V_{s,j}) \ , \quad 1 \le j \le m \ .$$

For any r–tuple $a \in \mathbb{Z}^r$ such that $a_{s_{j-1}+1} = \ldots = a_{s_j}$, $1 \le j \le m$, we set

$$Q_s^a = Q_{s,1}^{a_{s_1}} \otimes \ldots \otimes Q_{s,m}^{a_{s_m}} \ .$$

If $\eta : M(V) \to M_s(V)$ is the natural projection, then

$$(2.6)\qquad \eta^\star V_{s,j} = V_{s_j} \quad , \quad \eta^\star Q_{s,j} = Q_{s_{j-1}+1} \otimes \ldots \otimes Q_{s_j} \quad , \quad \eta^\star Q_s^a = Q^a \ .$$

On the other hand, one has the identification

$$M_s(V) = \mathrm{Isom}(\mathbb{C}^r, V)/B_s$$

where B_s is the parabolic subgroup of matrices $(z_{\lambda\mu})$ with $z_{\lambda\mu} = 0$ for all λ, μ such that there exists an integer $j = 1, \ldots, m-1$ with $\lambda \le s_j$ and $\mu > s_j$. We define

U_s as the unipotent subgroup of lower triangular matrices $(z_{\lambda\mu})$ with $z_{\lambda\mu} = 0$ for all λ, μ such that $s_{j-1} < \lambda \neq \mu \leq s_j$ for some j , and we set $B^s = {}^tB_s$, $U^s = {}^tU_s$. In the same way as above, we get

(2.7) $$TM_s(V) = \mathrm{Hom}(V,V)/W_s \quad , \quad W_s = \{g \ ; \ g(V_{s_{j-1}}) \subset V_{s_j}\} \quad ,$$

(2.8) $$T^\star M_s(V) = W_s^\perp \quad ,$$

(2.9) $$K_{M_s(V)} = Q_{s,1}^{s_1-r} \otimes \ldots \otimes Q_{s,j}^{s_{j-1}+s_j-r} \otimes \ldots \otimes Q_{s,m}^{s_{m-1}} = Q_s^{c(s)}$$

where $c(s) = (s_1 - r, \ldots, s_1 - r, \ldots, s_{j-1} + s_j - r, \ldots, s_{j-1} + s_j - r, \ldots)$ is the canonical weight of $M_s(V)$.

• *Curvature form of the line bundle Q^a .*

Assume now that V is a *hermitian* vector space. Then all our bundles carry a natural hermitian metric. We are going to compute the curvature of Q^a at any point $[e] \in M(V)$. Choose an orthonormal basis $(e_1, \ldots, e_r)$ of V which corresponds to the given point $[e]$. It is clear that $eB_re^{-1} \subset \mathrm{Gl}(V)$ is the isotropy subgroup of $[e]$, whereas $eU^re^{-1}.[e] = [eU^r]$ is an affine open subset of $M(V)$, corresponding to bases $(\zeta_1, \ldots, \zeta_r)$ of V such that

$$\zeta_\mu = e_\mu + \sum_{\lambda<\mu} z_{\lambda\mu} e_\lambda \quad , \quad 1 \leq \mu \leq r \quad , \quad z_{\lambda\mu} \in \mathbb{C} \quad .$$

Then $(z_{\lambda\mu})_{1\leq\lambda<\mu\leq r}$ is a coordinate system on $[eU^r]$ and the map

$$M(V) \ni [\zeta] \longmapsto \tilde{\zeta}_\mu := \zeta_\mu \mod V_\mu$$

is a local section of $Q_\mu = V_{\mu-1}/V_\mu$. Hence

$$c(Q_\mu) = -d'd'' \log |\tilde{\zeta}_\mu|^2 \quad .$$

Let us identify $\tilde{\zeta}_\mu$ with the orthogonal projection of ζ_μ on $V_{\mu-1} \cap V_\mu^\perp$. Then Gram-Schmidt's orthogonalization process yields

$$\tilde{\zeta}_\mu = \zeta_\mu - \sum_{\nu>\mu} \frac{\langle \zeta_\mu, \tilde{\zeta}_\nu \rangle}{|\tilde{\zeta}_\nu|^2} \tilde{\zeta}_\nu \quad , \quad |\tilde{\zeta}_\mu|^2 = |\zeta_\mu|^2 - \sum_{\nu>\mu} \frac{|\langle \zeta_\mu, \tilde{\zeta}_\nu \rangle|^2}{|\tilde{\zeta}_\nu|^2} \quad .$$

Since $\langle \zeta_\mu, \zeta_\nu \rangle = \overline{z}_{\mu\nu} + \sum_{\lambda<\mu} z_{\lambda\mu}\overline{z}_{\lambda\nu}$, it follows by backward induction on ν that $\langle \zeta_\mu, \tilde{\zeta}_\nu \rangle = \overline{z}_{\mu\nu} + \mathrm{O}(|z|^2)$ for $\nu > \mu$, hence

$$|\tilde{\zeta}_\mu|^2 = 1 + \sum_{\lambda<\mu} |z_{\lambda\mu}|^2 - \sum_{\nu>\mu} |z_{\mu\nu}|^2 + \mathrm{O}(|z|^3) \quad .$$

We obtain therefore

(2.10) $$c(Q_\mu)_{[e]} = -\sum_{\lambda<\mu} dz_{\lambda\mu} \wedge d\overline{z}_{\lambda\mu} + \sum_{\nu>\mu} dz_{\mu\nu} \wedge d\overline{z}_{\mu\nu} \quad ,$$

(2.11) $$c(Q^a)_{[e]} = \sum_{\mu} a_\mu c(Q_\mu) = \sum_{\lambda<\mu} (a_\lambda - a_\mu) dz_{\lambda\mu} \wedge d\overline{z}_{\lambda\mu} \quad .$$

Corollary 2.12. — *Q^a is ≥ 0 if and only if $a_1 \geq a_2 \geq \ldots \geq a_r$. If there exists an index j such that $a_j < a_{j+1}$, then $H^0(M(V), Q^a) = 0$.*

Proof. — Only the second statement and the "only if" part of the first remain to be proved. Let us observe that the projection $\eta \ : \ M(V) \longrightarrow M_s(V)$,

$s = (0, \ldots, j-1, j+1, \ldots, r)$, is a bundle with fibers $P(V_{j-1}/V_{j+1}) \simeq \mathbb{P}^1$. The restriction of Q_λ to each fiber is trivial if $\lambda \neq j, j+1$ whereas $Q_{j\restriction\mathbb{P}^1} \simeq \mathrm{O}(1)$ and $Q_{j+1\restriction\mathbb{P}^1} \simeq \mathrm{O}(-1)$. Therefore $Q^a_{\restriction\mathbb{P}^1} \simeq \mathrm{O}(a_j - a_{j+1})$ cannot have any non-zero section or any semi-positive metric if $a_j < a_{j+1}$. ∎

