Regularization of closed positive currents and Intersection Theory

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Abstract. — *Let X be a compact complex manifold and let T be a closed positive current of bidegree $(1,1)$ on X. It is shown that T is the weak limit of a sequence (T_k) of smooth closed real $(1,1)$-currents with small negative part. The negative part of the T_k 's can be bounded in terms of the Lelong numbers of T, once a lower bound for the curvature of the tangent bundle TX is known. Moreover, Kiselman's procedure for killing Lelong numbers of a plurisubharmonic function is extended to manifolds by an alternative method based on Hörmander's L^2 estimates for $\overline{\partial}$. These results are then applied to derive various results concerning divisors or intersection theory in the context of analytic geometry. Especially, we obtain a relation between effective and numerically effective divisors on arbitrary compact manifolds, and we show that every manifold X in the Fujiki class $\mathcal{C}$ with nef tangent bundle is Kähler. If D is an effective divisor in a Kähler manifold, we also obtain a general self-intersection inequality giving a bound of the degrees of the constant multiplicity strata of D, in terms of a polynomial in the cohomology class $\{D\} \in H^2(X, \mathbb{R})$.*

Contents

Key words: Closed positive current, plurisubharmonic function, Lelong number, L^2 estimates for $\overline{\partial}$, effective and numerically effective divisors, Fujiki class $\mathcal{C}$, intersection theory, self-intersection inequality.

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1. Introduction and main results

Let X be a compact complex manifold of dimension n. According to [Le 57], a current T of bidimension (p,p) $\big($that is, of bidegree $(n-p,n-p)\big)$ over X is said to be *positive* if $T\wedge i\alpha_1\wedge\overline{\alpha}_1\wedge\ldots\alpha_p\wedge\overline{\alpha}_p$ is a positive measure for all smooth $(1,0)$-forms $\alpha_1,\ldots,\alpha_p$ on X. We say that T is *almost positive* if there exists a smooth form v of bidegree $(n-p,n-p)$ such that $T+v\geq 0$. Similarly, a function φ on X is said to be *almost psh* if φ is locally equal to the sum of a psh (plurisubharmonic) function and of a smooth function; then the $(1,1)$-current $i\partial\overline{\partial}\varphi$ is almost positive; conversely, if a locally integrable function φ is such that $i\partial\overline{\partial}\varphi$ is almost positive, then φ is equal a.e. to an almost psh function. If T is closed and almost positive, the Lelong numbers (or density numbers) $\nu(T,x)$ are well defined, since the negative part always contributes for zero. For every $c>0$, we introduce the Lelong number upperlevel sets

$$E_c(T)=\big\{x\in X\,;\,\nu(T,x)\geq c\big\}.$$

A well-known theorem of Siu [Siu 74] asserts that $E_c(T)$ is an analytic subset of X. Our main result is the following approximation-regularization theorem:

Main Theorem 1.1. — *Let T be a closed almost positive $(1,1)$-current and let α be a smooth real $(1,1)$-form in the same $\partial\overline{\partial}$-cohomology class as T, i.e. $T=\alpha+\frac{i}{\pi}\partial\overline{\partial}\psi$ where ψ is an almost psh function. Let γ be a continuous real $(1,1)$-form such that $T\geq\gamma$. Suppose that $\mathcal{O}_{TX}(1)$ is equipped with a smooth hermitian metric such that the Chern curvature form satisfies*

$$c\big(\mathcal{O}_{TX}(1)\big)+\pi_X^\star u\geq 0$$

with $\pi_X:P(T^\star X)\to X$ and with some nonnegative smooth $(1,1)$-form u on X. Fix a hermitian metric ω on X. Then for every $c>0$, there is a sequence of closed almost positive $(1,1)$-currents $T_{c,k}=\alpha+\frac{i}{\pi}\partial\overline{\partial}\psi_{c,k}$ such that $\psi_{c,k}$ is smooth on $X\smallsetminus E_c(T)$ and decreases to ψ as k tends to $+\infty$ (in particular, the current $T_{c,k}$ is smooth on $X\smallsetminus E_c(T)$ and converges weakly to T on X), and such that

- (i) $T_{c,k}\geq\gamma-\min\{\lambda_k,c\}u-\varepsilon_k\omega$ *where:*
- (ii) $\lambda_k(x)$ *is a decreasing sequence of continuous functions on X such that* $\lim_{k\to+\infty}\lambda_k(x)=\nu(T,x)$ *at every point,*
- (iii) ε_k *is positive decreasing and* $\lim_{k\to+\infty}\varepsilon_k=0$,
- (iv) $\nu(T_{c,k},x)=\big(\nu(T,x)-c\big)_+$ *at every point $x\in X$.*

Here $\mathcal{O}_{TX}(1)$ is the tautological line bundle associated to the tangent bundle TX over the projectivized hyperplane bundle $P(T^\star X)$. Observe that the theorem gives in particular approximants $T_{c,k}$ which are smooth everywhere on X if we take $c>\max_{x\in X}\nu(T,x)$. Such a regularization result was already proved in [De 82] under somewhat stronger curvature hypotheses (namely, X has a Kähler metric for which the induced metric on $\mathcal{O}_{TX}(1)$ satisfies the above curvature assumption). Properties (i) and (ii) show that the presence of positive Lelong numbers is an obstruction to smoothing when no loss of positivity is admitted; the old regularization theorem of Richberg [Ri 68] precisely solved the special case when the potential ψ is finite and continuous (the Lelong numbers are then always zero). For a general choice of the constant c, the inequalities (iv) tell us that the procedure kills all Lelong numbers that are $\leq c$ and shifts all others by $-c$. Thus we obtain an analogue over arbitrary manifolds of Kiselman's procedure [Ki 78] for killing Lelong numbers (here we assumed X compact only for the sake of simplicity; our method can be easily extended to noncompact manifolds, but uniform estimates only hold on relatively compact subsets). Unfortunately, Kiselman's method, which depends on the Legendre transformation for psh functions, only works with the stronger curvature assumption $c(TX) + u \otimes \mathrm{Id}_{TX} \geq 0$ in the sense of Griffiths. Since this method is simpler and more natural, we first explain it in Section 2. With the $\mathcal{O}_{TX}(1)$ lower bound assumption (which is much better from the point of view of algebraic geometry), we have to use more complicated arguments based on a combination of different types of L^2 estimates for $\overline{\partial}$, due chronologically to Hörmander [Hö 66], Skoda [Sk 72b] and Ohsawa-Takegoshi [OT 87, Oh 88]. The proof occupies Sections 3,4,5. Hörmander's estimates are used in Section 4 to construct hermitian metrics on high symmetric powers S^mTX and $S^mT^\star X$, when starting from a metric on $\mathcal{O}_{TX}(1)$. The Ohsawa-Takegoshi estimates are needed to approximate psh functions locally by logarithms of holomorphic functions, while Skoda's L^2 estimates are used in the gluing process, to compare the size of high order jets of functions produced in different coordinate patches.

The Main Theorem can be applied in many ways to get interesting results of intersection theory. Since we consider non necessarily algebraic manifolds, it is better to work with $\partial\overline{\partial}$-cohomology classes instead of De Rham classes: we define

$$H^{p,q}_{\partial\overline{\partial}}(X) = \big\{d\text{-closed } (p,q)\text{-forms}\big\}/\big\{\partial\overline{\partial}\text{-exact } (p,q)\text{-forms}\big\}. \tag{1.2}$$

By means of the Frölicher spectral sequence, it is easily shown that these cohomology groups are finite dimensional and can be computed either with spaces of smooth forms or with currents. In both cases, the quotient topology of $H^{p,q}_{\partial\overline{\partial}}(X)$ induced by the Fréchet topology of smooth forms or by the weak topology of currents is Hausdorff. Clearly $H^{\bullet}_{\partial\overline{\partial}}(X)$ is a bigraded algebra. This algebra is isomorphic to the usual De Rham cohomology algebra $H^\bullet(X,\mathbb{C})$ if X is Kähler or more generally if X is in the Fujiki class $\mathcal{C}$ of manifolds bimeromorphic to Kähler manifolds. Our first application concerns the relation between pseudo-effective and nef cohomology classes (nef = numerically effective).

Definition 1.3. — *A cohomology class $\{\alpha\} \in H^{1,1}_{\partial\overline{\partial}}(X)$ is said to be pseudo-effective if it can be represented by a closed positive $(1,1)$-current, and nef if for some fixed hermitian metric ω on X and for every $\varepsilon > 0$ there is a smooth form $\alpha_\varepsilon \in \{\alpha\}$ such that $\alpha_\varepsilon \geq -\varepsilon\omega$. We denote respectively by $H^{1,1}_{\mathrm{psef}}(X)$ and $H^{1,1}_{\mathrm{nef}}(X)$ the cones of pseudo-effective and nef cohomology classes.*

It is not difficult to see (cf. section 6) that $H^{1,1}_{\mathrm{psef}}(X)$ and $H^{1,1}_{\mathrm{nef}}(X)$ are closed convex cones in $H^{1,1}_{\partial\overline{\partial}}(X)$ and that $H^{1,1}_{\mathrm{nef}}(X) \subset H^{1,1}_{\mathrm{psef}}(X)$. When X is Kähler,

$H^{1,1}_{\mathrm{nef}}(X)$ coincides with the closure of the Kähler cone, that is, the convex cone generated by Kähler classes. Recall that the real Neron-Severi space $NS_{\mathbb{R}}(X)$ is the real subspace of $H^{1,1}_{\partial\overline{\partial}}(X)$ generated by integral classes, i.e. by Chern classes of holomorphic line bundles. The algebraic situation is then described by the following properties (these properties are often taken as a definition by algebraists; compare with [De 90a]).

PROPOSITION 1.4. — *When X is projective algebraic, $H^{1,1}_{\mathrm{psef}}(X) \cap NS_{\mathbb{R}}(X)$ is the closed convex cone generated by cohomology classes of effective divisors, and $H^{1,1}_{\mathrm{nef}}(X) \cap NS_{\mathbb{R}}(X)$ is the closed convex cone generated by ample divisors. This last cone is characterized by the following property: a class $\{\alpha\} \in NS_{\mathbb{R}}(X)$ is nef if and only if $\{\alpha\} \cdot C = \int_C \alpha \geq 0$ for every curve $C \subset X$.*

It is of course always true that $\{\alpha\} \in H^{1,1}_{\mathrm{nef}}(X)$ implies $\int_C \alpha \geq 0$ for every curve, but even for $\{\alpha\} \in NS_{\mathbb{R}}(X)$ the converse is not always true when X is not projective, because X might not have enough curves; similarly, a Kähler manifold X may have closed positive $(1,1)$-currents but no divisors; this is why we felt preferable to choose 1.3 as the general definition of pseudo-effectivity and nefness. It follows from Prop. 1.4 that the cone $H^{1,1}_{\mathrm{nef}}(X)$ is in general strictly smaller than $H^{1,1}_{\mathrm{psef}}(X)$, even on projective manifolds: for example, when X is the blow-up of a manifold X' at a point and $E \simeq \mathbb{P}^{n-1}$ is the exceptional divisor, then $\{E\} = c_1(\mathcal{O}(E))$ and $\mathcal{O}(E)_{\restriction E} \simeq \mathcal{O}(-1)$, so $\{E\} \cdot C = -\deg C < 0$ for every curve $C \subset E$; therefore $\{E\} \in H^{1,1}_{\mathrm{psef}}(X) \smallsetminus H^{1,1}_{\mathrm{nef}}(X)$. The Main Theorem gives the following interesting relation between pseudo-effective and nef classes.

COROLLARY 1.5. — *Suppose that there is a nef cohomology class $\{u\}$ in $H^{1,1}_{\partial\overline{\partial}}(X)$ such that $c_1\big(\mathcal{O}_{TX}(1)\big) + \pi_X^\star\{u\}$ is nef over $P(T^\star X)$. Then for any closed positive $(1,1)$-current T on X and $b_n = \inf\{c > 0\,; \operatorname{codim} E_c(T) \geq n\}$, the cohomology class $\{T\} + b_n\{u\}$ is nef. In particular, if TX is nef, we have $H^{1,1}_{\mathrm{nef}}(X) = H^{1,1}_{\mathrm{psef}}(X)$.*

By definition, we say that a vector bundle E is nef if $c_1\big(\mathcal{O}_E(1)\big)$ is nef over the projectivized bundle $P(E^\star)$ of hyperplanes of E. We refer to [DPS 91] for the basic properties of nef vector bundles over general compact complex manifolds. The following related result is an easy consequence of the Main Theorem.

COROLLARY 1.6. — *Let X be a compact complex manifold with TX nef. Then X is Kähler if and only if X is in the Fujiki class $\mathcal{C}$, and X is projective if and only if X is Moishezon.*

Finally, it is possible to extend the self-intersection inequality proved in [De 90b] to an arbitrary closed positive $(1,1)$-current T on a Kähler manifold X (in the original version, we had to suppose X projective algebraic and $\{T\}$ in $H^2(X,\mathbb{Z})$). To the current T we associate the sequence $0 = b_1 \leq \ldots \leq b_n \leq b_{n+1}$ of "*jumping values*" b_p such that the dimension of $E_c(T)$ drops by one unit when

c gets larger than b_p, namely $\operatorname{codim} E_c(T) = p$ when $c \in \,]b_p, b_{p+1}]$ (by this, we mean that all components have codimension $\geq p$, with equality for at least one of them). Let $(Z_{p,k})_{k\geq 1}$ be the collection of p-codimensional components of all sets $E_c(T)$, $c \in \,]b_p, b_{p+1}]$, and let $\nu_{p,k} = \min_{x\in Z_{p,k}} \nu(T,x) \in \,]b_p, b_{p+1}]$ be the generic Lelong number of T along $Z_{p,k}$. We are interested in the following question: is it possible to derive a bound for the degrees of the Lelong number strata $Z_{p,k}$ in terms of the cohomology class $\{T\} \in H^2_{DR}(X, \mathbb{R})$? The answer is affirmative:

THEOREM 1.7. — *Suppose that X is Kähler and that $\mathcal{O}_{TX}(1)$ has a hermitian metric such that $c\big(\mathcal{O}_{TX}(1)\big) + \pi_X^\star u \geq 0$, where u is a smooth closed nonnegative $(1,1)$-form. For each $p = 1, \ldots, n$, the De Rham cohomology class $(\{T\} + b_1\{u\}) \cdots (\{T\} + b_p\{u\})$ can be represented by a closed positive current Θ_p of bidegree (p,p) such that*

$$\Theta_p \geq \sum_{k\geq 1} (\nu_{p,k} - b_1) \ldots (\nu_{p,k} - b_p)\, [Z_{p,k}] + (T_{\mathrm{abc}} + b_1 u) \wedge \ldots \wedge (T_{\mathrm{abc}} + b_p u)$$

where $T_{\mathrm{abc}} \geq 0$ is the absolutely continuous part in the Lebesgue decomposition of the coefficients of T into absolutely continuous and singular measures. As a consequence, if ω is a Kähler metric on X and if $\{u\}$ is a nef cohomology class such that $c_1\big(\mathcal{O}_{TX}(1)\big) + \pi_X^\star\{u\}$ is nef, the degrees of the components $Z_{p,k}$ with respect to ω satisfy the estimate

$$\sum_{k=1}^{+\infty} (\nu_{p,k} - b_1) \ldots (\nu_{p,k} - b_p) \int_X [Z_{p,k}] \wedge \omega^{n-p} \leq \big(\{T\} + b_1\{u\}\big) \cdots \big(\{T\} + b_p\{u\}\big) \cdot \{\omega\}^{n-p}.$$

To give the flavour of inequality 1.7 in a simple case, suppose that X is a Kähler surface and that $T = [C]$ is the current of integration on an irreducible curve $C \subset X$. Then $b_1 = 0$, $b_2 = 1$ and b_3 is the maximum of the multiplicities $\nu_k \geq 2$ of the singular points of C. For $p = 2$, the inequality gives $\sum \nu_k(\nu_k - 1) \leq \{C\} \cdot (\{C\} + \{u\})$ (in that case, the inequality is well-known and is related to the genus formula for curves). The general proof follows essentially the same lines as the algebraic case considered in [De 90b] (this case was an essential ingredient in our proof of numerical criteria for very ample line bundles). The main idea is to kill the Lelong numbers of T up to the level b_j ; then the singularities of the resulting current T_j occur only in codimension j and the wedge product $T_1 \wedge \ldots \wedge T_p$ is well defined by means of Bedford-Taylor's definition of the Monge-Ampère operator and suitable integration by parts (see [De 91]).

The above results, especially corollaries 1.5 and 1.6, have been strongly motivated by our common work with Thomas Peternell and Michael Schneider on the classification of compact Kähler varieties with nef tangent bundle ([DPS 91], to appear). The author wishes to thank both of them for very fruitful discussions.

2. Kiselman's singularity attenuation technique

Let us first recall some basic facts concerning Chern connection and curvature. If E is a hermitian holomorphic vector bundle on a complex n-dimensional manifold M, we denote by D_E (or simply D when no confusion can occur) the Chern connection on E. We denote by $c(E) = \frac{i}{2\pi} D_E^2$ the associated Chern curvature tensor, which is a hermitian $(1,1)$-form with values in $\mathrm{Hom}(E,E)$. With respect to a local orthonormal frame $(e_\lambda)_{1\le\lambda\le r}$ of E and to local coordinates $(z_j)_{1\le j\le n}$ on M, such a form can be written

$$c(E) = i \sum c_{jk\lambda\mu}\, dz_j \wedge d\overline{z}_k \otimes e_\lambda^\star \otimes e_\mu$$

with $c_{kj\mu\lambda} = \overline{c}_{jk\lambda\mu}$ for $1 \le j,k \le n$ and $1 \le \lambda,\mu \le r$. As usual, we identify $c(E)$ with the hermitian form on $TM \otimes E$ defined by

$$\langle c(E)\bullet,\bullet\rangle = \sum c_{jk\lambda\mu}\, (dz_j \otimes e_\lambda^\star) \otimes (\overline{dz_k \otimes e_\mu^\star}), \tag{2.1}$$

and we say that $c(E)$ is *semipositive (resp. positive) in the sense of Griffiths* if $c(E)(t\otimes v\,,\, t\otimes v) \ge 0$ (resp. > 0) for all non zero vectors $t \in TM$, $v \in E$.

Now, let X be a compact n-dimensional complex manifold and let T be a closed almost positive current of bidegree $(1,1)$ on X. We select a smooth closed $(1,1)$-form α representing the same $\partial\overline{\partial}$-cohomology class as T and an almost psh function ψ on X such that $T = \alpha + \frac{i}{\pi}\partial\overline{\partial}\psi$. Such a decomposition exists even when X is non-Kähler, since we can always find an open covering (U_j) of X such that $T = \frac{i}{\pi}\partial\overline{\partial}\psi_j$ over U_j, and construct a global function $\psi = \sum \theta_j\psi_j$ by means of a partition of unity (θ_j) subordinate to U_j (observe that $\psi - \psi_k$ is smooth on U_k because all differences $\psi_j - \psi_k$ are smooth in the intersections $U_j \cap U_k$). By replacing T with $T - \alpha$ and γ with $\gamma - \alpha$ in the Main Theorem, we can suppose without loss of generality that $\{T\} = 0$, i.e. that $T = \frac{i}{\pi}\partial\overline{\partial}\psi$ with an almost psh function ψ on X. Thus we have to devise a singularity attenuation technique for almost psh functions. We first describe Kiselman's method [Ki 78]. An alternative method working under slightly more general hypotheses will be explained in section 5.

