INSTITUT DE FRANCE
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Entire curves in complex projective varieties and differential equations

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Introduction and goals

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Assume that X is of general type, i.e. $\kappa(X) = n = \dim X$ where

$$\kappa(X) := \limsup_{m \to +\infty} \frac{\log h^0(X, K_X^{\otimes m})}{\log m}.$$

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Arithmetic counterpart (Lang 1987) – very optimistic ?

For X projective defined over a number field $\mathbb{K}_0$, the exceptional locus $Y = \mathrm{Exc}(X)$ in GGL's conjecture equals $\mathrm{Mordell}(X) =$ smallest Y such that $X(\mathbb{K}) \smallsetminus Y$ is finite, $\forall \mathbb{K}$ number field $\supset \mathbb{K}_0$.

Category of directed varieties

More generally, we are interested in entire curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$, where V is a (possibly singular) linear subspace of X, i.e. a closed irreducible analytic subspace such that $\forall x \in X$, $V_x := V \cap T_{X,x}$ is linear.

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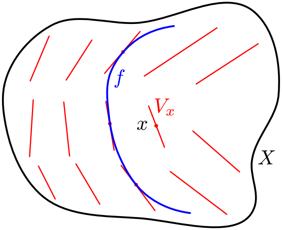

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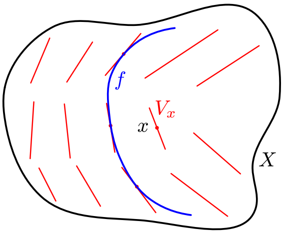

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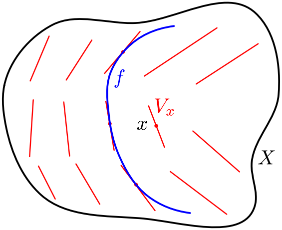

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– Arrows $\psi : (X, V) \to (Y, W)$ holomorphic s.t. $d\psi(V) \subset W$
– "Absolute case" (X, T_X), i.e. $V = T_X$
– "Relative case" $(X, T_{X/S})$ where $X \to S$
– "Integrable case" when $[\mathcal{O}(V), \mathcal{O}(V)] \subset \mathcal{O}(V)$ (foliations)

Canonical sheaf of a directed variety (X,V)

Canonical sheaf of a directed manifold (X, V)

When V is nonsingular, i.e. a subbundle, one simply sets

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$$\mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*) \to \mathcal{L}_V := \text{invert. sheaf } \mathcal{O}(\Lambda^r V^*)^{**}$$

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Caution

One may have to first blow up X, otherwise ${}^b\mathcal{K}_V$ need not always provide the appropriate geometric information.

Canonical sheaf of a directed variety (X,V) [sequel]

Blow up process for a directed variety

If $\mu : \widetilde{X} \to X$ is a modification, then $\widetilde{X}$ is equipped with the pull-back directed structure $\widetilde{V} = \overline{\tilde{\mu}^{-1}(V_{|X'})}$, where $X' \subset X$ is a Zariski open set over which μ is a biholomorphism.

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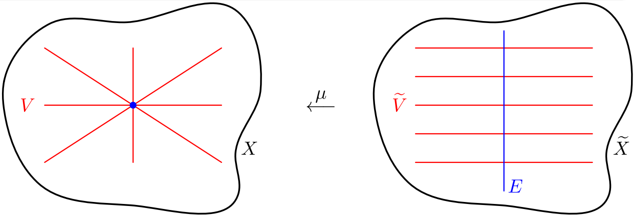

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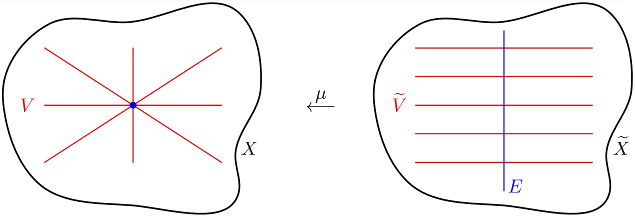

Observation

One always has

$$^b\mathcal{K}_V \subset \mu_*({}^b\mathcal{K}_{\widetilde{V}}) \subset \mathcal{L}_V = \mathcal{O}(\det V^*)^{**},$$

and $\mu_*({}^b\mathcal{K}_{\widetilde{V}})$ "increases" with μ (taking $\widetilde{\widetilde{X}} \to \widetilde{X} \to X$).

