L^2-Methods and Effective Results in Algebraic Geometry

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Abstract. One important problem arising in algebraic geometry is the computation of effective bounds for the degree of embeddings in a projective space of given algebraic varieties. This problem is intimately related to the following question: Given a positive (or ample) line bundle L on a projective manifold X, can one compute explicitly an integer m_0 such that mL is very ample for $m \geqslant m_0$? It turns out that the answer is much easier to obtain in the case of adjoint line bundles $2(K_X + mL)$, for which universal values of m_0 exist. We indicate here how such bounds can be derived by a combination of powerful analytic methods: theory of positive currents and plurisubharmonic functions (Lelong), L^2 estimates for $\overline{\partial}$ (Andreotti-Vesentini, Hörmander, Bombieri, Skoda), Nadel vanishing theorem, Aubin-Calabi-Yau theorem, and holomorphic Morse inequalities.

1. Basic concepts of hermitian differential geometry

Let X be a complex manifold of dimension n and let F be a C^∞ complex vector bundle of rank r over X. A *connection D* on F is a linear differential operator D acting on spaces $C^\infty(X, \Lambda^{p,q}T_X^\star \otimes F)$ of F-valued differential forms, increasing the degree by 1 and satisfying Leibnitz' rule

$$D(f \wedge u) = df \wedge u + (-1)^{\deg f} f \wedge Du$$

for all forms $f \in C^\infty(X, \Lambda^{a,b}T_X^\star)$, $u \in C^\infty(X, \Lambda^{p,q}T_X^\star \otimes F)$. As usual, we split $D = D' + D''$ into its $(1,0)$ and $(0,1)$ parts, where

$$D' + D'' : C^\infty(X, \Lambda^{p,q}T_X^\star \otimes F) \longrightarrow C^\infty(X, \Lambda^{p+1,q}T_X^\star \otimes F) \oplus C^\infty(X, \Lambda^{p,q+1}T_X^\star \otimes F).$$

With respect to a trivialization $\tau : F_{\restriction\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}^r$, a connection D can be written $Du \simeq_\tau du + \Gamma \wedge u$, where $\Gamma = \Gamma' + \Gamma''$ is an arbitrary $(r \times r)$-matrix of 1-forms and d acts componentwise. A standard computation shows that $D^2 u \simeq_\tau \Theta(D) \wedge u$, where $\Theta(D) = d\Gamma + \Gamma \wedge \Gamma$ is a global 2-form on X with values in $\mathrm{Hom}(F, F)$. This form is called the *curvature tensor* of F. In the important case of rank 1 bundles, $\Theta(F) = d\Gamma$ is a d-closed form with complex values; it is well known that the De Rham cohomology class of $\theta(F) := \frac{i}{2\pi}\Theta(F) = \frac{i}{2\pi}D^2$ is the image in De Rham cohomology of the first Chern class $c_1(F) \in H^2(M, \mathbb{Z})$. For any line bundles $F_1, \ldots, F_p$ on X and any compact p-dimensional analytic set Y in X, we set

$$F_1 \cdot \ldots \cdot F_p \cdot Y = \int_Y c_1(F_1) \wedge \ldots \wedge c_1(F_p).$$

If F is equipped with a C^∞ hermitian metric h, the connection D is said to be *compatible with h* if

$$d\langle u, v\rangle_h = \langle Du, v\rangle_h + \langle u, Dv\rangle_h$$

for all smooth sections u, v of F. This is equivalent to the antisymmetry condition $\Gamma^\star = -\Gamma$ (in a unitary frame), i.e. $\Gamma'' = -\Gamma'^\star$. In particular, a compatible connection D is uniquely determined by its $(0,1)$-component D''. If F has a *holomorphic structure*, we precisely have a canonical $(0,1)$-connection $D'' = \overline{\partial}$ obtained by letting $\overline{\partial}$ act componentwise. Hence, there exists a unique $(1,0)$-connection D' that makes $D = D' + \overline{\partial}$ compatible with the hermitian metric. This connection is called the *Chern connection*. Let (e_λ) be a local holomorphic frame of $F_{\restriction \Omega}$ and let $H = (h_{\lambda\mu})$, $h_{\lambda\mu} = \langle e_\lambda, e_\mu\rangle$ be the hermitian matrix representing the metric. Standard computations show that the Chern connection and curvature are given by

$$D' \simeq_\tau \partial + \overline{H}^{-1}\partial\overline{H} \wedge \bullet = \overline{H}^{-1}\partial(\overline{H}\bullet), \qquad \Theta(F) = \overline{\partial}(\overline{H}^{-1}\partial\overline{H}).$$

