On the geometry of positive cones of projective and Kähler varieties

Jean-Pierre Demailly
Institut Fourier, Université de Grenoble I
38402 Saint-Martin d'Hères, France

To the memory of Guido Fano

Abstract. The goal of these notes is to give a short introduction to several works by Sébastien Boucksom, Mihai Paun, Thomas Peternell and myself on the geometry of positive cones of projective or Kähler manifolds. Mori theory has shown that the structure of projective algebraic manifolds is – up to a large extent – governed by the geometry of its cones of divisors or curves. In the case of divisors, two cones are of primary importance: the cone of ample divisors and the cone of effective divisors (and the closure of these cones as well). We introduce here the analogous transcendental cones for arbitrary compact Kähler manifolds, and show that these cones depend only on analytic cycles and on the Hodge structure of the base manifold. Also, we obtain new very precise duality statements connecting the cones of curves and divisors via Serre duality. As a consequence, we are able to prove one of the basic conjectures in the classification of projective algebraic varieties – a subject which Guido Fano contributed to in many ways : a projective algebraic manifold X is uniruled (i.e. covered by rational curves) if and only if its canonical class $c_1(K_X)$ does not lie in the closure of the cone spanned by effective divisors.

§1. Nef and pseudo-effective cones

Let X be a compact complex manifold and $n = \dim_{\mathbb{C}} X$. We are especially interested in *closed positive currents* of type $(1,1)$

$$T = i \sum_{1 \leqslant j,k \leqslant n} T_{jk}(z) dz_j \wedge d\overline{z}_k, \qquad dT = 0.$$

Recall that a current is a differential form with distribution coefficients, and that a current is said to be positive if the distribution $\sum \lambda_j \overline{\lambda}_k T_{jk}$ is a positive measure for all complex numbers λ_j. The coefficients T_{jk} are then complex measures. Important examples of closed positive $(1,1)$-currents are currents of integration over divisors :

$$D = \sum c_j D_j, \qquad [D] = \sum c_j [D_j]$$

where the current $[D_j]$ is defined by duality as

$$\langle [D_j], u \rangle = \int_{D_j} u_{|D_j}$$

for every $(n-1,n-1)$ test form u on X. Another important example is the Hessian form $T = i\partial\overline{\partial}\varphi$ of a plurisubharmonic function on an open set $\Omega \subset X$. A *Kähler metric* on X is a positive definite hermitian $(1,1)$-form

$$\omega(z) = i \sum_{1\leqslant j,k\leqslant n} \omega_{jk}(z) dz_j \wedge d\overline{z}_k \qquad \text{such that } d\omega = 0,$$

with smooth coefficients. To every closed real $(1,1)$-form (or current) α is associated its cohomology class

$$\{\alpha\} \in H^{1,1}_{\mathbb{R}}(X) \subset H^2(X,\mathbb{R}).$$

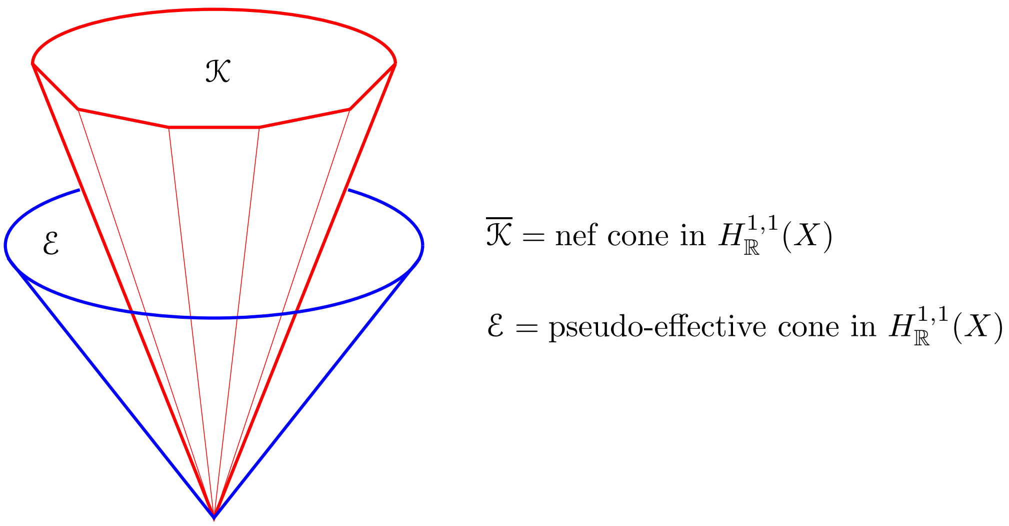
This assertion hides a nontrivial fact, namely the fact that all cohomology groups involved (De Rham, Dolbeault, ...) can be defined either in terms of smooth forms or in terms of currents. In fact, if we consider the associated complexes of sheaves, forms and currents both provide acyclic resolutions of the same sheaf (locally constant functions, resp. holomorphic sections).