When $a_1 \geq \ldots \geq a_r$, the bundle Q^a is not necessarily > 0 on $M(V)$; in fact one can write Q^a as the induced bundle $\eta^\star Q^a_s$ where $s_1 < \ldots < s_{m-1}$ is the sequence of integers $\lambda = 1, \ldots, r-1$ such that $a_{\lambda+1} > a_\lambda$. The affine open subset $[eU^s] \subset M_s(V)$ is a neighborhood of $[e]$, and $M_s(V)$ has local coordinates $(z_{\lambda\mu})$ where λ, μ are such that $\lambda \leq s_{j-1} < s_j \leq \mu$ for some j , *i.e.* $a_\lambda > a_\mu$. The curvature of Q^a_s is given formally by the same expression as (2.11) :

$$c(Q^a_s)_{[e]} = \sum_{a_\lambda > a_\mu} (a_\lambda - a_\mu) dz_{\lambda\mu} \wedge d\overline{z}_{\lambda\mu} \quad . \tag{2.13}$$

We see therefore that $Q^a_s > 0$ on $M_s(V)$.

• *Cohomology groups of Q^a .*

It remains now to compute $H^0(M_s(V), Q^a_s) \simeq H^0(M(V), Q^a)$ when $a_1 \geq \ldots \geq a_r$. Without loss of generality we may assume that $a_r \geq 0$, because $Q_1 \otimes \ldots \otimes Q_r = \det V$ is a trivial bundle.

PROPOSITION 2.14. — *For all integers $a_1 \geq a_2 \geq \ldots \geq a_r \geq 0$, there is a canonical isomorphism*

$$H^0(M(V), Q^a) = \Gamma^a V$$

where $\Gamma^a V \subset S^{a_1} V \otimes \ldots \otimes S^{a_r} V$ is the set of polynomials $f(\zeta_1^\star, \ldots, \zeta_r^\star)$ on $(V^\star)^r$ which are homogeneous of degree a_λ with respect to $\zeta^\star_\lambda$ and invariant under the left action of U_r on $(V^\star)^r = \mathrm{Hom}(V, \mathbb{C}^r)$:

$$f(\zeta_1^\star, \ldots, \zeta_{\lambda-1}^\star, \zeta_\lambda^\star + \zeta_\nu^\star, \ldots, \zeta_r^\star) = f(\zeta_1^\star, \ldots, \zeta_r^\star) \quad , \quad \forall \nu < \lambda \quad .$$

Proof. — To any section $\sigma \in H^0(M(V), Q^a)$ we associate the holomorphic function f on $\mathrm{Isom}(V, \mathbb{C}^r) \subset (V^\star)^r$ defined by

$$f(\zeta_1^\star, \ldots, \zeta_r^\star) = (\zeta_1^\star)^{a_1} \otimes \ldots \otimes (\zeta_r^\star)^{a_r} . \sigma([\zeta_1, \ldots, \zeta_r])$$

where $(\zeta_1, \ldots, \zeta_r)$ is the dual basis of $(\zeta_1^\star, \ldots, \zeta_r^\star)$, and where the linear form induced by $\zeta^\star_\lambda$ on $Q_\lambda = V_{\lambda-1}/V_\lambda \simeq \mathbb{C}\zeta_\lambda$ is still denoted $\zeta^\star_\lambda$. Let us observe that f is homogeneous of degree a_λ in $\zeta^\star_\lambda$ and locally bounded in a neighborhood of every r–tuple of $(V^\star)^r \setminus \mathrm{Isom}(V, \mathbb{C}^r)$ (because $M(V)$ is compact and $a_\lambda \geq 0$) . Therefore f can be extended to a polynomial on all $(V^\star)^r$. The invariance of f under U_r is clear. Conversely, such a polynomial f obviously defines a unique section σ on $M(V)$. ∎

From the definition of $\Gamma^a V$, we see that

$$S^k V = \Gamma^{(k,0,\ldots,0)} V \quad , \tag{2.15}$$

$$\Lambda^k V = \Gamma^{(1,\ldots,1,0,\ldots,0)} V \quad . \tag{2.16}$$

For arbitrary $a \in \mathbb{Z}^r$, proposition 2.14 remains true if we set

$$\Gamma^a V = \Gamma^{(a_1 - a_r, \ldots, a_{r-1} - a_r, 0)} V \otimes (\det V)^{a_r} \quad \text{when} \quad a \quad \text{is non-increasing} \quad ,$$

$$\Gamma^a V = 0 \qquad \text{otherwise} \quad .$$

The elements $a \in \mathbb{Z}^r$ will be ordered according to the partial ordering :

$$a \succcurlyeq b \quad \text{iff} \quad \sum_{1\le\lambda\le\mu} a_\lambda \ge \sum_{1\le\lambda\le\mu} b_\lambda \quad , \quad 1 \le \mu \le r \quad .$$

Bott's theorem [3] shows that $\Gamma^a V$ is an irreducible representation of $\mathrm{Gl}(V)$ of highest weight a; all irreducible representations of $\mathrm{Gl}(V)$ are in fact of this type (*cf.* Kraft [10]). In particular, since the weights of the action of a maximal torus $T^r \subset \mathrm{Gl}(V)$ on $V^{\otimes k}$ verify $a_1 + \ldots + a_r = k$ and $a_\lambda \ge 0$, we have a canonical $\mathrm{Gl}(V)$–isomorphism

$$V^{\otimes k} = \bigoplus_{\substack{a_1+\ldots+a_r=k \\ a_1\ge\ldots\ge a_r\ge 0}} \mu(a,k)\,\Gamma^a V \tag{2.17}$$

where $\mu(a,k) > 0$ is the multiplicity of the isotypical factor $\Gamma^a V$ in $V^{\otimes k}$.

Bott's formula (*cf.* also Demazure [6] for a very simple proof) gives in fact the expression of all cohomology groups $H^q(M(V), Q^a)$, but we will need them here only in the case of *dominant* weights $a_1 \ge \ldots \ge a_r$.

PROPOSITION 2.18. — *Set* $N = \dim M(V)$, $N(s) = \dim M_s(V)$. *If* $a_{s_j} - a_{s_{j+1}} \ge 1$, *then*

(a) $H^{N(s),q}(M_s(V), Q_s^a) = 0$ *for all* $q \ge 1$

(b) $H^{N(s),0}(M_s(V), Q_s^a) = \Gamma^{a+c(s)} V$.