Local procedure. — To begin with, let φ be a psh function on an open set $U \subset \mathbb{C}^n$. Suppose that the trivial vector bundle $E = U \times \mathbb{C}^n$ is equipped with a smooth hermitian metric $||\xi||_z$ for $\xi \in \{z\} \times \mathbb{C}^n$ (later on, the hermitian structure of $U \times \mathbb{C}^n$ will come from a trivialization of $TX_{\restriction U}$). For $w \in \mathbb{C}$, we set

$$\Phi_\infty(z,w) = \sup_{||\xi||_z \le 1} \varphi(z + e^w\xi). \tag{2.2}$$

Clearly $\Phi_\infty(z,w)$ depends only on the real part $\mathrm{Re}\, w$ and is defined on the open set Ω of points $(z,w) \in U \times \mathbb{C}$ such that $\mathrm{Re}\, w < \log d_z(z, \partial U)$, where d_z denotes euclidean distance with respect to $||\ ||_z$. Moreover $\Phi_\infty(z,w)$ is a convex increasing function of $\mathrm{Re}\, w$ by the well known properties of plurisubharmonic functions. Next, for $(z,w) \in \Omega$, we introduce the *Legendre transform*

$$\Phi_c(z,w) = \inf_{t \le 0}\ \Phi_\infty(z, w+t) - ct, \qquad c > 0. \tag{2.3}$$

It is easy to see that the family (Φ_c) is increasing in c and that

$$\lim_{c\to 0}\Phi_c(z,w)=\varphi(z),\qquad \lim_{c\to+\infty}\Phi_c=\Phi_\infty. \tag{2.4}$$

When $||\xi||_z$ is taken to be a constant metric, we know by Kiselman [Ki 78] that Φ_∞ and Φ_c are plurisubharmonic functions of the pair (z,w), and that the Lelong numbers of $\Phi_c(\bullet,w)$ are given by

$$\nu\big(\Phi_c(\bullet,w),z\big)=\big(\nu(\varphi,z)-c\big)_+,\qquad \forall(z,w)\in\Omega. \tag{2.5}$$

In general, Φ_∞ is continuous on Ω and its right derivative $\partial\Phi_\infty(z,w)/\partial\mathrm{Re}\,w_+$ is upper semicontinuous; indeed, this partial derivative is the decreasing limit of $\big(\Phi_\infty(z,w+t)-\Phi_\infty(z,w)\big)/t$ as $t\downarrow 0_+$. It follows that Φ_c is continuous on $\Omega\smallsetminus(E_c(\varphi)\times\mathbb{C})$: in fact, we have $\nu(\varphi,z)=\lim_{t\to-\infty}\partial\Phi_\infty(z,t)/\partial t_+<c$ on every compact set $K\subset\Omega\smallsetminus(E_c(\varphi)\times\mathbb{C})$, so by the upper semicontinuity there is a constant t_0 such that $\partial\Phi_\infty(z,w+t)/\partial t_+<c$ for $(z,w)\in K$ and $t<t_0$. Therefore

$$\Phi_c(z,w)=\inf_{t_0\le t\le 0}\Phi_\infty(z,w+t)-ct\quad\text{ on } K,$$

and this infimum with compact range is continuous. Our first goal is to investigate the plurisubharmonicity of Φ_c when $||\xi||_z$ is a variable hermitian metric.

PROPOSITION 2.6. — *Suppose that the Chern curvature of the hermitian metric on $E=U\times\mathbb{C}^n$ satisfies $c(E)+u\otimes\mathrm{Id}_E\ge 0$ in the sense of Griffiths, for some nonnegative $(1,1)$-form u on X. Then*

(i) *For all $v\ge 0$, we have*

$$\Phi_c(z,w-v)\ge\Phi_c(z,w)-\min\Big\{\frac{\partial\Phi_\infty(z,w)}{\partial\mathrm{Re}\,w_-}\,,\,c\Big\}v\,;$$

(ii) *For $(\zeta,\eta)\in TU\times\mathbb{C}$ and $c\in\,]0,+\infty]$, the Hessian of Φ_c satisfies*

$$\frac{i}{\pi}\partial\overline{\partial}(\Phi_c)_{(z,w)}(\zeta,\eta)\ge-\min\Big\{\frac{\partial\Phi_\infty(z,w)}{\partial\mathrm{Re}\,w_+}\,,\,c\Big\}\,u_z(\zeta).$$

Proof. — (i) For $v\ge 0$ and $t\le 0$, we have

$$\Phi_\infty(z,w+t-v)\ge\Phi_\infty(z,w+t)-v\frac{\partial\Phi_\infty(z,w+t)}{\partial\mathrm{Re}\,w_-}$$

by convexity of $\Phi_\infty(z,w)$ in $\mathrm{Re}\,w$. As $\partial\Phi_\infty(z,w)/\partial\mathrm{Re}\,w_-$ is increasing in $\mathrm{Re}\,w$, the infimum of both sides minus ct gives

$$\Phi_c(z,w-v)\ge\Phi_c(z,w)-v\frac{\partial\Phi_\infty(z,w)}{\partial\mathrm{Re}\,w_-}.$$

On the other hand, the change of variables $t=t'+v$ yields

$$\Phi_c(z,w-v)\ge\inf_{t'\le -v}\Phi_\infty(z,w+t')-c(t'+v)\ge\Phi_c(z,w)-cv$$

and (i) follows. The proof of (ii) requires the following simple lemma.

LEMMA 2.7. — *Let $z_0 \in U$ be fixed and let v be the hermitian quadratic function with constant Hessian $\frac{i}{\pi}\partial\overline{\partial}v = u_{z_0}$. Then there is a holomorphic $n \times n$ matrix $g(z)$ near z_0 such that $g(z_0) = \mathrm{Id}$ and*

$$||g(z) \cdot \xi||_z \le ||\xi||_{z_0} e^{v(z-z_0)+O(|z-z_0|^3)}.$$

Proof. — Without loss of generality, we may suppose that $z_0 = 0$ and that $||\xi||_{z_0}$ is given by the unit hermitian matrix (otherwise take any orthonormal basis (e_λ) as the new basis of $\mathbb{C}^n$). We have a Taylor expansion

$$||\xi||_z^2 = \sum_{1\le\lambda\le n} |\xi_\lambda|^2 + \sum_{1\le j,\lambda,\mu\le n} (a_{j\lambda\mu} z_j + \overline{a}_{j\mu\lambda}\overline{z}_j)\xi_\lambda\overline{\xi}_\mu + O(|z|^2|\xi|^2)$$

at first order. We can get rid of the first order terms by setting

$$g_1(z) \cdot \xi = \xi - \sum_{j,\lambda,\mu} a_{j\lambda\mu} z_j \xi_\lambda \, e_\mu.$$

A simple calculation shows indeed that $||g_1(z) \cdot \xi||_z^2 = \sum |\xi_\lambda|^2 + O(|z|^2|\xi|^2)$. We then look at the second order terms in the Taylor expansion

$$\begin{aligned} ||g_1(z) \cdot \xi||_z^2 = \sum_\lambda |\xi_\lambda|^2 &+ \sum_{j,k,\lambda,\mu} (a_{jk\lambda\mu} z_j z_k + \overline{a}_{jk\mu\lambda}\overline{z}_j\overline{z}_k)\xi_\lambda\overline{\xi}_\mu \\ &+ \sum_{j,k,\lambda,\mu} b_{jk\lambda\mu} z_j\overline{z}_k\xi_\lambda\overline{\xi}_\mu + O(|z|^3|\xi|^2) \end{aligned}$$

and kill the coefficients $a_{jk\lambda\mu}$ by setting

$$g(z) \cdot \xi = g_1(z) \cdot \xi - \sum_{j,k,\lambda,\mu} a_{jk\lambda\mu} z_j z_k \xi_\lambda e_\mu.$$

The remaining coefficients $(b_{jk\lambda\mu})$ cannot be killed and are related to the curvature tensor $c(E)$ by $c_{jk\lambda\mu} = -\frac{1}{\pi} b_{jk\lambda\mu}$. If we write $v_{jk} = \partial^2 v/\partial z_j\partial\overline{z}_k(0)$, we have $u(0) = \frac{i}{\pi}\sum v_{jk} dz_j \wedge d\overline{z}_k$ and the curvature assumption $c(E) + u \otimes \mathrm{Id}_E \ge 0$ at the origin implies

$$\sum_{j,k,\lambda,\mu} -b_{jk\lambda\mu} z_j\overline{z}_k\xi_\lambda\overline{\xi}_\mu + \sum_{j,k} v_{jk} z_j\overline{z}_k \sum_\lambda |\xi_\lambda|^2 \ge 0.$$

Therefore we get

$$\begin{aligned} ||g(z) \cdot \xi||_z^2 &= \sum_\lambda |\xi_\lambda|^2 + \sum_{j,k,\lambda,\mu} b_{jk\lambda\mu} z_j\overline{z}_k\xi_\lambda\overline{\xi}_\mu + O(|z|^3|\xi|^2) \\ &\le ||\xi||_0^2\big(1 + v(z) + O(|z|^3)\big). \end{aligned}$$

The asserted inequality 2.7 is equivalent to this. □

Proof of 2.6 (ii). — Fix $(z_0, w_0) \in \Omega$ and an arbitrary quadratic function v such that $\frac{i}{\pi}\partial\overline{\partial}v > u_{z_0}$. Then Lemma 2.7 yields an invertible matrix $g(z)$ such that $||g(z)\cdot\xi||_z \le ||\xi||_{z_0} e^{v(z-z_0)}$ near z_0. A change of variable $\xi \mapsto g(z)\cdot\xi$ in (2.2) implies

$$\begin{aligned}\Phi_\infty(z,w) &= \sup_{||g(z)\cdot\xi||_z\le 1} \varphi(z+e^w g(z)\cdot\xi)\\ &\ge \sup_{||\xi||_{z_0}\le e^{-v(z-z_0)}} \varphi(z+e^w g(z)\cdot\xi) \qquad \text{near } (z_0,w_0),\end{aligned}$$

with the inequality being an equality for $z=z_0$. The function

$$h(z,w) = \sup_{||\xi||_{z_0}\le 1} \varphi(z+e^w g(z)\cdot\xi)$$

is plurisubharmonic and, if h_c is its Legendre transform, we obtain successively

$$\begin{aligned}\Phi_\infty(z,w) &\ge h\big(z,w-v(z-z_0)\big),\\ \Phi_c(z,w) &\ge h_c\big(z,w-v(z-z_0)\big)\end{aligned}$$

near (z_0,w_0), with equality for $z=z_0$. For small $(\zeta,\eta)\in TU\times\mathbb{C}$, we combine this with the mean value inequality to get

$$\begin{aligned}\int_0^{2\pi} \Phi_c(z_0+e^{i\theta}\zeta, w_0+e^{i\theta}\eta)\,\frac{d\theta}{2\pi} &\ge \int_0^{2\pi} h_c\big(z_0+e^{i\theta}\zeta, w_0+e^{i\theta}\eta - v(\zeta)\big)\,\frac{d\theta}{2\pi}\\ &\ge h_c\big(z_0, w_0-v(\zeta)\big) = \Phi_c\big(z_0,w_0-v(\zeta)\big).\end{aligned}$$

This inequality is still valid at points (z,w) near (z_0,w_0), provided that $u_z < \frac{i}{\pi}\partial\overline{\partial} v$ and that $|\zeta|^2+|\eta|^2 < r_{z,w}^2$ is small enough. By 2.6 (i), we get the inequality

$$\int_0^{2\pi} \Phi_c(z+e^{i\theta}\zeta, w+e^{i\theta}\eta)\,\frac{d\theta}{2\pi} \ge \Phi_c(z,w) - \min\Big\{\frac{\partial\Phi_\infty(z,w)}{\partial \mathrm{Re}\, w_+}\,,\, c\Big\} v(\zeta)$$

near (z_0,w_0). For $A > \partial\Phi_\infty(z_0,w_0)/\partial\mathrm{Re}\,w_+$, we still have $A>\partial\Phi_\infty(z,w)/\partial\mathrm{Re}\,w_+$ in a neighborhood by the upper semicontinuity, and we conclude that the function $\Phi_c(z,w)+\min\{A,c\}v(z)$ satisfies the mean value inequality near (z_0,w_0). Hence $\Phi_c(z,w)+\min\{A,c\}v(z)$ is psh near (z_0,w_0) and (ii) is proved. $\square$

Gluing process. — The next step is to describe a gluing process for the construction of global regularizations of almost psh functions. We suppose that X is equipped with a hermitian metric ω such that its Chern curvature satisfies $c(TX)+u\otimes\mathrm{Id}_{TX}\ge 0$ in the sense of Griffiths, where u is a smooth semipositive $(1,1)$-form on X. Let $T=\frac{i}{\pi}\partial\overline{\partial}\psi$ where ψ is an almost psh function on X, and let γ be a real continuous $(1,1)$-form such that $T\ge\gamma$. We select a finite covering (W_ν) of X with open coordinate charts. Given $\delta>0$, we take in each W_ν a maximal family of points with (coordinate) distance to the boundary $\ge 3\delta$ and mutual distance $\ge\delta$. In this way, we get for $\delta>0$ small a finite covering of X by open balls U'_j of radius δ, such that the concentric ball U_j of radius 2δ is relatively compact in the corresponding chart W_ν. Let $\tau_j : U_j \longrightarrow B(a_j,2\delta)$ be the isomorphism given by the coordinates of W_ν. Let $\varepsilon(\delta)$ be a modulus of continuity for γ on the sets U_j, such that $\lim_{\delta\to 0}\varepsilon(\delta)=0$ and $\gamma_x-\gamma_{x'}\le\frac12\varepsilon(\delta)\,\omega_x$ for all $x,x'\in U_j$. We denote by γ_j the $(1,1)$-form with constant coefficients on $B(a_j,2\delta)$ such that $\tau_j^\star\gamma_j$ coincides with $\gamma-\varepsilon(\delta)\,\omega$ at $\tau_j^{-1}(a_j)$. Then we have

$$0\le\gamma-\tau_j^\star\gamma_j\le 2\varepsilon(\delta)\,\omega \quad \text{on } U'_j \tag{2.8}$$

for $\delta > 0$ small. We set $\psi_j = \psi \circ \tau_j^{-1}$ on $B(a_j, 2\delta)$ and let $\widetilde{\gamma}_j$ be the homogeneous quadratic function in $z - a_j$ such that $\frac{i}{\pi}\partial\overline{\partial}\widetilde{\gamma}_j = \gamma_j$ on $B(a_j, 2\delta)$. Finally, we set

(2.9 a) $$\varphi_j(z) = \psi_j(z) - \widetilde{\gamma}_j(z) \quad \text{on } B(a_j, 2\delta).$$

Then φ_j is plurisubharmonic, since

$$\frac{i}{\pi}\partial\overline{\partial}(\varphi_j \circ \tau_j) = T - \tau_j^\star \gamma_j \geq \gamma - \tau_j^\star \gamma_j \geq 0.$$

Let $||\xi||_{j,z}$ be the metric induced by ω at a point $z \in B(a_j, 2\delta)$ in the trivialization $d\tau_j : TX_{\restriction U_j} \to B(a_j, 2\delta) \times \mathbb{C}^n$. We combine (2.2) and (2.3) to define functions

(2.9 b) $$\Phi_{j,c}(z, w) = \inf_{t \leq 0} \Big(\sup_{||\xi||_{j,z} \leq 1} \varphi_j(z + e^{w+t}\xi) - ct \Big), \qquad z \in B(a_j, \sqrt{2}\,\delta),$$

(2.9 c) $$\Psi_{j,c}(z, w) = \Phi_{j,c}(z, w) + \widetilde{\gamma}_j(z) - \delta|z - a_j|^2, \qquad z \in B(a_j, \sqrt{2}\,\delta),$$

(2.9 d) $$\Psi_c(x, w) = \sup_{U_j'' \ni x} \Psi_{j,c}(\tau_j(x), w), \qquad x \in X,$$

where $U_j'' = \tau_j^{-1}(B(a_j, \sqrt{2}\,\delta))$. Note that $U_j' \subset\subset U_j'' \subset\subset U_j$ by construction and that there exists $w_0 = \log\delta - C_1 < 0$ such that $z + e^{w+t}\xi \in B(a_j, 2\delta)$ when $z \in B(a_j, \sqrt{2}\,\delta)$ and $\mathrm{Re}\, w < w_0$. Then $\Psi_c(x, w)$ is well defined on $X \times \{\mathrm{Re}\, w < w_0\}$. We have to check that the sup used in the definition of Ψ_c does not create discontinuities when x passes through a boundary $\partial U_j''$. For this, we must compare $\Psi_{k,c}(z, w)$ and $\Psi_{j,c}(\tau_{jk}(z), w)$ for $z \in \tau_k(U_j'' \cap U_k'')$, where $\tau_{jk} = \tau_j \circ \tau_k^{-1}$. By (2.9 a,b) there is a uniform estimate

$$\Phi_{j,c}(z, w) = \inf_{t \leq 0} \Big(\sup_{||\xi||_{j,z} \leq 1} \psi_j(z + e^{w+t}\xi) - ct \Big) - \widetilde{\gamma}_j(z) + O(|e^w|),$$

therefore (2.9 c) gives

(2.10) $$\Psi_{j,c}(z, w) = \inf_{t \leq 0} \Big(\sup_{||\xi||_{j,z} \leq 1} \psi_j(z + e^{w+t}\xi) - ct \Big) - \delta|z - a_j|^2 + O(|e^w|).$$

The error terms $O(|e^w|)$ come from the variation of the quadratic function $\widetilde{\gamma}_j(z)$ when z is replaced by $z + e^{w+t}\xi$. This variation is bounded by $C_2|e^w|$ with a constant C_2 independent of δ.

When we make the substitution $z \mapsto \tau_{jk}(z)$, the trivialization of the tangent bundle is submitted to a corresponding transformation $\xi \mapsto d\tau_{jk}(z) \cdot \xi$, in particular $||\xi||_{k,z} = ||d\tau_{jk}(z) \cdot \xi||_{j,\tau_{jk}(z)}$. As $\psi_k = \psi_j \circ \tau_{jk}$, we get

(2.11$_k$) $$\sup_{||\xi||_{k,z} \leq 1} \psi_k(z + e^{w+t}\xi) = \sup_{||\xi||_{k,z} \leq 1} \psi_j \circ \tau_{jk}(z + e^{w+t}\xi),$$

to be compared with

(2.11$_j$) $$\sup_{||\xi||_{j,\tau_{jk}(z)} \leq 1} \psi_j(\tau_{jk}(z) + e^{w+t}\xi) = \sup_{||\xi||_{k,z} \leq 1} \psi_j\big(\tau_{jk}(z) + e^{w+t} d\tau_{jk}(z) \cdot \xi\big).$$

However $\tau_{jk}(z + \xi) - [\tau_{jk}(z) + d\tau_{jk}(z) \cdot \xi] = O(|\xi|^2)$, thus if $B_j(z, r)$ denotes the euclidean ball of center z and radius r with respect to $||\ \ ||_{j,z}$, there is a constant $C_3 > 0$ such that

$$B_j(\tau_{jk}(z), r - C_3 r^2) \subset \tau_{jk}(B_k(z, r)) \subset B_j(\tau_{jk}(z), r + C_3 r^2)$$

for all j,k and $r < r_0$. By taking $r = |e^{w+t}|$, these inclusions show that

$$\begin{aligned}|(2.11_j) - (2.11_k)| &\le \Psi_{j,\infty}\big(\tau_{jk}(z), \log(r + C_3 r^2)\big) - \Psi_{j,\infty}\big(\tau_{jk}(z), \log(r - C_3 r^2)\big)\\ &\le C_4\big(\log(r + C_3 r^2) - \log(r - C_3 r^2)\big) \le C_5 |e^{w+t}|,\end{aligned}$$

because $\partial\Psi_{j,\infty}(z,w)/\partial w_+$ is uniformly bounded on $U_j \times \,]-\infty, w_0[$ (the bound is independent of δ). Therefore (2.10) implies

$$\Psi_{j,c}(\tau_{jk}(z), w) - \Psi_{k,c}(z,w) = \delta(|z - a_k|^2 - |\tau_{jk}(z) - a_j|^2) + O(|e^w|).$$

Now, if $x \in X$, there exists by construction an index k such that $x \in U'_k$, i.e. $z = \tau_k(x) \in B(a_k, \delta)$. If $x \in U''_j$ tends to a point in the boundary $\partial U''_j$, then $|\tau_{jk}(z) - a_j|$ tends to $\sqrt{2}\,\delta$ and

$$\Psi_{j,c}(\tau_{jk}(z), w) - \Psi_{k,c}(z,w) \longrightarrow \delta(|z - a_k|^2 - 2\delta^2) + O(|e^w|) \le -\delta^3 + O(|e^w|).$$

