Compact complex manifolds whose tangent bundles satisfy numerical effectivity properties

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Dedicated to Prof. M.S. Narasimhan and C.S. Seshadri on their sixtieth birthday

0. Introduction

A compact Riemann surface always has a hermitian metric with constant curvature, in particular the curvature sign can be taken to be constant: the negative sign corresponds to curves of general type (genus ≥ 2), while the case of zero curvature corresponds to elliptic curves (genus 1), positive curvature being obtained only for $\mathbb{P}^1$ (genus 0). In higher dimensions the situation is much more subtle and it has been a long standing conjecture due to Frankel to characterize $\mathbb{P}_n$ as the only compact Kähler manifold with positive holomorphic bisectional curvature. Hartshorne strengthened Frankel's conjecture and asserted that $\mathbb{P}_n$ is the only compact complex manifold with ample tangent bundle. In his famous paper [Mo79], Mori solved Hartshorne's conjecture by using characteristic p methods. Around the same time Siu and Yau [SY80] gave an analytic proof of the Frankel conjecture. Combining algebraic and analytic tools Mok [Mk88] classified all compact Kähler manifolds with semi-positive holomorphic bisectional curvature.

From the point of view of algebraic geometry, it is natural to consider the class of projective manifolds X whose tangent bundle is numerically effective (nef). This has been done by Campana and Peternell [CP91] and –in case of dimension 3– by Zheng [Zh90]. In particular, a complete classification is obtained for dimension at most three.

The main purpose of this work is to investigate compact (most often Kähler) manifolds with nef tangent or anticanonical bundles in arbitrary dimension. We first discuss some basic properties of nef vector bundles which will be needed in the sequel in the general context of compact complex manifolds. We refer to [DPS91] and [DPS92] for detailed proofs. Instead, we put here the emphasis on some unsolved questions.

1. Numerically effective vector bundles

In algebraic geometry a powerful and flexible notion of semi-positivity is *numerical effectivity* ("nefness"). We will explain here how to extend this notion to arbitrary compact complex manifolds.

DEFINITION 1.1. — *A line bundle L on a projective manifold X is said to be numerically effective (nef for short) if $L \cdot C \geq 0$ for all compact curves $C \subset X$.*

It is clear that a line bundle with semi-positive curvature is nef. The converse had been conjectured by Fujita [Fu83]. Unfortunately this is not true, a simple counterexample can be obtained as follows:

Example 1.2. — Let Γ be an elliptic curve and let E be a rank 2 vector bundle over Γ which is a non split extension of $\mathcal{O}$ by $\mathcal{O}$; such a bundle E can be described as the locally constant vector bundle over Γ whose monodromy is given by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

associated to a pair of generators of $\pi_1(\Gamma)$. We take $L = \mathcal{O}_E(1)$ over the ruled surface $X = \mathbb{P}(E)$. Then L is nef and it can be checked that, up to a positive constant factor, there is only one (possibly singular) hermitian metric on L with semi-positive curvature; this metric is unfortunately singular and has logarithmic poles along a curve. Thus L cannot be semi-positive for any smooth hermitian metric. □

DEFINITION 1.3. — *A vector bundle E is called nef if the line bundle $\mathcal{O}_E(1)$ is nef on $\mathbb{P}(E)$ (= projectivized bundle of hyperplanes in the fibres of E).*

Again it is clear that a vector bundle E which admits a metric with semi-positive curvature (in the sense of Griffiths) is nef. A compact Kähler manifold X having semi-positive holomorphic bisectional curvature has by definition a tangent bundle TX with semi-positive curvature. Again the converse does not hold. One difficulty in carrying over the algebraic definition of nefness to the Kähler case is the possible lack of curves. This is overcome by the following:

DEFINITION 1.4. — *Let X be a compact complex manifold with a fixed hermitian metric ω. A line bundle L over X is nef if for every $\varepsilon > 0$ there exists a smooth hermitian metric h_ε on L such that the curvature satisfies*

$$\Theta_{h_\varepsilon} \geq -\varepsilon\, \omega\, .$$

This means that the curvature of L can have an arbitrarily small negative part. Clearly a nef line bundle L satisfies $L \cdot C \geq 0$ for all curves $C \subset X$, but the converse is not true (X may have no curves at all, as is the case for instance for generic complex tori). For projective algebraic X both notions coincide; this is an easy consequence of Seshadri's ampleness criterion: take L to be a nef line bundle in the sense of Def. 1 and let A be an ample line bundle; then $L^{\otimes k} \otimes A$ is ample for every integer k and thus L has a smooth hermitian metric with curvature form $\Theta(L) \geq -\frac{1}{k}\Theta(A)$.

Definition 1.3 can still be used to define the notion of nef vector bundles over arbitrary compact manifolds. If (E, h) is a hermitian vector bundle, recall that the Chern curvature tensor

$$\Theta_h(E) = \frac{i}{2\pi} D^2_{E,h} = i \sum_{\substack{1 \leq j,k \leq n \\ 1 \leq \lambda,\mu \leq r}} a_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^\star_\lambda \otimes e_\mu$$

is a hermitian (1,1)-form with values in $\mathrm{Hom}(E, E)$. We say that (E, h) is *semi-positive in Griffiths' sense* [Gr69] and write $\Theta_h(E) \geq 0$ if $\Theta_h(E)(\xi \otimes t) = \sum a_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda \overline{v}_\mu \geq 0$ for every $\xi \in T_xX$, $v \in E_x$, $x \in X$. We write $\Theta_h(E) > 0$ in case there is strict inequality for $\xi \neq 0, v \neq 0$. Numerical effectivity can then be characterized by the following differential geometric criterion (see [De91]).