In the special case where F has rank 1, it is convenient to write the unique coefficient $H = h_{11}$ of the hermitian metric in the form $H = e^{-2\varphi}$. The function φ is called the *weight* of the metric in the local coordinate patch Ω. We then find $\Theta(F) = 2\partial\overline{\partial}\varphi$. It is important to observe that this formula still makes sense in the context of distribution theory if φ is just an arbitrary L^1_{loc} function. As we shall see later, the case of logarithmic poles is very important for the applications.

(1.1) Definition. *A singular hermitian metric on a line bundle F is a metric given in any trivialization $\tau : F_{\restriction \Omega} \xrightarrow{\;\simeq\;} \Omega \times \mathbb{C}$ by*

$$\|\xi\| = |\tau(\xi)|\, e^{-\varphi(x)}, \qquad x \in \Omega,\ \xi \in F_x,$$

where $\varphi \in L^1_{\text{loc}}(\Omega)$ is an arbitrary function. The associated curvature current is

$$\theta(F) = \frac{\mathrm{i}}{\pi}\partial\overline{\partial}\varphi.$$

The *Lelong-Poincaré* equation states that $\frac{\mathrm{i}}{\pi}\partial\overline{\partial}\log|f| = [D_f]$, where f is a holomorphic or meromorphic function and $[D_f]$ is the current of integration over the divisor of f. More generally, we have

$$\frac{\mathrm{i}}{\pi}\partial\overline{\partial}\log\|\sigma\| = [D_\sigma] - \theta(F)$$

for every section $\sigma \in H^0(X, F)$, as follows from the equality $\|\sigma\| = |f|e^{-\varphi}$, if $f = \tau(\sigma)$. As a consequence, the De Rham cohomology class of $[D_\sigma]$ coincides with the first Chern class $c_1(F)_{\mathbb{R}} \in H^2_{\text{DR}}(X, \mathbb{R})$.

2. Positivity and ampleness

Let $(z_1, \ldots, z_n)$ be holomorphic coordinates on X and let $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ be an orthonormal frame of F. Let the curvature tensor of F be

$$\Theta(F) = \sum_{1\leqslant j,k\leqslant n,\ 1\leqslant\lambda,\mu\leqslant r} c_{jk\lambda\mu}dz_j \wedge dz_k \otimes e_\lambda^\star \otimes e_\mu.$$

Clearly, this tensor can be identified with a hermitian form on $T_X \otimes F$, namely

$$\widetilde{\Theta}(F)(t) = \sum c_{jk\lambda\mu} t_{j\lambda} \overline{t_{k\mu}}, \qquad t = \sum t_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in T_X \otimes F.$$

(2.1) Definition (Kodaira, Nakano, Griffiths). *A holomorphic vector bundle F is*

- *positive in the sense of Nakano if $\widetilde{\Theta}(F)(t) > 0$ for all nonzero tensors $t \in T_X \otimes F$;*
- *positive in the sense of Griffiths if $\widetilde{\Theta}(F)(\xi \otimes v) > 0$ for all nonzero decomposable tensors $\xi \otimes v \in T_X \otimes F$.*

In particular, a holomorphic line bundle F is positive if and only if its weights φ are strictly plurisubharmonic (psh), i.e. if $(\partial^2\varphi/\partial z_j\overline{\partial}z_k)$ is positive definite.

(2.2) Example. Let $D = \sum \alpha_j D_j$ be a divisor with coefficients $\alpha_j \in \mathbb{Z}$ and let $F = \mathcal{O}(D)$ be the associated invertible sheaf of meromorphic functions u such that $\operatorname{div}(u) + D \geqslant 0$; the corresponding line bundle can be equipped with the singular metric defined by $\|u\| = |u|$. If g_j is a generator of the ideal of D_j on an open set $\Omega \subset X$, then $\tau(u) = u \prod g_j^{\alpha_j}$ defines a trivialization of $\mathcal{O}(D)$ over Ω; thus, our singular metric is associated with the weight $\varphi = \sum \alpha_j \log|g_j|$. By the Lelong-Poincaré equation, we find

$$\frac{\mathrm{i}}{\pi}\Theta\big(\mathcal{O}(D)\big) = \frac{\mathrm{i}}{\pi}\partial\overline{\partial}\varphi = [D] \geqslant 0,$$

where $[D] = \sum \alpha_j[D_j]$ denotes the current of integration over D.