This implies that formula (2.9 d) does not involve values of $\Psi_{j,c}$ near $\partial U''_j$ when $w \le w_1 = 3\log\delta - C_6$. In particular Ψ_c is continuous on $X \setminus E_c(T)$, since each function $\Psi_{j,c}$ has this property. Also, by Kiselman's result 2.5, we get

$$\nu\big(\Psi_c(\bullet, w), x\big) = \nu\big(\Phi_{j,c}(\bullet, w), x\big) = \big(\nu(\varphi, x) - c\big)_+ = \big(\nu(\psi, x) - c\big)_+ \tag{2.12}$$

when $x \in U'_j$. Clearly $\Psi_c(x,w)$ is convex and increasing in w, for each function $\Phi_{j,c}$ has this property. Finally, let $x \in X$ with $x \in U''_k$, let $z = \tau_k(x)$ and let v_k be a positive quadratic function on $B(a_k, 2\delta)$ with $\frac{i}{\pi}\partial\overline{\partial} v_k > (\tau_k^{-1})^\star u_x$. As $\tau_{jk}(z + e^{i\theta}\zeta) = \tau_{jk}(z) + e^{i\theta} d\tau_{jk}(z)\cdot\zeta + O(|\zeta|^2)$, the definition of $\Phi_{j,c}$, $\Psi_{j,c}$ in (2.9 b,c) and the mean value argument used in the proof of 2.6 imply

$$\int_0^{2\pi} \Phi_{j,c}\big(\tau_{jk}(z + e^{i\theta}\zeta), w + e^{i\theta}\eta\big)\,\frac{d\theta}{2\pi} \ge \Phi_{j,c}\big(\tau_{jk}(z), w - v_k(\zeta)\big),$$

$$\begin{aligned}\int_0^{2\pi} \Psi_{j,c}\big(\tau_{jk}(z + e^{i\theta}\zeta), w + e^{i\theta}\eta\big)\,\frac{d\theta}{2\pi} &\ge \Phi_{j,c}\big(\tau_{jk}(z), w - v_k(\zeta)\big)\\ &+ \widetilde{\gamma}_j(\tau_{jk}(z)) + i\partial\overline{\partial}\widetilde{\gamma}_j(d\tau_{jk}(z)\cdot\zeta) - O(|\zeta|^3) - \delta|\tau_{jk}(z) - a_j|^2 - O(\delta)|\zeta|^2\end{aligned} \tag{2.13}$$

for every j such that $U''_j \ni x$; here we use the fact that for a suitable quadratic function v_j on $B(a_j, x)$ approximating closely $(\tau_j^{-1})^\star u_x$ (more accurately than v_k did), we have

$$v_j\big(e^{i\theta} d\tau_{jk}(z)\cdot\zeta + O(|\zeta|^2)\big) \le v_k(\zeta)$$

for ζ small. Since $\frac{i}{\pi}\partial\overline{\partial}\widetilde{\gamma}_j \circ \tau_j = \tau_j^\star\gamma_j \ge \gamma - 2\varepsilon(\delta)\omega$ by construction, we find

$$\frac{i}{\pi}\partial\overline{\partial}\widetilde{\gamma}_j(d\tau_{jk}(z)\cdot\zeta) \ge (\gamma_x - 2\varepsilon(\delta)\omega_x)\big(d\tau_k^{-1}(z)\cdot\zeta\big).$$

When we take the supremum over j in (2.13) and use definition (2.9 d), we get

$$\int_0^{2\pi} \Psi_c\big(\tau_k^{-1}(z + e^{i\theta}\zeta), w + e^{i\theta}\eta\big)\,\frac{d\theta}{2\pi} \ge \Psi_c(x, w - v_k(\zeta)) + \pi(\gamma_x - \widetilde{\varepsilon}(\delta)\omega_x)\big(d\tau_k^{-1}(z)\cdot\zeta\big)$$

with $\widetilde{\varepsilon}(\delta) = 2\varepsilon(\delta) + O(\delta)$. By the arguments already explained in the proof of 2.6 (ii), we infer

$$\frac{i}{\pi}\partial\overline{\partial}(\Psi_c)_{(x,w)}(\xi,\eta) \ge -\min\Big\{\frac{\partial\Psi_\infty(x,w)}{\partial \mathrm{Re}\, w_+}\,,\, c\Big\}\, u_x(\xi) + \gamma_x(\xi) - \widetilde{\varepsilon}(\delta)\omega_x(\xi) \tag{2.14}$$

when $w < w_1(\delta) = 3\log\delta - C_6$. Finally, we obtain the sequence $\psi_{c,k}$ needed in the Main Theorem by setting

$$\psi_{c,k}(x) = \Psi_c(x, w_k) + \frac{1}{k} \qquad \text{with } \delta = \delta_k := 2^{-k}$$

and with a rapidly decreasing sequence $w_k \to -\infty$. The covering (U_j) changes with the parameter δ when δ tends to 0, so it is not a priori clear that $\psi_{c,k}$ is decreasing; this is the reason why we introduced the additional term $1/k$; the change of covering between $\psi_{c,k}$ and $\psi_{c,k+1}$ can only modify our functions by terms of order of magnitude $\delta_k^2 = 2^{-2k}$ (coming from the change of quadratic functions) and $|e^{w_k}|$ (coming from the change of chart in the definition of $\Phi_{j,c}$). These terms are negligible with respect to $1/k - 1/(k+1)$ if we take for instance $w_k \le -3k \le 4\log\delta_k$. Property (2.12) shows that $\nu(\psi_{c,k}, x) = (\nu(\psi, x) - c)_+$ at every point. Moreover, (2.14) implies

$$\frac{i}{\pi}\partial\overline{\partial}\psi_{c,k}(x) \ge -\min\Big\{\frac{\partial\Psi_\infty(x, w_k)}{\partial \mathrm{Re}\, w_+}\,,\, c\Big\}\, u_x + \gamma_x - \widetilde{\varepsilon}(\delta_k)\omega_x.$$

Since $\partial\Psi_\infty(x, w)/\partial\mathrm{Re}\, w_+$ is upper semi-continuous and converges to $\nu(\frac{i}{\pi}\partial\overline{\partial}\psi, x)$ for each δ fixed, the following claim is easy to show: given any strictly decreasing sequence of continuous functions $\lambda_k \ge 0$ on X such that $\lim\lambda_k(x) = \nu(\frac{i}{\pi}\partial\overline{\partial}\psi, x)$, there is a choice of w_k such that $\partial\Psi_\infty(x, w_k)/\partial\mathrm{Re}\, w_+ \le \lambda_k(x)$ for $\delta = \delta_k$. Therefore the currents $T_{c,k} = \frac{i}{\pi}\partial\overline{\partial}\psi_{c,k}$ satisfy all properties 1.1 (i)-(iv).

Our construction only showed a priori that $T_{c,k}$ is continuous on $X \smallsetminus E_c(T)$. In fact $T_{c,k}$ can be made smooth on $X \smallsetminus E_c(T)$ by Richberg's approximation theorem [Ri 68]: by Lemma 2.15 below, there are smooth approximants $\psi'_{c,k}$ of $\psi_{c,k}$ on $X \smallsetminus E_c(T)$ with $|\psi'_{c,k} - \psi_{c,k}| < 2^{-k}$ and with an arbitrary small loss of positivity on the continuous lower bound of the Hessian, say a loss $\le 2^{-k}\omega$. It is then clear that $\psi'_{c,k}$ extends to an almost psh function on X and that $T'_{c,k} = \alpha + \frac{i}{\pi}\partial\overline{\partial}\psi'_{c,k}$ satisfies essentially the same estimates as $T_{c,k}$. Theorem 1.1 is thus proved under the more restrictive condition that $c(TX) + u \otimes \mathrm{Id}_{TX} \ge 0$ in the sense of Griffiths.

Lemma 2.15 (Richberg [Ri 68], Greene-Wu [GW 75]). — *Let ψ be a quasi psh function on a complex manifold M such that $i\partial\overline{\partial}\psi \ge \gamma$ where γ is a continuous $(1,1)$-form. For any hermitian metric ω and any continuous function $\delta > 0$ on M, there is a smooth function ψ' such that $\psi < \psi' < \psi + \delta$ and $i\partial\overline{\partial}\psi \ge \gamma - \delta\,\omega$ on M.*

Sketch of proof. — Take a locally finite covering (B_j) of M by open balls and take a convolution $\psi \star \rho_{\varepsilon_j}$ by a regularizing kernel on a neighborhood of each $\overline{B}_j$. Let z^j be a local coordinate identifying B_j to the ball $B(0, r_j)$. The smoothing process consists in gluing the functions $\psi \star \rho_{\varepsilon_j}(z) + \eta_j(r_j^2 - |z^j|^2)$ by taking the maximum with respect to all overlapping balls at the given point z. A smooth function can be obtained by means of regularized max functions. For a suitable choice of $\varepsilon_j \ll \inf_{B_k \cap B_j \ne \emptyset} \eta_k$, the boundary values of the functions do not contribute to the maximum. The estimates are then clearly satisfied if the ε_j's and η_j's are small enough. □

3. Approximation of plurisubharmonic functions by logarithms of holomorphic functions

Let T be a closed almost positive current of bidegree $(1,1)$ on a compact complex manifold X. We write $T = \alpha + \frac{i}{\pi}\partial\overline{\partial}\psi$ with an almost psh function ψ on X. The first step in the proof of Th. 1.1 consists in showing that ψ can be replaced locally by the logarithm of a sum of squares of holomorphic functions, in such a way that the Lelong numbers and the almost positive lower bound of T are preserved. For this, we use basically the same L^2 technique as in Section 9 of [De 90b] (namely the Ohsawa-Takegoshi L^2 extension theorem [OT 87]), but we work with functions on pseudoconvex open sets rather than with global sections of ample line bundles. We begin by a description of this local procedure.

Proposition 3.1. — *Let φ be a plurisubharmonic function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi} d\lambda < +\infty$ and let $\varphi_m = \frac{1}{2m}\log\sum|\sigma_\ell|^2$ where (σ_ℓ) is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that*

(i) $\varphi(z) - \dfrac{C_1}{m} \le \varphi_m(z) \le \sup\limits_{|\zeta - z| < r} \varphi(\zeta) + \dfrac{1}{m}\log\dfrac{C_2}{r^n}$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$ and

(ii) $\nu(\varphi, z) - \dfrac{n}{m} \le \nu(\varphi_m, z) \le \nu(\varphi, z)$ *for every $z \in \Omega$.*

Remark 3.2. — Proposition 3.1 can be used to derive a very simple proof of Siu's result [Siu 74] that $E_c(T)$ is an analytic set, at least in the case of a bidegree $(1,1)$-current T (the case of an arbitrary bidegree can be actually reduced to the $(1,1)$ case by a standard argument due to P. Lelong). To see this, it is enough to consider the case of a current $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$ on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. The inequalities in (ii) imply that $E_c(T) = \bigcap_{m \ge m_0} E_{c-n/m}(T_m)$, where $T_m = \frac{i}{\pi}\partial\overline{\partial}\varphi_m$. Now, it is clear that $E_c(T_m)$ is the analytic set defined by the equations $\sigma_\ell^{(\alpha)}(z) = 0$ for all multi-indices α such that $|\alpha| < mc$. Thus $E_c(T)$ is analytic. $\square$

Proof. — Note that $\sum|\sigma_\ell(z)|^2$ is the square of the norm of the evaluation linear form $f \mapsto f(z)$ on $\mathcal{H}_\Omega(m\varphi)$. As φ is locally bounded above, the L^2 topology is actually stronger than the topology of uniform convergence on compact subsets of Ω. It follows that the series $\sum|\sigma_\ell|^2$ converges uniformly on Ω and that its sum is real analytic. Moreover we have

$$\varphi_m(z) = \sup_{f \in B(1)} \frac{1}{m}\log|f(z)|$$

where $B(1)$ is the unit ball of $\mathcal{H}_\Omega(m\varphi)$. For $r < d(z, \partial\Omega)$, the mean value inequality applied to the psh function $|f|^2$ implies

$$|f(z)|^2 \le \frac{1}{\pi^n r^{2n}/n!}\int_{|\zeta-z|<r}|f(\zeta)|^2 d\lambda(\zeta)$$
$$\le \frac{1}{\pi^n r^{2n}/n!}\exp\Big(2m\sup_{|\zeta-z|<r}\varphi(\zeta)\Big)\int_\Omega |f|^2 e^{-2m\varphi}d\lambda.$$

If we take the supremum over all $f \in B(1)$ we get

$$\varphi_m(z) \le \sup_{|\zeta-z|<r}\varphi(\zeta) + \frac{1}{2m}\log\frac{1}{\pi^n r^{2n}/n!}$$

and the second inequality in (i) is proved. Conversely, the Ohsawa-Takegoshi extension theorem [OT 87, Oh 88] (Appendix, A.3) applied to the 0-dimensional subvariety $\{z\} \subset \Omega$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on Ω such that $f(z) = a$ and

$$\int_\Omega |f|^2 e^{-2m\varphi}d\lambda \le C_3|a|^2 e^{-2m\varphi(z)},$$

where C_3 only depends on n and $\operatorname{diam}\Omega$. We fix a such that the right hand side is 1. This gives the other inequality

$$\varphi_m(z) \ge \frac{1}{m}\log|a| = \varphi(z) - \frac{\log C_3}{2m}.$$

The above inequality implies $\nu(\varphi_m, z) \le \nu(\varphi, z)$. In the opposite direction, we find

$$\sup_{|x-z|<r}\varphi_m(x) \le \sup_{|\zeta-z|<2r}\varphi(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n}.$$

Divide by $\log r$ and take the limit as r tends to 0. The quotient by $\log r$ of the supremum of a psh function over $B(x, r)$ tends to the Lelong number at x. Thus we obtain

$$\nu(\varphi_m, x) \ge \nu(\varphi, x) - \frac{n}{m}. \quad \square$$

Gluing process. — When $T = \frac{i}{\pi}\partial\overline{\partial}\psi$ is a global current on X, we use a covering of X by coordinate open balls U_j of radius 2δ constructed as in section 2, and we let $U'_j \subset\subset U''_j \subset\subset U_j$, γ_j, $\widetilde{\gamma}_j$ be defined in the same way. On each open set U_j the function $\varphi_j := \psi - \widetilde{\gamma}_j \circ \tau_j$ defined in (2.9 a) is plurisubharmonic, so Proposition 3.1 gives functions

$$\varphi_{j,m} = \frac{1}{2m}\log\sum_\ell |\sigma_{j,\ell}|^2, \qquad (\sigma_{j,\ell}) = \text{basis of } \mathcal{H}_{U_j}(m\varphi_j), \tag{3.3}$$

approximating φ_j as m tends to $+\infty$ and satisfying the inequalities

$$\varphi_j(x) - \frac{C_1}{m} \le \varphi_{j,m}(x) \le \sup_{|\zeta-x|<r}\varphi_j(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n}. \tag{3.4}$$

The functions $\varphi_{j,m} + \widetilde{\gamma}_j \circ \tau_j$ on U_j are then glued together by a partition of unity technique developed in the following lemma.

LEMMA 3.5. — *Let $U'_j \subset\subset U''_j$ be locally finite open coverings of a complex manifold X by relatively compact open sets, and let θ_j be smooth nonnegative functions with support in U''_j, such that $\theta_j \le 1$ on U''_j and $\theta_j = 1$ on U'_j. Let $A_j \ge 0$ be such that*

$$i(\theta_j \partial\overline{\partial}\theta_j - \partial\theta_j \wedge \overline{\partial}\theta_j) \ge -A_j\omega \quad \text{on } U''_j \smallsetminus U'_j.$$

Finally, let w_j be almost psh functions on U_j with the property that $i\partial\overline{\partial}w_j \ge \gamma$ for some real $(1,1)$-form γ on M, and let C_j be constants such that

$$w_j(x) \le C_j + \sup_{k\ne j,\, U'_k \ni x} w_k(x) \quad \text{on } U''_j \smallsetminus U'_j.$$

Then the function $w = \log\big(\sum \theta_j^2 e^{w_j}\big)$ is almost psh and satisfies

$$i\partial\overline{\partial}w \ge \gamma - 2\Big(\sum_j \mathbb{1}_{U''_j \smallsetminus U'_j} A_j e^{C_j}\Big)\omega.$$

Proof. — If we set $\alpha_j = \theta_j \partial w_j + 2\partial\theta_j$, a straightforward computation shows that

$$\partial w = \frac{\sum(\theta_j^2 \partial w_j + 2\theta_j\partial\theta_j)e^{w_j}}{\sum \theta_j^2 e^{w_j}} = \frac{\sum \theta_j e^{w_j}\alpha_j}{\sum \theta_j^2 e^{w_j}},$$

$$\partial\overline{\partial}w = \frac{\sum\big(\alpha_j \wedge \overline{\alpha}_j + \theta_j^2\partial\overline{\partial}w_j + 2\theta_j\partial\overline{\partial}\theta_j - 2\partial\theta_j\wedge\overline{\partial}\theta_j\big)e^{w_j}}{\sum\theta_j^2 e^{w_j}} - \frac{\sum_{j,k}\theta_j e^{w_j}\theta_k e^{w_k}\alpha_j\wedge\overline{\alpha}_k}{\big(\sum\theta_j^2 e^{w_j}\big)^2}$$

$$= \frac{\sum_{j<k}\big|\theta_j\alpha_k - \theta_k\alpha_j\big|^2 e^{w_j}e^{w_k}}{\big(\sum\theta_j^2 e^{w_j}\big)^2} + \frac{\sum\theta_j^2 e^{w_j}\partial\overline{\partial}w_j}{\sum\theta_j^2 e^{w_j}} + \frac{\sum\big(2\theta_j\partial\overline{\partial}\theta_j - 2\partial\theta_j\wedge\overline{\partial}\theta_j\big)e^{w_j}}{\sum\theta_j^2 e^{w_j}}$$

by using the Legendre identity. The first term in the last line is nonnegative and the second one is $\ge \gamma$. In the third term, if x is in the support of $\theta_j\partial\overline{\partial}\theta_j - \partial\theta_j\wedge\overline{\partial}\theta_j$, then $x \in U''_j \smallsetminus U'_j$ and so $w_j(x) \le C_j + w_k(x)$ for some $k \ne j$ with $U'_k \ni x$ and $\theta_k(x) = 1$. This gives

$$i\frac{\sum\big(2\theta_j\partial\overline{\partial}\theta_j - 2\partial\theta_j\wedge\overline{\partial}\theta_j\big)e^{w_j}}{\sum\theta_j^2 e^{w_j}} \ge -2\sum_j \mathbb{1}_{U''_j\smallsetminus U'_j} e^{C_j} A_j\omega.$$

The expected lower bound follows. □

We now apply Lemma 3.5 to the functions

$$w_j(x) = 2m\widetilde{\gamma}_j(z^j) + \sqrt{m}(\delta^2 - |z^j|^2) + \log\sum_\ell |\sigma_{j,\ell}(x)|^2$$

where $z^j = \tau_j(x)$ is a local coordinate identifying U_j to $B(0,2\delta)$. The associated function w is defined by

$$w(x) = \log\Big(\sum_j \theta_j(x)^2 \exp\big(2m\widetilde{\gamma}_j(z^j) + \sqrt{m}(\delta^2 - |z^j|^2)\big)\sum_\ell |\sigma_{j,\ell}(x)|^2\Big)$$

and we set $\psi_m = \frac{1}{2m} w$. On U'_j we have $\theta_j = 1$ and $|z^j| < \delta$. Hence on every compact subset of U'_j there is a small constant $c(\delta) > 0$ such that

$$w(x) \geq 2m\widetilde{\gamma}_j(z^j) + 2c(\delta)\sqrt{m} + \log \sum_\ell |\sigma_{j,\ell}(x)|^2,$$

$$\psi_m(x) \geq \widetilde{\gamma}_j(z^j) + \varphi_{j,m}(x) + \frac{c(\delta)}{\sqrt{m}} \geq \widetilde{\gamma}_j(z^j) + \varphi_j(x) - \frac{C_1}{m} + \frac{c(\delta)}{\sqrt{m}} > \psi(x)$$

for m large, in view of (3.4) and the definition of φ_j. Therefore $\psi_m > \psi$ for $m \geq m_0(\delta)$ large. In the opposite direction, if $N(\delta)$ is the maximum number of overlapping balls U_j, we have

$$w(x) \leq \log N(\delta) + \delta^2\sqrt{m} + \max_j \Big\{2m\widetilde{\gamma}_j(z^j) + \log \sum_\ell |\sigma_{j,\ell}(x)|^2\Big\}.$$

By definition of φ_j we have $\sup_{|\zeta - x| < r} \varphi_j(\zeta) \leq \sup_{|\zeta-x|<r} \psi(\zeta) - \widetilde{\gamma}_j(z^j) + C_4 r$, thus by (3.4) again we find

$$\begin{aligned}\psi_m(x) &\leq \max_j \big\{\widetilde{\gamma}_j(z^j) + \varphi_{j,m}(x)\big\} + \frac{1}{2m}\log N(\delta) + \frac{\delta^2}{\sqrt{m}} \\ &\leq \sup_{|\zeta - x|<r} \psi(\zeta) + C_4 r + \frac{\delta^2}{\sqrt{m}} + \frac{1}{m}\log \frac{C_2 N(\delta)^{1/2}}{r^n}.\end{aligned}$$

By taking $r = 1/m$ for instance, we see that ψ_m converges to ψ. On the other hand (2.8) implies $\frac{i}{\pi}\partial\overline{\partial}\widetilde{\gamma}_j(z^j) = \tau_j^\star \gamma_j \geq \gamma - 2\varepsilon(\delta)\omega$, thus for m large

$$\frac{i}{\pi}\partial\overline{\partial} w_j \geq 2m\tau_j^\star\gamma_j - O(\sqrt{m}) \geq 2m(\gamma - 3\varepsilon(\delta)\omega).$$

Lemma 3.6 below will be used to show that $w_j(x) \leq C_j + \sup_{k\neq j,\, U'_k \ni x} w_k(x)$ for $C_j = 0$ and m large enough. Hence Lemma 3.5 gives an additional bounded negative term in the Hessian of w and we finally get

$$\frac{i}{\pi}\partial\overline{\partial}\psi_m \geq \gamma - 4\varepsilon(\delta)\omega$$

for $m \geq m'_0(\delta)$ large.