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By Noetherianity, one can define a sequence of rank 1 sheaves

$$\mathcal{K}_V^{[m]} = \lim_{\mu} \uparrow \mu_*({}^b\mathcal{K}_{\widetilde{V}})^{\otimes m}, \quad \mu_*({}^b\mathcal{K}_V)^{\otimes m} \subset \mathcal{K}_V^{[m]} \subset \mathcal{L}_V^{\otimes m}$$

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Definition

We say that (X, V) is of general type if the pluricanonical sheaf sequence $\mathcal{K}_V^{[\bullet]}$ is big, i.e. $H^0(X, \mathcal{K}_V^{[m]})$ provides a generic embedding of X for a suitable $m \gg 1$.

Generalized Green-Griffiths-Lang conjecture

Generalized GGL conjecture

If (X, V) is directed manifold of general type, i.e. $\mathcal{K}_V^{[\bullet]}$ is big, then there exists an algebraic locus $Y \subsetneq X$ such that for every $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one has $f(\mathbb{C}) \subset Y$.

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Basic strategy

Show that the entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ must satisfy nontrivial algebraic differential equations $P(f\,;\, f', f'', \ldots, f^{(k)}) = 0$, and actually, many such equations.

Definition of algebraic differential operators

Let $(\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, $t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$ be a curve, $f(0) = x$, and pick local holomorphic coordinates $(z_1, \ldots, z_n)$ centered at x on a coordinate open set $U \simeq U' \times U'' \subset \mathbb{C}^r \times \mathbb{C}^{n-r}$ such that $\pi' : U \to U'$ induces an isomorphism $d\pi' : V \to U \times \mathbb{C}^r$.

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$$\pi' \circ f(t) = t\xi_1 + \ldots + t^k \xi_k + O(t^{k+1}), \quad \xi_s = \frac{1}{s!} \nabla^s f(0),$$

where ∇ is the trivial connection on $V \simeq U \times \mathbb{C}^r$.

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One considers the Green-Griffiths bundle $E_{k,m}^{\mathrm{GG}} V^*$ of polynomials of weighted degree m, written locally in coordinate charts as

$$P(x\,;\, \xi_1, \ldots, \xi_k) = \sum a_{\alpha_1 \alpha_2 \ldots \alpha_k}(x) \xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k}, \quad \xi_s \in V.$$

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These can also be viewed as algebraic differential operators

$$\begin{aligned} P(f_{[k]}) &= P(f\,;\,f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k}. \end{aligned}$$

Definition of algebraic differential operators [sequel]

Here $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet, and $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are supposed to holomorphic functions on X.

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The reparametrization action : $f \mapsto f \circ \varphi_\lambda$, $\varphi_\lambda(t) = \lambda t$, $\lambda \in \mathbb{C}^*$ yields $(f \circ \varphi_\lambda)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$, whence a $\mathbb{C}^*$-action

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Direct image formula

If $J^{\mathrm{nc}}_k V$ is the set of non constant k-jets, one defines the Green-Griffiths bundle to be $X^{\mathrm{GG}}_k = J^{\mathrm{nc}}_k V/\mathbb{C}^*$ and $\mathcal{O}_{X^{\mathrm{GG}}_k}(1)$ to be the associated tautological rank 1 sheaf. Then we have

$$\pi_k : X^{\mathrm{GG}}_k \to X, \qquad E^{\mathrm{GG}}_{k,m} V^* = (\pi_k)_* \mathcal{O}_{X^{\mathrm{GG}}_k}(m).$$

Main cohomology estimates

As an application of holomorphic Morse inequalities, one can get the following fundamental estimates.