(2.3) Example. Assume that $\sigma_1, \ldots, \sigma_N$ are nonzero holomorphic sections of F. Then we can define a natural (possibly singular) hermitian metric on F by

$$\|\xi\|^2 = \frac{|\tau(\xi)|^2}{\sum_{1 \leqslant j \leqslant N} |\tau(\sigma_j(x))|^2}$$

with respect to any trivialization τ. The associated weight function is $\varphi(x) = \log\big(\sum |\tau(\sigma_j(x))|^2\big)^{1/2}$. In this case φ is a psh function; thus, $\mathrm{i}\Theta(F)$ is a closed positive current. Let us denote by Σ the linear system defined by $\sigma_1, \ldots, \sigma_N$ and by $B_\Sigma = \bigcap \sigma_j^{-1}(0)$ its base locus. Let

$$\Phi_\Sigma : X \setminus B_\Sigma \to \mathbb{P}^{N-1}, \qquad x \mapsto (\sigma_1(x) : \sigma_2(x) : \ldots : \sigma_N(x))$$

be the associated map. Then $\theta(F) = \frac{\mathrm{i}}{2\pi}\log(|\sigma_1|^2 + \cdots + |\sigma_N|^2)$ is the pullback over $X \setminus B_\Sigma$ of the Fubini-Study metric ω_{FS} on $\mathbb{P}^{N-1}$.

(2.4) Definition. *A holomorphic line bundle F over a compact complex manifold X is*

- *very ample, if the map $\Phi_{|F|} : X \to \mathbb{P}^{N-1}$ defined by the complete linear system $|F| = P(H^0(X, F))$ is a regular embedding (this means in particular that $B_{|F|} = \emptyset$);*
- *ample, if mF is very ample for some positive integer m.*

Here we used an additive notation for $\operatorname{Pic}(X) = H^1(X, \mathcal{O}^\star)$, i.e. $mF = F^{\otimes m}$. By Example (2.2), every ample line bundle F has a smooth hermitian metric with positive definite curvature form; indeed, if $\Phi_{|mF|}$ is an embedding, then we get a positive definite

curvature form $\theta(F^{\otimes m}) = \Phi^{\star}_{|mF|}(\omega_{\mathrm{FS}})$ and we need only extract the mth root of this metric to get the desired smooth metric on F. The converse is also true:

(2.5) Kodaira embedding theorem (1954). *A line bundle F is ample if and only if F can be equipped with a smooth hermitian metric of positive curvature.*

In this context, Fujita [Fuj87] has raised the following important conjecture.

(2.6) Conjecture (Fujita, 1987). *If L is an ample line bundle on a projective n-fold X, then $K_X + (n+1)L$ is globally generated and $K_X + (n+2)L$ is very ample.*

Here $K_X = \Lambda^n T_X^{\star}$ is the canonical bundle. The example of curves shows that K_X is needed to get a uniform answer (if L is a bundle of degree 1 on a curve, then in general mL does not have any nonzero section unless $m \geqslant g = $ genus). Also, the example of projective spaces show that Fujita's bounds would be optimal, because $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$.

Such questions have attracted a lot of attention in recent years. First, the case of surfaces has been completely settled by Reider; in [Rei88] he obtains a very sharp criterion for global generation and very ampleness of line bundles in dimension 2. In higher dimensions, let us mention [Dem90, 93, 94, 95] and the works of Fujita [Fuj87, 94], Kollár [Kol93], Ein-Lazarsfeld [EL92, 93], Lazarsfeld [Laz93], and Siu [Siu93, 94]. Our goal is to describe a few powerful analytic methods that are useful in this context.