Lemma 3.6. — *There are constants $C_{j,k} = C_{j,k}(\delta)$ independent of m such that the almost psh functions*

$$\widetilde{w}_j(x) = 2m\,\widetilde{\gamma}_j(z^j) + \log \sum_\ell \big|\sigma_{j,\ell}(x)\big|^2, \qquad x \in U''_j$$

satisfy

$$|\widetilde{w}_j - \widetilde{w}_k| \leq (2n+2)\log m + C_{j,k} \quad \text{on } U''_j \cap U''_k.$$

Proof. — The proof will be given in a more general context in the next section: we apply 4.6 (i) when E is the trivial bundle $\mathcal{O}_X$ equipped with the weight $e^{-\psi}$. In that case we simply have $P(E^\star) = X$, $p = \mathrm{Id}_X$, $\mathcal{O}_E(1) = \mathcal{O}_X$ and $v = \gamma$. Note that the smoothness of the metric of E is not used in the proof of 4.6 (i). The main idea is as follows: for any holomorphic function h on U_j, a $\overline{\partial}$ equation can be solved on U_k, after some truncation, to get a holomorphic function assuming the same value as h at any prescribed point of $U''_j \cap U''_k$; this can be done with a uniform control on L^2 norms. $\square$

By Lemma 3.6, we see that $w_j(x) = \widetilde{w}_j(x) + \sqrt{m}(\delta^2 - |z^j|^2)$ satisfies $w_j(x) \le w_k(x)$ for $x \in (\overline{U}''_j \smallsetminus U'_j) \cap U'_k$ and m large enough (note that $\delta^2 - |z^j|^2 \le 0$ and $\delta^2 - |z^k|^2 > 0$ at x). Since X is covered by a collection of compact sets contained in (U'_k), we see that the assumption of Lemma 3.5 on w_j is actually satisfied with $C_j = 0$ for $m \ge m''_0(\delta)$ large enough. If we take $\delta = \delta_m$ decreasing to 0 very slowly with m, we obtain a sequence ψ_m satisfying the following properties.

PROPOSITION 3.7. — *Let ψ be an almost psh function on a compact complex manifold X such that $\frac{i}{\pi}\partial\overline{\partial}\psi \ge \gamma$ for some continuous $(1,1)$-form γ. Then there is a sequence of almost psh functions ψ_m such that ψ_m has the same singularities as a logarithm of a sum of squares of holomorphic functions and*

(i) $\psi(x) < \psi_m(x) \le \sup_{|\zeta - x| < r} \psi(\zeta) + C\Big(\frac{|\log r|}{m} + r + m^{-1/2}\Big)$
with respect to coordinate open sets covering X. In particular, ψ_m converges to ψ pointwise and in $L^1(X)$ and

(ii) $\nu(\psi, x) - \frac{n}{m} \le \nu(\psi_m, x) \le \nu(\psi, x)$ *for every* $x \in X$;

(iii) $\frac{i}{\pi}\partial\overline{\partial}\psi_m \ge \gamma - \varepsilon_m\omega$ *with* $\varepsilon_m > 0$ *decreasing to* 0.

4. Construction of hermitian metrics by L^2 techniques

Let E be a holomorphic vector bundle of rank r over a complex manifold M, and let $\mathcal{O}_E(1)$ be the associated canonical line bundle over the projectivized bundle $P(E^\star)$ of hyperplanes of E. By definition there is an exact sequence

$$0 \longrightarrow H \longrightarrow \pi^\star E \longrightarrow \mathcal{O}_E(1) \longrightarrow 0$$

where $\pi : P(E^\star) \to M$ is the projection and $H \subset \pi^\star E$ the tautological hyperplane subbundle. Suppose that $\mathcal{O}_E(1)$ is equipped with a hermitian metric. As the total space of $\mathcal{O}_E(-1)$ is the blow-up of the total space of $E^\star$ along the zero section, a hermitian metric on $\mathcal{O}_E(-1)$ corresponds to a *Finsler metric* on $E^\star$, that is, to an arbitrary positive homogeneous function of degree 1 along the fibers of $E^\star$. Our goal is to construct hermitian metrics on all high symmetric powers S^mE which induce approximations of the given metric on $\mathcal{O}_E(1)$ with a good control of the curvature. For this, we use the fact that $\pi_\star\mathcal{O}_E(m) \simeq S^mE$ and we apply Hörmander's L^2 estimates for $\overline{\partial}$ to construct holomorphic sections of $\mathcal{O}_E(m)$ locally over M (of course, no global holomorphic section need exist in general).

THEOREM 4.1. — *Suppose that $\mathcal{O}_E(1)$ is equipped with a smooth hermitian metric such that $c\big(\mathcal{O}_E(1)\big) \ge \pi^\star v$ for some continuous real $(1,1)$-form v*

on M. Then, for each positive integer m, there is a smooth hermitian metric on S^mE such that

(i) *the sequence of metrics on $\mathcal{O}_E(1)$ induced by the surjective morphisms $\pi^\star S^mE \to \mathcal{O}_E(m)$ converges locally uniformly to the original metric;*

(ii) *for every relatively compact subset $M' \subset\subset M$ and every $\varepsilon > 0$, there exists an integer m_0 such that $m \geq m_0$ implies*

$$c(S^mE) \geq m(v - \varepsilon\omega) \otimes \mathrm{Id}_{S^mE} \quad \text{over } \ M'$$

in the sense of Griffiths, where ω is a fixed hermitian metric on M.

Proof. — For every $M' \subset\subset M$ and $\varepsilon > 0$ fixed, we are going to construct a sequence of metrics on S^mE satisfying (i) and (ii) with this choice of ε and M'. This will be enough to prove the result, by taking some suitable diagonal subsequence with ε_k decreasing to 0 and M'_k increasing to M. Let (U_j) be a locally finite covering of M by open balls such that each of these balls is relatively compact in a coordinate open chart. Let $z^j = (z_1, \ldots, z_n)$ be coordinates near $\overline{U}_j$ and let v_j be the $(1,1)$-form with constant coefficients in U_j which coincides with $v - \varepsilon\,\omega$ at the center of U_j. Finally, let $\widetilde{v}_j(z^j) = \pi\sum z_k\overline{z}_\ell\big(\partial^2 v/\partial z_k\partial\overline{z}_\ell(0) - \varepsilon\omega_{k\ell}(0)\big)$ be the associated quadratic function such that $v_j = \frac{i}{\pi}\partial\overline{\partial}\widetilde{v}_j$. By choosing the radius of U_j small enough, we can suppose that $0 \leq v - v_j \leq 2\varepsilon\,\omega$ on U_j for every j. Now, we consider the Hilbert space $\mathcal{H}_j(m)$ of holomorphic sections $f \in H^0\big(\pi^{-1}(U_j), \mathcal{O}_E(m)\big)$ with the L^2 norm

$$||f||_j^2 = \int_{\pi^{-1}(U_j)} |f|^2 e^{2m\,\widetilde{v}_j(z^j)} dV,$$

where dV is a volume element on $P(E^\star)$ (fixed once for all) and $|f|^2$ is the pointwise norm on $\mathcal{O}_E(m)$ induced by the given hermitian metric on $\mathcal{O}_E(1)$. This means that we have multiplied the original metric of $\mathcal{O}_E(1)$ by the weight $\exp(2\widetilde{v}_j)$. The corresponding curvature form is

$$c\big(\mathcal{O}_E(1)\big) - \frac{i}{\pi}\partial\overline{\partial}\widetilde{v}_j(z^j) = c\big(\mathcal{O}_E(1)\big) - v_j \geq \pi^\star(v - v_j) \geq 0 \tag{4.2}$$

thanks to the curvature hypothesis. Finally, let $U'_j \subset\subset U''_j \subset\subset U_j$ be concentric balls such that (U'_j) still cover M and let θ_j be smooth functions with support in U''_j, such that $0 \leq \theta_j \leq 1$ on U''_j and $\theta_j = 1$ on $\overline{U}'_j$. We define a hermitian metric on $S^mE^\star$ as follows: for all $x \in M$ and $\xi \in S^mE^\star_x$ we set

$$||\xi||^2_{(m)} = \sum_j \theta_j^2(x)\exp\big(2m\widetilde{v}_j(z^j) + \sqrt{m}(r'^2_j - |z^j|^2)\big)\sum_\ell \big|\sigma_{j,\ell}(x)\cdot\xi\big|^2, \tag{4.3}$$

where r'_j is the radius of U'_j and $(\sigma_{j,\ell})_{\ell\geq 1}$ is an orthonormal basis of $\mathcal{H}_j(m)$. Here, we consider $\sigma_{j,\ell}$ as a section in $H^0(U_j, S^mE)$ and we use the obvious pairing between S^mE and $S^mE^\star$ to compute $\sigma_{j,\ell}(x)\cdot\xi$. It is easy to see that the sum $\sum_\ell |\sigma_{j,\ell}(x)\cdot\xi|^2$ does not depend on the choice of the orthonormal basis and converges locally uniformly above every compact subset of U_j : in fact the evaluation linear form $f \mapsto f(x)\cdot\xi$ is continuous on $\mathcal{H}_j(m)$ and the above sum is the square of its norm; the local uniform convergence comes from the fact that the linear form varies continuously with (x,ξ) when $x \in U_j$. The metric on $S^mE^\star$ defined by (4.3) is thus smooth. It induces by duality a metric on S^mE with the desired properties. To check this, we use similar techniques to those of section 3. The following elementary lemma is needed.

LEMMA 4.4. — *Given a hermitian holomorphic vector bundle F and a real $(1,1)$-form γ on M, we have the equivalent properties*

$$c(F) \ge \gamma \otimes \mathrm{Id}_F \Longleftrightarrow -c(F^\star) \ge \gamma \otimes \mathrm{Id}_{F^\star} \Longleftrightarrow \frac{i}{2\pi}\partial\overline{\partial}\log||\xi||^2 \ge p^\star\gamma, \quad \xi \in F^\star$$

where $\log||\xi||^2$ is seen as a function on $F^\star$ and where $p : F^\star \to M$ is the projection.

Proof. — The first equivalence is obvious, for $-c(F^\star)$ is the transposition of $c(F)$. The second equivalence follows from the fact that for any local holomorphic section σ of $F^\star$, we have

$$\frac{i}{2\pi}\partial\overline{\partial}\log||\sigma||^2 \ge -\frac{\langle c(F^\star)\sigma,\sigma\rangle}{||\sigma||^2},$$

with equality at any point $x \in M$ where $D\sigma(x) = 0$ and $\sigma(x) \ne 0$. $\square$

In order to prove the curvature estimate 4.1 (ii), we apply Lemma 3.5 with $w(x,\xi) = \log||\xi||^2_{(m)}$ and

$$w_j(x,\xi) = 2m\,\widetilde{v}_j(z^j) + \sqrt{m}(r_j'^2 - |z^j|^2) + \log\sum_\ell \big|\sigma_{j,\ell}(x)\cdot\xi\big|^2, \tag{4.5}$$

on the total space $X = S^mE^\star$ covered by $p^{-1}(U_j'')$. For this, we need to estimate the differences $|w_j(\xi) - w_k(\xi)|$ over $p^{-1}(U_j'' \cap U_k'')$. This is the crucial point where L^2 estimates for $\overline{\partial}$ are needed.

LEMMA 4.6. — *There are constants $C_{j,k}$ and C_j' independent of m such that the almost psh functions*

$$\widetilde{w}_j(x,\xi) = 2m\,\widetilde{v}_j(z^j) + \log\sum_\ell \big|\sigma_{j,\ell}(x)\cdot\xi\big|^2, \qquad (x,\xi) \in p^{-1}(U_j'') \subset S^mE^\star$$

satisfy

(i) $|\widetilde{w}_j - \widetilde{w}_k| \le (2n+2)\log m + C_{j,k}$ on $p^{-1}(U_j'' \cap U_k'')$;

(ii) $\big|\widetilde{w}_j(x,\eta^m) - 2m\log|\eta|\,\big| \le (2n+2r-2)\log m + C_j'$ for $x \in U_j''$, $\eta \in E_x^\star$,

where $|\eta|$ denotes the original Finsler metric on $E^\star$.

Proof. — As $E_{\restriction\overline{U}_j} \simeq \overline{U}_j \times \mathbb{C}^r$ is trivial near $\overline{U}_j$, we can define a hermitian metric on $E_{\restriction\overline{U}_j}$ with positive definite curvature by taking $|\xi|^2 = \sum|\xi_\lambda|^2e^{-|z^j|^2}$. The associated curvature form on $\mathcal{O}_E(1)$ is positive and defines a Kähler metric

ω_j on $\pi^{-1}(U_j) \simeq U_j \times \mathbb{P}^{r-1}$, equal to the direct sum of the euclidean metric of U_j and of the Fubini-Study metric of $\mathbb{P}^{r-1}$. In particular, the Ricci curvature of ω_j is nonnegative (it is even positive along each fiber $\mathbb{P}^{r-1}$). Let $e^{-\rho_j}$ be the ratio between the new metric on $\mathcal{O}_E(1)_{\restriction U_j}$ and the original one, which will be kept for the calculation of the curvature form $c(\mathcal{O}_E(1))$. By definition of ρ_j we have

$$c\big(\mathcal{O}_E(1)\big) + \frac{i}{\pi}\partial\overline{\partial}\rho_j = \omega_j > 0. \tag{4.7}$$

Let τ be an arbitrary almost psh function on a neighborhood of $\pi^{-1}(\overline{U}_j)$ and let $\mu \geq 0$ be a large integer such that

$$\frac{i}{\pi}\partial\overline{\partial}\tau + \mu\omega_j \geq \omega_j \quad \text{ on } \ \pi^{-1}(U_j). \tag{4.8}$$

For each $q = 0, 1, \ldots, n$, we consider the Hilbert space $\mathcal{F}_j^{0,q}(m)$ of $(0,q)$-forms f on $\pi^{-1}(U_j)$ with values in $\mathcal{O}_E(m)$, equipped with the L^2 norm

$$||f||_{j,q}^2 = \int_{\pi^{-1}(U_j)} |f|_j^2 e^{2(m-\mu)\widetilde{v}_j(z^j) - 2\mu\rho_j - 2\tau} dV_j\,,$$

where $dV_j = \omega_j^{n+r-1}/(n+r-1)!$ and where the pointwise norm $|f|_j$ is computed by means of ω_j and of the original metric on $\mathcal{O}_E(m)$. Thanks to (4.2), (4.7), (4.8), the weight involved in the norm $||\ \ ||_{j,q}$ corresponds to a curvature form on $\mathcal{O}_E(m)$ given by

$$\begin{aligned} &m\,c\big(\mathcal{O}_E(1)\big) - (m-\mu)\frac{i}{\pi}\partial\overline{\partial}\widetilde{v}_j + \mu\frac{i}{\pi}\partial\overline{\partial}\rho_j + \frac{i}{\pi}\partial\overline{\partial}\tau \\ &\qquad\qquad = \frac{i}{\pi}\partial\overline{\partial}\tau + \mu\omega_j + (m-\mu)\big(c\big(\mathcal{O}_E(1)\big) - v_j\big) \geq \omega_j. \end{aligned}$$

Hence, by Hörmander's L^2 estimates (Appendix, Theorem A.1), for every $(0,q)$-form $g \in \mathcal{F}_j^{0,q}(m)$ with $q \geq 1$ and $\overline{\partial} g = 0$, there is a $(0,q-1)$-form $f \in \mathcal{F}_j^{0,q-1}(m)$ such that

$$\overline{\partial} f = g, \quad ||f||_{j,q-1} \leq q^{-1/2}||g||_{j,q}. \tag{4.9}$$

Observe that this estimate is valid for $(0,q)$-forms because $\mathrm{Ricci}(\omega_j) \geq 0$. Moreover, if τ is bounded, the norm $||\ \ ||_{j,0}$ is equivalent to the norm $||f||_j$ of $\mathcal{H}_j(m)$, up to constants independent of m.

(i) Fix a point $x \in U_j'' \cap U_k''$ and $\xi \in S^m E_x^\star$. There exists $h \in \mathcal{H}_k(m)$ with $||h||_k = 1$ such that

$$|h(x)\cdot\xi|^2 = \sum_\ell |\sigma_{k,\ell}(x)\cdot\xi|^2\,;$$

in fact, h satisfies this relation if and only if h is orthogonal to the hyperplane of sections $\sigma \in \mathcal{H}_k(m)$ such that $\sigma(x)\cdot\xi = 0$. Let χ be a cut-off function with support in the ball $B(x, 1/m)$, equal to 1 on $B(x, 1/2m)$ and with $|\overline{\partial}\chi| \leq C_1 m$. For $m \geq m_0$ large enough (independent of x) we have $B(x, 1/m) \subset U_j \cap U_k$, and (4.9) allows us to solve the equation

$$\overline{\partial} f = h\,\overline{\partial}(\chi\circ\pi) \quad \text{on } \ \pi^{-1}(U_j), \qquad ||f||_{j,0} \leq ||h\,\overline{\partial}(\chi\circ\pi)||_{j,1}.$$

We then obtain a holomorphic section

$$h' = h(\chi \circ \pi) - f \in H^0\big(\pi^{-1}(U_j), \mathcal{O}_E(m)\big).$$

We want h' to coincide with h along the fiber $\pi^{-1}(x)$, so we take $\tau(z) = n\log|z^j - z^j(x)|$ depending only on the base variables with a logarithmic pole at x: the L^2 condition on f then implies that f vanishes identically along $\pi^{-1}(x)$. As τ is psh, we can take in fact $\mu = 1$ here. By the bound on $|\overline{\partial}\chi|$, we have

$$\begin{aligned}
||h\,\overline{\partial}(\chi\circ\pi)||^2_{j,1} &\le C_2 m^2 \int_{\pi^{-1}\big(B(x,1/m)\smallsetminus B(x,1/2m)\big)} \frac{|h|^2 e^{2(m-1)\widetilde{v}_j(z^j)-2\rho_j}}{|z^j - z^j(x)|^{2n}} dV_j \\
&\le C_3 m^{2n+2} \int_{\pi^{-1}B(x,1/m)} |h|^2 e^{2m\widetilde{v}_j(z^j)} dV_j \\
&\le C_4 m^{2n+2} e^{2m\big(\widetilde{v}_j(z^j(x)) - \widetilde{v}_k(z^k(x))\big)} ||h||^2_k
\end{aligned}$$

with constants C_2, C_3, C_4 independent of m and x. In the last inequality, we used the fact that the oscillation of $\widetilde{v}_j$ and $\widetilde{v}_k$ on $B(x,1/m)$ is $O(1/m)$. Hence we get the estimate

$$||f||^2_j \le C_5 ||f||^2_{j,0} \le C_4 C_5 m^{2n+2} e^{2m\big(\widetilde{v}_j(z^j(x)) - \widetilde{v}_k(z^k(x))\big)} ||h||^2_k$$

where $||h||_k = 1$ (the constant C_5 exists because τ is bounded above). The norm $||h(\chi\circ\pi)||_j$ satisfies a similar estimate. This implies

$$\begin{aligned}
||h'||^2_j &\le C_6 m^{2n+2} e^{2m\big(\widetilde{v}_j(z^j(x)) - \widetilde{v}_k(z^k(x))\big)}, \\
\sum_\ell \big|\sigma_{j,\ell}(x)\cdot\xi\big|^2 &\ge C_6^{-1} m^{-(2n+2)} e^{-2m\big(\widetilde{v}_j(z^j(x)) - \widetilde{v}_k(z^k(x))\big)} |h'(x)\cdot\xi|^2 \\
&\ge C_6^{-1} m^{-(2n+2)} e^{-2m\big(\widetilde{v}_j(z^j(x)) - \widetilde{v}_k(z^k(x))\big)} \sum_\ell \big|\sigma_{k,\ell}(x)\cdot\xi\big|^2,
\end{aligned}$$

since by construction $h'(x) = h(x)$ and $|h(x)\cdot\xi|^2 = \sum\big|\sigma_{k,\ell}(x)\cdot\xi\big|^2$. By taking logarithms and reversing the roles of j and k, we obtain the desired inequality (i).