The manifold X is said to be *Kähler* if it possesses at least one Kähler metric ω. It is well known that every projective manifold $X \subset \mathbb{P}^N_{\mathbb{C}}$ is Kähler (the restriction of the Fubini-Study metric ω_{FS} to X is a Kähler metric with integral cohomology class $\{\omega_{FS}\} \in H^2(X,\mathbb{Z})$). Conversely, the Kodaira embedding theorem [Kod54] states that every compact Kähler manifold X possessing a Kähler metric ω with an integral cohomology class $\{\omega\} \in H^2(X,\mathbb{Z})$ can be embedded in projective space as a projective algebraic subvariety.

1.1. Definition. *Let X be a compact Kähler manifold.*

(i) *The Kähler cone is the set $\mathcal{K} \subset H^{1,1}_{\mathbb{R}}(X)$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an* <u>*open convex cone*</u>.

(ii) *The pseudo-effective cone is the set $\mathcal{E} \subset H^{1,1}_{\mathbb{R}}(X)$ of cohomology classes $\{T\}$ of closed positive currents of type $(1,1)$. This is a* <u>*closed convex cone*</u>.



The openness of $\mathcal{K}$ is clear by definition, and the closedness of $\mathcal{E}$ follows from the fact that bounded sets of currents are weakly compact (as follows from the similar weak compacteness property for bounded sets of positive measures). It is then clear that $\overline{\mathcal{K}} \subset \mathcal{E}$.

In spite of the fact that cohomology groups can be defined either in terms of forms or currents, it turns out that the cones $\overline{\mathcal{K}}$ and $\mathcal{E}$ are in general different. To see this, it is enough to observe that a Kähler class $\{\alpha\}$ satisfies $\int_Y \alpha^p > 0$ for every p-dimensional analytic set. On the other hand, if X is the surface obtained by blowing-up $\mathbb{P}^2$ in one point, then the exceptional divisopr $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

In case X is projective, it is interesting to consider also the algebraic analogues of our "transcendental cones" $\mathcal{K}$ and $\mathcal{E}$, which consist of suitable integral divisor classes. Since the cohomology classes of such divisors live in $H^2(X,\mathbb{Z})$, we are led to introduce the Neron-Severi lattice and the associated Neron-Severi space

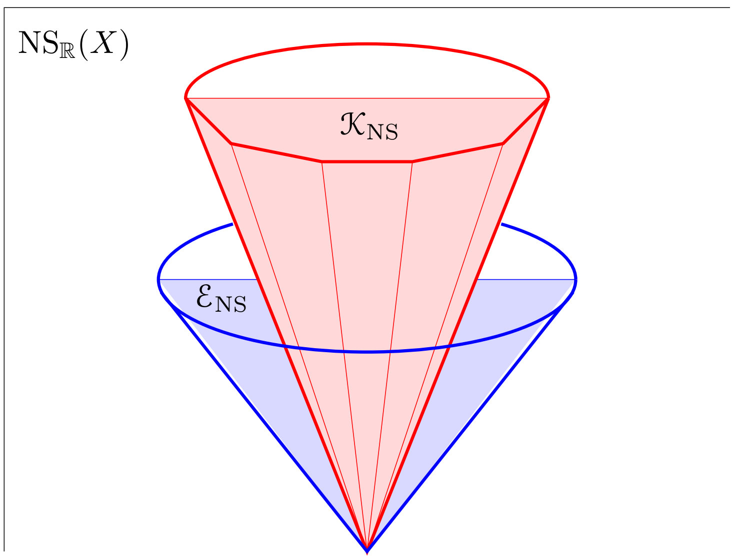
$$\begin{aligned} \mathrm{NS}(X) &:= H^{1,1}_{\mathbb{R}}(X) \cap \big(H^2(X,\mathbb{Z})/\{\text{torsion}\}\big), \\ \mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}. \end{aligned}$$