Theorem (D-, 2010)

Let (X, V) be a directed manifold, $A \to X$ an ample $\mathbb{Q}$-line bundle, (V, h) and (A, h_A) hermitian, $\Theta_{A,h_A} > 0$. Define

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^* \mathcal{O}\Big(-\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)A\Big),$$

$$\eta = \Theta_{\det V^*, \det h^*} - \Theta_{A,h_A}.$$

Then for all $q \geq 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have upper and lower bounds [$q = 0$ is most useful!]

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \leq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r} \Big(\int_{X(\eta,q)} (-1)^q \eta^n + \frac{C}{\log k}\Big)$$

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Holomorphic Morse inequalities: main statement

The *q*-index set of a real $(1,1)$-form θ is defined to be

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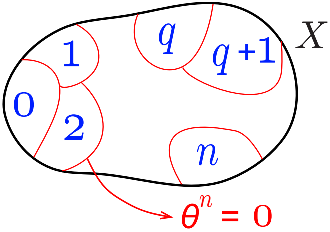
Set also $X(\theta, \leq q) = \bigcup_{0 \leq j \leq q} X(\theta, j)$.

$X(\theta, q)$ and $X(\theta, \leq q)$ are open sets.

$\mathrm{sign}(\theta^n) = (-1)^q$ on $X(\theta, q)$.



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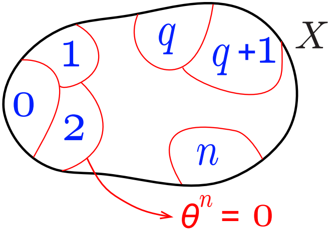
Set also $X(\theta,\le q) = \bigcup_{0\le j\le q} X(\theta,j)$.

$X(\theta,q)$ and $X(\theta,\le q)$ are open sets.

$\mathrm{sign}(\theta^n) = (-1)^q$ on $X(\theta,q)$.



Theorem (D-, 1985)

Let (L,h) be a hermitian line bundle on X, $\mathcal{F}$ a coherent sheaf, $\theta = \Theta_{L,h}$ and $r = \mathrm{rank}\,\mathcal{F}$. Then, as $m \to +\infty$

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes \mathcal{F}) \le r\,\frac{m^n}{n!}\int_{X(\theta,\le q)} (-1)^q \theta^n + o(m^n).$$

1^{st} step: define a Finsler metric on k-jet bundles

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$$\Psi_{h_k}(f) := \Big(\sum_{1\le s\le k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s}\Big)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

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Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k^{\rm GG}}(1)$, with curvature form $(x, \xi_1, \ldots, \xi_k) \mapsto$

$$\Theta_{L_k,h_k} = \omega_{{\rm FS},k}(\xi) + \frac{i}{2\pi} \sum_{1 \le s \le k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor Θ_{V^*,h^*} and $\omega_{{\rm FS},k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\rm GG} \to X$.

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The expression gets simpler by using polar coordinates

$$x_s = |\xi_s|_h^{2p/s}, \qquad u_s = \xi_s / |\xi_s|_h = \nabla^s f(0) / |\nabla^s f(0)|.$$

2^{nd} step: probabilistic interpretation of the curvature

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi} \sum_{1 \le s \le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha} \overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors u_s in the unit sphere bundle $SV \subset V$.

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The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum|\xi_s|^{2p/s}=1$, so we can take here $x_s\ge 0$, $\sum x_s=1$. This is essentially a sum of the form $\sum\frac{1}{s}Q(u_s)$ where $Q(u)=\langle\Theta_{V^*,h^*}u,u\rangle$ and u_s are random points of the sphere, and so as $k\to+\infty$ this can be estimated by a "Monte-Carlo" integral

$$\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\int_{u\in SV}Q(u)\,du.$$

As Q is quadratic, $\displaystyle\int_{u\in SV}Q(u)\,du=\frac{1}{r}\,\mathrm{Tr}(Q)=\frac{1}{r}\,\mathrm{Tr}(\Theta_{V^*,h^*})=\frac{1}{r}\Theta_{\det V^*}.$