(ii) The proof is similar, except that we take h to be a local holomorphic section of $\mathcal{O}_E(m)$ near $[\eta] \in P(E^\star_x)$ such that $|h([\eta])| = 1$; for this, we select a vector $a \in E^\star$ such that $\langle a, \eta\rangle = |\eta|$ and $\sum|a_\lambda|^2 \le C_7$, and we take h to be the image of the constant section $a^m \in H^0(U_j, S^m E)$ in $\mathcal{O}_E(m)$. Denote by $B([\eta], r)$ the geodesic ball of center $[\eta]$ and radius r in $P(E^\star)$, and let χ be a cut-off function with support in $B([\eta], 1/m)$ equal to 1 on $B([\eta], 1/2m)$. If $x \in U''_j$ and m is large, then $B([\eta], 1/m) \subset \pi^{-1}(U_j)$. By (4.9), we solve the equation

$$\overline{\partial} f = h\,\overline{\partial}\chi \quad \text{on} \quad \pi^{-1}(U_j), \qquad ||f||_{j,0} \le ||h\,\overline{\partial}\chi||_{j,1},$$

and we choose a weight τ with an isolated logarithmic pole of order $n + r - 1 = \dim P(E^\star)$ at $[\eta]$; the associated constant μ can be chosen independent of x, $[\eta]$ and m. Then the L^2 condition gives $f([\eta]) = 0$, thus

$$h' = h\chi - f \in H^0\big(\pi^{-1}(U_j), \mathcal{O}_E(m)\big)$$

satisfies $h'([\eta]) = h([\eta])$. Moreover we have

$$||h\,\overline{\partial}\chi||^2_{j,1} \le C_8 m^{2(n+r-1)+2}\int_{B([\eta],1/m)} |h|^2 e^{2m\widetilde{v}_j(z^j)}dV_j \le C_9 m^2 e^{2m\widetilde{v}_j(z^j(x))},$$

$$||h'||^2_j \le C_{10} m^2 e^{2m\widetilde{v}_j(z^j(x))}.$$

This implies

$$\begin{aligned}\sum_\ell \big|\sigma_{j,\ell}(x)\cdot\eta^m\big|^2 &\ge C_{10}^{-1} m^{-2} e^{-2m\widetilde{v}_j(z^j(x))}|h'(x)\cdot\eta^m|^2\\ &= C_{10}^{-1} m^{-2} e^{-2m\widetilde{v}_j(z^j(x))}|\eta|^{2m} \qquad (4.10)\end{aligned}$$

because $|h'(x)\cdot\eta^m| = |h([\eta])\cdot\eta^m| = |\eta|^m$ by construction. Conversely, if $h \in \mathcal{H}_j(m)$ is a section of norm $||h||_j = 1$ such that

$$\sum_\ell \big|\sigma_{j,\ell}(x)\cdot\eta^m\big|^2 = |h(x)\cdot\eta^m|^2 = |h([\eta])\cdot\eta^m|^2,$$

the mean value inequality on the ball $B([\eta],1/m) \subset P(E^\star)$ shows that

$$\begin{aligned}\big|h([\eta])\big|^2 &\le C_{11} m^{2(n+r-1)} e^{-2\widetilde{v}_j(z^j(x))}\int_{B([\eta],1/m)} |h|^2 e^{2m\widetilde{v}_j(z^j)}dV_j\\ &\le C_{11} m^{2(n+r-1)} e^{-2\widetilde{v}_j(z^j(x))},\end{aligned}$$

because the oscillation of the weight of $\mathcal{O}_E(m)$ is bounded by a constant on a ball of radius $1/m$. Hence

$$\sum_\ell \big|\sigma_{j,\ell}(x)\cdot\eta^m\big|^2 \le C_{11} m^{2(n+r-1)} e^{-2\widetilde{v}_j(z^j(x))}|\eta|^{2m} \qquad (4.11)$$

and inequality (ii) follows from the combination of (4.10) and (4.11). □

End of proof of Theorem 4.1. — Thanks to Lemma 4.6 (i), the functions $w_j(x,\xi)$ defined in (4.5) satisfy $w_j(x,\xi) \le w_k(x,\xi)$ for any $x \in (\overline{U}''_j \smallsetminus U'_j) \cap U'_k$ and m large enough (note that $r'^2_j - |z^j|^2 \le 0$ and $r'^2_k - |z^k|^2 > 0$ at x). By an obvious compactness argument, we see that the assumption of Lemma 3.5 on w_j is satisfied with $C_j = 0$ for $m \ge m'_0$ large enough:

$$w_j(x,\xi) \le \sup_{k\ne j,\,U'_k\ni x} w_k(x,\xi) \quad \text{for } x \in U''_j \smallsetminus U'_j.$$

Moreover, as $v_j \ge v - 2\varepsilon\omega$ and $\sqrt{m} \ll m\varepsilon$ for m large, we have

$$\frac{i}{2\pi}\partial\overline{\partial}w_j \ge m\,p^\star v_j - \sqrt{m}\frac{i}{2\pi}\partial\overline{\partial}|z^j|^2 \ge m\,p^\star(v - 3\varepsilon\,\omega)$$

for $m \ge m''_0$. Lemma 3.5 then shows that $w(x,\xi) = \log||\xi||^2_{(m)}$ satisfies

$$\frac{i}{2\pi}\partial\overline{\partial}w \ge m\,p^\star(v - 3\varepsilon\,\omega) - p^\star\Big(2\sum_j \mathbb{1}_{U''_j\smallsetminus U'_j}A_j\omega\Big) \ge m\,p^\star(v - 4\varepsilon\,\omega)$$

for $m \ge m'''_0$ large. By Lemma 4.4 applied to $F = S^mE$, we infer

$$c(S^mE) \ge m(v - 4\varepsilon\,\omega)\otimes \mathrm{Id}_{S^mE} \quad \text{for } m \ge m_0 = \max(m'_0, m''_0, m'''_0),$$

hence 4.1 (ii) is proved with 4ε instead of ε. Now, Lemma 4.6 (ii) shows that $\frac{1}{m}w_j(x,\eta^m)$ converges uniformly to $\log|\eta|$ on $E^\star_{\restriction U''_j}$. Therefore the global weight $\frac{1}{m}w(x,\eta^m)=\frac{1}{m}\log||\eta^m||^2_{(m)}$ converges locally uniformly to $\log|\eta|$ on $E^\star$, i.e., the inclusions $\mathcal{O}_E(-m)\subset\pi^\star S^mE^\star$ induce metrics on $\mathcal{O}_E(-1)$ which converge to the original metric. This statement is equivalent to 4.1 (i) by duality. □

5. An alternative singularity attenuation technique

As already remarked in section 2, it is sufficient to prove the Main Theorem when $T=\frac{i}{\pi}\partial\overline{\partial}\psi$ has zero cohomology class. Let $U'_j\subset\subset U''_j\subset\subset U_j$ be concentric balls of respective radius δ, $\sqrt{2}\delta$, 2δ such that the family (U'_j) covers X, and let γ_j, $\widetilde{\gamma}_j$ be as in sections 2,3. We set $\varphi_j:=\psi-\widetilde{\gamma}_j\circ\tau_j$ on U_j. By (3.3), (3.4) and Lemma 3.5 we obtain local approximations $\varphi_{j,m}$ satisfying the following properties:

(5.1) $\varphi_{j,m}=\dfrac{1}{2m}\log\sum_\ell|\sigma_{j,\ell}|^2$ with $(\sigma_{j,\ell})_{\ell\geq 0}=$ basis of $\mathcal{H}_{U_j}(m\varphi_j)$,

(5.2) $\Big(\nu(\psi,x)-\dfrac{n}{m}\Big)_+\leq\nu(\varphi_{j,m}\,,x)\leq\nu(\psi,x)$ on U_j,

(5.3) $\big|(\varphi_{j,m}+\widetilde{\gamma}_j\circ\tau_j)-(\varphi_{k,m}+\widetilde{\gamma}_k\circ\tau_k)\big|\leq(n+1)\dfrac{\log m+C(\delta)}{m}$ on $U''_j\cap U''_k$.

The inequalities (5.2) show that $\varphi_{j,m}$ roughly has the same singularities as ψ. We want to "attenuate" these singularities, i.e. to shift the Lelong numbers by $-c$ for some positive number c. For this, the idea is to replace the functions $\sigma_{j,\ell}$ in $\varphi_{j,m}$ by their derivatives of order $q\leq[mc]$. Since we must do this in such a way that the resulting functions defined on various sets U_j can be patched together, it is preferable to deal with jet sections $J^q\sigma_{j,\ell}\in H^0(U_j,J^q\mathcal{O}_X)$. Here $J^q\mathcal{O}_X$ denotes the global vector bundle of q-jets of holomorphic functions on X.

Lemma 5.4. — *Suppose that $\mathcal{O}_{TX}(1)$ is equipped with a smooth hermitian metric such that $c\big(\mathcal{O}_{TX}(1)\big)+\pi_X^\star u\geq 0$ for some nonnegative smooth $(1,1)$-form u on X. Let ω be a fixed hermitian metric on X. Then there are smooth hermitian metrics on $J^q\mathcal{O}_X$ for all $q\geq 0$, such that*

$$c(J^q\mathcal{O}_X)\leq q(u+\varepsilon_q\omega)\otimes\mathrm{Id}_{J^q\mathcal{O}_X}\quad\text{with}\quad\lim_{q\to+\infty}\varepsilon_q=0.$$

Proof. — By Theorem 4.1 applied to $E=TX$ and $v=-u$, there are hermitian metrics on S^qTX such that

$$c(S^qTX)\geq -q(u+\delta_q\omega)\otimes\mathrm{Id}_{S^qTX}\quad\text{with}\quad\lim_{q\to+\infty}\delta_q=0.$$

We argue by induction on q and use the canonical exact sequence

$$0\longrightarrow S^qT^\star X\xrightarrow{\ j\ }J^q\mathcal{O}_X\xrightarrow{\ \pi\ }J^{q-1}\mathcal{O}_X\longrightarrow 0.$$

When $q=0$, we equip $J^q\mathcal{O}_X=\mathcal{O}_X$ with the trivial flat metric. Assume that $q\geq 1$ and that a metric has already been constructed on $J^{q-1}\mathcal{O}_X$. Fix a C^∞-splitting

$j' : J^q\mathcal{O}_X \to S^qT^\star X$ and the corresponding right inverse $\pi' : J^{q-1}\mathcal{O}_X \to J^q\mathcal{O}_X$ onto the kernel of j'. For every $\eta > 0$, we define a hermitian metric on $J^q\mathcal{O}_X$ by

$$|\xi|_\eta^2 = \eta^2 |j'\xi|^2 + |\pi\xi|^2, \qquad \forall \xi \in J^q\mathcal{O}_X.$$

With respect to the given metrics on $S^qT^\star X$, $J^{q-1}\mathcal{O}_X$ and to $|\ \ |_\eta$ on $J^q\mathcal{O}_X$, the adjoints $j_\eta^\star$ and $\pi_\eta^\star$ of j,π are given by $j_\eta^\star = \eta j'$ and $\pi_\eta^\star = \pi'$. It follows that the second fundamental form with values in $\mathrm{Hom}(S^qT^\star X, J^{q-1}\mathcal{O}_X)$ is equal to $\eta\beta$ for some fixed matrix valued $(1,0)$-form β (apply the standard formula $\beta^\star = -j^\star\overline{\partial}\pi^\star$). In the orthogonal decomposition $(j_\eta^\star, \pi) : J^q\mathcal{O}_X \longrightarrow S^{q-1}T^\star X \oplus J^{q-1}\mathcal{O}_X$, the curvature of $J^q\mathcal{O}_X$ is then given by

$$c(J^q\mathcal{O}_X)_\eta = \begin{pmatrix} c(S^qT^\star X) - \eta^2 \frac{i}{2\pi}\beta^\star \wedge \beta & -\frac{i}{2\pi}\eta\overline{\partial}\beta^\star \\ \frac{i}{2\pi}\eta\partial\beta & c(J^{q-1}\mathcal{O}_X) - \eta^2\frac{i}{2\pi}\beta\wedge\beta^\star \end{pmatrix}.$$

It follows that

$$\begin{aligned} c(J^q\mathcal{O}_X)_\eta &\le \begin{pmatrix} q(u+\delta_q\omega)\otimes \mathrm{Id}_{S^qT^\star X} & 0 \\ 0 & (q-1)(u+\varepsilon_{q-1}\omega)\otimes \mathrm{Id}_{J^{q-1}\mathcal{O}_X} \end{pmatrix} + O(\eta)\,\omega\otimes \mathrm{Id}_{J^q\mathcal{O}_X} \\ &\le q(u+\varepsilon_q\omega)\otimes \mathrm{Id}_{J^q\mathcal{O}_X} \end{aligned}$$

where ε_q is given by the induction formula

$$\varepsilon_q = \max\Big\{\delta_q, \Big(1-\frac{1}{q}\Big)\varepsilon_{q-1}\Big\} + O(\eta).$$

We take $\eta = \eta_q$ so small that $O(\eta) \le 2^{-q}$. Since $\prod(1-1/q) = 0$, it is then easy to check that $\lim \varepsilon_q = 0$.

Remark 5.5. — This construction can be reformulated as follows: take a C^∞-splitting $J^q\mathcal{O}_X = \bigoplus_{0\le k\le q} S^kT^\star X$, and rescale the metric on each summand $S^kT^\star X$ by a positive factor η_k. In general, the sequence (η_k) will have to decay very quickly to 0 (in a way that we do not try to control). In order to emphasize this dependence on the sequence (η_k), we denote by $|\ \ |_{(\eta)}$ the resulting metric on $J^q\mathcal{O}_X$. □

Now, we would like to compare the jet norms $\sum_\ell |J^q\sigma_{j,\ell}|^2_{(\eta)}$ for various open sets U_j. Inequality (5.3) does this for $q = 0$. Unfortunately, it seems very difficult to derive precise estimates for q-jets from estimates on the functions (the Cauchy inequalities are not precise enough because they involve high powers of the radius). We shall therefore modify our functions $\sigma_{j,\ell}$ in such a way that the jets can be estimated more easily. Our method relies heavily on Skoda's L^2 estimates for ideals of holomorphic functions [Sk 72b]. We set $\sigma_j = (\sigma_{j,\ell})_{\ell\ge 0}$ and $|\sigma_j|^2 = \sum_\ell |\sigma_{j,\ell}|^2$. For all integers $m,p \ge 1$, we consider the Hilbert space $\mathcal{H}_{U_j''}(mp\varphi_{j,m})$ of holomorphic functions $f \in H^0(U_j'', \mathcal{O}_X)$ equipped with the L^2-norm

$$||f||^2_{j,m,p} = \int_{U_j''} |f|^2 e^{-2mp\varphi_{j,m}} d\lambda = \int_{U_j''} |f|^2 |\sigma_j|^{-2p} d\lambda,$$

and we introduce the functions

$$\varphi''_{j,m,p} = \frac{1}{2mp}\log\sum_\ell |\sigma''_{j,\ell}|^2 \quad \text{on } \ U_j'',$$

where $(\sigma''_{j,\ell})$ is an orthonormal basis of $\mathcal{H}_{U''_j}(mp\varphi_{j,m})$. By 3.1 (ii) applied to $\varphi = \log|\sigma_j| = m\,\varphi_{j,m}$ and m replaced by p, we get

$$\nu(\varphi_{j,m}\,,x) - \frac{n}{mp} \le \nu(\varphi''_{j,m,p}\,,x) \le \nu(\varphi_{j,m}\,,x) \quad \text{on } U''_j. \tag{5.6}$$

Thus, for p large, the Lelong numbers have not been substantially modified. Our goal is to compare the norms $\sum_\ell |J^q\sigma''_{j,\ell}|^2_{(\eta)}$ for different indices j. By (2.8) the Hessian of $\widetilde{\gamma}_j\circ\tau_j - \widetilde{\gamma}_k\circ\tau_k$ is bounded by $4\varepsilon(\delta)\omega$ on $U_j\cap U_k$. Since $\widetilde{\gamma}_j$ is quadratic, all its translates differ by a real affine function. Hence

$$\big[\widetilde{\gamma}_j(\tau_j(x)) - \widetilde{\gamma}_j(\tau_j(x)-b_j)\big] - \big[\widetilde{\gamma}_k(\tau_k(x)) - \widetilde{\gamma}_k(\tau_k(x)-b_k)\big]$$

is a pluriharmonic function, equal to the real part of a holomorphic function h_{jk} on $U_j\cap U_k$. We choose b_j and b_k such that the critical points of $\widetilde{\gamma}_j(\tau_j(x)-b_j)$ and $\widetilde{\gamma}_k(\tau_k(x)-b_k)$ coincide at some point of $U_j\cap U_k$. Then

$$|\widetilde{\gamma}_j\circ\tau_j - \widetilde{\gamma}_k\circ\tau_k - \mathrm{Re}\,h_{jk}| = |\widetilde{\gamma}_j(\tau_j-b_j) - \widetilde{\gamma}_k(\tau_k-b_k)| \le C_1\varepsilon(\delta)\delta^2 \quad \text{on } U_j\cap U_k, \tag{5.7}$$

because the Hessian of this function is $O(\varepsilon(\delta))$ and $\mathrm{diam}(U_j\cap U_k) = O(\delta)$. The required estimates for jets will be obtained by the following lemmas when $p \gg m$.