All classes of real divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}$, lie by definition in $\mathrm{NS}_{\mathbb{R}}(X)$. Notice that the integral lattice $H^2(X,\mathbb{Z})/\{\text{torsion}\}$ need not hit at all the subspace $H^{1,1}_{\mathbb{R}}(X) \subset H^2(X,\mathbb{R})$ in the Hodge decomposition, hence in general the Picard number

$$\rho(X) = \mathrm{rank}_{\mathbb{Z}} \mathrm{NS}(X) = \dim_{\mathbb{R}} \mathrm{NS}_{\mathbb{R}}(X)$$

satisfies $\rho(X) \leqslant h^{1,1} = \dim_{\mathbb{R}} H^{1,1}_{\mathbb{R}}(X)$, but the equality can be strict (actually, it is well known that a generic complex torus $X = \mathbb{C}^n/\Lambda$ satisfies $\rho(X) = 0$ and $h^{1,1} = n^2$). In order to deal with the case of algebraic varieties we introduce

$$\mathcal{K}_{\mathrm{NS}} = \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X), \qquad \mathcal{E}_{\mathrm{NS}} = \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X).$$



A very important fact is that the "Neron-Severi part" of any of the open or closed transcendental cones $\mathcal{K}$, $\mathcal{E}$, $\overline{\mathcal{K}}$, $\mathcal{E}^\circ$ is algebraic, i.e. can be characterized in simple algebraic terms.

1.2. Theorem. *Let X be a projective manifold. Then*

(i) *$\mathcal{E}_{\rm NS}$ is the closure of the cone generated by classes of effective divisors, i.e. divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}_+$.*

(ii) *$\mathcal{K}_{\rm NS}$ is the open cone generated by classes of ample (or very ample) divisors A (Recall that a divisor A is said to be very ample if the linear system $H^0(X, \mathcal{O}(A))$ provides an embedding of X in projective space).*

(iii) *The interior $\mathcal{E}^\circ_{\rm NS}$ is the cone generated by classes of big divisors, namely divisors D such that $h^0(X, \mathcal{O}(kD)) \geqslant c\,k^{\dim X}$ for k large.*

(iv) *The closed cone $\overline{\mathcal{K}}_{\rm NS}$ consists of the closure of the cone generated by nef divisors D (or nef line bundles L), namely effective integral divisors D such that $D\cdot C \geqslant 0$ for every curve C.*

Sketch of proof. These results were already observed (maybe in a slightly different terminology) in [Dem90]. If we denote by $\mathcal{K}_{\rm alg}$ the open cone generated by ample divisors, resp. by $\mathcal{E}_{\rm alg}$ the closure of the cone generated by effective divisors, it is obvious that

$$K_{\rm alg} \subset K_{\rm NS}, \qquad \mathcal{E}_{\rm alg} \subset \mathcal{E}_{\rm NS}.$$

As was to be expected, the interesting part lies in the converse inclusions. The inclusion $K_{\rm NS} \subset \mathcal{K}_{\rm alg}$ is more or less equivalent to the Kodaira embedding theorem : if a rational class $\{\alpha\}$ is in $\mathcal{K}$, then some multiple of $\{\alpha\}$ is the first Chern class of a hermitian line bundle L whose curvature form is Kähler. Therefore L is ample and $\{\alpha\} \in \mathcal{K}_{\rm alg}$; property (ii) follows.