3. Bochner technique and vanishing theorems

Let X be a compact complex n-fold equipped with a *Kähler metric*, namely a positive $(1,1)$-form $\omega = \mathrm{i} \sum \omega_{jk} dz_j \wedge d\overline{z}_k$ with $d\omega = 0$. Let F be a holomorphic vector bundle on X equipped with a hermitian metric, and let

$$\Delta' = D'D'^{\star} + D'^{\star}D', \qquad \Delta'' = D''D''^{\star} + D''^{\star}D'',$$

be the complex Laplace operators associated with the Chern connection D. Here the adjoints $D'^{\star}$, $D''^{\star}$ are the formal adjoints computed with respect to the L^2 norm $\|u\|^2 = \int_X |u(x)|^2 \, dV_{\omega}(x)$, where $|u|$ is the pointwise hermitian norm and $dV_{\omega} = \omega^n/n!$ is the volume form. The fundamental results of Hodge theory imply isomorphisms

$$H^q(X, \Omega_X^p \otimes F) = H^{p,q}_{\overline{\partial}}(X, F) \simeq \mathcal{H}^{p,q}(X, F)$$

between sheaf cohomology groups, Dolbeault $\overline{\partial}$-cohomology groups, and the space $\mathcal{H}^{p,q}$ of harmonic (p,q)-forms $\Delta''u = 0$. The next fundamental fact is an identity originally used by Bochner to prove vanishing results for Betti numbers. Slightly later, the identity was extended to the complex situation by Kodaira and Nakano.

(3.1) Bochner-Kodaira-Nakano formula (1954). *For all $u = \sum u_{J,K,\lambda} dz_I \wedge d\overline{z}_J \otimes e_{\lambda}$ of class C^{∞} and type (p,q), we have*

$$\Delta''u = \Delta'u + A_{F,\omega}^{p,q}u,$$

where $A_{F,\omega}^{p,q}$ is the hermitian endomorphism such that $\langle A_{F,\omega}^{p,q}u, u \rangle =$

$$\sum c_{jk\lambda\mu} u_{J,jS,\lambda} \overline{u_{J,kS,\mu}} + \sum c_{jk\lambda\mu} u_{kR,K,\lambda} \overline{u_{jR,K,\mu}} - \sum c_{jj\lambda\mu} u_{J,K,\lambda} \overline{u_{J,K,\mu}},$$

and the summations are extended to all relevant indices $1 \leqslant j, k \leqslant n$, $1 \leqslant \lambda, \mu \leqslant r$, and all relevant multiindices $|J| = p$, $|K| = q$, $|R| = p-1$, $|S| = q-1$.

As $\langle \Delta' u, u \rangle = \|D'u\|^2 + \|D'^\star u\|^2 \geqslant 0$ the Bochner-Kodaira-Nakano formula implies

$$\langle \Delta'' u, u \rangle \geqslant \int_X \langle A^{p,q}_{F,\omega} u, u \rangle \, dV_\omega.$$

If $A^{p,q}_{F,\omega}$ is positive definite, every (p,q)-harmonic form has to vanish and we conclude that $H^q(X, \Omega^p_X \otimes F) = 0$. In the special case of rank 1 bundles, we can take at each point $x \in X$ simultaneous diagonalizations

$$\omega(x) = i \sum dz_j \wedge d\overline{z}_j, \qquad \Theta(F)(x) = i \sum \gamma_j(x) dz_j \wedge d\overline{z}_j,$$

where $\gamma_1(x) \leqslant \cdots \leqslant \gamma_n(x)$ are the curvature eigenvalues. Then $c_{jj\lambda\mu} = \gamma_j$ and

$$\langle A^{p,q}_{F,\omega} u, u \rangle = \sum_{J,K} \Big(\sum_{j \in K} \gamma_j - \sum_{j \notin J} \gamma_j \Big) |u_{JK}|^2 \geqslant (\gamma_1 + \cdots + \gamma_q - \gamma_{n-p+1} - \cdots - \gamma_n) |u|^2.$$

Assume now that $i\Theta(F)$ is positive. The choice $\omega = i\Theta(F)$ yields $\gamma_j = 1$ for $j = 1, 2, \ldots, n$ and $\langle A^{p,q}_{F,\omega} u, u \rangle = (p + q - n)|u|^2$. From this, we immediately infer:

(3.2) Akizuki-Kodaira-Nakano vanishing theorem (1954). *If F is a positive line bundle on a compact complex manifold X, then*

$$H^{p,q}(X, F) = H^q(X, \Omega^p_X \otimes F) = 0 \qquad for \ p + q \geqslant n + 1.$$

The above vanishing result is optimal. Unfortunately, it cannot be extended to semipositive or numerically effective line bundles of bidegrees (p,q) with $p < n$, as shown by a counterexample of Ramanujam [Ram74].