Criterion 1.5. — *Let ω be a fixed hermitian metric on X. A vector bundle E on X is nef if and only if there is a sequence of hermitian metrics h_m on S^mE and a sequence ε_m of positive numbers decreasing to 0 such that*

$$\Theta_{h_m}(S^mE) \geq -m\varepsilon_m \omega \otimes \mathrm{Id}_{S^mE}$$

in the sense of Griffiths.

The main functorial properties of nef vector bundles are summarized in the following proposition.

Proposition 1.6. — *Let X be an arbitrary compact complex manifold and let E be a holomorphic vector bundle over X.*

(i) *If $f : Y \to X$ is a holomorphic map with equidimensional fibres, then E is nef if and only if $f^\star E$ nef.*

(ii) *Let $\Gamma^a E$ be the irreducible tensor representation of $Gl(E)$ of highest weight $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$, with $a_1 \geq \ldots \geq a_r \geq 0$. Then $\Gamma^a E$ is nef. In particular, all symmetric and exterior powers of E are nef.*

(iii) *let F be a holomorphic vector bundle over X. If E and F are nef, then $E \otimes F$ is nef.*

(iv) *If some symmetric power S^mE is nef ($m > 0$), then E is nef.*

(v) *Let $0 \to F \to E \to Q \to 0$ be an exact sequence of holomorphic vector bundles over X. Then*

(α) *E nef $\Rightarrow$ Q nef.*

(β) *F, Q nef $\Rightarrow$ E nef.*

(γ) *E nef, $(\det Q)^{-1}$ nef $\Rightarrow$ F nef.*

The proof of these properties in the general analytic context can be easily obtained by curvature computations. The arguments are parallel to those of the algebraic case and will therefore be omitted (see [Ha66] and [CP91] for that case). Another useful result which will be used over and over in the sequel is

PROPOSITION 1.7. — *Let E be a nef vector bundle over a connected compact n-fold X and let $\sigma \in H^0(X, E^\star)$ be a non zero section. Then σ does not vanish anywhere.*

Proof. We merely observe that if h_m is a sequence of hermitian metrics on $S^m E$ as in criterion 5, then

$$T_m = \frac{i}{\pi}\partial\overline{\partial}\frac{1}{m}\log||\sigma^m||_{h_m}$$

has zero $\partial\overline{\partial}$-cohomology class and satisfies $T_m \geq -\varepsilon_m\omega$. It follows that T_m converges to a weak limit $T \geq 0$ with zero cohomology class. Thus $T = i\partial\overline{\partial}\varphi$ for some global plurisubharmonic function φ on X. By the maximum principle this implies $T = 0$. However, if σ vanishes at some point x, then all T_m have Lelong number ≥ 1 at x. Therefore so has T, contradiction. □

One of our key results is a characterization of vector bundles E which are numerically flat, i.e. such that both E and $E^\star$ are nef.

THEOREM 1.8. — *Suppose that X is Kähler. Then a holomorphic vector bundle E over X is numerically flat iff E admits a filtration*

$$\{0\} = E_0 \subset E_1 \subset \ldots \subset E_p = E$$

by vector subbundles such that the quotients E_k/E_{k-1} are hermitian flat, i.e. given by unitary representations $\pi_1(X) \to U(r_k)$.

Sketch of proof. — It is clear by 1.6 (v) that every vector bundle which is filtrated with hermitian flat quotients is nef as well as its dual. Conversely, suppose that E is numerically flat. This assumption implies $c_1(E) = 0$. Fix a Kähler metric ω. If E is ω-stable, then E is Hermite-Einstein by the Uhlenbeck-Yau theorem [UY86]. Moreover we have $0 \leq c_2(E) \leq c_1(E)^2$ by Theorem 1.9 below, so $c_2(E) = 0$. Kobayashi's flatness criterion derived from Lübke's inequality on Chern classes then implies that E is hermitian flat. Now suppose that E is unstable and take $\mathcal{F} \subset \mathcal{O}(E)$ to be a destabilizing subsheaf of minimal rank p. We then have by definition $c_1(\mathcal{F}) = c_1(\det\mathcal{F}) = 0$ and the morphism $\det\mathcal{F} \to \Lambda^p E$ cannot have any zero (otherwise the zeroes would contribute to give a semi-negative non zero curvature current on the line bundle $\det\mathcal{F}$, contradiction). This implies easily that $\mathcal{F}$ is locally free, and we infer that $\mathcal{F}$ is also numerically flat. Since $\mathcal{F}$ is stable by definition, $\mathcal{F}$ must be hermitian flat. We set $E_1 = \mathcal{F}$, observe that $E' = E/E_1$ is again numerically flat and proceed by induction on the rank. □

Another key point, which has been indeed used in the above proof, is the fact that the Fulton-Lazarsfeld inequalities [FL83] for Chern classes of ample vector bundles still hold for nef vector bundles over compact Kähler manifolds:

Theorem 1.9. — *Let (X,ω) be a compact Kähler manifold and let E be a nef vector bundle on X. Then for all positive polynomials P the cohomology class $P(c(E))$ is numerically positive, that is, $\int_Y P(c(E))\wedge\omega^k \ge 0$ for any k and any subvariety Y of X.*

By a positive polynomial in the Chern classes, we mean as usual a homogeneous weighted polynomial $P(c_1,\ldots,c_r)$ with $\deg c_i = 2i$, such that P is a positive integral combination of Schur polynomials:

$$P_a(c) = \det(c_{a_i-i+j})_{1\le i,j\le r}, \qquad r\ge a_1\ge a_2\ge\ldots\ge a_r\ge 0$$