Proof. — Under the assumption of (a), Q_s^a is > 0 by (2.13). The result follows therefore from the Kodaira-Akizuki-Nakano theorem. Now (b) is a consequence of proposition 2.14 since

$$H^{N(s),q}(M_s(V), Q_s^a) = H^q(M_s(V), K_{M_s(V)} \otimes Q_s^a) = H^q(M_s(V), Q_s^{a+c(s)}) \quad . \blacksquare$$

3. An isomorphism theorem

Our aim here is to generalize Griffiths and Le Potier's isomorphism theorems ([8], [13]) in the case of arbitrary flag bundles, following the simple method of Schneider [14] .

Let X be a n–dimensional compact complex manifold and $E \longrightarrow X$ a holomorphic vector bundle of rank r . For every sequence $0 = s_0 < s_1 < \ldots < s_m = r$, we associate to E its flag bundle $Y = M_s(E) \longrightarrow X$. If $a \in \mathbb{Z}^r$ is such that $a_{s_{j-1}+1} = \ldots = a_{s_j}$, $1 \le j \le m$, we may define a line bundle $Q_s^a \longrightarrow Y$ just as we did in §2 . Let us set

$$\Omega_X^p = \Lambda^p T^\star X \quad , \quad \Omega_Y^p = \Lambda^p T^\star Y \quad .$$

One has an exact sequence

$$0 \longrightarrow \pi^\star \Omega_X^1 \longrightarrow \Omega_Y^1 \longrightarrow \Omega_{Y/X}^1 \longrightarrow 0 \tag{3.1}$$

where $\Omega_{Y/X}^1$ is by definition the bundle of relative differential 1–forms along the fibers of the projection $\pi : Y = M_s(E) \longrightarrow X$. One may then define a decreasing filtration of Ω_Y^t as follows :

$$F^{p,t} = F^p(\Omega_Y^t) = \pi^\star(\Omega_X^p) \wedge \Omega_Y^{t-p} \quad . \tag{3.2}$$

The corresponding graded bundle is given by

$$G^{p,t} = F^{p,t}/F^{p+1,t} = \pi^\star(\Omega^p_X) \otimes \Omega^{t-p}_{Y/X} \quad . \tag{3.3}$$

Over any open subset of X where E is a trivial bundle $X \times V$ with $\dim_{\mathbb{C}} V = r$, the exact sequence (3.1) splits as well as the filtration (3.2). Using proposition 2.18, we obtain the following lemma.

LEMMA. — *For every weight a such that $a_{s_j} - a_{s_{j+1}} \geq 1$, $1 \leq j \leq m-1$, the sheaf of sections of $\Omega^{N(s)}_{Y/X} \otimes Q^a_s$ has direct images*

$$\begin{cases} R^q\pi_\star\big(\Omega^{N(s)}_{Y/X} \otimes Q^a_s\big) = 0 \quad \text{for} \quad q \geq 1 \\ \pi_\star\big(\Omega^{N(s)}_{Y/X} \otimes Q^a_s\big) = \Gamma^{a+c(s)}E \quad . \end{cases} \tag{3.4}$$

Let L be an arbitrary line bundle on X. Under the hypothesis $a_{s_j} - a_{s_{j+1}} \geq 1$, formulas (3.3) and (3.4) yield

$$R^q\pi_\star(G^{p,p+N(s)} \otimes Q^a_s \otimes \pi^\star L) = 0 \quad \text{for} \quad q \geq 1 \quad ,$$
$$\pi_\star(G^{p,p+N(s)} \otimes Q^a_s \otimes \pi^\star L) = \Omega^p_X \otimes \Gamma^{a+c(s)}E \otimes L \quad .$$

The Leray spectral sequence implies therefore :

THEOREM 3.5. — *If $a_{s_j} - a_{s_{j+1}} \geq 1$, then for all $q \geq 0$*

$$H^q(Y, G^{p,p+N(s)} \otimes Q^a_s \otimes \pi^\star L) \simeq H^{p,q}(X, \Gamma^{a+c(s)}E \otimes L) \quad .$$

When $p = n$, $G^{n,n+N(s)}$ is the only non-vanishing quotient in the filtration of the canonical line bundle $\Omega^{n+N(s)}_Y$. We thus obtain the following generalization of Griffiths' isomorphism theorem [8] :

$$H^{n+N(s),q}(M_s(E), Q^a_s \otimes \pi^\star L) \simeq H^{n,q}(X, \Gamma^{a+c(s)}E \otimes L) \quad . \tag{3.6}$$

In order to carry results for line bundles over to vector bundles, one needs the following lemma.

LEMMA 3.7. — *Assume that $a_{s_1} > a_{s_2} > \ldots > a_{s_m} \geq 0$. Then*

(a) $E \geq 0$ (*resp.* > 0) $\Longrightarrow$ $Q^a_s \geq 0$ (*resp.* > 0) ;

(b) $E \geq 0$ *and* $L > 0$ $\Longrightarrow$ $Q^a_s \otimes \pi^\star L > 0$;

(c) E *ample* $\Longrightarrow$ Q^a_s *ample* .

Proof. — Part (a) will be proved in §4 (*cf.* formula (4.9)) and (b) follows from the fact that $c(Q^a_s) > 0$ along the fibers of π .

(c) By definition of an ample vector bundle (Hartshorne [9]) , S^kE is very ample for $k \geq k_0$ large enough. Hence $\Gamma^{ka}E$, which is a direct summand in $S^{ka_1}E \otimes \ldots \otimes S^{ka_r}E$, is also very ample for $k \geq k_0$. Now, formula (2.14) shows that $Q^a_s > 0$ along the fibers of π , hence Q^{ka}_s is very ample along each fiber for $k \geq k_1$. Since $\pi_\star(Q^{ka}_s) = \Gamma^{ka}E$, we conclude that Q^{ka}_s is very ample for $k \geq \max(k_0, k_1)$. ■

We are now ready to attack the proof of the main theorem. We study first the special case $p = n$.