4. Hörmander's L^2 estimates and existence theorems

The basic existence theorem is the following result, which is essentially due to Hörmander [Hö65] and, in a more geometric setting, to Andreotti-Vesentini [AV65].

(4.1) Theorem. *Let (X, ω) be a complete Kähler manifold. Let F be a hermitian vector bundle of rank r over X, and assume that $A = A^{p,q}_{F,\omega}$ is positive definite everywhere on $\Lambda^{p,q} T^\star_X \otimes F$, $q \geqslant 1$. Then for any form $g \in L^2(X, \Lambda^{p,q} T^\star_X \otimes F)$ with*

$$D'' g = 0 \quad and \quad \int_X \langle (A^{p,q}_{F,\omega})^{-1} g, g \rangle \, dV_\omega < +\infty,$$

there exists a $(p, q-1)$-form f such that $D'' f = g$ and

$$\int_X |f|^2 \, dV_\omega \leqslant \int_X \langle (A^{p,q}_{F,\omega})^{-1} g, g \rangle \, dV_\omega.$$

The proof can be ultimately reduced to a simple duality argument for unbounded operators on a Hilbert space, based on the a priori inequality

$$\|D'' u\|^2 + \|D''^\star u\|^2 \geqslant \int_X \langle A^{p,q}_{F,\omega} u, u \rangle \, dV_\omega.$$

The above L^2 existence theorem can be applied in the fairly general context of *weakly pseudoconvex* manifolds (i.e. manifolds possessing a weakly psh exhaustion function), thanks to the fact that every weakly pseudoconvex Kähler manifold (X, ω) carries a *complete* Kähler metric. In particular, the existence theorem can be applied on compact manifolds, pseudoconvex open sets in $\mathbb{C}^n$, Stein manifolds, etc. By regularization arguments, the existence theorem also applies when F is a line bundle and the hermitian metric is a singular metric with positive curvature in the sense of currents. In fact, the solutions obtained with the regularized metrics have weak L^2 limits satisfying the desired estimates. Especially, we get the following more tractable version in the case $p = n$.

(4.2) Corollary. *Let (X, ω) be a Kähler weakly pseudoconvex complex manifold of dimension n. Let F be a holomorphic line bundle on X, equipped with a singular metric whose local weights $\varphi \in L^1_{\mathrm{loc}}$ satisfy $\mathrm{i}\Theta(F) = 2\mathrm{i}\partial\overline{\partial}\varphi \geqslant \varepsilon\omega$ for some $\varepsilon > 0$. For every $g \in L^2(X, \Lambda^{n,q}T_X^\star \otimes F)$ with $D''g = 0$, there exists $f \in L^2(X, \Lambda^{p,q-1}T_X^\star \otimes F)$ such that $D''f = g$ and*

$$\int_X |f|^2 e^{-2\varphi}\, dV_\omega \leqslant \frac{1}{q\varepsilon} \int_X |g|^2 e^{-2\varphi}\, dV_\omega.$$

This result leads in a natural way to the concept of *multiplier ideal sheaves*, according to Nadel [Nad89]. The basic idea was already implicit in the work of Bombieri [Bom70] and Skoda [Sk72].

(4.3) Multiplier ideal sheaves. *Let φ be a psh function on an open subset $\Omega \subset X$. We define $\mathcal{I}(\varphi) \subset \mathcal{O}_X$ to be the sheaf of germs $f \in \mathcal{O}_{\Omega,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable on a small neighborhood V of x with respect to the Lebesgue measure.*

(4.4) Main property ([Nad89], [Dem93]). *The ideal sheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_X$ is a coherent analytic sheaf. Its zero variety $V(\mathcal{I}(\varphi))$ is the set of points in a neighborhood of which $e^{-2\varphi}$ is nonintegrable.*

A basic observation is that the zero variety $V(\mathcal{I}(\varphi))$ is closed related to the sublevel sets of Lelong numbers of φ.