Fundamental vanishing theorem and diff. equations

Passing to a "singular version" of holomorphic Morse inequalities to accommodate singular metrics ([Bonavero, 1996]), one gets

Corollary: existence of global jet differentials (D-, 2010)

Let (X, V) be of general type, i.e. ${}^b\mathcal{K}_V^{\otimes p}$ big rank 1 sheaf, and let

$$L_{k,\varepsilon} = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}(-\delta_k \varepsilon A), \quad \delta_k = \frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big),$$

with A ample. Then there exist many nontrivial global sections

$$P \in H^0(X_k^{\mathrm{GG}}, L_{k,\varepsilon}^{\otimes m}) \simeq H^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-m\delta_k \varepsilon A))$$

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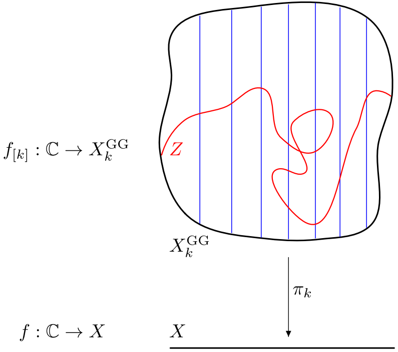
Fundamental vanishing theorem $\Rightarrow$ differential equations

[Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]
For all global differential operators $P \in H^0(X, E_{k,m}^{\rm GG} V^* \otimes \mathcal{O}(-qA))$, $q \in \mathbb{N}^*$, and all $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one has $P(f_{[k]}) \equiv 0$.

The base locus problem

Geometrically, this can interpreted by stating that the image $f_{[k]}(\mathbb{C})$ of the k-jet curve lies in the base locus

$$Z = \bigcap_{m \in \mathbb{N}^*} \bigcap_{\sigma \in H^0(X_k^{\mathrm{GG}}, L_{k,\varepsilon}^{\otimes m})} \sigma^{-1}(0) \subset X_k^{\mathrm{GG}}.$$



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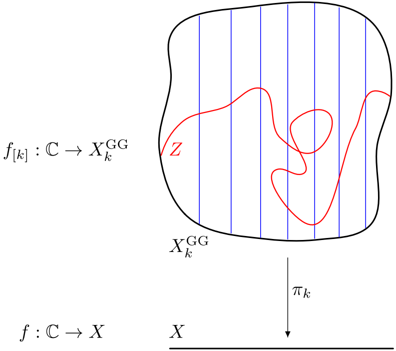

To prove the GGL conjecture, we would need to get $\pi_k(Z) \subsetneq X$.

General problem concerning base loci

Let (L, h) be a hermitian line bundle over X. If we assume that $\theta = \Theta_{L,h}$ satisfies $\int_{X(\theta,\leq 1)} \theta^n > 0$, then we know that L is big, i.e. that $h^0(X, L^{\otimes m}) \geq c\, m^n$, for $m \geq m_0$ and $c > 0$,

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Definition

The "iterated base locus" $\mathrm{IBs}(L)$ is obtained by picking inductively $Z_0 = X$ and $Z_k =$ zero divisor of a section σ_k of $L^{\otimes m_k}$ over the normalization of Z_{k-1}, and taking $\bigcap_{k, m_1, \ldots, m_k, \sigma_1, \ldots, \sigma_k} Z_k$.

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Find a condition, e.g. in the form of Morse integrals (or analogs) for $\theta=\Theta_{L,h}$, ensuring for instance that $\operatorname{codim}\mathrm{IBs}(L)>p$.

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We would need for instance to be able to check the positivity of Morse integrals $\int_{Z(\theta_{|Z}, \le 1)} \theta^{n-p}$ for Z irreducible, $\operatorname{codim} Z = p$.

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Let (X, V) be a directed variety of general type. Then there exists $k_0 \in \mathbb{N}$ and $\delta > 0$ with the following properties.