(by convention $c_0 = 1$ and $c_i = 0$ if $i \ne [0,r]$, $r = \operatorname{rank} E$). The proof of Theorem 1.9 is based essentially on the same arguments as the original proof of [FL83] for the ample case: the starting point is the nonnegativity of all Chern classes $c_k(E)$ (Bloch-Gieseker [BG71]); the general case then follows from a formula of Schubert calculus known as the Kempf-Laksov formula [KL74], which expresses any Schur polynomial $P_a(c(E))$ as a Chern class $c_k(F_a)$ of some related vector bundle F_a. The only change occurs in the proof of Gieseker's result, where the Hard Lefschetz theorem is needed for arbitrary Kähler metrics instead of hyperplane sections (fortunately enough, the technique then gets simplified, covering tricks being eliminated). Since $c_1c_{k-1}-c_k$ is a Schur polynomial, we get by induction

$$0\le c_k(E)\le c_1(E)^k \quad \text{for all } k.$$

Therefore all Chern monomials are bounded above by corresponding powers $c_1(E)^k$ of the same degree, and we infer:

Corollary 1.10. — *If E is nef and $c_1(E)^n = 0$, $n = \dim X$, then all Chern polynomials $P(c(E))$ of degree $2n$ vanish.*

2. Compact Kähler manifolds with nef anticanonical line bundle

Compact Kähler manifolds with zero or semi-positive Ricci curvature have been investigated by various authors (cf. [Ca57], [Ko61], [Li67, 71, 72], [Bo74a, b], [Be83], [Ko81] and [Kr86]). The purpose of this section is to discuss the following two conjectures.

Conjecture 2.1. — *Let X be a compact Kähler manifold with numerically effective anticanonical bundle K_X^{-1}. Then the fundamental group $\pi_1(X)$ has polynomial growth.*

Conjecture 2.2. — *Let X be a compact Kähler manifold with K_X^{-1} numerically effective. Then the Albanese map $\alpha : X\to \operatorname{Alb}(X)$ is a smooth fibration onto the Albanese torus.*

These conjectures are known to be true if K_X^{-1} is semi-positive. In both cases, the proof is based on differential geometric techniques (see e.g. [Bi63], [HK78] for Conjecture 2.1 and [Li71] for Conjecture 2.2). However, the methods of proof are not so easy to carry over to the nef case. Our main contributions to these conjectures are derived from Theorem 2.3 below.

THEOREM 2.3. — *Let X be a compact Kähler manifold with K_X^{-1} nef. Then $\pi_1(X)$ is a group of subexponential growth.*

The proof actually gives the following additional fact (this was already known before, see [Bi63]).

COROLLARY 2.4. — *If moreover $-K_X$ is hermitian semi-positive, then $\pi_1(X)$ has polynomial growth of degree $\le 2\dim X$, in particular $h^1(X, \mathcal{O}_X) \le \dim X$.*

As noticed by F. Campana (private communication), Theorem 2.3 also implies the following consequences.

COROLLARY 2.5. — *Let X be a compact Kähler manifold with K_X^{-1} nef. Let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map and set $n = \dim X$, $d = \dim \alpha(X)$. If $d = 0, 1$ or n, then α is surjective. The same is true if $d = n-1$ and if X is projective algebraic.*

COROLLARY 2.6. — *Let X be a Kähler surface or a projective 3-fold with K_X^{-1} nef. Then the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is surjective.*

We now explain the main ideas required in the proof of Theorem 2.3. If G is a finitely generated group with generators $g_1, \ldots, g_p$, we denote by $N(k)$ the number of elements $\gamma \in G$ which can be written as words

$$\gamma = g_{i_1}^{\varepsilon_1} \ldots g_{i_k}^{\varepsilon_k}, \qquad \varepsilon_j = 0,\ 1 \ \text{ or } \ -1$$

of length $\le k$ in terms of the generators. The group G is said to have *subexponential growth* if for every $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that

$$N(k) \le C(\varepsilon) e^{\varepsilon k} \quad \text{ for } \ k \ge 0.$$

This notion is independent of the choice of generators. In the free group with two generators, we have $N(k) = 1 + 4(1 + 3 + 3^2 + ... + 3^{k-1}) = 2 \cdot 3^k - 1$, thus a group with subexponential growth cannot contain a non abelian free subgroup.

The first step consists in the construction of suitable Kähler metrics by means of the Calabi-Yau theorem [Y77]. Let ω be a fixed Kähler metric on X. Since K_X^{-1} is nef, for every $\varepsilon > 0$ there exists a smooth hermitian metric h_ε on K_X^{-1} such that

$$u_\varepsilon = \Theta_{h_\varepsilon}(K_X^{-1}) \ge -\varepsilon\omega.$$

By [Y77] and [Y78] there exists a unique Kähler metric ω_ε in the cohomology class $\{\omega\}$ such that

$$(+) \qquad \mathrm{Ricci}(\omega_\varepsilon) = -\varepsilon\omega_\varepsilon + \varepsilon\omega + u_\varepsilon.$$

In fact u_ε belongs to the Ricci class $c_1(K_X^{-1}) = c_1(X)$, hence so does the right hand side $-\varepsilon\omega_\varepsilon + \varepsilon\omega + u_\varepsilon$. In particular there exists a function f_ε such that

$$u_\varepsilon = \mathrm{Ricci}(\omega) + i\partial\overline{\partial} f_\varepsilon.$$