Lemma 5.8. — *For every $f \in \mathcal{H}_{U''_j}(mp\varphi_{j,m})$ with $p > n$ and every $L = (\ell_1,\ldots,\ell_{p-n-1}) \in \mathbb{N}^{p-n-1}$, there are holomorphic functions g_L on $U''_j\cap U''_k$ such that*

$$f = e^{-mph_{jk}} \sum_{L\in\mathbb{N}^{p-n-1}} g_L\sigma_k^L, \qquad \sigma_k^L := \prod_{1\le s\le p-n-1} \sigma_{k,\ell_s},$$

satisfying the L^2 estimates

$$\int_{U''_j\cap U''_k} \sum_L |g_L|^2|\sigma_k|^{-2(n+1)}d\lambda \le C(\delta,m,p)\int_{U''_j} |f|^2 e^{-2mp\varphi_{j,m}}d\lambda,$$

$$C(\delta,m,p) = (p-n)\exp\Big(2mp\Big(C_1\varepsilon(\delta)\delta^2 + C_2(\delta)\frac{\log m}{m}\Big)\Big).$$

Proof. — By definition of $\varphi_{k,m}$ we have

$$\int_{U''_j\cap U''_k} |fe^{mph_{jk}}|^2|\sigma_k|^{-2p}d\lambda = \int_{U''_j\cap U''_k} |f|^2 e^{2mp(\mathrm{Re}\,h_{jk}-\varphi_{k,m})}d\lambda.$$

On the other hand, inequalities (5.7) and (5.3) imply

$$\begin{aligned}\mathrm{Re}\,h_{jk} - \varphi_{k,m} &\le (\widetilde{\gamma}_j\circ\tau_j - \widetilde{\gamma}_k\circ\tau_k) - \varphi_{k,m} + C_1\varepsilon(\delta)\delta^2\\ &\le -\varphi_{j,m} + C_1\varepsilon(\delta)\delta^2 + C_2(\delta)\frac{\log m}{m} \quad \text{on } U''_j\cap U''_k.\end{aligned}$$

Therefore

$$\int_{U''_j\cap U''_k} |fe^{mph_{jk}}|^2|\sigma_k|^{-2p}d\lambda \le \frac{1}{p-n}C(\delta,m,p)\int_{U''_j} |f|^2e^{-2mp\varphi_{j,m}}d\lambda$$

and the desired functions g_L follow from Skoda's L^2 existence theorem [Sk 72b] (see Appendix, corollary A.5). $\square$

Corollary 5.9. — *Let $U_j^{(3)}$ be a ball such that $U'_j \subset\subset U_j^{(3)} \subset\subset U''_j$. For every $f \in \mathcal{H}_{U''_j}(mp\varphi_{j,m})$ with $p > n$, there are holomorphic functions b_ℓ on $U_j^{(3)} \cap U_k^{(3)}$ such that*

$$f = e^{-mph_{jk}} \sum_{\ell \in \mathbb{N}} b_\ell \sigma''_{k,\ell} \quad \text{on} \;\; U_j^{(3)} \cap U_k^{(3)},$$

$$\sum_{\ell \in \mathbb{N}} \sup_{U_j^{(3)} \cap U_k^{(3)}} |b_\ell|^2 \le C'(\delta, m, p) \int_{U''_j} |f|^2 e^{-2mp\varphi_{j,m}} d\lambda$$

with

$$C'(\delta, m, p) = (p-n) \exp\Big(2mp\Big(C_1\varepsilon(\delta)\delta^2 + C_2(\delta)\frac{\log m}{m}\Big) + C_3(\delta, m)\Big).$$

Proof. — By Nadel's coherence theorem [Na 89] (see also [De 90b], Lemma 4.4), the sheaf of germs of holomorphic functions h such that $|h|^2|\sigma_k|^{-2(n+1)} \in L^2_{\mathrm{loc}}$ is coherent on U_k. Let $v_1, \ldots, v_N$ be generators of this sheaf on a neighborhood of $\overline{U}''_k$, chosen in such a way that $\int_{U''_k} |v_s|^2 |\sigma_k|^{-2(n+1)} d\lambda \le 1$ (note that the covering depends on δ and that σ_k depends on m, so $N = N(\delta, m)$ is independent of p). Since the functions g_L constructed in Lemma 5.8 are sections of this sheaf on $U''_j \cap U''_k$, Cartan's theorem A shows that there are holomorphic functions $w_{L,1}, \ldots, w_{L,N}$ with $g_L = \sum_s v_s w_{L,s}$ on $U''_j \cap U''_k$, such that the family $(w_{L,s})_{1 \le s \le N}$ depends continuously on g_L in the topology of uniform convergence on compact subsets. In particular, there is a constant $C_4(\delta, m)$ independent of p such that

$$\sum_{1 \le s \le N} \sup_{U_j^{(3)} \cap U_k^{(3)}} |w_{L,s}|^2 \le C_4(\delta, m) \int_{U''_j \cap U''_k} |g_L|^2 |\sigma_k|^{-2(n+1)} d\lambda.$$

From Lemma 5.8, we infer

$$f = e^{-mph_{jk}} \sum_{L,s} v_s w_{L,s} \sigma_k^L \qquad \text{where}$$

$$\sum_{|L| = p-n-1} \int_{U''_k} \big|v_s \sigma_k^L\big|^2 |\sigma_k|^{-2p} d\lambda = \int_{U''_k} |v_s|^2 |\sigma_k|^{-2(n+1)} d\lambda \le 1.$$

Therefore, there are scalars $a_{L,s,\ell} \in \mathbb{C}$ with $\sum_{L,\ell} |a_{L,s,\ell}|^2 \le 1$, such that

$$v_s \sigma_k^L = \sum_\ell a_{L,s,\ell} \sigma''_{k,\ell}$$

in terms of the basis $(\sigma''_{k,\ell})$. On $U_j^{(3)} \cap U_k^{(3)}$, we finally get

$$f = e^{-mph_{jk}} \sum_{L,s,\ell} a_{L,s,\ell} w_{L,s} \sigma''_{k,\ell} = e^{-mph_{jk}} \sum_\ell b_\ell \sigma''_{k,\ell}$$

where $b_\ell = \sum_{L,s} a_{L,s,\ell} w_{L,s}$ satisfies the estimate

$$\begin{aligned}\sum_{\ell\in\mathbb{N}} \sup_{U_j^{(3)}\cap U_k^{(3)}} |b_\ell|^2 &\le \sum_{L,s,\ell} |a_{L,s,\ell}|^2 \sum_{L,s} \sup_{U_j^{(3)}\cap U_k^{(3)}} |w_{L,s}|^2 \\ &\le N\, C_4(\delta,m) \sum_L \int_{U_j''\cap U_k''} |g_L|^2 |\sigma_k|^{-2(n+1)} d\lambda \\ &\le N\, C_4(\delta,m) C(\delta,m,p) \int_{U_j''} |f|^2 e^{-2mp\varphi_{j,m}} d\lambda.\end{aligned}$$

The last inequality is a consequence of Lemma 5.8. Corollary 5.9 follows. □

Corollary 5.10. — *Let $U_j^{(4)}$ be balls such that $U_j' \subset\subset U_j^{(4)} \subset\subset U_j^{(3)}$. For every δ, m, p, q there is a sequence $\eta = (\eta_k)$ such that for all j,k*

$$e^{2mp\widetilde{\gamma}_j\circ\tau_j} \sum_\ell |J^q\sigma''_{j,\ell}|^2_{(\eta)} \le C''(\delta,m,p,q) e^{2mp\widetilde{\gamma}_k\circ\tau_k} \sum_\ell |J^q\sigma''_{k,\ell}|^2_{(\eta)} \quad \text{on } U_j^{(4)}\cap U_k^{(4)},$$

with

$$C''(\delta,m,p,q) = 2(p-n)\binom{q+n}{n} \exp\Big(2mp\Big(2C_1\varepsilon(\delta)\delta^2 + C_2(\delta)\frac{\log m}{m}\Big) + C_3(\delta,m)\Big).$$

Proof. — Let $f \in \mathcal{H}_{U_j''}(mp\varphi_{j,m})$. We compute the q-jet $J^q f$ in local coordinates by means of the identity obtained in Corollary 5.9. Leibnitz' rule implies

$$J^q f = e^{-mph_{jk}} \sum_{\ell\in\mathbb{N}} b_\ell J^q\sigma''_{k,\ell} + \sum_{\ell\in\mathbb{N}} \sum_{0<|\alpha|<q} \frac{1}{\alpha!} \partial^\alpha(e^{-mph_{jk}} b_\ell)\, A^\alpha(J^q\sigma''_{k,\ell})$$

where $A^\alpha : J^q\mathcal{O}_X \to J^q\mathcal{O}_X$ denotes the (local) bundle morphism which maps a jet $z\mapsto \sum a_\beta(z-x)^\beta$ to $z\mapsto \sum a_\beta(z-x)^{\alpha+\beta}$ mod $O(|z-x|^{q+1})$. In the natural decreasing filtration $F^tJ^q\mathcal{O}_X = \ker(J^q\mathcal{O}_X \to J^{t-1}\mathcal{O}_X)$ whose graded module is $\bigoplus S^tT^\star X$, we have $A^\alpha(F^tJ^q\mathcal{O}_X) \subset F^{t+|\alpha|}J^q\mathcal{O}_X$. It follows from this and the definition of $|\ \ |_{(\eta)}$ that the operators A^α have a very small $|\ \ |_{(\eta)}$-norm when the quotients η_{t+1}/η_t are taken small enough. Moreover, for every holomorphic function b on $U_j^{(3)}\cap U_k^{(3)}$, the Cauchy inequalities yield constants $C_5(\delta,\alpha,m,p,q)$ such that

$$\sup_{U_j^{(4)}\cap U_k^{(4)}} |\partial^\alpha(be^{-mph_{jk}})| \le C_5(\delta,\alpha,m,p,q) \sup_{U_j^{(3)}\cap U_k^{(3)}} |b|.$$

Hence by taking the quotients η_{t+1}/η_t small enough, we get at each point of $U_j^{(4)}\cap U_k^{(4)}$ the estimate

$$\begin{aligned}|J^q f|^2_{(\eta)} &\le 2e^{-2mp\mathrm{Re}\, h_{jk}} \Big(\sum_{\ell\in\mathbb{N}} \sup_{U_j^{(3)}\cap U_k^{(3)}} |b_\ell|^2\Big) \sum_{\ell\in\mathbb{N}} |J^q\sigma''_{k,\ell}|^2_{(\eta)} \\ &\le 2e^{-2mp\mathrm{Re}\, h_{jk}} C'(\delta,m,p) ||f||^2_{j,m,p} \sum_{\ell\in\mathbb{N}} |J^q\sigma''_{k,\ell}|^2_{(\eta)},\end{aligned}$$

thanks to corollary 5.9. We take the supremum over all elements f in the unit ball of $\mathcal{H}_{U''_j}(mp\varphi_{j,m})$ and combine with (5.7) to obtain

$$e^{2mp\widetilde{\gamma}_j\circ\tau_j}\sup_{f\in B(1)}|J^qf|^2_{(\eta)}\le 2e^{2mpC_1\varepsilon(\delta)\delta^2}C'(\delta,m,p)e^{2mp\widetilde{\gamma}_k\circ\tau_k}\sum_{\ell\in\mathbb{N}}|J^q\sigma''_{k,\ell}|^2_{(\eta)}.$$

Now, for every point $x\in X$, the quadratic form $f\mapsto|J^qf(x)|^2_{(\eta)}$ has rank equal to the rank $\binom{q+n}{n}$ of the bundle $J^q\mathcal{O}_X$. By the elementary Lemma 5.11 below, we thus get

$$\sum_{\ell\in\mathbb{N}}|J^q\sigma''_{j,\ell}(x)|^2_{(\eta)}\le\binom{q+n}{n}\sup_{f\in B(1)}|J^qf(x)|^2_{(\eta)}$$

and corollary 5.10 follows. □

Lemma 5.11. — *Let Q be a semi-positive quadratic form of finite rank r on a Hilbert space $\mathcal{H}$. Then, for any orthonormal basis (e_ℓ) of $\mathcal{H}$ we have*

$$\sum_\ell Q(e_\ell)\le r\sup_{x\in B(1)}Q(x).$$

Proof. — If $Q=|f|^2$ is the square of a linear form ($r=1$), both sides coincide with the square of the norm of f. If $Q=\sum_{1\le j\le r}|f_j|^2$ has rank r, then

$$\sum_\ell Q(e_\ell)\le\sum_{1\le j\le r}\sup_{x\in B(1)}|f_j(x)|^2\le r\sup_{x\in B(1)}Q(x).\qquad\square$$

Lemma 5.12. — *Suppose that the concentric balls $U_k^{(5)}\subset\subset U'_k$ of radius $\delta/2$ still cover X. Fix a nonnegative real number c and take $q=\lceil mpc\rceil$ equal to the smallest integer $\ge mpc$. Then there are constants $C_6,C_7\ge 0$ such that the functions*

$$w_j(x)=2mp\Big(\widetilde{\gamma}_j(\tau_j(x))+\sqrt{\varepsilon(\delta)}\big(\delta^2-|\tau_j(x)|^2\big)+\frac{1}{\sqrt{m}}\Big)+\log\sum_\ell|J^q\sigma''_{j,\ell}|^2_{(\eta)}\quad\text{on } U_j^{(4)}$$

enjoy the following properties for $p\gg m\gg 1/\delta$ and $\eta_{t+1}/\eta_t\ll 1/p$:

(i) $\frac{i}{\pi}\partial\overline{\partial}w_j\ge 2mp\big(\gamma-C_6\sqrt{\varepsilon(\delta)}\,\omega\big)-2q(u+\varepsilon_q\omega)$ *on* $U_j^{(4)}$,

(ii) $\psi(x)\le\frac{1}{2mp}w_j(x)\le\sup_{|z-x|<r}\psi(z)+C_7\Big(\frac{|\log r|}{m}+r+m^{-1/2}+\delta^2\Big)$ *on* U'_j,

(iii) $w_j(x)\le\sup_{k\ne j,\,U'_k\ni x}w_k(x)$ on $U_j^{(4)}\smallsetminus U'_j$,

(iv) $\Big(\nu(\psi,x)-c-\frac{2n+1}{m}\Big)_+\le\frac{1}{2mp}\nu(w_j,x)\le\big(\nu(\psi,x)-c\big)_+$ *on* $U_j^{(4)}$.

Proof. — (i) We have $\frac{i}{\pi}\partial\overline{\partial}(\widetilde{\gamma}_j\circ\tau_j)\ge\gamma-2\varepsilon(\delta)\omega$ by (2.8). On the other hand, Lemma 5.4 implies

$$\frac{i}{2\pi}\partial\overline{\partial}\log|J^q\sigma''_{j,\ell}|^2_{(\eta)}\ge-\frac{\langle c(J^q\mathcal{O}_X)\,J^q\sigma''_{j,\ell}\,,J^q\sigma''_{j,\ell}\rangle}{|J^q\sigma''_{j,\ell}|^2_{(\eta)}}\ge-q(u+\varepsilon_q\omega).$$

These inequalities imply (i).

(ii) By definition of $|\ \ |_{(\eta)}$, we have $|J^q\sigma''_{j,\ell}|_{(\eta)} \geq |\sigma''_{j,\ell}|$. Proposition 3.1 (i) applied first to $\varphi_j = \psi - \widetilde{\gamma}_j \circ \tau_j$ and then to $\log|\sigma_j| = m\varphi_{j,m}$ implies the existence of $C_8, C_9 \geq 0$ such that

$$\varphi_j(x) - \frac{C_8}{m} \leq \frac{1}{m}\log|\sigma_j(x)| \leq \sup_{|z-x|<r} \varphi_j(z) + \frac{1}{m}\log\frac{C_8}{r^n},$$
$$\log|\sigma_j(x)| - \frac{C_9}{p} \leq \frac{1}{p}\log|\sigma''_j(x)| \leq \sup_{|z-x|<r} \log|\sigma_j(z)| + \frac{1}{p}\log\frac{C_9}{r^n}.$$

Thus on U'_j we have

$$\begin{aligned} w_j &\geq 2mp\,\widetilde{\gamma}_j \circ \tau_j + 2\log|\sigma''_j| \\ &\geq 2mp\Big(\widetilde{\gamma}_j \circ \tau_j + \varphi_j + \frac{1}{\sqrt{m}} - \frac{C_8}{m} - \frac{C_9}{mp}\Big) \geq 2mp\,\psi \end{aligned}$$

for $p \gg m \gg 1$. In the opposite direction, we find

$$\frac{1}{mp}\log|\sigma''_j(x)| \leq \sup_{|z-x|<2r} \varphi_j(z) + C_{10}\frac{|\log r|}{m}$$

and therefore

$$\frac{1}{2mp}\log\sum_\ell |J^q\sigma''_{j,\ell}(x)|^2_{(\eta)} \leq \sup_{|z-x|<2r} \varphi_j(z) + C_{10}\frac{|\log r| + 1}{m}$$

for small η (the left hand side decreases to $\frac{1}{mp}\log|\sigma''_j(x)|$ when all η_t decrease to 0, and we can apply Dini's lemma since the right hand side is continuous). The right hand inequality in (ii) follows by adding $\widetilde{\gamma}_j \circ \tau_j + \sqrt{\varepsilon(\delta)}\delta^2 + m^{-1/2}$ to this.

(iii) As $\binom{q+n}{n} \leq (q+1)^n \leq (mpc+2)^n$, corollary (5.10) gives

$$\begin{aligned} \Big|\Big(\widetilde{\gamma}_j \circ \tau_j + \frac{1}{2mp}\log\sum_\ell |J^q\sigma''_{j,\ell}|^2_{(\eta)}\Big) - \Big(\widetilde{\gamma}_k \circ \tau_k + \frac{1}{2mp}\log\sum_\ell |J^q\sigma''_{k,\ell}|^2_{(\eta)}\Big)\Big| \\ \leq 2C_1\varepsilon(\delta)\delta^2 + C_2(\delta)\frac{\log m}{m} + \frac{1}{2mp}\big(C_3(\delta,m) + \log 2p(mpc+2)^n\big). \end{aligned}$$

When we take $p \gg m \gg 1/\delta$, this upper bound is less than $3C_1\varepsilon(\delta)\delta^2$. On the other hand, the additional terms $\sqrt{\varepsilon(\delta)}(\delta^2 - |\tau_j(x)|^2)$ create a discrepancy at least equal to $\frac{3}{4}\sqrt{\varepsilon(\delta)}\delta^2$ when $x \in (U^{(4)}_j \smallsetminus U'_j) \cap U^{(5)}_k$. This is sufficient to compensate $3C_1\varepsilon(\delta)\delta^2$ and (iii) follows.

(iv) A combination of inequalities (5.2) and (5.6) gives

$$\nu(\psi,x) - \frac{2n}{m} \leq \nu(\varphi_{j,m,p}\,,x) \leq \nu(\psi,x) \quad \text{on } U''_j,$$

where $2mp\varphi_{j,m,p} = \log\sum_\ell |\sigma''_{j,\ell}|^2$. The Lelong numbers of $\log\sum_\ell |J^q\sigma''_{j,\ell}|^2_{(\eta)}$ are shifted down by $2q$ with $mpc \leq q < mpc+1$. Hence we get

$$\nu(\psi,x) - \frac{mpc+1}{mp} - \frac{2n}{m} \leq \frac{1}{2mp}\nu(w_j,x) \leq \big(\nu(\psi,x) - c\big)_+$$

and (iv) is proved. □

Proof of Theorem 1.1. — We use Lemma 3.5 to glue the functions w_j of Lemma 5.12 into a single function $w = \log \sum \theta_j^2 e^{w_j}$, by means of cut-off functions θ_j such that $\theta_j = 1$ on U_j' and $\operatorname{Supp} \theta_j \subset U_j^{(4)}$. We then have a loss of positivity $-C_{11}(\delta)\omega$ in the Hessian of w with respect to 5.12 (i). Since the differences $|w_j - w_k|$ are bounded, we get $\nu(w, x) = \nu(w_j, x)$ on $U_j^{(4)}$. Also, the obvious inequalities

$$w_j(x) \le w(x) \le \sup_{U_k^{(4)} \ni x} w_k(x) + \log N(\delta) \qquad \text{on } \ U_j'$$

(with $N(\delta)$ = maximal number of overlapping sets $U_k^{(4)}$) show that 5.12 (ii) is essentially preserved for $p \gg \log N(\delta)$. Therefore $\Psi_{c,\delta,m,p} := \frac{1}{2mp} w$ satisfies on X the global estimates

$$\text{(5.13 i)} \qquad \frac{i}{\pi}\partial\overline{\partial}\Psi_{c,\delta,m,p} \ge \gamma - c\,u - 2C_6\sqrt{\varepsilon(\delta)}\,\omega,$$

$$\text{(5.13 ii)} \qquad \psi(x) \le \Psi_{c,\delta,m,p}(x) \le \sup_{|z-x|<r} \psi(z) + C_7\Big(\frac{|\log r|}{m} + r + \delta\Big),$$

$$\text{(5.13 iv)} \qquad \Big(\nu(\psi, x) - c - \frac{2n+1}{m}\Big)_+ \le \nu(\Psi_{c,\delta,m,p}\,, x) \le \big(\nu(\psi, x) - c\big)_+$$

when $p \gg m \gg 1/\delta$. Moreover $\Psi_{c,\delta,m,p}$ has the same singularities as the logarithm of a sum of squares of holomorphic functions, thus $\Psi_{c,\delta,m,p}$ is smooth on $X \smallsetminus E_c(\psi)$.