THEOREM 3.8. — *Let $a \in \mathbb{Z}^r$ be such that*

$$a_1 \geq a_2 \geq \ldots \geq a_h > a_{h+1} = \ldots = a_r = 0 \quad , \quad 1 \leq h \leq r-1 \quad .$$

Assume that E is ample and $L \geq 0$, or $E \geq 0$ and L ample. Then

$$H^{n,q}(X, \Gamma^a E \otimes (\det E)^l \otimes L) = 0 \quad \textit{for} \quad q \geq 1 \quad , \quad l \geq h \quad .$$

Proof of theorem 3.8 and of the main theorem. —

Let $s_1 > \ldots > s_{m-1}$ be the sequence of integers $\lambda = 1, \ldots, r-1$ such that $a_{\lambda+1} > a_\lambda$. Then theorem 3.5 implies

$$(3.9) \qquad H^{p,q}(X, \Gamma^a E \otimes (\det E)^l \otimes L) \simeq H^q(Y, G^{p,p+N(s)} \otimes Q_s^b \otimes \pi^\star L)$$

where $b = a - c(s) + (l, \ldots, l)$. The canonical weight $c(s)$ is non-decreasing and $c(s)_r = s_{m-1} = h$, hence

$$b_{s_1} > \ldots > b_{s_m} = l - h \quad .$$

Lemma 3.7 shows that $Q_s^b \otimes \pi^\star L > 0$ if $l \geq h$. Now, it is clear that $F^{p,p+N(s)} = \Omega_Y^{p+N(s)}$. One gets thus an exact sequence

$$(3.10) \qquad 0 \longrightarrow F^{p+1,p+N(s)} \longrightarrow \Omega_Y^{p+N(s)} \longrightarrow G^{p,p+N(s)} \longrightarrow 0 \quad .$$

The Kodaira-Akizuki-Nakano vanishing theorem (1.4) applied to $Q_s^b \otimes \pi^\star L$ with $\dim Y = n + N(s)$ yields

$$H^q(Y, \Omega_Y^{p+N(s)} \otimes Q_s^b \otimes \pi^\star L) = 0 \quad \text{for} \quad p + q \geq n+1 \quad .$$

The cohomology groups in (3.9) will therefore vanish if and only if

$$(3.11) \qquad H^{q+1}(Y, F^{p+1,p+N(s)} \otimes Q_s^b \otimes \pi^\star L) = 0 \quad .$$

This is obvious if $p = n$, for $F^{n+1,n+N(s)} = 0$. In the general case $p < n$, we will establish in §4 that (3.11) holds for $p + q \geq n$ and $b_r = l - h \geq A(n,p,q)$. This will be done by means of a curvature estimate for the bundle $F^{p+1,p+N(s)}$. ∎

Remark 3.12. — If $p + q = n$, we still obtain some result, namely that the canonical map

$$H^{p+N(s),q}(Y, Q_s^b \otimes \pi^\star L) \longrightarrow H^{p,q}(X, \Gamma^a E \otimes (\det E)^l \otimes L)$$

is onto when $l \geq h + A(n,p,q)$.

Remark 3.13. — If the exact sequence (3.10) splits, then (3.11) is an immediate consequence of the Kodaira-Nakano-Akizuki theorem. However, Peternell-Le Potier and Schneider [11] , [12] have shown that in general the filtration $F^\bullet(\Omega_Y^t)$ does not split, and this is the reason why we have to introduce additional considerations in order to prove (3.11).

4. A curvature estimate for the subbundle $F^{p+1,p+N(s)}$.

We assume here that $E, L \longrightarrow X$ are hermitian vector bundles of respective ranks $r, 1$ and that $E \geq 0$ and $L > 0$, or $E > 0$ and $L \geq 0$. Let $a \in \mathbb{Z}^r$ be such that $a_1 \geq \ldots \geq a_r \geq 0$ and let $s_1 < \ldots < s_{m-1}$ be the sequence of integers $\lambda = 1, \ldots, r-1$ such that $a_{\lambda+1} > a_\lambda$. We set for simplicity

$$Y = M_s(E) \quad , \quad \Omega = \Omega_Y^{p+N(s)} \quad , \quad F = F^{p+1,p+N(s)} \quad , \quad G = G^{p,p+N(s)} = \Omega/F \quad .$$

Our aim is to prove that the analogue of (3.11) :

$$H^{q+1}(Y, F \otimes Q_s^a \otimes \pi^\star L) = 0 \tag{4.1}$$

holds when $p+q \geq n$ and $a_r \geq A(n,p,q)$. Let us consider the exact sequences of vector bundles over Y :

$$\begin{cases} 0 \longrightarrow \quad F \quad \longrightarrow \quad \Omega \quad \longrightarrow \quad G \quad \longrightarrow 0 \quad , \\ 0 \longrightarrow F(a) \longrightarrow \Omega(a) \longrightarrow G(a) \longrightarrow 0 \quad , \end{cases} \tag{4.2}$$

where the second sequence arises from the first one after taking tensor products with the line bundle $Q_s^a \otimes \pi^\star L$. Then Y can be equipped with the Kähler metric $\omega = ic(Q_s^a \otimes \pi^\star L)$; the positivity of ω is a consequence of (4.9) below. To every smooth form v of type $(p+N(s), q+1)$ with values in $Q_s^a \otimes \pi^\star L$, let us apply the Akizuki-Nakano inequality (1.3) , where $\dim Y = n + N(s)$:

$$||D''_{\Omega(a)} v||^2 + ||D''^\star_{\Omega(a)} v||^2 \geq (p+q-n+1))\ ||v||^2 \quad . \tag{4.3}$$

With respect to the orthogonal C^∞–splitting $\Omega \simeq F \oplus G$, the Chern connections of Ω , F , G are related by the well-known formula (*cf.* [8]) :

$$D_\Omega = \begin{pmatrix} D_F & -\beta^\star \wedge \bullet \\ \beta \wedge \bullet & D_G \end{pmatrix} \quad , \quad \beta \in C^\infty(\Lambda^{1,0} T^\star Y \otimes \mathrm{Hom}(F,G)) \ ;$$

$\beta^\star$ is a D''–closed (0,1)–form with values in $\mathrm{Hom}(G,F)$, and its cohomology class is the obstuction to the existence of a global splitting of (4.2). We obtain therefore

$$D''_{\Omega(a)} = \begin{pmatrix} D''_{F(a)} & -\beta^\star \wedge \bullet \\ 0 & D''_{G(a)} \end{pmatrix} \quad , \quad D''^\star_{\Omega(a)} = \begin{pmatrix} D''^\star_{F(a)} & 0 \\ -\beta \lrcorner \bullet & D''^\star_{G(a)} \end{pmatrix} \quad ,$$

where $\lrcorner$ denotes the interior product of differential forms combined with the evaluation map $\mathrm{Hom}(F,G) \times F \to G$ (note that $\beta \lrcorner \bullet = (\beta^\star \wedge \bullet)^\star$). For every $(0, q+1)$–form f with values in $F(a)$ we get

$$D''_{F(a)} f = D''_{\Omega(a)} f \quad , \quad ||D''^\star_{F(a)} f||^2 = ||D''^\star_{\Omega(a)} f||^2 - ||\beta \lrcorner f||^2 \quad . \tag{4.4}$$

From (4.3) and (4.4), we see that the vanishing property (4.1) will hold if

$$|\beta \lrcorner f|^2 < (p+q-n+1)\,|f|^2 \tag{4.5}$$

at every point of Y . We are going to compute β explicitly in suitable coordinate systems on Y . Let $y^0 \in Y$ be an arbitrary point and $(x_1, \ldots, x_n)$ local coordinates on X centered at the point $x^0 = \pi(y^0)$.