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with $k \ge k_0$, $\mathrm{ord}(P_j) = s_j$, $1 \le s_1 < \cdots < s_\ell \le k$, $\displaystyle\sum_{1 \le j \le \ell} \frac{1}{s_j} \le \delta \log k$,

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Unfortunately, this seems insufficient to prove the GGL conjecture.

Further geometric structures: Semple jet bundles

- **Functor "1-jet" :** $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :

  $\tilde{X} = P(V) =$ bundle of projective spaces of lines in V

  $\pi : \tilde{X} = P(V) \to X, \quad (x, [v]) \mapsto x, \quad v \in V_x$

  $\tilde{V}_{(x,[v])} = \big\{\xi \in T_{\tilde{X},(x,[v])}\,;\ \pi_*\xi \in \mathbb{C}v \subset T_{X,x}\big\}$

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- **Basic exact sequences.** On $X_k = P(V_{k-1})$, one has

$$0 \to T_{X_k/X_{k-1}} \to V_k \xrightarrow{d\pi_k} \mathcal{O}_{X_k}(-1) \to 0 \qquad \Rightarrow \text{rank } V_k = r$$
$$0 \to \mathcal{O}_{X_k} \to \pi_k^\star V_{k-1} \otimes \mathcal{O}_{X_k}(1) \to T_{X_k/X_{k-1}} \to 0 \ \ \text{(Euler)}$$

Direct image formula for Semple bundles

For $n = \dim X$ and $r = \operatorname{rank} V$, one gets a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\operatorname{rank} V_k = r$,
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X_k is a smooth compactification of $X_k^{\mathrm{GG,reg}}/\mathbb{G}_k = J_k^{\mathrm{GG,reg}}/\mathbb{G}_k$, where $\mathbb{G}_k$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization: $(f, \varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k-jets of regular curves.

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Direct image formula for invariant differential operators

$E_{k,m}V^* := (\pi_{k,0})_* \mathcal{O}_{X_k}(m) =$ sheaf of algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ such that $P((f \circ \varphi)_{[k]}) = \varphi'^m P(f_{[k]}) \circ \varphi$.

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We define an induced directed structure $(Z, W) \hookrightarrow (X_k, V_k)$ by taking the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure of $T_{Z'} \cap V_k$ taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$.

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Alternatively, one could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage (X_k^a, V_k^a) of the "absolute Semple tower" associated with $(X_0^a, V_0^a) = (X, T_X)$
(so as to deal only with nonsingular ambient Semple bundles).

Induced directed structure on a subvariety

Let Z be an irreducible algebraic subset of some Semple k-jet bundle X_k over X (k arbitrary).

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This produces an induced directed subvariety

$$(Z, W) \subset (X_k, V_k).$$

It is easy to show that
$\pi_{k,k-1}(Z) = X_{k-1} \Rightarrow \operatorname{rank} W < \operatorname{rank} V_k = \operatorname{rank} V$.

Some tautological morphisms

Denote $\mathcal{O}_{X_k}(\underline{a}) = \pi_{k,1}^*\mathcal{O}_{X_1}(a_1) \otimes \cdots \otimes \pi_{k,k-1}^*\mathcal{O}_{X_{k-1}}(a_{k-1}) \otimes \mathcal{O}_{X_k}(a_k)$ for every k-tuple $\underline{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$, and let $\underline{1} = (1, \ldots, 1) \in \mathbb{Z}^k$.

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Absolute and induced tautological morphisms

- For all $p = 1, ..., n$, there is a tautological morphism

$$\Phi_{k,p}^X : \pi_{k,0}^*\Lambda^p T_X^* \to \Lambda^p(V_k^a)^* \otimes \mathcal{O}_{X_k^a}((p-1)\underline{1})$$

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$$\Phi_k^{Z,W} : {}^b\Lambda^{r_0} W_0^* \to {}^b\mathcal{K}_W \otimes \mathcal{O}_{X_k}(\underline{a})_{|Z}$$

where ${}^b\mathcal{K}_W \subset (\Lambda^{r'} W^*)^{**}$, ${}^b\Lambda^{r_0} W_0^*$ is a quotient of the sheaf $\pi_{k,0}^* {}^b\Lambda^{r_0} V^*$ of bounded r_0-forms on V, and $\underline{a} \in \mathbb{N}^k$.