(4.5) Definition. *The Lelong number of a psh function φ at a point $x \in X$ is the limit $\nu(\varphi, x) := \liminf_{z \to x} \varphi(z)/\log|z - x|$. The function φ is said to have a logarithmic pole of coefficient γ if $\gamma = \nu(\varphi, x) > 0$.*

(4.6) Lemma ([Sk72]). *Let φ be psh on Ω and let $x \in \Omega$.*

- *If $\nu(\varphi, x) < 1$, then $e^{-2\varphi}$ is integrable near $x \Rightarrow \mathcal{I}(\varphi)_x = \mathcal{O}_{\Omega,x}$.*
- *If $\nu(\varphi, x) \geqslant n + s$, $s \in \mathbb{N}$, then $e^{-2\varphi} \geqslant C|z - x|^{-2n-2s}$ near x and $\mathcal{I}(\varphi)_x \subset m_{\Omega,x}^{s+1}$.*

(4.7) Simple algebraic case. *Let $\varphi = \sum \alpha_j \log|g_j|$, $\alpha_j \in \mathbb{Q}^+$, be associated with a normal crossing $\mathbb{Q}$-divisor $D = \sum \alpha_j D_j$, $D_j = g_j^{-1}(0)$. An easy computation gives*

$$\mathcal{I}(\varphi) = \mathcal{O}\left(-\sum \lfloor \alpha_j \rfloor D_j\right) = \mathcal{O}(-\lfloor D \rfloor),$$

where $\lfloor \alpha_j \rfloor = $ the integral part of α_j. If the assumption on normal crossings is omitted, a desingularization of D has to be used in combination with the following fonctoriality property for direct images.

(4.8) Basic fonctoriality property. *Let $\mu : X' \to X$ be a modification (i.e. a proper generically $1 : 1$ holomorphic map), and let φ be a psh function on X. Then*

$$\mu_\star\big(\mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi \circ \mu)\big) = \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi).$$

Let us now consider the case of general algebraic singularities

$$\varphi \sim \frac{\alpha}{2} \log \left(|f_1|^2 + \cdots + |f_N|^2 \right)$$

with $\alpha \in \mathbb{Q}^+$ and f_j holomorphic on an open set $\Omega \subset X$. By Hironaka's theorem, there exists a smooth modification $\mu : \widetilde{X} \to X$ of X such that $\mu^\star(f_1, \ldots, f_N)$ is an invertible sheaf $\mathcal{O}(-D)$ associated with a normal crossing divisor $D = \sum \lambda_j D_j$. Then

$$\mathcal{I}(\varphi) = \mu_\star \mathcal{O}_{\widetilde{X}} \left(\sum (\rho_j - \lfloor \alpha \lambda_j \rfloor) D_j \right),$$

where $R = \sum \rho_j D_j$ is the zero divisor of the jacobian J_μ of the blow-up map. In this context, we get the following important vanishing theorem, which can be seen as a generalization of the Kawamata-Viehweg vanishing theorem (see [Kaw82], [Vie82], [EV86]).

(4.9) Nadel vanishing theorem ([Nad89], [Dem93]). *Let (X, ω) be a Kähler weakly pseudoconvex manifold, and let F be a holomorphic line bundle over X equipped with a singular hermitian metric of weight φ. Assume that $\mathrm{i}\Theta(F) \geqslant \varepsilon \omega$ for some continuous positive function ε on X. Then*

$$H^q \left(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(\varphi) \right) = 0 \qquad \text{for all } q \geqslant 1.$$

Proof. In virtue of Hörmander's L^2 estimates applied on small balls, the $\overline{\partial}$-complex of L^2_{loc} (n, q)-forms is a (fine) resolution of the sheaf $\mathcal{O}(K_X + F) \otimes \mathcal{I}(\varphi)$. The global L^2 cohomology is also zero by the L^2 estimates applied globally on X. $\square$

(4.10) Corollary. *Let $x_1, \ldots, x_N$ be isolated points in the zero variety $V(\mathcal{I}(\varphi))$. Then there is a surjective map*

$$H^0(X, K_X + F) \longrightarrow \bigoplus_{1 \leqslant j \leqslant N} \left(\mathcal{O}(K_X + L) \otimes \mathcal{O}_X / \mathcal{I}(\varphi) \right)_{x_j}.$$

In particular, if the weight function φ is such that $\nu(\varphi, x) \geqslant n + s$ at some point $x \in X$ and $\nu(\varphi, y) < 1$ at nearby points, then $H^0(X, K_X + F)$ generates all s-jets at x.

(4.11) Remark. It is an easy exercise (left to the reader!) to show that Corollary (4.10) implies the Kodaira embedding theorem.