Lemma. — *There exists a local holomorphic frame $(e_1, \ldots, e_r)$ of E such that y^0 coincides with the flag $[e_1(x^0), \ldots, e_r(x^0)]$ and*

$$\langle e_\lambda(x) \, , e_\mu(x) \rangle = \delta_{\lambda\mu} - \sum_{i,j} c_{ij\lambda\mu}\, x_i \overline{x}_j + \mathrm{O}(|x|^3) \quad , \tag{4.6}$$

where $(c_{ij\lambda\mu})$ is the curvature tensor of E .

Proof. — Choose a holomorphic frame $(\varepsilon_1, \ldots, \varepsilon_r)$ of E such that $(\varepsilon_1(x^0), \ldots, \varepsilon_r(x^0))$ is orthonormal and $[\varepsilon_1(x^0), \ldots, \varepsilon_r(x^0)] = y^0$. Then the inner product $\langle \varepsilon_\lambda(x), \varepsilon_\mu(x)\rangle$ has a Taylor expansion of the type

$$\begin{aligned}\langle \varepsilon_\lambda(x), \varepsilon_\mu(x)\rangle = \delta_{\lambda\mu} + \sum_i (\gamma_{i\lambda\mu} x_i + \overline{\gamma}_{i\mu\lambda}\overline{x}_i) \\ + \sum_{i,j}(\gamma_{ij\lambda\mu} x_i \overline{x}_j + \gamma'_{ij\lambda\mu} x_i x_j + \overline{\gamma}'_{ij\mu\lambda}\overline{x}_i\overline{x}_j) + \mathrm{O}(|x|^3) \quad .\end{aligned}$$

This expansion can be reduced to (4.6) (with suitable coefficients $c_{ij\lambda\mu}$) if one sets

$$e_\lambda(x) = \varepsilon_\lambda(x) - \sum_{i,\mu} \gamma_{i\lambda\mu} x_i \, e_\mu - \sum_{i,j,\mu} \gamma'_{ij\lambda\mu} x_i x_j \, e_\mu \quad .$$

Now (4.6) implies

$$\begin{aligned} De_\lambda &= -\sum_{i,j,\mu} c_{ij\lambda\mu}\overline{x}_j dx_i \otimes e_\mu + \mathrm{O}(|x|^2) \quad , \\ D^2 e_\lambda &= \sum_{i,j,\mu} c_{ij\lambda\mu} dx_i \wedge d\overline{x}_j \otimes e_\mu + \mathrm{O}(|x|) \quad ,\end{aligned}$$

showing that the $c_{ij\lambda\mu}$'s are precisely the curvature coefficients at x^0 . ■

Let us denote by $z = (z_{\lambda\mu})$ the affine coordinates on the fiber $M_s(E_x) \subset Y$ associated to the basis $(e_1(x), \ldots, e_r(x))$. Then $(x_1, \ldots, x_n, z_{\lambda\mu})$ define local coordinates on Y in a neighborhood of y^0 . Assume first that $Y = M_s(E) = M(E)$ is the manifold of complete flags of E . Then we have tautological subbundles $V_\lambda = \mathrm{Vect}(e_{\lambda+1}, \ldots, e_r) \subset \pi^\star E$ and the map

$$Y \ni (x, z) \longmapsto \zeta_\mu = e_\mu(x) + \sum_{\lambda<\mu} z_{\lambda\mu} e_\lambda(x) \tag{4.7}$$

is a local section of $V_{\mu-1}$. Let us denote by $\widetilde{\zeta}_\mu$ the image of ζ_μ in $Q_\mu = V_{\mu-1}/V_\mu$, represented by the orthogonal projection of ζ_μ on $V_{\mu-1} \cap (V_\mu)^\perp$. As in §2, one finds

$$\begin{cases} \widetilde{\zeta}_\lambda = \zeta_\lambda - \displaystyle\sum_{\mu>\lambda} \frac{\langle \zeta_\lambda, \widetilde{\zeta}_\mu\rangle}{|\widetilde{\zeta}_\mu|^2}\widetilde{\zeta}_\mu \quad , \quad |\widetilde{\zeta}_\lambda|^2 = |\zeta_\lambda|^2 - \displaystyle\sum_{\mu>\lambda} \frac{|\langle \zeta_\lambda, \widetilde{\zeta}_\mu\rangle|^2}{|\widetilde{\zeta}_\mu|^2} \quad , \\ \langle \zeta_\lambda, \widetilde{\zeta}_\mu\rangle = \overline{z}_{\lambda\mu} - \displaystyle\sum_{i,j} c_{ij\lambda\mu} x_i \overline{x}_j \quad \mathrm{mod}(z^2, x^3, x^2 z) \ \text{ for } \ \lambda < \mu \quad . \end{cases} \tag{4.8}$$

We need a Taylor expansion of $c(Q_\mu) = -d'd'' \log|\widetilde{\zeta}_\mu|^2$ up to order 1, hence of $|\widetilde{\zeta}_\mu|^2$ up to order 3. Moreover, pure terms x^3 , z^3 will not play any role because the Kähler property of $c(Q_\mu)$ enables one to get rid of the terms $\mathrm{O}(|x| dx \wedge d\overline{x})$, $\mathrm{O}(|z| dz \wedge d\overline{z})$. Therefore, we are interested only in terms of degree ≤ 2 and in mixed terms xz^2 , $x^2 z$. Thanks to formulas (4.7) and (4.8), we get the following equalities modulo the ideal (x^3, z^3, x^2z^2) :

$$|\widetilde{\zeta}_\lambda|^2 \sim |\zeta_\lambda|^2 - \sum_{\mu>\lambda} |\langle \zeta_\lambda, \widetilde{\zeta}_\mu\rangle|^2 \quad ,$$

$$|\zeta_\lambda|^2 \sim 1 - \sum_{i,j} c_{ij\lambda\lambda} x_i \overline{x}_j + \sum_{\mu<\lambda} |z_{\mu\lambda}|^2 - \sum_{i,j,\mu<\lambda} c_{ij\lambda\mu} x_i \overline{x}_j \overline{z}_{\mu\lambda} - \sum_{i,j,\mu<\lambda} \overline{c}_{ij\lambda\mu} \overline{x}_i x_j z_{\mu\lambda} \quad ,$$

$$|\langle \zeta_\lambda, \widetilde{\zeta}_\mu\rangle|^2 \sim |z_{\lambda\mu}|^2 - \sum_{i,j} c_{ij\lambda\mu} x_i \overline{x}_j z_{\lambda\mu} - \sum_{i,j} \overline{c}_{ij\lambda\mu} \overline{x}_i x_j \overline{z}_{\lambda\mu} \quad .$$