Geometric use of the tautological morphisms

Theorem (D-, 2021)

Let (X, V) be a directed variety. Assume that ${}^b\Lambda^p V^*$ is strongly big for some $p \leq r = \operatorname{rank} V$, in the sense that for $A \in \mathrm{Pic}(X)$ ample, the symmetric powers $S^m({}^b\Lambda^p V^*) \otimes \mathcal{O}(-A)$ are generated by their sections over a Zariski open set of X, for $m \gg 1$.

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- If $p \geq 2$, there exists a subvariety $Y \subsetneq X$ and finitely many induced directed subvarieties $(Z_\alpha, W_\alpha) \subset (X_k, V_k)$ with $\operatorname{rank} W_\alpha \leq p - 1$, such that all curves $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ satisfy either $f(\mathbb{C}) \subset Y$ or $f_{[k]} : (\mathbb{C}, T_\mathbb{C}) \to \bigcup (Z_\alpha, W_\alpha)$.

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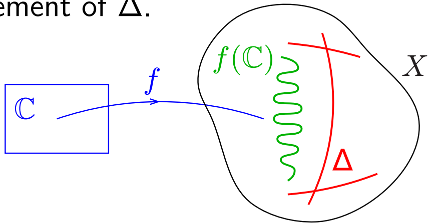
- In particular, if $p = 2$, all entire curves $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ are either contained in $Y \subsetneq X$, or they are tangent to a rank 1 foliation on a subvariety $Z \subset X_k$. This implies that the latter curves are parametrized by a finite dimensional space.

Logarithmic version

More generally, if $\Delta = \sum \Delta_j$ is a reduced normal crossing divisor in X, we want to study entire curves $f : \mathbb{C} \to X \smallsetminus \Delta$ drawn in the complement of Δ.



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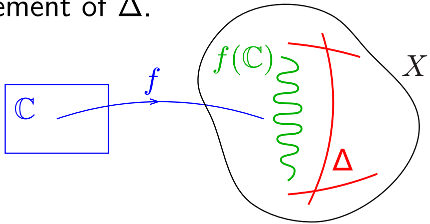

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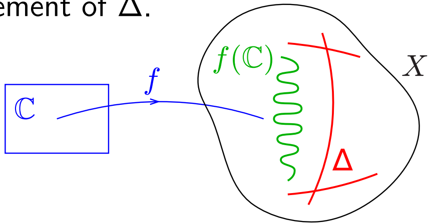

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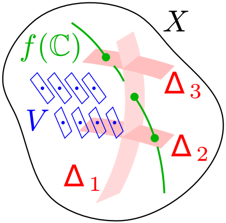
Logarithmic/orbifold directed versions

(Work in progress with F. Campana, L. Darondeau & E. Rousseau)

There are also more general versions dealing with entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ and avoiding a normal crossing divisor Δ transverse to V ("logarithmic case"), or meeting $\Delta = \sum(1 - \frac{1}{\rho_j})\Delta_j$ with multiplicities $\geq \rho_j$ along Δ_j ("orbifold case").



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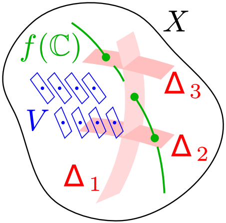

At this step, positivity is to be expressed for a sequence of orbifold cotangent bundles $V^*\langle\Delta^{(s)}\rangle, \quad \Delta^{(s)} = \sum_j \left(1 - \frac{s}{\rho_j}\right)_+ \Delta_j.$

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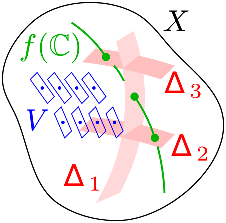

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In all cases, proving the GGL conjecture with optimal positivity conditions (i.e. only assuming bigness of the logarithmic/orbifold canonical sheaf) seems to require a better use of stability properties.

Thank you for your attention!