In order to complete the proof of Theorem 1.1, we still have to organize the functions $\Psi_{c,\delta,m,p}$ into a decreasing sequence $(\psi_{c,k})_{k\ge1}$ converging to ψ and satisfying properties 1.1 (i,ii) (which are stronger than inequality (5.13 i)). The arguments are almost purely formal. We first note that for given δ, m, p there is a choice of (η) which works for all $c' \in [0, c]$ and $q' = \lceil mpc' \rceil \le q$. Then by construction, we see that w_j, w and $\Psi_{c',\delta,m,p}$ are increasing in c'. Fix a strictly decreasing sequence of continuous functions $\lambda_k > 0$ on X such that $\lim_{k\to+\infty} \lambda_k(x) = \nu(\psi, x)$ at every point. We select inductively a sequence $\delta_\ell \downarrow 0$ and strictly increasing sequences (m_ℓ), (p_ℓ) such that the above estimates (5.13) are satisfied and such that for all $k = 1, \ldots, \ell$

$$\text{(5.14)} \qquad \Psi_{(1-k/(\ell+1))c,\delta_{\ell+1},m_{\ell+1},p_{\ell+1}} + \frac{1}{\ell+1} < \Psi_{(1-k/\ell)c,\delta_\ell,m_\ell,p_\ell} + \frac{1}{\ell} \qquad \text{on } \ A_{k,\ell},$$

where $A_{k,\ell} := \lambda_\ell^{-1}\big([0, (1-k/\ell)c]\big)$. This is actually possible by Dini's lemma, because $A_{k,\ell}$ is a compact set contained in $X \smallsetminus E_{(1-k/\ell)c}(\psi)$: thus, the right hand side of (5.14) is smooth and $\ge \psi + 1/\ell$ on $A_{k\ell}$, whereas the left hand side converges to the smaller limit $\psi + 1/(\ell+1)$ as $\delta_{\ell+1} \to 0$ and $m_{\ell+1}, p_{\ell+1} \to +\infty$. For a reason that will be apparent later, we also take $m_\ell \ge (2n+1)\ell/c$. We claim that the (upper semi-continuous) regularized upper envelope

$$\psi_{c,k} = \Big(\sup_{\ell\ge k} \Psi_{(1-k/\ell)c,\delta_\ell,m_\ell,p_\ell} + \frac{1}{\ell}\Big)^{\star}$$

has all required properties. In fact, it is clear that the sequence $(\psi_{c,k})_{k\ge 1}$ is decreasing and (5.13 ii) shows that it converges to ψ. Now, we have by definition $\psi_{c,k} \ge \Psi_{(1-k/\ell)c,\delta_\ell,m_\ell,p_\ell}$ and therefore

$$\nu(\psi_{c,k}, x) \le \nu(\Psi_{(1-k/\ell)c,\delta_\ell,m_\ell,p_\ell}, x) \le \big(\nu(\psi, x) - (1 - k/\ell)c\big)_+$$

for all $\ell \ge k$. This implies $\nu(\psi_{c,k}, x) \le (\nu(\psi, x) - c)_+$ when ℓ tends to $+\infty$. On the other hand, all terms in the supremum have Lelong numbers at least equal to

$$\nu(\psi, x) - \Big(1 - \frac{k}{\ell}\Big)c - \frac{2n+1}{m_\ell} \ge \nu(\psi, x) - c$$

thanks to our choice $(2n+1)/m_\ell \le c/\ell$. By standard arguments, we get therefore the converse inequality $\nu(\psi_{c,k}, x) \ge \nu(\psi, x) - c$. The monotonicity condition (5.14) implies

$$\psi_{c,k} = \sup_{k\le\ell\le s} \Psi_{(1-k/\ell)c,\delta_\ell,m_\ell,p_\ell} + \frac{1}{\ell} \quad \text{ on } \ A_{k,s}$$

(note that $A_{k,\ell}$ is increasing in ℓ and that $\bigcup_\ell A_{k,\ell} = X \smallsetminus E_c(\psi)$). Therefore the Hessian of $\psi_{c,k}$ satisfies the weakest bound valid for each term in this finite envelope, namely

$$\frac{i}{\pi}\partial\overline{\partial}\psi_{c,k} \ge \gamma - (1 - k/s)c\,u - 2C_6\sqrt{\varepsilon(\delta_k)}\omega \quad \text{ on } \ \{\lambda_s < (1 - k/s)c\} \subset A^\circ_{k,s}$$

thanks to (5.13 i). If $x \in X$ with $\lambda_k(x) < c$ is given, we select the smallest $s > k$ such that $\lambda_s(x) < (1 - k/s)c$. Then $(1 - k/(s-1))c \le \lambda_{s-1}(x)$ and we see that

$$(1 - k/s)c \le \lambda_{s-1}(x) + kc/s(s-1) < \lambda_k(x) + c/k.$$

We infer from this

$$\frac{i}{\pi}\partial\overline{\partial}\psi_{c,k} \ge \gamma - (\lambda_k + c/k)u - 2C_6\sqrt{\varepsilon(\delta_k)}\omega \quad \text{ on } \ \{\lambda_k < c\}.$$

However, all terms in $\psi_{c,k}$ have a Hessian bounded below by $\gamma - c\,u - 2C_6\sqrt{\varepsilon(\delta_k)}\omega$, so we get in all cases

$$\frac{i}{\pi}\partial\overline{\partial}\psi_{c,k} \ge \gamma - \min\{\lambda_k, c\}u - \big(2C_6\sqrt{\varepsilon(\delta_k)} + O(1/k)\big)\omega \quad \text{ on } \ X.$$

Finally, as $\psi_{c,k}$ is a finite supremum on each set $A_{k,\ell}$ in $X \smallsetminus E_c(\psi) = \bigcup_\ell A_{k,\ell}$, we conclude that $\psi_{c,k}$ is continuous on $X \smallsetminus E_c(\psi)$. Richberg's theorem 2.15 can be applied to make $\psi_{c,k}$ smooth on $X \smallsetminus E_c(\psi)$. Theorem 1.1 is thus completely proved. □

Remark 5.15. — By a further application of Proposition 3.7 to each function $\psi_{c,k}$, one could achieve that $\psi_{c,k}$ has the same singularities along $E_c(\psi)$ as a logarithm of a sum of squares of holomorphic functions, divided by some large integer $2m$. Of course, the Lelong numbers are then rational numbers with denominator m, so one cannot expect in that case that condition 1.1 (iv) holds exactly. Only a weaker condition of the type

$$\big(\nu(\psi, x) - c - 1/k\big)_+ \le \nu(\psi_{c,k}, x) \le \big(\nu(\psi, x) - c\big)_+$$

will be obtained.

6. Application to numerically effective divisors

We first prove the list of easy facts mentioned in the introduction about pseudoeffective and nef cohomology classes.

Proposition 6.1. — *Let X be a compact complex manifold. Then*

(i) *$H^{1,1}_{\rm psef}(X)$ and $H^{1,1}_{\rm nef}(X)$ are closed convex cones in $H^{1,1}_{\partial\overline\partial}(X)$.*

(ii) *$H^{1,1}_{\rm nef}(X) \subset H^{1,1}_{\rm psef}(X)$.*

(iii) *If X is Kähler, then $H^{1,1}_{\rm nef}(X)$ is the closure of the Käkler cone.*

(iv) *For any $\{\alpha\} \in H^{1,1}_{\rm nef}(X)$ and any curve C, then $\{\alpha\}\cdot C = \int_C \alpha \ge 0$.*

(v) *If X is projective and if $\{\alpha\} \in NS_{\rm I\!R}(X)$ is such that $\{\alpha\}\cdot C \ge 0$ for every curve C, then $\{\alpha\} \in H^{1,1}_{\rm nef}(X)$.*

(vi) *If X is projective, $H^{1,1}_{\rm psef}(X) \cap NS_{\rm I\!R}(X)$ is the closure of the cone of effective divisors in $NS_{\rm I\!R}(X)$.*

(vii) *If X is projective, $H^{1,1}_{\rm nef}(X) \cap NS_{\rm I\!R}(X)$ is the closure of the cone of ample divisors in $NS_{\rm I\!R}(X)$.*

Proof. — Unless otherwise specified, we take ω to be a special hermitian metric such that $\partial\overline\partial(\omega^{n-1}) = 0$ (where $n = \dim X$). Such a metric always exists by Gauduchon [Ga 77]. Note the following useful fact: if T is a closed positive $(1,1)$-current, the total mass $\int_X T\wedge\omega^{n-1}$ only depends on the $\partial\overline\partial$-cohomology class of T.

(i) Let T_k be closed positive currents such that the classes $\{T_k\}$ converge to a limit $\{\Theta\}$. Then $\int_X T_k\wedge\omega^{n-1}$ converges to $\int_X \Theta\wedge\omega^{n-1}$. In particular the sequence T_k is bounded in mass, and therefore weakly compact. If $T_{k_\nu} \to T$ is a weakly convergent subsequence, the limit T is a closed positive current, so $\{T\} = \{\Theta\}$ is pseudoeffective. Therefore $H^{1,1}_{\rm psef}(X)$ is closed. Similarly, if $\{\alpha_k\}$ converges to $\{\alpha\}$, we can select smooth representatives $\beta_k \in \{\alpha\} - \{\alpha_k\}$ converging to 0 in the C^∞ topology. Fix $\varepsilon > 0$. If $\{\alpha_k\}$ is nef, there is a smooth representative $\alpha_{k,\varepsilon}$ such that $\alpha_{k,\varepsilon} \ge -\varepsilon\omega$. For k large, we have $\beta_k \ge -\varepsilon\omega$, thus $\{\alpha\} = \{\alpha_k\} + \{\beta_k\}$ contains $\alpha_{k,\varepsilon} + \beta_k \ge -2\varepsilon\omega$. Hence $\{\alpha\}$ is nef and $H^{1,1}_{\rm nef}(X)$ is closed.

(ii) Let $\alpha_\varepsilon \in \{\alpha\}$ be a smooth form such that $\alpha_\varepsilon \ge -\varepsilon\omega$. Then $\alpha_\varepsilon + \varepsilon\omega \ge 0$ and the mass $\int_X (\alpha_\varepsilon + \varepsilon\omega)\wedge\omega^{n-1} = \int_X \alpha\wedge\omega^{n-1} + \varepsilon\int_X \omega^n$ is bounded. Therefore the family $\alpha_\varepsilon + \varepsilon\omega$ admits a weakly convergent subsequence in the sense of currents. If T is the limit, then T is closed positive and $\{\alpha\} = \{T\}$, so $\{\alpha\}$ is pseudoeffective.

(iii) Obviously, the Kähler cone is contained in $H^{1,1}_{\rm nef}(X)$. Conversely, if ω is a Kähler metric and if $\alpha_\varepsilon \in \{\alpha\}$ is a smooth representative with $\alpha_\varepsilon \ge -\varepsilon\omega$, then $\alpha_\varepsilon + 2\varepsilon\omega$ is a Kähler form whose cohomology class converges to $\{\alpha\}$.

(iv) We have $\int_C \alpha = \int_C \alpha_\varepsilon \ge -\varepsilon\int_C \omega$ for every $\varepsilon > 0$.

(v) If X is projective, there is a basis of $NS_{\mathbb{R}}(X)$ whose members are ample divisor classes $\gamma_1, \ldots, \gamma_r$ (r = Picard number of X). Then for any $\{\alpha\} \in NS_{\mathbb{R}}(X)$, there are arbitrary small numbers $\varepsilon_j > 0$ such that $\alpha + \sum \varepsilon_j \gamma_j$ is a rational class. Let p be an integer such that $p(\alpha + \sum \varepsilon_j \gamma_j)$ is integral. Then $p(\alpha + \sum \varepsilon_j \gamma_j)$ is the first Chern class of a line bundle L. Suppose that $\{\alpha\} \cdot C \geq 0$ for every curve C. We have

$$c_1(L) \cdot C \geq p \sum \varepsilon_j \{\gamma_j\} \cdot C \geq \mathrm{const} \int_C \omega$$

and Kleiman's criterion shows that L is ample. Hence L is positive and there is a representative $\alpha' \in \{\alpha\}$ such that $p(\alpha' + \sum \varepsilon_j \gamma_j) > 0$. In particular $\alpha' \geq -\sum \varepsilon_j \gamma_j \geq -\varepsilon\omega$ for ε_j small enough, and we see that $\{\alpha\}$ is nef.

(vii) The proof of (v) actually shows that every nef class $\{\alpha\} \in NS_{\mathbb{R}}(X)$ can be approximated by ample rational classes $\{\alpha' + \sum \varepsilon_j \gamma_j\}$.

(vi) By definition $H^{1,1}_{\mathrm{psef}}(X) \cap NS_{\mathbb{R}}(X)$ contains all classes of effective divisors. Conversely, if T is a closed positive current such that $\{T\} \in NS_{\mathbb{R}}(X)$, the arguments of (v) show that there are arbitrary small numbers ε_j, an integer p and a line bundle L such that $c_1(L) = p\big\{T + \sum \varepsilon_j \gamma_j\big\}$. This means that we can equip L with a singular metric (locally of the form $e^{-\varphi}$ for some psh function φ), such that the Chern curvature form of L is precisely

$$c(L) = p\Big(T + \sum \varepsilon_j \gamma_j\Big),$$

in particular $c(L) \geq \varepsilon\omega$ for some Kähler metric ω. We can then apply Hörmander's estimates (cf. corollary A.5) to construct sections of $K_X \otimes L^m$ for $m \geq m_0$ large. Hence $c_1(K_X \otimes L^m)$ contains the class $\{D\}$ of an effective divisor. We infer

$$\{T\} = \frac{1}{p} c_1(L) - \sum \varepsilon_j \{\gamma_j\} = \frac{1}{mp}\big(\{D\} - c_1(K_X)\big) - \frac{1}{p} \sum \varepsilon_j \{\gamma_j\}.$$

By taking m large and ε_j small, we see that $\{T\} - \frac{1}{mp}\{D\}$ can be made arbitrarily small. □

We now use our Main Theorem to give a proof of corollaries 1.5 and 1.6. The following elementary lemmas are needed.

Lemma 6.2. — *Let $\{u\}$ be a nef cohomology class on X such that $c_1(\mathcal{O}_{TX}(1)) + \pi_X^\star\{u\}$ is nef over $P(T^\star X)$. Let ω be a hermitian metric on X. Then for every $\varepsilon > 0$ there exists a smooth form $u_\varepsilon \in \{u\}$ with $u_\varepsilon \geq -\varepsilon\omega$ and a smooth hermitian metric on $\mathcal{O}_{TX}(1)$ such that*

$$c\big(\mathcal{O}_{TX}(1)\big) + \pi_X^\star(u_\varepsilon + \varepsilon\omega) \geq 0.$$

Proof. — Let h_ω be the metric on $\mathcal{O}_{TX}(1)$ associated to the metric of TX defined by ω, and let $c(\mathcal{O}_{TX}(1))_\omega$ be the curvature of h_ω. Since $c(\mathcal{O}_{TX}(1))_\omega$ is positive along the fibers of $\pi_X : P(T^\star X) \to X$, there is a constant $C \geq 1$ such that

$$\beta := c\big(\mathcal{O}_{TX}(1)\big)_\omega + \pi_X^\star(u + C\omega) > 0 \quad \text{on} \quad P(T^\star X).$$

Now $(1+\varepsilon)u_\varepsilon - \varepsilon u \in \{u\}$. Thus, by definition of nefness, there is for each $\varepsilon > 0$ a hermitian metric h_ε on $\mathcal{O}_{TX}(1)$ such that the associated curvature form satisfies

$$c\big(\mathcal{O}_{TX}(1)\big)_\varepsilon + \pi_X^\star\big((1+\varepsilon)u_\varepsilon - \varepsilon u\big) \geq -\varepsilon\beta.$$

Hence we get

$$c\big(\mathcal{O}_{TX}(1)\big)_\varepsilon + \varepsilon\, c\big(\mathcal{O}_{TX}(1)\big)_\omega + \pi_X^\star\big((1+\varepsilon)u_\varepsilon + \varepsilon C\omega\big) \geq 0.$$

This implies that the curvature of the barycenter metric $(h_\varepsilon h_\omega^\varepsilon)^{1/(1+\varepsilon)}$ satisfies

$$c\big(\mathcal{O}_{TX}(1)\big) + \pi_X\Big(u_\varepsilon + \frac{\varepsilon C}{1+\varepsilon}\omega\Big) \geq 0.$$

The lemma follows by replacing ε with ε/C. □

Lemma 6.3. — *Let T be a closed positive $(1,1)$-current which is smooth on $X \setminus \Sigma$ for some finite subset $\Sigma \subset X$. Then $\{T\}$ is nef.*

Proof. — For all $x_j \in \Sigma$, fix disjoint small coordinate neighborhoods V_j of x_j and write $T = \frac{i}{\pi}\partial\overline{\partial}\varphi_j$ with psh functions $\varphi_j \in C^\infty(\overline{V}_j \setminus \{x_j\})$. Let θ_j be a cut-off function equal to 1 near x_j with support in V_j. Then $\theta_j(z)\log|z-x_j|$ has a Hessian which is bounded below by $-C\omega$ for some positive constant C. We denote by $\max_\varepsilon = \max\star\rho_\varepsilon$ a regularized max function and set

$$\varphi'_j(z) = \max\nolimits_\varepsilon\big(\varphi_j(z) + \varepsilon\theta_j(z)\log|z-x_j|\ ,\ -A\big) \quad \text{on}\ \ V_j,$$

with a large constant $A > 0$. Then φ'_j is smooth on $\overline{V}_j$, coincides with φ_j near ∂V_j and satisfies $\frac{i}{\pi}\partial\overline{\partial}\varphi'_j \geq -C\varepsilon\,\omega$. Therefore the current T' equal to T on $X \setminus \bigcup V_j$ and to $\frac{i}{\pi}\partial\overline{\partial}\varphi'_j$ on V_j is smooth. We have $T' \in \{T\}$ and $T' \geq -C\varepsilon\,\omega$, thus $\{T\}$ is nef. □

Proof of corollary 1.5. — Let T be a closed positive $(1,1)$-current and let $b_n = \inf\{c > 0\,;\, \operatorname{codim} E_c(T) \geq n\}$. Fix u_ε as in Lemma 6.2 and $c > b_n$. Then Theorem 1.1 applied with $\gamma = 0$ and $u = u_\varepsilon + \varepsilon\omega$ shows that there is $T_{c,k} \in \{T\}$ such that $T_{c,k}$ is smooth on $X \setminus E_c(T)$ and

$$T_{c,k} \geq -c(u_\varepsilon + \varepsilon\omega) - \varepsilon\omega,$$

thus $T_{c,k} + c\,u_\varepsilon \geq -(c+1)\varepsilon\omega$. Now, $E_c(T)$ is a finite set. The arguments given in Lemma 6.3 show that $\{T_{c,k} + c\,u_\varepsilon\} = \{T\} + c\{u\}$ is nef. Since this is true for all $c > b_n$, we conclude that $\{T\} + b_n\{u\}$ is nef and corollary 1.5 is proved. When $b_n = 0$ we get in particular the following much stronger version of Lemma 6.3 (the existence of a nef class $\{u\}$ is not needed in that statement). □

Corollary 6.4. — *Let T be a closed positive current which has zero Lelong numbers except perhaps on a countable subset of X. Then $\{T\}$ is nef.*

Proof of corollary 1.6. — Let X be a manifold in the Fujiki class $\mathcal{C}$ such that TX is nef. We know by Fujiki [Fu 78] that X has a smooth Kähler modification

$\mu : \widetilde{X} \to X$. Let α be a Kähler metric on $\widetilde{X}$ and let $T = \mu_\star\alpha$ be the direct image current. If ω is a hermitian metric on X, we have $\alpha \ge \delta\,\mu^\star\omega$ on $\widetilde{X}$ for some small constant $\delta > 0$, thus by taking the direct image we get $T = \mu_\star\alpha \ge \delta\omega$ on X. Now Lemma 6.2 shows the existence of smooth hermitian metrics on $\mathcal{O}_{TX}(1)$ such that $c(\mathcal{O}_{TX}(1)) + \pi_X^\star(\varepsilon\omega) \ge 0$. Fix $c > \max_X \nu(T,x)$. The Main Theorem produces a smooth approximation $T_{c,k}$ of T such that $T_{c,k} \ge \delta\omega - c\varepsilon\,\omega - \varepsilon_k\omega$. Therefore $T_{c,k} \ge \frac{\delta}{2}\omega$ if ε is small enough and k large. Thus $T_{c,k}$ is a Kähler metric and X is Kähler. In particular, if X is Moishezon with TX nef, we can take α to be the curvature form of an ample line bundle $\mathcal{O}(D)$ over $\widetilde{X}$. Then T and $T_{c,k}$ are cohomologous to $\mu_\star([D]) = [D']$ where $D' = \mu(D)$ is the image of D in X. This implies that $\mathcal{O}(D')$ is ample and therefore X is projective algebraic (we could also have applied the well-known result of Moishezon [Mo 67] that every Kähler Moishezon manifold is projective algebraic). The converse implications in corollary 1.6 are trivial. □

7. Application to self-intersection inequalities

An interesting application of our regularization technique is the estimation of a global intersection number $\int_X \{T_1\} \wedge \ldots \wedge \{T_n\}$ of closed positive currents of type $(1,1)$, when the pointwise product $T_1 \wedge \ldots \wedge T_n$ is not well defined as a current. In fact, explicit examples show that $T_1 \wedge \ldots \wedge T_n$ cannot be defined in a reasonable way without additional hypotheses (see e.g. [Ki 84] and [De 91]); in the case of a self-intersection with $T_1 = \ldots = T_n$ equal to the current of integration over an exceptional divisor E, it may even happen that $\int_X \{E\}^n < 0$. In order to overcome this difficulty, the idea is to replace the currents T_j by smooth approximations and to compute the limit. Such a method has been already used in [De 90a] to get asymptotic bounds for the dimensions of cohomology groups with values in line bundles $\mathcal{O}(kD)$, when D is an effective divisor.