We have now

$$c(Q^a) = \sum a_\lambda c(Q_\lambda) = d'd''\Big(-\sum a_\lambda \log|\widetilde{\zeta}_\lambda|^2\Big) \quad ,$$

$$-\sum_\lambda a_\lambda \log|\widetilde{\zeta}_\lambda|^2 \sim \sum_{i,j,\lambda} a_\lambda c_{ij\lambda\lambda} x_i \overline{x}_j + \sum_{\lambda<\mu} (a_\lambda - a_\mu)|z_{\lambda\mu}|^2 - \sum_{i,j,\lambda<\mu} (a_\lambda - a_\mu) c_{ij\lambda\mu} x_i \overline{x}_j z_{\lambda\mu} - \sum_{i,j,\lambda<\mu} (a_\lambda - a_\mu) \overline{c}_{ij\lambda\mu} \overline{x}_i x_j \overline{z}_{\lambda\mu} \quad .$$

We find therefore

$$\begin{aligned} c(Q^a) = {} & \sum_{i,j} \Big(\sum_\lambda a_\lambda c_{ij\lambda\lambda} + \mathrm{O}(|x|)\Big) dx_i \wedge d\overline{x}_j \\ & + \sum_{\lambda<\mu} (a_\lambda - a_\mu) dz_{\lambda\mu} \wedge d\overline{z}_{\lambda\mu} + \mathrm{O}(|z| dz \wedge d\overline{z}) \\ & - \sum_{i,j,\lambda<\mu} (a_\lambda - a_\mu) c_{ij\lambda\mu} \big(z_{\lambda\mu} dx_i \wedge d\overline{x}_j + x_i dz_{\lambda\mu} \wedge d\overline{x}_j\big) \\ & - \sum_{i,j,\lambda<\mu} (a_\lambda - a_\mu) \overline{c}_{ij\lambda\mu} \big(\overline{z}_{\lambda\mu} dx_j \wedge d\overline{x}_i + \overline{x}_i dx_j \wedge d\overline{z}_{\lambda\mu}\big) + \mathrm{O}(|x|^2 + |z|^2) \quad . \end{aligned}$$

Since $Q^a = \pi_s^\star Q_s^a$, the same identity holds for Q_s^a . At the point y^0 we get

$$c(Q_s^a)_{y^0} = \sum_{i,j,\lambda} a_\lambda c_{ij\lambda\lambda} dx_i \wedge d\overline{x}_j + \sum_{a_\lambda > a_\mu} (a_\lambda - a_\mu) dz_{\lambda\mu} \wedge d\overline{z}_{\lambda\mu} \quad . \tag{4.9}$$

Now, $\omega = i\big(\pi^\star c(L) + c(Q_s^a)\big)$ is Kähler on $Y = M_s(E)$, thus in particular along the fiber $x = 0$ and along the local section $z = 0$. It follows that one can find coordinate changes $x \mapsto x'$, $z'_{\lambda\mu} = \sqrt{a_\lambda - a_\mu}\, z_{\lambda\mu} \mod z^2$ such that the terms $\mathrm{O}(|x| dx \wedge d\overline{x})$ and $\mathrm{O}(|z| dz \wedge d\overline{z})$ disappear in the expansion of ω , and such that

$$\omega_{ij}(y^0) = c(L)_{ij}(x^0) + \sum_\lambda a_\lambda c_{ij\lambda\lambda} = \delta_{ij} \quad . \tag{4.10}$$

We obtain therefore

$$\begin{aligned} \frac{1}{i}\omega = {} & \sum_j dx'_j \wedge d\overline{x}'_j + \sum_{\lambda<\mu} dz'_{\lambda\mu} \wedge d\overline{z}'_{\lambda\mu} \\ & - \sum_{i,j,\lambda<\mu} \sqrt{a_\lambda - a_\mu}\; c_{ij\lambda\mu} \big(z'_{\lambda\mu} dx'_i \wedge d\overline{x}'_j + x'_i dz'_{\lambda\mu} \wedge d\overline{x}'_j\big) \\ & - \sum_{i,j,\lambda<\mu} \sqrt{a_\lambda - a_\mu}\; \overline{c}_{ij\lambda\mu} \big(\overline{z}'_{\lambda\mu} dx'_j \wedge d\overline{x}'_i + \overline{x}'_i dx'_j \wedge d\overline{z}'_{\lambda\mu}\big) + \mathrm{O}(|x'|^2 + |z'|^2) \quad . \end{aligned}$$

Omitting the primes in the coordinates x' , z' for simplicity, we see that the norms of the basis elements of TY with respect to ω are given modulo $\mathrm{O}(|x|^2+|z|^2)$ by

$$\begin{aligned}\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\rangle &\sim \delta_{ij}-\sum_{\lambda<\mu}\sqrt{a_\lambda-a_\mu}\,\big(c_{ij\lambda\mu}z_{\lambda\mu}+c_{ij\mu\lambda}\overline{z}_{\lambda\mu}\big)\quad,\\ \langle\frac{\partial}{\partial z_{\lambda\mu}},\frac{\partial}{\partial z_{\lambda'\mu'}}\rangle &\sim \delta_{\lambda\lambda'}\delta_{\mu\mu'}\quad,\\ \langle\frac{\partial}{\partial z_{\lambda\mu}},\frac{\partial}{\partial x_j}\rangle &\sim -\sum_i\sqrt{a_\lambda-a_\mu}c_{ij\lambda\mu}x_i\quad.\end{aligned}$$

By duality, we get

$$\begin{aligned}\langle dx_i,dx_j\rangle &\sim \delta_{ij}+\sum_{\lambda<\mu}\sqrt{a_\lambda-a_\mu}\,\big(\overline{c}_{ij\lambda\mu}\overline{z}_{\lambda\mu}+\overline{c}_{ij\mu\lambda}z_{\lambda\mu}\big)\quad,\\ \langle dz_{\lambda\mu},dz_{\lambda'\mu'}\rangle &\sim \delta_{\lambda\lambda'}\delta_{\mu\mu'}\quad,\\ \langle dz_{\lambda\mu},dx_j\rangle &\sim \sum_i\sqrt{a_\lambda-a_\mu}\,\overline{c}_{ij\lambda\mu}\overline{x}_i\quad.\end{aligned}$$