If we set $\omega_\varepsilon = \omega + i\partial\overline{\partial}\varphi$ (where φ depends on ε), equation (+) is equivalent to the Monge-Ampère equation

$$(++) \qquad \frac{(\omega + i\partial\overline{\partial}\varphi)^n}{\omega^n} = e^{\varepsilon\varphi - f_\varepsilon}$$

because

$$\begin{aligned} i\partial\overline{\partial}\log(\omega + i\partial\overline{\partial}\varphi)^n/\omega^n &= \mathrm{Ricci}(\omega) - \mathrm{Ricci}(\omega_\varepsilon) \\ &= \varepsilon(\omega_\varepsilon - \omega) + \mathrm{Ricci}(\omega) - u_\varepsilon \\ &= i\partial\overline{\partial}(\varepsilon\varphi - f_\varepsilon). \end{aligned}$$

It follows from the general results of [Y78] that (++) has a unique solution φ, thanks to the fact the right hand side of (++) is increasing in φ. Since $u_\varepsilon \geq -\varepsilon\omega$, equation (+) implies in particular that $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega$.

Now, recall the well-known differential geometric technique for bounding $N(k)$ (this technique has been explained to us in a very efficient way by S. Gallot). Let (M, g) be a compact Riemannian m-fold and let $E \subset \widetilde{M}$ be a fundamental domain for the action of $\pi_1(M)$ on the universal covering $\widetilde{M}$. Fix $a \in E$ and set $\beta = \operatorname{diam} E$. Since $\pi_1(M)$ acts isometrically on $\widetilde{M}$ with respect to the pull-back metric $\widetilde{g}$, we infer that

$$E_k = \bigcup_{\gamma\in\pi_1(M),\ \mathrm{length}(\gamma)\leq k} \gamma(E)$$

has volume equal to $N(k)\,\mathrm{Vol}(M)$ and is contained in the geodesic ball $B(a, \alpha k+\beta)$, where α is the maximum of the length of loops representing the generators g_j. Therefore

$$(*) \qquad N(k) \leq \frac{\mathrm{Vol}\big(B(a, \alpha k + \beta)\big)}{\mathrm{Vol}(M)}$$

and it is enough to bound the volume of geodesic balls in $\widetilde{M}$. For this we use the following fundamental inequality due to R. Bishop [Bi63], Heintze-Karcher [HK78] and M. Gage [Ga80].

Lemma 2.7. — *Let*

$$\Phi : T_a\widetilde{M} \to \widetilde{M}, \qquad \Phi(\zeta) = \exp_a(\zeta)$$

be the (geodesic) exponential map. Denote by

$$\Phi^* dV_g = a(t,\zeta)\, dt\, d\sigma(\zeta)$$

the expression of the volume element in spherical coordinates with $t \in \mathbb{R}_+$ and $\zeta \in S_a(1) =$ unit sphere in $T_a\widetilde{M}$. Suppose that $a(t,\zeta)$ does not vanish for

$t \in \,]0,\tau(\zeta)[\,$, with $\tau(\zeta) = +\infty$ or $a(\tau(\zeta),\zeta) = 0$. Then $b(t,\zeta) = a(t,\zeta)^{1/(m-1)}$ satisfies on $]0,\tau(\zeta)[$ the inequality

$$\frac{\partial^2}{\partial t^2}\,b(t,\zeta) + \frac{1}{m-1}\,\mathrm{Ricci}_g(v(t,\zeta),v(t,\zeta))b(t,\zeta) \le 0$$

where

$$v(t,\zeta) = \frac{d}{dt}\,\exp_a(t\zeta) \in S_{\Phi(t\zeta)}(1) \subset T_{\Phi(t\zeta)}\widetilde{M}.$$

If $\mathrm{Ricci}_g \ge -\varepsilon g$, we infer in particular

$$\frac{\partial^2 b}{\partial t^2} - \frac{\varepsilon}{m-1}\,b \le 0$$

and therefore $b(t,\zeta) \le \alpha^{-1}\sinh(\alpha t)$ with $\alpha = \sqrt{\varepsilon/(m-1)}$ (to check this, observe that $b(t,\zeta) = t + o(t)$ at 0 and that $\sinh(\alpha t)\,\partial b/\partial t - \alpha\cosh(\alpha t)\,b$ has a negative derivative). Now, every point $x \in B(a,r)$ can be joined to a by a minimal geodesic arc of length $< r$. Such a geodesic arc cannot contain any focal point (i.e. any critical value of Φ), except possibly at the end point x. It follows that $B(a,r)$ is the image by Φ of the open set

$$\Omega(r) = \big\{(t,\zeta) \in [0,r[\,\times S_a(1)\,;\, t < \tau(\zeta)\big\}.$$

Therefore

$$\mathrm{Vol}_g(B(a,r)) \le \int_{\Omega(r)} \Phi^* dV_g = \int_{\Omega(r)} b(t,\zeta)^{m-1} dt\, d\sigma(\zeta).$$

As $\alpha^{-1}\sinh(\alpha t) \le t\,e^{\alpha t}$, we get

$$(**) \qquad \mathrm{Vol}_g(B(a,r)) \le \int_{S_a(1)} d\sigma(\zeta) \int_0^r t^{m-1} e^{(m-1)\alpha t} dt \le v_m r^m e^{\sqrt{(m-1)\varepsilon}\,r}$$

where v_m is the volume of the unit ball in $\mathbb{R}^m$. □

In our application, the difficulty is that the metric $g = \omega_\varepsilon$ varies with ε as well as the constants $\alpha = \alpha_\varepsilon$, $\beta = \beta_\varepsilon$ in $(*)$, and $\alpha_\varepsilon\sqrt{(m-1)\varepsilon}$ need not converge to 0 as ε tends to 0. We overcome this difficulty by the following lemma.