5. Numerical criteria for very ample line bundles

The simplest approach to this problem is a recent technique due to [Siu94], which rests merely on Nadel's vanishing theorem and the Riemann-Roch formula. We formulate here a slightly improved version (see also [Dem94, 95]).

(5.1) Theorem. *Let L be an ample line bundle on a projective n-fold X. Let $x_j \in X$ and $s_j \in \mathbb{N}$ be given, $1 \leqslant j \leqslant N$. For*

$$m \geqslant m_0 = 2 + \sum_{1 \leqslant j \leqslant N} \binom{3n + 2s_j - 1}{n}$$

$H^0(X, 2K_X + mL)$ generates simultaneously jets of order s_j at all points x_j. In particular, $2K_X + mL$ is very ample for $m \geqslant 2 + \binom{3n+1}{n}$.

Proof. By a result of Fujita, $K_X + mL$ is ample for $m \geqslant m_0$ (in fact Fujita has shown that $K_X + mL$ is nef for $m \geqslant m + 1$ and ample for $m \geqslant n + 2$). The idea is to use a recursion procedure for the construction of psh weights $(\varphi_\nu)_{\nu \geqslant 1}$ on $K_X + m_0 L$ such that

(α) the curvature of $K_X + m_0 L$ is positive definite: $i\partial\overline{\partial}\varphi_\nu \geqslant \varepsilon_\nu \omega$ for some $\varepsilon_\nu > 0$, where ω is the Kähler metric;

(β) $\nu(\varphi_\nu, x_j) \geqslant n + s_j$ for all j;

(γ) $\mathcal{I}(\varphi_{\nu+1}) \supsetneq \mathcal{I}(\varphi_\nu)$ whenever $\dim V(\mathcal{I}(\varphi_\nu)) > 0$.

Indeed, Nadel's vanishing theorem implies

$$H^q(X, \mathcal{O}(2K_X + mL) \otimes \mathcal{O}/\mathcal{I}(\varphi_\nu)) = 0 \qquad \text{for } m \geqslant m_0 \text{ and } q \geqslant 1.$$

Hence, $h^0 = \chi$ is large for some $m \in [m_0, 2m_0 - 1]$, and the existence of a section σ vanishing at order $2(n + s_j)$ at all points x_j follows by the Riemann-Roch formula and an elementary count of dimensions. We then set inductively

$$\varphi_{\nu+1} = \log(e^{\varphi_\nu} + e^{(1-m/2m_0)\psi}|\sigma|^{1/2}),$$

where ψ is a weight for a smooth metric of positive definite curvature on L. Condition (γ) guarantees that the process stops after a finite number of steps. $\qquad\square$

One weak point of the above result is that large multiples of L are required. Instead, we would like to find conditions on L implying that $2(K_X + L)$ is very ample. For this, we need a convenient measurement of how large L is.

(5.2) Definition. *Let L be a numerically effective line bundle, i.e. a line bundle such that $L^p \cdot Y \geqslant 0$ for all p-dimensional subvarieties Y. For every $S \subset X$, we set*

$$\mu(L, S) = \min_{Y \cap S \neq \emptyset} (L^p \cdot Y)^{1/p},$$

where Y runs over all p-dimensional subvarieties intersecting S. The main properties of this invariant are:

• *Linearity: $\forall k \geqslant 0, \ \mu(kL, S) = k\,\mu(L, S)$;*

• *Nakai-Moǐshezon criterion: L is ample if and only if $\mu(L, X) > 0$.*

(5.3) Theorem ([Dem93]). *Let $s, m \in \mathbb{N}$, $s \geqslant 1$, $m \geqslant 2$. If L is ample and satisfies*

$$(m - 1)\,\mu(L, X) \geqslant 6(n + s)^n - s,$$

then $2K_X + mL$ generates s-jets. Moreover, the result still holds with $6(n + s)^n$ replaced by $12n^n$ if $s = 1$; in particular, $2K_X + 12n^n L$ is always very ample.