Taking the exterior derivative in the above estimates, we find that the Chern connection D on $\Omega^1_Y=T^\star Y$ is given in terms of the basis vectors dx_i , $dz_{\lambda\mu}$ by

$$\begin{aligned}D(dx_j) &= \sum_{i,\lambda<\mu}\sqrt{a_\lambda-a_\mu}\;c_{ij\lambda\mu}(dz_{\lambda\mu}\otimes dx_i+dx_i\otimes dz_{\lambda\mu})+\mathrm{O}(|x|+|z|)\quad,\\ D(dz_{\lambda\mu}) &= \quad 0\quad+\mathrm{O}(|x|+|z|)\quad.\end{aligned}$$

The subbundle $F=F^{p+1,N(s)}$ (resp. the quotient bundle $G=G^{p,p+N(s)}$) admits at y^0 the orthonormal basis

$$dx_I\wedge dz_J\ \text{ with }\ |I|+|J|=p+N(s)\ ,\ |I|\geq p+1\ \big(\text{resp. } |I|\leq p\ ,\ |J|=N(s)\big)\quad.$$

Let $v=\sum v_{I,J}\,dx_I\wedge dz_J$ be a C^∞ section of F . The $(1,0)$–form $\beta\wedge v$ is nothing else than the projection of Dv on $G=\Omega/F$. From this observation, one obtains the expression of β at y^0 :

$$\beta\wedge v=\sum_{i,j,\lambda<\mu}\sqrt{a_\lambda-a_\mu}\;c_{ij\lambda\mu}\,dx_i\otimes\big(dz_{\lambda\mu}\wedge\big(\frac{\partial}{\partial x_j}\;\lrcorner\;v\big)\big)\mod F\quad,\tag{4.11}$$

where $\xi\;\lrcorner\;v$ means contraction of the differential form v by the tangent vector ξ . In fact any differentiation of a factor dx_j in a term $D(v_{I,J}\,dx_I\wedge dz_J)$ decreases of one unity the partial degree $|I|$ when dx_j is differentiated into $c_{ij\lambda\mu}\,dx_i\otimes dz_{\lambda\mu}$. The corresponding part of the differential is thus in G if $|I|=p+1$. For every $(0,q+1)$–form $f=\sum f_{I,J,K,L}\,dx_I\wedge dz_J\wedge d\overline{x}_K\wedge d\overline{z}_L$ with values in $F(a)$, $|I|+|J|=p+N(s)$, $|I|\geq p+1$, $|K|+|L|=q+1$, we obtain consequently

$$\beta\;\lrcorner\;f=\sum_{i,j,\lambda<\mu}\sqrt{a_\lambda-a_\mu}\;c_{ij\lambda\mu}\frac{\partial}{\partial\overline{x}_i}\;\lrcorner\;\big(dz_{\lambda\mu}\wedge\big(\frac{\partial}{\partial x_j}\;\lrcorner\;f\big)\big)\mod F(a)\quad.\tag{4.12}$$

The only terms of f that contribute to the expression of $\beta\;\lrcorner\;f$ are those for which $|I|=p+1$ and $|J|=N(s)-1$. Let us write $g=\beta\;\lrcorner\;f$ under the form

$$g=\sum g_{I',J',K',L'}\,dx_{I'}\wedge dz_{J'}\wedge d\overline{x}_{K'}\wedge d\overline{z}_{L'}\quad,$$

where $|I'| = p$, $|J'| = N(s)$, $|K'| + |L'| = q$. Formula (4.12) implies

$$g_{I',J',K',L'} = \sum_{i,j,\lambda<\mu} \pm\sqrt{a_\lambda - a_\mu}\, c_{ij\lambda\mu}\, f_{jI',J'\setminus\{\lambda\mu\},iK',L'} \quad ,$$

$$|g_{I',J',K',L'}|^2 \le \Big(\sum_{i,j,\lambda<\mu} (a_\lambda - a_\mu)|c_{ij\lambda\mu}|^2\Big) \sum_{i,j,\lambda<\mu} |f_{jI',J'\setminus\{\lambda\mu\},iK',L'}|^2 \quad ,$$

and $\sum_{I',J',K',L'} \sum_{i,j,\lambda<\mu} |f_{jI',J'\setminus\{\lambda\mu\},iK',L'}|^2 \le (p+1)(q+1) \sum_{I,J,K,L} |f_{I,J,K,L}|^2$. We obtain therefore the inequality

$$(4.13) \qquad |\beta \lrcorner f|^2 \le (p+1)(q+1)\Big(\sum_{i,j,\lambda<\mu} (a_\lambda - a_\mu)|c_{ij\lambda\mu}|^2\Big)\, |f|^2 \quad .$$

The main point now is to find an estimate of the sum $\sum_{i,j,\lambda<\mu}(a_\lambda - a_\mu)|c_{ij\lambda\mu}|^2$ under condition (4.10).

LEMMA 4.14. — *Let $(h_{\lambda\mu})_{1\le\lambda,\mu\le r}$ be a semi-positive hermitian matrix and let $\alpha_1 \le \ldots \le \alpha_r$ be real numbers. Then*

$$\sum_{\lambda<\mu}(\alpha_\mu - \alpha_\lambda)|h_{\lambda\mu}|^2 \le \frac{1}{4}(\alpha_r - \alpha_1)\Big(\sum_\lambda h_{\lambda\lambda}\Big)^2 \quad .$$

Proof. — Use Cauchy-Schwarz inequality $|h_{\lambda\mu}|^2 \le h_{\lambda\lambda}h_{\mu\mu}$ and take $t_\lambda = h_{\lambda\lambda}$ in the identity

$$\frac{1}{4}(\alpha_r - \alpha_1)\Big(\sum_\lambda t_\lambda\Big)^2 - \sum_{\lambda<\mu}(\alpha_\mu - \alpha_\lambda)\, t_\lambda t_\mu$$
$$= \frac{1}{4} \sum_{1\le\lambda<r} (\alpha_{\lambda+1} - \alpha_\lambda)(t_1 + \ldots + t_\lambda - t_{\lambda+1} - \ldots - t_r)^2 \ge 0 \quad . \ \blacksquare$$

LEMMA 4.15. — *Under condition (4.10) one has*

$$\sum_{i,j,\lambda<\mu} (a_\lambda - a_\mu)|c_{ij\lambda\mu}|^2 \le \frac{1}{4}n(n+1)\Big(\frac{1}{a_r} - \frac{1}{a_1}\Big) \quad .$$