Similarly, if we take a rational class $\{\alpha\} \in \mathcal{E}^\circ_{\rm NS}$, then it is still in $\mathcal{E}$ by subtracting a small multiple $\varepsilon\omega$ of a Kähler class, hence some multiple of $\{\alpha\}$ is the first Chern class of a hermitian line bundle (L, h) with curvature form

$$T = \Theta_h(L) := -\frac{i}{2\pi} i\partial\overline{\partial} \log h \geqslant \varepsilon\omega.$$

The standard theory of L^2 estimates for the $\overline{\partial}$-operator ([AV65], [Dem82]) then shows that large multiples kL admit a large number of sections, hence kL can be represented by a big divisor. This implies (iii) and also that $\mathcal{E}^\circ_{\rm NS} \subset \mathcal{E}_{\rm alg}$. Therefore $\mathcal{E}_{\rm NS} \subset \mathcal{E}_{\rm alg}$ by passing to the closure ; (i) follows. The statement (iv) about nef divisors follows e.g. from [Kle66], [Har70], since every nef divisor is a limit of a sequence of ample rational divisors. □

As a natural extrapolation of the algebraic situation, we say that $\overline{\mathcal{K}}$ is the cone of *nef* $(1,1)$-cohomology classes (even though these classes are not necessarily integral). Property 1.2 (i) also explains the terminology used for the pseudo-effective cone.

§2. Numerical characterization of the Kähler cone

We describe here the main results obtained in [DP03]. The upshot is that the Kähler cone depends only on the intersection product of the cohomology ring, the Hodge structure and the homology classes of analytic cycles. More precisely, we have :

2.1. Theorem. *Let X be a compact Kähler manifold. Let $\mathcal{P}$ be the set of real $(1,1)$ cohomology classes $\{\alpha\}$ which are numerically positive on analytic cycles, i.e. such that $\int_Y \alpha^p > 0$ for every irreducible analytic set Y in X, $p = \dim Y$. Then the Kähler cone $\mathcal{K}$ of X is one of the connected components of $\mathcal{P}$.*

2.2. Special case. *If X is projective algebraic, then $\mathcal{K} = \mathcal{P}$.*

These results (which are new even in the projective case) can be seen as a generalization of the well-known Nakai-Moishezon criterion. Recall that the Nakai-Moishezon criterion provides a necessary and sufficient criterion for a line bundle to be ample: *a line bundle $L \to X$ on a projective algebraic manifold X is ample if and only if*

$$L^p \cdot Y = \int_Y c_1(L)^p > 0,$$

for every algebraic subset $Y \subset X$, $p = \dim Y$.

It turns out that the numerical conditions $\int_Y \alpha^p > 0$ also characterize arbitrary transcendental Kähler classes when X is projective: this is precisely the meaning of the special case 2.2.

2.3. Example. The following example shows that the cone $\mathcal{P}$ need not be connected (and also that the components of $\mathcal{P}$ need not be convex, either). Let us consider for instance a complex torus $X = \mathbb{C}^n/\Lambda$. It is well-known that a generic torus X does not possess any analytic subset except finite subsets and X itself. In that case, the numerical positivity is expressed by the single condition $\int_X \alpha^n > 0$. However, on a torus, $(1,1)$-classes are in one-to-one correspondence with constant hermitian forms α on $\mathbb{C}^n$. Thus, for X generic, $\mathcal{P}$ is the set of hermitian forms on $\mathbb{C}^n$ such that $\det(\alpha) > 0$, and Theorem 2.1 just expresses the elementary result of linear algebra saying that the set $\mathcal{K}$ of positive definite forms is one of the connected components of the open set $\mathcal{P} = \{\det(\alpha) > 0\}$ of hermitian forms of positive determinant (the other components, of course, are the sets of forms of signature (p,q), $p + q = n$, q even. They are not convex when $p > 0$ and $q > 0$).