Lemma 2.8. — *Let U_1, U_2 be compact subsets of $\widetilde{X}$. Then for every $\delta > 0$, there are closed subsets $U_{1,\varepsilon,\delta} \subset U_1$ and $U_{2,\varepsilon,\delta} \subset U_2$ with $\mathrm{Vol}_\omega(U_j \smallsetminus U_{j,\varepsilon,\delta}) < \delta$, such that any two points $x_1 \in U_{1,\varepsilon,\delta}$, $x_2 \in U_{2,\varepsilon,\delta}$ can be joined by a path of length $\le C\,\delta^{-1/2}$ with respect to ω_ε, where C is a constant independent of ε and δ.*

We will not explain the details. The basic observation is that

$$\int_X \omega_\varepsilon \wedge \omega^{n-1} = \int_X \omega^n$$

does not depend on ε, therefore $\|\omega_\varepsilon\|_{L^1(X)}$ is uniformly bounded. This is enough to imply the existence of sufficiently many paths of bounded length between random points taken in X (this is done for example by computing the average length of piecewise linear paths). □

We let U be a compact set containing the fundamental domain E, so large that $U^\circ \cap g_j(U^\circ) \neq \emptyset$ for each generator g_j. We apply Lemma 2.8 with $U_1 = U_2 = U$ and $\delta > 0$ fixed such that

$$\delta < \frac{1}{2}\mathrm{Vol}_\omega(E), \qquad \delta < \frac{1}{2}\mathrm{Vol}_\omega\big(U \cap g_j(U)\big).$$

We get $U_{\varepsilon,\delta} \subset U$ with $\mathrm{Vol}_\omega(U \smallsetminus U_{\varepsilon,\delta}) < \delta$ and $\mathrm{diam}_{\,\omega_\varepsilon}(U_{\varepsilon,\delta}) \le C\delta^{-1/2}$. The inequalities on volumes imply that $\mathrm{Vol}_\omega(U_{\varepsilon,\delta} \cap E) \ge \frac{1}{2}\mathrm{Vol}_\omega(E)$ and $U_{\varepsilon,\delta} \cap g_j(U_{\varepsilon,\delta}) \neq \emptyset$ for every j (note that all g_j preserve volumes). It is then clear that

$$W_{k,\varepsilon,\delta} := \bigcup_{\gamma \in \pi_1(X),\ \mathrm{length}(\gamma) \le k} \gamma(U_{\varepsilon,\delta})$$

satisfies

$$\mathrm{Vol}_\omega(W_{k,\varepsilon,\delta}) \ge N(k)\,\mathrm{Vol}_\omega(U_{\varepsilon,\delta} \cap E) \ge N(k)\,\frac{1}{2}\,\mathrm{Vol}_\omega(E) \quad \text{and}$$

$$\mathrm{diam}_{\,\omega_\varepsilon}(W_{k,\varepsilon,\delta}) \le k\,\mathrm{diam}_{\,\omega_\varepsilon} U_{\varepsilon,\delta} \le kC\delta^{-1/2}.$$

Since $m = \dim_{\mathbb{R}} X = 2n$, inequality $(**)$ implies

$$\mathrm{Vol}_{\omega_\varepsilon}(W_{k,\varepsilon,\delta}) \le \mathrm{Vol}_{\omega_\varepsilon}\big(B(a, kC\delta^{-1/2})\big) \le C_4 k^{2n} e^{C_5\sqrt{\varepsilon}\,k}.$$

Now X is compact, so there is a constant $C(\varepsilon) > 0$ such that $\omega^n \le C(\varepsilon)\omega_\varepsilon^n$. We conclude that

$$N(k) \le \frac{2\,\mathrm{Vol}_\omega(W_{k,\varepsilon,\delta})}{\mathrm{Vol}_\omega(E)} \le C_6 C(\varepsilon) k^{2n} e^{C_5\sqrt{\varepsilon}\,k}.$$

The proof of Theorem 2.3 is complete. □

Remark 2.9. — It is well known and easy to check that equation (++) implies

$$C(\varepsilon) \le \exp\Big(\max_X f_\varepsilon - \min_X f_\varepsilon\Big).$$

Therefore it is reasonable to expect that $C(\varepsilon)$ has polynomial growth in ε^{-1} ; this would imply that $\pi_1(X)$ has polynomial growth by taking $\varepsilon = k^{-2}$. When K_X^{-1} has a semipositive metric, we can even take $\varepsilon = 0$ and find a metric ω_0 with $\mathrm{Ricci}(\omega_0) = u_0 \ge 0$. This implies Corollary 2.4.

Proof of Corollary 2.5. — If $d = 0$, then by definition $H^0(X, \Omega_X^1) = 0$ and $\mathrm{Alb}(X) = \{0\}$.

If $d = n$, the Albanese map has generic rank n, so there exist holomorphic 1-forms $u_1, \ldots, u_n$ such that $u_1 \wedge \ldots \wedge u_n \not\equiv 0$. However $u_1 \wedge \ldots \wedge u_n$ is a section of K_X which has nef dual, so $u_1 \wedge \ldots \wedge u_n$ cannot vanish by Proposition 1.7 and K_X is trivial. Therefore $u_1 \wedge \ldots \hat{u}_k \ldots \wedge u_n \wedge v$ must be a constant for every holomorphic 1-form v and $(u_1, \ldots, u_n)$ is a basis of $H^0(X, \Omega_X^1)$. This implies $\dim A(X) = n$, hence α is surjective.