Proof. — Let us apply lemma 4.14 to

$$h_{\lambda\mu} = \sqrt{a_\lambda a_\mu} \sum_{i,j} c_{ij\lambda\mu} t_i \bar{t}_j \quad , \quad \alpha_\lambda = \frac{1}{a_\lambda} \quad ,$$

where $t = (t_1, \ldots, t_n)$ are arbitrary complex numbers. The Griffiths semi-positivity assumption on $c(E)$ means that $(h_{\lambda\mu})$ is semi-positive for all t . We get

$$\sum_\lambda h_{\lambda\lambda} = \sum_{i,j,\lambda} a_\lambda c_{ij\lambda\lambda} t_i \bar{t}_j \le |t|^2$$

by condition (4.10), thus

$$(4.16) \qquad \sum_{\lambda<\mu}(a_\lambda - a_\mu)\Big|\sum_{i,j} c_{ij\lambda\mu} t_i \bar{t}_j\Big|^2 \le \frac{1}{4}\Big(\frac{1}{a_r} - \frac{1}{a_1}\Big)|t|^4 \quad .$$

Apply now inequality (4.16) to $t = (e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbb{T}^n$ and integrate the result over $\mathbb{T}^n$. Parseval's identity for Fourier series yields

$$\sum_{\lambda<\mu}(a_\lambda - a_\mu)\Big(\sum_{i\neq j}|c_{ij\lambda\mu}|^2 + \big|\sum_i c_{ii\lambda\mu}\big|^2\Big) \leq \frac{1}{4}\Big(\frac{1}{a_r} - \frac{1}{a_1}\Big)n^2 \quad .$$

Inequality (4.16) applied to each vector of the standard basis of $\mathbb{C}^n$ yields in the same way

$$\sum_{\lambda<\mu}(a_\lambda - a_\mu)|c_{ii\lambda\mu}|^2 \leq \frac{1}{4}\Big(\frac{1}{a_r} - \frac{1}{a_1}\Big)$$

for all i , and lemma 4.15 follows. ■

Combining inequality (4.13) with lemma 4.15 we get

$$|\beta \lrcorner f|^2 < \frac{1}{4a_r}\, n(n+1)(p+1)(q+1)|f|^2 \quad ,$$

and using criterion (4.5) we see that $H^{q+1}(Y, F \otimes Q_s^a \otimes \pi^\star L) = 0$ for

$$a_r \geq \frac{n(n+1)(p+1)(q+1)}{4(p+q-n+1)} \quad , \quad p+q \geq n \quad .$$

The proof of the main theorem is therefore achieved.

5. On the Borel-Le Potier spectral sequence.

Denote as before $\pi : Y = M_s(E) \longrightarrow X$ the projection. To every integer t and every coherent analytic sheaf $\mathcal{S}$ on Y , one may associate the complex

$$D'' \ : \ \mathcal{K}^q = \Gamma\big(Y, \mathcal{C}^\infty_Y(\Omega^t_Y \otimes \overline{\Omega}^q_Y) \otimes_{\mathcal{O}_Y} \mathcal{S}\big) \longrightarrow \mathcal{K}^{q+1}$$

of C^∞–differential forms of type (t,q) with values in $\mathcal{S}$. This Dolbeault complex is filtered by the decreasing sequence of subcomplexes

$$D'' \ : \ \mathcal{K}^q_p = \Gamma\big(Y, \mathcal{C}^\infty_Y(F^p(\Omega^t_Y) \otimes \overline{\Omega}^q_Y) \otimes_{\mathcal{O}_Y} \mathcal{S}\big) \longrightarrow \mathcal{K}^{q+1}_p \quad .$$

This gives rise to a spectral sequence which we shall name after Borel and Le Potier, whose E_0 , E_1 terms are

$$(5.1) \qquad \begin{cases} E_0^{p,q-p} = \Gamma\big(Y, \mathcal{C}^\infty_Y(G^{p,t} \otimes \overline{\Omega}^q_Y) \otimes_{\mathcal{O}_Y} \mathcal{S}\big) \\ E_1^{p,q-p} = H^q(Y, G^{p,t} \otimes \mathcal{S}) \quad . \end{cases}$$

The limit term $E_\infty^{p,q-p}$ is the p–graded module corresponding to the filtration of $H^q(\mathcal{K}^\bullet) = H^q(Y, \Omega^t_Y \otimes \mathcal{S})$ by the canonical images of the groups $H^q(\mathcal{K}^\bullet_p)$. Assume that the spectral sequence degenerates in E_2 , *i.e.* $d_r : E_r^{p,q-p} \to E_r^{p+r,q+1-(p+r)}$ is zero for all $r \geq 2$ (by Peternell, Le Potier and Schneider [12], the spectral sequence does not degenerate in general in E_1). Then $E_2^{p,q-p} = E_\infty^{p,q-p}$. This equality means that the q–th cohomology group of the E_1–complex

$$d_1 \ : \ H^q(Y, G^{p,t} \otimes \mathcal{S}) \longrightarrow H^{q+1}(Y, G^{p+1,t} \otimes \mathcal{S})$$

is the p-graded module corresponding to a filtration of $H^q(Y, \Omega^t_Y \otimes \mathcal{S})$. By Kodaira-Akizuki-Nakano, we get therefore :

Proposition 5.1. — *Assume that E is ample and $L \geq 0$, or $E \geq 0$ and L ample, and that the E_2–degeneracy occurs for the ample invertible sheaf $\mathcal{S} = Q_s^a \otimes \pi^\star L$ on Y . Then the complex*

$$d_1 \ : \ H^q(Y, G^{p,t} \otimes Q_s^a \otimes \pi^\star L) \longrightarrow H^{q+1}(Y, G^{p+1,t} \otimes Q_s^a \otimes \pi^\star L)$$

is exact in degree $q \geq n + N(s) + 1 - t$.

This result would be a considerable help for the proof of vanishing theorems. For example, it is shown in [5] that the main vanishing theorem would be true with $l \geq r - 1 + \min\{n-p, n-q\}$.

Since d_1 is the coboundary operator associated to the exact sequence $0 \longrightarrow G^{p+1,t} \longrightarrow F^{p,t}/F^{p+2,t} \longrightarrow G^{p,t} \longrightarrow 0$, it is easy to see that $d_1 = -\beta^\star \wedge \bullet$ where $\beta^\star$ is the D''–closed (0,1)–form of §4, reinterpreted as a (0,1)–section of $\mathrm{Hom}(G^{p,t}, G^{p+1,t})$. Our hope is that the E_2–degeneracy can be proved in all cases by a suitable deepening of the analytic method of §4.

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