Proof. By Corollary (4.10), the main point is to construct psh weights φ that achieve the desired ideals $\mathcal{I}(\varphi)_{x_j}$ for the jets. This is done by solving a complex Monge-Ampère equation

$$\left(\omega + \frac{i}{\pi}\partial\overline{\partial}\varphi\right) = f, \qquad \omega = \theta(L),$$

where f is a linear combination of Dirac measures δ_{x_j} and of a uniform density with respect to ω^n. The solution φ does exist by the Aubin-Calabi-Yau theorem, but in general, the poles of φ are not isolated. Hence, the Lelong numbers have to be estimated precisely: this is indeed possible by means of intersection inequalities for positive currents. We refer to [Dem93, 94] for details. □

6. Holomorphic Morse inequalities

The starting point is the following differential geometric asymptotic inequality, in which $X(\leqslant q, L)$ denotes the set of points $x \in X$ at which $\theta_h(L)(x)$ has at most q negative eigenvalues. The proof is obtained by a careful study of the spectrum of the complex Laplace operator Δ''. See [Dem85, 91] for details.

(6.1) Strong Morse inequalities ([Dem85]). *Let X be a compact complex n-fold and (L, h) a hermitian line bundle. Then, as $k \to +\infty$,*

$$\sum_{0 \leqslant j \leqslant q} (-1)^{q-j} h^j(X, kL) \leqslant \frac{k^n}{n!} \int_{X(\leqslant q, L)} (-1)^q (\theta_h(L))^n + o(k^n).$$

(6.2) Special case (algebraic version). *Let $L = F - G$, where F and G are numerically effective. Then for all $q = 0, 1, \ldots, n$,*

$$\sum_{0 \leqslant j \leqslant q} (-1)^{q-j} h^j(X, kL) \leqslant \frac{k^n}{n!} \sum_{0 \leqslant j \leqslant q} (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j + o(k^n).$$

In particular, for $q = 1$ we get

$$h^0(X, kL) - h^1(X, kL) \geqslant \frac{k^n}{n!} (F^n - n F^{n-1} \cdot G) - o(k^n).$$

(6.3) Corollary. *If F, G are nef and $F^n > 0$, then $k(mF - G)$ has sections as soon as $m > n F^{n-1} \cdot G / F^n$ and $k \gg 0$.*

(6.4) Corollary. *If F, G are nef and $F^n > 0$, then $H^0(X, K_X + mF - G) \neq 0$ for some $m \leqslant n F^{n-1} \cdot G / F^n + n + 1$.*

Proof. Set $m_0 := \lfloor n F^{n-1} \cdot G / F^n \rfloor + 1$. By Corollary (6.3), $m_0 F - G$ has a psh weight φ with $i\partial\overline{\partial}\varphi \gg 0$; thus, $H^q(X, \mathcal{O}(K_X + mF - G) \otimes \mathcal{I}(\varphi)) = 0$ for $q \geqslant 1$ and $m \geqslant m_0$. The Hilbert polynomial is thus equal to

$$h^0(X, \mathcal{O}(K_X + mF - G) \otimes \mathcal{I}(\varphi)) \geqslant 0,$$

and it must be nonzero for some $m \in [m_0, m_0 + n]$ because there are at most n roots. □

A similar proof yields

(6.5) Corollary. *If F, G are nef with $F^n > 0$, and Y is a p-dimensional subvariety, then $H^0(Y, \omega_Y \otimes \mathcal{O}_Y(mF - G)) \neq 0$ for some $m \leqslant p F^{p-1} \cdot G \cdot Y / F^p \cdot Y + p + 1$, where ω_Y is the L^2 dualizing sheaf of Y.*

A proof by backward induction on $\dim Y$ then yields the following effective version of the big Matsusaka theorem ([Mat72], [KoM83]), improving Siu's result [Siu93].

(6.6) Theorem ([Siu93], [Dem94, 95]) *Let F and G be nef line bundles on a projective n-fold X. Assume that F is ample and set $H = \lambda_n(K_X + (n+2)F)$ with $\lambda_2 = 1$ and $\lambda_n = \binom{3n+1}{n} - 2n$ for $n \geqslant 3$. Then $mF - G$ is very ample for*

$$m \geqslant (2n)^{(3^{n-1}-1)/2} \frac{(F^{n-1} \cdot (G+H))^{(3^{n-1}+1)/2}(F^{n-1} \cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(F^n)^{3^{n-2}(n/2-1/4)+1/4}}.$$

In particular mF is very ample for

$$m \geqslant C_n \ (F^n)^{3^{n-2}} \left(n+2+\frac{F^{n-1} \cdot K_X}{F^n} \right)^{3^{n-2}(n/2+3/4)+1/4}$$

with $C_n = (2n)^{(3^{n-1}-1)/2}(\lambda_n)^{3^{n-2}(n/2+3/4)+1/4}$.

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