If $d = 1$, the image $C = \alpha(X)$ is a smooth curve. The genus g of C cannot be ≥ 2, otherwise $\pi_1(X)$ would be mapped onto a subgroup of finite index in $\pi_1(C)$, and thus would be of exponential growth, contradicting Theorem 2.3. Therefore C is an elliptic curve and is a subtorus of $\mathrm{Alb}(X)$. By the universal property of the Albanese map, this is possible only if $C = \mathrm{Alb}(X)$.

The case $d = n - 1$ is more subtle and uses Mori theory (this is why we have to assume X projective algebraic). We refer to [DPS92] for the details. □

3. Compact complex manifolds with nef tangent bundles

Several interesting classes of such manifolds are produced by the following simple observation.

PROPOSITION 3.1. — *Every homogeneous compact complex manifold has a nef tangent bundle.*

Indeed, if X is homogeneous, the Killing vector fields generate TX, so TX is a quotient of a trivial vector bundle. In particular, we get the following

EXAMPLES 3.2 (homogeneous case). —

(i) Rational homogeneous manifolds: $\mathbb{P}_n$, flag manifolds, quadrics Q_n
(all are Fano manifolds, i.e. projective algebraic with K_X^{-1} ample).

(ii) Tori $\mathbb{C}^n/\Lambda$
(Kähler, possibly non algebraic).

(iii) Hopf manifolds $\mathbb{C}^n \smallsetminus \{0\}/H$ where H is a discrete group of homotheties
(non Kähler for $n \geq 2$).

(iv) Iwasawa manifolds G/Λ where G is the group of unipotent upper triangular $p \times p$ matrices and Λ the subgroup of matrices with entries in the ring of integers of some imaginary quadratic field, e.g. $\mathbb{Z}[i]$
(non Kähler for $p \geq 3$, although TX is trivial). □

We must remark at this point that not all manifolds X with nef tangent bundles are homogeneous, the automorphism group may even be discrete:

EXAMPLE 3.3. — Let $\Gamma = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, $Im\,\tau > 0$, be an elliptic curve. Consider the quotient space $X = (\Gamma \times \Gamma \times \Gamma)/G$ where $G = \{1, g_1, g_2, g_1 g_2\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (z_1 + \tfrac{1}{2}, -z_2, -z_3), \\ g_2(z_1, z_2, z_3) &= (-z_1, z_2 + \tfrac{1}{2}, -z_3 + \tfrac{1}{2}), \\ g_1 g_2(z_1, z_2, z_3) &= (-z_1 + \tfrac{1}{2}, -z_2 + \tfrac{1}{2}, z_3 + \tfrac{1}{2}). \end{aligned}$$

Then G acts freely, so X is smooth. It is clear also that TX is nef (in fact TX is unitary flat). Since the pull-back of TX to $\Gamma \times \Gamma \times \Gamma$ is trivial, we easily conclude that TX has no sections, thanks to the change of signs in g_1, g_2, g_1g_2. Therefore the automorphism group $\mathrm{Aut}(X)$ is discrete. The same argument shows that $H^0(X, \Omega^1_X) = 0$.

EXAMPLE 3.4. — Let X be the ruled surface $\mathbb{P}(E)$ over the elliptic curve $\Gamma = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ defined in Example 1.2. Then the relative tangent bundle of $\mathbb{P}(E) \to \Gamma$ (=relative anticanonical line bundle) is $\pi^\star(\det E^\star) \otimes \mathcal{O}_E(2) \simeq \mathcal{O}_E(2)$ and $T\Gamma$ is trivial, so TX is nef. Moreover X is almost homogeneous, with automorphisms induced by

$$(x, z_1, z_2) \mapsto (x + a, z_1 + b, z_2), \quad (a, b) \in \mathbb{C}^2$$

and a single closed orbit equal to the curve $\{z_2 = 0\}$. Here, no finite étale cover of X can be homogeneous, otherwise $K_X^{-1} = \mathcal{O}_E(2)$ would be semi-positive. Observe that no power of K_X^{-1} is generated by sections, although K_X^{-1} is nef. □

Our main result is a structure theorem on the Albanese map of compact Kähler manifolds with nef tangent bundles.

MAIN THEOREM 3.5. — *Let X be a compact Kähler manifold with nef tangent bundle TX. Let $\widetilde{X}$ be a finite étale cover of X of maximum irregularity $q = q(\widetilde{X}) = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Then*

(i) $\pi_1(\widetilde{X}) \simeq \mathbb{Z}^{2q}$.

(ii) *The Albanese map $\alpha : \widetilde{X} \to A(\widetilde{X})$ is a smooth fibration over a q-dimensional torus with nef relative tangent bundle.*

(iii) *The fibres F of α are Fano manifolds with nef tangent bundles.*

Recall that a Fano manifold is by definition a compact complex manifold with ample anticanonical bundle K_X^{-1}. It is well known that Fano manifolds are always simply connected (Kobayashi [Ko61]). As a consequence we get

COROLLARY 3.6. — *With the assumtions of* 3.5, *the fundamental group $\pi_1(X)$ is an extension of a finite group by $\mathbb{Z}^{2q}$.*

In order to complete the classification of compact Kähler manifolds with nef tangent bundles (up to finite étale cover), a solution of the following two conjectures would be needed.

CONJECTURE 3.7 (Campana-Peternell [CP91]). — *Let X be a Fano manifold. Then X has a nef tangent bundle if and only if X is rational homogeneous.*

The evidence we have for Conjecture 3.7 is that it is true up to dimension 3. In dimension 3 there are more than 100 different types of Fano manifolds, but only five types have a nef tangent bundle, namely $\mathbb{P}_3$, Q_3 (quadric), $\mathbb{P}_1 \times \mathbb{P}_2$, $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ and the flag manifold $F_{1,2}$ of lines and planes in $\mathbb{C}^3$; all five are homogeneous.