Sketch of proof of Theorems 2.1 and 2.2. By definition a Kähler current is a closed positive current T of type $(1,1)$ such that $T \geqslant \varepsilon\omega$ for some smooth Kähler metric ω and $\varepsilon > 0$ small enough. The singularities of a closed positive current T are measured by its Lelong numbers

$$\nu(T,x) = \liminf_{z \to x} \frac{\varphi(z)}{\log|z - x|},$$

where $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$ near x. A fundamental theorem of Siu [Siu74] states that the Lelong sublevel sets $E_c(T) := \{x \in X\,;\, \nu(T,x) \geqslant c\}$ are *analytic sets* for every $c > 0$. The crucial steps of the proof of Theorem 2.1 are contained in the following statements.

2.4. Proposition (Paun [Pau98a, 98b]). *Let X be a compact complex manifold (or more generally a compact complex space). Then*

(i) *The cohomology class of a closed positive $(1,1)$-current $\{T\}$ is nef if and only if the restriction $\{T\}_{|Z}$ is nef for every irreducible component Z in any of the Lelong sublevel sets $E_c(T)$.*

(ii) *The cohomology class of a Kähler current* $\{T\}$ *is a Kähler class* (*i.e. the class of a* smooth *Kähler form*) *if and only if the restriction* $\{T\}_{|Z}$ *is a Kähler class for every irreducible component* Z *in any of the Lelong sublevel sets* $E_c(T)$.

The proof of Proposition 2.4 is not extremely hard if we take for granted the fact that Kähler currents can be approximated by Kähler currents with logarithmic poles, a fact which was proved in [Dem92] (see also Theorem 5.1 below). The main point then consists in an induction on dimension and a standard gluing procedure : if $T = \alpha + \frac{i}{\pi}\partial\overline{\partial}\varphi$ where φ is smooth on $X \smallsetminus Z$ and has $-\infty$ poles along Z, then we can remove the poles of φ by replacing φ with $\max(\varphi, \psi - C)$, provided ψ is smooth and defined near Z and C is a large constant. □

The next (and more substantial step) consists of the following result which is reminiscent of the Grauert-Riemenschneider conjecture ([Siu84], [Dem85]).

2.5. Theorem ([DP03])**.** *Let* X *be a compact Kähler manifold and let* $\{\alpha\}$ *be a nef class* (*i.e.* $\{\alpha\} \in \overline{\mathcal{K}}$). *Assume that* $\int_X \alpha^n > 0$. *Then* $\{\alpha\}$ *contains a Kähler current* T, *in other words* $\{\alpha\} \in \mathcal{E}^\circ$.

Proof. The basic argument is to prove that for every irreducible analytic set $Y \subset X$ of codimension p, the class $\{\alpha\}^p$ contains a closed positive (p,p)-current Θ such that $\Theta \geqslant \delta[Y]$ for some $\delta > 0$. We check this by observing that $\alpha + \varepsilon\omega$ is a Kähler class, hence by the Calabi-Yau theorem [Yau78] the Monge-Ampère equation

$$(\alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon)^n = f_\varepsilon$$

can be solved with an arbitrary right-hand side $f_\varepsilon > 0$ such that

$$\int_X f_\varepsilon = C_\varepsilon = \int_X (\alpha + \varepsilon\omega)^n.$$

However, by our assumption that $\int_X \alpha^n > 0$, the constant C_ε is bounded away from 0. We use this fact in order to concentrate a fixed amount of volume of the volume form f_ε in an ε-tubular neighborhood of Y. We then show that the sequence of (p,p)-forms $(\alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon)^p$ converges weakly to the desired current Θ (this part relies heavily on the theory of currents). The second and final part uses a "diagonal trick": apply the result just proved to

$$\widetilde{X} = X \times X, \qquad \widetilde{Y} = \text{diagonal} \subset \widetilde{X}, \qquad \widetilde{\alpha} = \mathrm{pr}_1^*\,\alpha + \mathrm{pr}_2^*\,\alpha.$$

It is then clear that $\widetilde{\alpha}$ is nef on $\widetilde{X}$ and that $\int_{\widetilde{X}} (\widetilde{\alpha})^{2n} > 0$. It follows by the above that the class $\{\widetilde{\alpha}\}^n$ contains a Kähler current Θ such that $\Theta \geqslant \delta[\widetilde{Y}]$ for some $\delta > 0$. Therefore the push-forward

$$T := (\mathrm{pr}_1)_*(\Theta \wedge \mathrm{pr}_2^*\,\omega)$$

is numerically equivalent to a multiple of α and dominates $\delta\omega$, and we see that T is a Kähler current. □