The self-intersection inequality 1.7 presented here is another direct application which uses the full force of Theorem 1.1 (see also [De 90b] for an independent proof in the special case where X is projective algebraic and $T \in H^2(X, \mathbb{Z})$). In fact, inequality 1.7 can be somewhat improved if the cohomology class $\{T\}$ is supposed to be nef. In that case, it becomes possible to kill the singularities "locally" on the manifold. More precisely, suppose we are given an arbitrary subset $\Xi \subset X$. We define the sequence of *jumping values of T with respect to Ξ* by

$$(7.1) \qquad b_p = b_p(T,\Xi) = \inf\big\{c > 0\,;\, \operatorname{codim}\big(E_c(T), x\big) \ge p,\ \forall x \in \Xi\big\},$$

with the convention that a germ has codimension $> n$ if and only if it is empty. Then $0 = b_1 \le b_2 \le \ldots \le b_n \le b_{n+1}$ with $b_{n+1} = \max_{x\in\Xi} \nu(T,x) < +\infty$, and for $c \in \,]b_p, b_{p+1}]$, the set $E_c(T)$ has codimension $\ge p$ at every point of Ξ and has at least one component of codimension p exactly intersecting Ξ.

THEOREM 7.2. — *If $\{T\}$ is nef, the inequalities of Theorem 1.7 hold with $b_p = b_p(T,\Xi)$ and $Z_{p,k}$ associated to an arbitrary subset Ξ of X.*

Before entering into the proof, let us recall a few basic facts of current theory (see [Siu 74] and [De 91] for details).

LEMMA 7.3 (Siu's decomposition formula [Siu 74]). — *Let Θ be a closed positive current of bidimension (p,p) and let Z_k be the collection of all p-dimensional components appearing in the upperlevel sets $E_c(T)$. Then*

$$\Theta = \sum \lambda_k [Z_k] + R$$

where $\lambda_k = \inf_{x \in Z_k} \nu(\Theta, x)$ is the generic Lelong number of Θ along Z_k and R is a closed positive current such that $\dim E_c(R) < p$ for every $c > 0$.

We also need a few results about intersection theory of closed positive currents. Let $T_1, \ldots, T_q$ be closed positive currents of bidegree $(1,1)$ on a complex manifold M. We can write $T_j = \frac{i}{\pi} \partial\overline{\partial} \varphi_j$ locally on every contractible Stein open subset $U \subset M$, for some plurisubharmonic function φ_j. If φ is psh or almost psh, we define the *unbounded locus* $L(\varphi)$ to be the set of points x such that φ is not bounded below near x.

LEMMA 7.4 ([De 91]). — *Let Θ be a closed positive current of bidimension (p,p). The wedge product $T_1 \wedge \ldots \wedge T_q \wedge \Theta$ is well defined as soon as*

$$L(\varphi_{j_1}) \cap \ldots \cap L(\varphi_{j_m}) \cap \operatorname{Supp} \Theta$$

has zero Hausdorff measure $\mathcal{H}_{2p-2m+1}$ for every choice of indices $j_1 < \ldots < j_m$ in $\{1, \ldots, q\}$. In particular, this is the case if $L(\varphi_{j_1}) \cap \ldots \cap L(\varphi_{j_m})$ is contained in an analytic subset of dimension $\leq p - m$. Moreover, the product is weakly continuous with respect to monotone decreasing sequences of plurisubharmonic functions $\varphi_j^k \to \varphi_j$ when k tends to $+\infty$.

The proof of this result is obtained essentially by using Bedford and Taylor's definition of Monge-Ampère operators (see [De 91]): we set $i\partial\overline{\partial}\varphi \wedge \Theta := i\partial\overline{\partial}(\varphi\Theta)$ when Θ is a closed positive current and φ is a locally bounded psh function. Suitable integration by parts then give inequalities of Chern-Levine-Nirenberg type and show that the product $i\partial\overline{\partial}\varphi_1 \wedge \ldots \wedge i\partial\overline{\partial}\varphi_q \wedge \Theta$ can be extended through the singularities of the potentials φ_j. Finally, when the product $i\partial\overline{\partial}\varphi \wedge \Theta$ is well defined, we have at every point x

$$\nu(i\partial\overline{\partial}\varphi \wedge \Theta, x) \geq \nu(i\partial\overline{\partial}\varphi, x)\, \nu(\Theta, x). \tag{7.5}$$

Proof of Theorem 1.7. — We argue by induction on p. For $p = 1$, Siu's decomposition formula shows that

$$T = \sum \nu_{1,k} [Z_{1,k}] + R,$$

where R is a closed positive current such that $\operatorname{codim} E_c(R) \geq 2$ for every $c > 0$. We have $R \geq T_{\mathrm{abc}}$ since the other part has singular measures as coefficients. The result is thus true with $\Theta_1 = T$.

Now, suppose that Θ_{p-1} has been constructed. For $c > b_p$, the current $T_{c,k} = \alpha + \frac{i}{\pi}\partial\overline{\partial}\psi_{c,k}$ produced by the Main Theorem is such that $T_{c,k}+c\,u+\varepsilon_k\omega \geq 0$ for some sequence $\varepsilon_k > 0$ converging to 0, thanks to (1.1 i, ii). Since $L(\psi_{c,k}) \subset E_c(T)$ and $\operatorname{codim} E_c(T) \geq p$, Lemma 7.4 shows that

$$\Theta_{p,c,k} = \Theta_{p-1} \wedge (T_{c,k} + c\,u + \varepsilon_k\omega)$$

is a well defined closed positive current. As $\{T_{c,k}\} = \{T\}$, the cohomology class $\{\Theta_{p,c,k}\}$ converges to $\{\Theta_{p-1}\}\cdot(\{T\}+c\{u\})$. In particular, the mass $\int_X \Theta_{p,c,k}\wedge\omega^{n-p}$ remains uniformly bounded and the family $(\Theta_{p,c,k})_{c\in]b_p,b_p+1],k\geq 1}$ is relatively compact in the weak topology (observe that the Kähler assumption has been used here). We define

$$\Theta_p = \lim_{c\to b_p+0}\ \lim_{k\to+\infty} \Theta_{p,c,k},$$

possibly after extracting some weakly convergent subsequence. Then $\{\Theta_p\} = \{\Theta_{p-1}\}\cdot(\{T\}+b_p\{u\})$, and so $\{\Theta_p\} = (\{T\}+b_1\{u\})\cdots(\{T\}+b_p\{u\})$. Moreover, we have

$$\begin{aligned}
\nu(\Theta_p, x) &\geq \limsup_{c\to b_p+0}\ \limsup_{k\to+\infty}\ \nu\big(\Theta_{p-1}\wedge(T_{c,k}+c\,u+\varepsilon_k\omega), x\big)\\
&\geq \nu(\Theta_{p-1}, x) \times \limsup_{c\to b_p+0}\ \limsup_{k\to+\infty}\ \nu(T_{c,k}, x)\\
&\geq \nu(\Theta_{p-1}, x)\big(\nu(T,x)-b_p\big)_+
\end{aligned}$$

by application of (1.1 iv) and (7.5) (and by the fact that the Lelong number of a weak limit is always at least equal to the limsup of the Lelong numbers). By induction we thus get

$$\nu(\Theta_p, x) \geq \big(\nu(T,x)-b_1\big)_+ \ldots \big(\nu(T,x)-b_p\big)_+,$$

in particular, the generic Lelong number of Θ_p along $Z_{p,k}$ is at least equal to $(\nu_{p,k}-b_1)\ldots(\nu_{p,k}-b_p)$. Siu's decomposition formula implies

$$\Theta_p \geq \sum_{k\geq 1}(\nu_{p,k}-b_1)\ldots(\nu_{p,k}-b_p)\,[Z_{p,k}].$$

Since the right hand side is Lebesgue singular, the desired inequality will be proved if we show in addition that

$$\Theta_{p,\mathrm{abc}} \geq (T_{\mathrm{abc}}+b_1u)\wedge\ldots\wedge(T_{\mathrm{abc}}+b_pu),$$

or inductively, that $\Theta_{p,\mathrm{abc}} \geq \Theta_{p-1,\mathrm{abc}}\wedge(T_{\mathrm{abc}}+b_pu)$. In order to do this, we simply have to make sure that $\lim_{k\to+\infty} T_{c,k,\mathrm{abc}} = T_{\mathrm{abc}}$ almost everywhere and to use induction again. But our arguments are not affected if we replace $\psi_{c,k}$ by $\psi'_{c,k} = \max\{\psi, \psi_{c,k}-A_k\}$ with A_k converging quickly to $+\infty$. It is then easy to show that a suitable choice of A_k gives $\lim(i\partial\overline{\partial}\psi'_{c,k})_{\mathrm{abc}} = (i\partial\overline{\partial}\psi)_{\mathrm{abc}}$ almost everywhere (see Lemma 10.12 in [De 90b]). □

Proof of Theorem 7.2. — We apply a similar induction on p. When $\Xi \neq X$ and $c > b_p = b_p(T,\Xi)$, the current $T_{c,k}$ may have singularities along some components of $E_c(T)$ of codimension $< p$ which do not intersect Ξ. In view of

Lemma 7.3, this might create trouble in the definition of $\Theta_{p,c,k}$. When $\{T\}$ is supposed to be nef, we can modify the inductive construction of Θ_p as follows. Let $\alpha_k \in \{T\}$ be a smooth representative such that $\alpha_k \geq -\varepsilon_k\omega$. Then we have $\alpha_k = \alpha + \frac{i}{\pi}\partial\overline{\partial}\varphi_k$ for some smooth function φ_k. We set

$$\psi_{c,k,A} = \max\{\psi_{c,k}\,,\, \varphi_k - A\}$$

with a large constant $A > 0$, and replace $T_{c,k}$, $\Theta_{p,c,k}$ by

$$T_{c,k,A} = \alpha + \frac{i}{\pi}\partial\overline{\partial}\psi_{c,k,A}, \qquad \Theta_{p,c,k,A} = \Theta_{p-1} \wedge (T_{c,k,A} + c\,u + \varepsilon_k\omega).$$

We have $\frac{i}{\pi}\partial\overline{\partial}\psi_{c,k} \geq -\alpha - c\,u - \varepsilon_k\omega$ by (1.1 i, ii) and $\frac{i}{\pi}\partial\overline{\partial}\varphi_k = \alpha_k - \alpha \geq -\alpha - \varepsilon_k\omega$. Therefore $\frac{i}{\pi}\partial\overline{\partial}\psi_{c,k,A}$ satisfies the same lower bound as $\psi_{c,k}$ and we get the inequality $T_{c,k,A} + c\,u + \varepsilon_k\omega \geq 0$ as before. Since the potential $\psi_{c,k,A}$ is bounded everywhere, the product $\Theta_{p,c,k,A}$ is well defined on X. We then define $\Theta_{p,c,k}$ by extracting a weak limit

$$\Theta_{p,c,k} = \lim_{A\to+\infty} \Theta_{p,c,k,A} \quad \text{ on } X.$$

By definition of $b_p = b_p(T,\Xi)$, there is a neighborhood V of Ξ such that $E_c(T) \cap V$ only contains components of codimension $\geq p$ when $c > b_p$. As $\lim_{A\to+\infty} \psi_{c,k,A} = \psi_{c,k}$, the continuity result of Lemma 7.4 implies

$$\Theta_{p,c,k} = \Theta_{p-1} \wedge (T_{c,k} + c\,u + \varepsilon_k\omega) \quad \text{ on } V.$$

The rest of the proof is identical to that of Th. 1.7. □

In the case $b_n = 0$, we get the following interesting complement to Corollary 6.4 by applying inequality 1.7 with $p = n$.

Corollary 7.6. — *Let T be a closed positive $(1,1)$-current which has zero Lelong numbers except perhaps on a countable subset of X. Then we have*

$$\int_X \{T\}^n \geq \sum_{x\in X} \nu(T,x)^n + \int_X T^n_{\mathrm{abc}}.$$

A. Appendix: basic results on L^2 estimates

We state here the basic L^2 existence theorems used in the above sections, concerning $\overline{\partial}$ equations or holomorphic functions. The first of these is the intrinsic manifold version of Hörmander's L^2 estimates [Hö 65,66], based on the Bochner-Kodaira-Nakano technique (see also Andreotti-Vesentini [AV 65]).

Theorem A.1. — *Let L be a holomorphic line bundle on a weakly pseudoconvex n-dimensional manifold X equipped with a Kähler metric ω. Suppose that L has a smooth hermitian metric whose curvature form satisfies*

$$2\pi\, c(L) + i\partial\overline{\partial}\varphi \geq A\,\omega$$

where φ is an almost psh function and A a positive continuous function on X. Then for every form v of type (n,q), $q \geq 1$, with values in L, such that $\overline{\partial} v = 0$ and

$$\int_X \frac{1}{A}|v|^2 e^{-\varphi} dV_\omega < +\infty,$$

there exists a form u of type $(n,q-1)$ with values in L such that $\overline{\partial} u = v$ and

$$\int_X |u|^2 e^{-\varphi} dV_\omega \leq \frac{1}{q}\int_X \frac{1}{A}|v|^2 e^{-\varphi} dV_\omega.$$

A *weakly pseudoconvex manifold* is by definition a complex manifold possessing a smooth weakly pseudoconvex exhaustion function (examples: Stein manifolds, compact manifolds, the total space of a Griffiths weakly negative vector bundle, ...). Suppose that φ has Lelong number $\nu(\varphi, x) = 0$ at a given point x. Then for every m the weight $e^{-m\varphi}$ is integrable in a small neighborhood V of x (see [Sk 72a]). Let θ be a cut-off function equal to 1 near x, with support in V. Let z be coordinates and let e be a local frame of L on V. For ε small enough, the curvature form

$$2\pi\, c(L) + i\partial\overline{\partial}\big(\varphi(z) + 2\varepsilon\theta(z)\log|z-x|\big)$$

is still positive definite. We apply A.1 to the bundle L^m equipped with the corresponding weight $m(\varphi(z) + 2\varepsilon\theta(z)\log|z-x|)$, and solve the equation $\overline{\partial} u = v$ for the $(n,1)$-form $v = \overline{\partial}(\theta(z)P(z)dz_1 \wedge \ldots \wedge dz_n \otimes e^m)$ associated to an arbitrary polynomial P. The L^2 estimate shows that the solution u has to vanish at order $\geq q+1$ at x where $q = [m\varepsilon] - n$, hence

$$\theta(z)P(z)dz_1 \wedge \ldots \wedge dz_n \otimes e^m - u(z)$$

is a holomorphic section of $K_X \otimes L^m$ with prescribed jet of order q at x.

Corollary A.2. — *Suppose that $2\pi\, c(L) + i\partial\overline{\partial}\varphi \geq \delta\,\omega$ for some $\delta > 0$. Let $x \in X$ be such that $\nu(\varphi, x) = 0$. Then there exists $\varepsilon > 0$ such that the sections in $H^0(X, K_X \otimes L^m)$ generate all jets of order $\leq m\varepsilon$ at x for m large.* □

The Ohsawa-Takegoshi estimates were needed in section 3 in the following very special case ([Oh 88], Cor. 2 p. 266).

Proposition A.3. — *Let Y be a pure dimensional closed complex submanifold of $\mathbb{C}^n$, let Ω be a bounded pseudoconvex open set and let φ be a plurisubharmonic function on Ω. Then for any holomorphic function f on $Y \cap \Omega$ with*

$$\int_{Y\cap\Omega} |f|^2 e^{-\varphi} dV_Y < +\infty,$$

there exists a holomorphic extension F to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} dV \leq A \int_{Y\cap\Omega} |f|^2 e^{-\varphi} dV_Y < +\infty.$$

Here A depends only on Y and on the diameter of Ω.

Finally, a crucial application of Skoda's L^2 estimates [Sk 72b] for ideals of holomorphic functions was made in section 5:

THEOREM A.4. — *Let φ be a plurisubharmonic function on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ and let $\sigma_1, \ldots, \sigma_N$ be holomorphic functions on Ω (the sequence σ_j can be infinite). Set $q = \min\{N-1, n\}$ and $|\sigma|^2 = \sum |\sigma_j|^2$. Then, for every holomorphic function f on Ω such that*

$$\int_\Omega |f|^2 |\sigma|^{-2(q+1+\alpha)} e^{-\varphi} dV < +\infty, \qquad \alpha > 0,$$

there exist holomorphic functions $g_1, \ldots, g_N$ on Ω such that $f = \sum_{1 \le j \le N} g_j \sigma_j$ and

$$\int_\Omega |g|^2 |\sigma|^{-2(q+\alpha)} e^{-\varphi} dV \le \frac{\alpha+1}{\alpha} \int_\Omega |f|^2 |\sigma|^{-2(q+1+\alpha)} e^{-\varphi} dV < +\infty.$$

COROLLARY A.5. — *With the same notations, suppose that*

$$\int_\Omega |f|^2 |\sigma|^{-2(q+m+\alpha)} e^{-\varphi} dV < +\infty$$

for some $\alpha > 0$ and some integer $m \ge 1$. Then there exist holomorphic functions g_L for all $L = (\ell_1, \ldots, \ell_m) \in \{1, \ldots, N\}^m$ such that

$$f = \sum_L g_L \sigma^L \quad \text{with} \quad \sigma^L = \sigma_{\ell_1} \sigma_{\ell_2} \ldots \sigma_{\ell_m},$$

$$\int_\Omega \sum_L |g_L|^2 |\sigma|^{-2(q+\alpha)} e^{-\varphi} dV \le \frac{\alpha+m}{\alpha} \int_\Omega |f|^2 |\sigma|^{-2(q+m+\alpha)} e^{-\varphi} dV < +\infty.$$

Proof. — Use induction on m: if the result is true for $(m-1, \alpha+1)$ then $f = \sum_\Lambda g_\Lambda \sigma^\Lambda$ with Λ of length $m-1$, and each function g_Λ can be written $g_\Lambda = \sum_{\ell_m} g_L \sigma_{\ell_m}$ with $L = (\Lambda, \ell_m)$ and

$$\int_\Omega \sum_{\ell_m} |g_L|^2 |\sigma|^{-2(q+\alpha)} e^{-\varphi} dV \le \frac{\alpha+1}{\alpha} \int_\Omega |G_\Lambda|^2 |\sigma|^{-2(q+1+\alpha)} e^{-\varphi} dV < +\infty,$$

$$\int_\Omega \sum_\Lambda |g_\Lambda|^2 |\sigma|^{-2(q+1+\alpha)} e^{-\varphi} dV \le \frac{\alpha+m}{\alpha+1} \int_\Omega |f|^2 |\sigma|^{-2(q+m+\alpha)} e^{-\varphi} dV < +\infty. \quad \square$$

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