A positive solution to Conjecture 3.7 would clarify the structure of fibers in the Albanese map of Theorem 3.5. To get a complete picture of the situation, we would still need to know how the fibers are deformed and glued together to yield a holomorphic family over the Albanese torus. We note that $K_{\widetilde{X}}^{-1}$ is relatively ample, thus for m large the fibers can be embedded in the projectivized bundle of the direct image bundle $\alpha_\star(K_{\widetilde{X}}^{-m})$.

Conjecture 3.8. — *In the situation of Theorem 3.5, all direct image bundles $\alpha_\star(K_{\widetilde{X}}^{-m})$ are numerically flat over the Albanese torus.*

It is easy to check that Conjecture 3.8 holds if the fibers of α are projective spaces or quadrics, but the general case seems to be much more involved. We now explain the main steps in the proof of Theorem 3.5. One of the key points is the following

Proposition 3.9. — *Let X be a compact Kähler n-fold with TX nef. Then*

(i) *If $c_1(X)^n > 0$, then X is a Fano manifold.*

(ii) *If $c_1(X)^n = 0$, then $\chi(\mathcal{O}_X) = 0$ and there exists a non zero holomorphic p-form, p suitable odd and a finite étale cover $\widetilde{X} \to X$ such that $q(\widetilde{X}) > 0$.*

Proof. We first check that every effective divisor D of X is nef. In fact, let $\sigma \in H^0(X, \mathcal{O}(D))$ be a section with divisor D. Then for k larger than the maximum vanishing order of σ on X, the k-jet section $J^k\sigma \in H^0(X, J^k\mathcal{O}(D))$ has no zeroes. Therefore, there is an injection $\mathcal{O} \to J^k\mathcal{O}(D)$ and a dual surjection

$$\big(J^k\mathcal{O}(D)\big)^\star \otimes \mathcal{O}(D) \to \mathcal{O}(D).$$

Now, $J^k\mathcal{O}(D)$ has a filtration whose graded bundle is $\bigoplus_{0\le p\le k} S^pT^\star X \otimes \mathcal{O}(D)$, so $(J^k\mathcal{O}(D))^\star \otimes \mathcal{O}(D)$ has a dual filtration with graded bundle $\bigoplus_{0\le p\le k} S^pTX$. By 1.6 (ii) and 1.6 (v)(β), we conclude that $(J^k\mathcal{O}(D))^\star \otimes \mathcal{O}(D)$ is nef, so its quotient $\mathcal{O}(D)$ is nef by 1.6 (v)(α).

Part (i) is based on the solution of the Grauert-Riemenschneider conjecture as proved in [De85]. Namely, $L = K_X^{-1} = \Lambda^n TX$ is nef and satisfies $c_1(L)^n > 0$, so L has Kodaira dimension n (holomorphic Morse inequalities are needed at that point because X is not supposed a priori to be algebraic). It follows that X is Moishezon, thus projective algebraic, and for $m > 0$ large we have $L^m = \mathcal{O}(D+A)$ with divisors D, A such that D is effective and A ample. Since D must be in fact nef, it follows that $L = K_X^{-1}$ is ample, as desired.

The most difficult part is (ii). Since $c_1(X)^n = 0$, Corollary 1.10 implies $\chi(\mathcal{O}_X) = 0$. By Hodge symmetry, we get $h^0(X, \Omega_X^p) = h^p(X, \mathcal{O}_X)$ and

$$\chi(\mathcal{O}_X) = \sum_{0\le p\le n} (-1)^p h^0(X, \Omega_X^p) = 0.$$

From this and the fact that $h^0(X, \mathcal{O}_X) = 1$, we infer the existence of a non zero p-form u for some suitable odd number p. Let

$$\sigma : \Lambda^{p-1}TX \longrightarrow \Omega_X^1$$

be the bundle morphism obtained by contracting $(p-1)$-vectors with u. For every $k>0$, the morphism $\Lambda^k\sigma$ can be viewed as a section of the bundle $\Lambda^k(\Lambda^{p-1}T^\star X)\otimes\Lambda^kT^\star X$ which has nef dual. Hence by Proposition 1.7 we know that $\Lambda^k\sigma$ is either identically zero or does not vanish. This means that σ has constant rank. Let E be the image of σ. Then E is a quotient bundle of $\Lambda^{p-1}TX$, so E is nef, and E is a subbundle of $\Omega^1_X=T^\star X$, so $E^\star$ is likewise nef. Theorem 1.8 implies the existence of a hermitian flat subbundle $E_1\subset E$. If E_1 would be trivial after pulling-back to some finite étale cover $\widetilde{X}$, we would get a trivial subbundle of $\Omega^1_{\widetilde{X}}$, hence $q(\widetilde{X})>0$ and the proposition would be proved. Otherwise E_1 is given by some infinite representation of $\pi_1(X)$ into some unitary group. Let Γ be the monodromy group (i.e. the image of $\pi_1(X)$ by the representation). We use a result of Tits [Ti72] according to which every subgroup of a linear group contains either a non abelian free subgroup or a solvable subgroup of finite index. The first case cannot occur by Theorem 2.?. In the second case, we may assume Γ solvable by taking some finite étale cover. We consider the series of derived groups

$$\Gamma\supset\Gamma_1\supset\ldots\supset\Gamma_N=\{0\}$$

and the largest index k such that Γ_k has finite index in Γ. Then the inverse image of Γ_k in $\pi_1(X)$ defines a finite étale cover $\widetilde{X}$ of X with infinite first homology group (the representation maps this group onto Γ_k/Γ_{k+1} which is infinite). Hence $q(\widetilde{X})>0$, as desired. □

Proof of the Main Theorem. — Let X be compact Kähler manifold with nef tangent bundle. Since a non zero holomorphic form $u\in H^0(X,\Omega^1_X)$ can never vanish by Proposition 1.7, it follows immediately that the Albanese map α has rank equal to $q(X)$ at every point, hence α is a submersion and $q(X)\le n$. Let $\widetilde{X}$ be a finite étale cover with maximum irregularity $q=q(\widetilde{X})$ (note that $\widetilde{X}$ also has a nef tangent bundle, so $q(\widetilde{X})\le n$). Let F denote the fibers of the Albanese map $\alpha:\widetilde{X}\to A(\widetilde{X})$. The relative tangent bundle exact sequence

$$0\to TF\to TX\xrightarrow{d\alpha}\alpha^\star TA(X)\to 0$$

in which $TA(X)$ is trivial shows by 1.6 (v)(γ) that TF is nef. Lemma 3.10 (iii) below implies that all finite étale covers $\widetilde{F}$ of F satisfy $q(\widetilde{F})=O$. Hence the fibers F must be Fano by Proposition 3.9 and the Main Theorem follows. □

Lemma 3.10. — *Let X,Y be compact Kähler manifolds and let $g:X\to Y$ be a smooth fibration with connected fibres. We let $q(X)$ be the irregularity of X and $\widetilde{q}(X)$ be the sup of the irregularity of all finite étale covers. If F denotes any fibre of g, then*

(i) $q(X)\le q(Y)+q(F)$,

(ii) $\widetilde{q}(X)\le\widetilde{q}(Y)+\widetilde{q}(F)$.

(iii) *Suppose that the boundary map $\pi_2(Y)\to\pi_1(F)$ is zero, that $\pi_1(F)$ contains an abelian subgroup of finite index and that Y contains a subvariety S with*

$\pi_1(S) \simeq \pi_1(Y)$, such that any two generic points in the universal covering $\widehat{S}$ can be joined through a chain of holomorphic images $\mathbb{C} \to \widehat{S}$. Then

$$\widetilde{q}(X) = \widetilde{q}(Y) + \widetilde{q}(F).$$

The proof is based on a use of the Leray spectral sequence and a study of the resulting monodromy on $H^1(F, \mathbb{C})$. Triviality of the monodromy is achieved in case (iii) because all Kähler deformations of tori over Y must be trivial. We refer the reader to [DPS91] for the details. In our application, Y is taken to be the Albanese torus, so assumption (iii) is satisfied with $S = Y$ ($\pi_1(F)$ contains an abelian subgroup of finite index thanks to Corollary 3.6, by using an induction on dimension).

4. Classification in dimension 2 and 3

By using the Kodaira classification of surfaces and the structure theorems of Section 3, it is not difficult to classify all Kähler surfaces with nef tangent bundles; except for tori, the Kähler classification is identical to the projective one. The projective case was already mentioned in [CP91] and [Zh90].

THEOREM 4.1. — *Let X be a smooth Kähler surface such that TX is nef. Then X is minimal and is exactly one of the surfaces in the following list:*

(i) *X is a torus;*

(ii) *X is hyperelliptic;*

(iii) $X = \mathbb{P}_2$;

(iv) $X = \mathbb{P}_1 \times \mathbb{P}_1$;

(v) *$X = \mathbb{P}(E)$, where E is a rank 2-vector bundle on an elliptic curve C with either*

  (α) *$E = \mathcal{O} \oplus L$, $L \in \mathrm{Pic}^0(C)$, or*

  (β) *E is given by a non split extension $0 \to \mathcal{O} \to E \to L \to 0$ with $L = \mathcal{O}$ or $\deg L = 1$.*

The list of non Kähler surfaces in the Kodaira classification is much smaller. It is then rather easy to check nefness in each case:

THEOREM 4.2. — *The smooth non Kähler compact complex surfaces with nef tangent bundles are precisely:*

(i) *Kodaira surfaces (that is, surfaces of Kodaira dimension 0 with $b_1(X)$ odd);*

(ii) *Hopf surfaces (that is, surfaces whose universal cover is $\mathbb{C}^2 \smallsetminus \{0\}$).*

A similar classification can be obtained for 3-dimensional compact Kähler manifolds.

THEOREM 7.1. — *Let X be a Kähler 3-fold. Then TX is nef if and only if X is up to finite étale cover one of manifolds in the following list:*

(i) $X = \mathbb{P}_3$;

(ii) $X = Q_3$, *the 3-dimensional quadric*;

(iii) $X = \mathbb{P}_1 \times \mathbb{P}_2$;

(iv) $X = \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$;

(v) $X = F_{1,2}$, *the flag manifold of subspaces of* $\mathbb{C}^3$;

(vi) $X = \mathbb{P}(E)$, *with E a numerically flat rank 3-bundle on an elliptic curve C* ;

(vii) $X = \mathbb{P}(E) \times_C \mathbb{P}(F)$, *with E, F numerically flat rank 2-bundles over an elliptic curve C* ;

(viii) $X = \mathbb{P}(E)$, *with E a numerically flat rank 2-bundle over a 2-dimensional complex torus*;

(ix) $X =$ *3-dimensional complex torus.*

The only non algebraic manifolds appear in classes (viii) and (ix) when the Albanese torus is not algebraic. Let us mention that the classification of projective 3-folds with nef tangent bundles was already carried out in [CP91] and [Zh90]. In addition to Theorem 3.5, the main ingredient is the classification of Fano 3-folds by Shokurov and Mori-Mukai. An inspection of the list yields the five first classes (i)-(v).

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(June 15, 1992)