End of Proof of Theorems 2.1 and 2.2. Clearly the open cone $\mathcal{K}$ is contained in $\mathcal{P}$, hence in order to show that $\mathcal{K}$ is one of the connected components of $\mathcal{P}$, we need only show that $\mathcal{K}$ is closed in $\mathcal{P}$, i.e. that $\overline{\mathcal{K}}\cap\mathcal{P}\subset\mathcal{K}$. Pick a class $\{\alpha\}\in\overline{\mathcal{K}}\cap\mathcal{P}$. In particular $\{\alpha\}$ is nef and satisfies $\int_X\alpha^n>0$. By Theorem 2.5 we conclude that $\{\alpha\}$ contains a Kähler current T. However, an induction on dimension using the assumption $\int_Y\alpha^p$ for all analytic subsets Y (we also use resolution of singularities for Y at this step) shows that the restriction $\{\alpha\}_{|Y}$ is the class of a Kähler current on Y. We conclude that $\{\alpha\}$ is a Kähler class by 2.4 (ii), therefore $\{\alpha\}\in\mathcal{K}$, as desired. □

The projective case 2.2 is a consequence of the following variant of Theorem 2.1.

2.6. Corollary. *Let X be a compact Kähler manifold. A $(1,1)$ cohomology class $\{\alpha\}$ on X is Kähler if and only if there exists a Kähler metric ω on X such that $\int_Y\alpha^k\wedge\omega^{p-k}>0$ for all irreducible analytic sets Y and all $k=1,2,\ldots,p=\dim Y$.*

Proof. The assumption clearly implies that

$$\int_Y(\alpha+t\omega)^p>0$$

for all $t\in\mathbb{R}_+$, hence the half-line $\alpha+(\mathbb{R}_+)\omega$ is entirely contained in the cone $\mathcal{P}$ of numerically positive classes. Since $\alpha+t_0\omega$ is Kähler for t_0 large, we conclude that the half-line in entirely contained in the connected component $\mathcal{K}$, and therefore $\alpha\in\mathcal{K}$. □

In the projective case, we can take $\omega=c_1(H)$ for a given very ample divisor H, and the condition $\int_Y\alpha^k\wedge\omega^{p-k}>0$ is equivalent to

$$\int_{Y\cap H_1\cap\ldots\cap H_{p-k}}\alpha^k>0$$

for a suitable complete intersection $Y\cap H_1\cap\ldots\cap H_{p-k}$, $H_j\in|H|$. This shows that algebraic cycles are sufficient to test the Kähler property, and the special case 2.2 follows. On the other hand, we can pass to the limit in 2.6 by replacing α by $\alpha+\varepsilon\omega$, and in this way we get also a characterization of nef classes.

2.7. Corollary. *Let X be a compact Kähler manifold. A $(1,1)$ cohomology class $\{\alpha\}$ on X is nef if and only if there exists a Kähler metric ω on X such that $\int_Y\alpha^k\wedge\omega^{p-k}\geqslant 0$ for all irreducible analytic sets Y and all $k=1,2,\ldots,p=\dim Y$.*

By a formal convexity argument, one can derive from 2.6 or 2.7 the following interesting consequence about the dual of the cone $\mathcal{K}$. We will not give the proof here, because it is just a simple tricky argument which does not require any new analysis.

2.8. Theorem. *Let X be a compact Kähler manifold. A $(1,1)$ cohomology class $\{\alpha\}$ on X is nef if and only for every irreducible analytic set Y in X, $p=\dim X$ and every Kähler metric ω on X we have $\int_Y\alpha\wedge\omega^{p-1}\geqslant 0$. In other words, the dual of the nef cone $\overline{\mathcal{K}}$ is the closed convex cone in $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by cohomology classes of currents of the form $[Y]\wedge\omega^{p-1}$ in $H^{n-1,n-1}(X,\mathbb{R})$, where Y runs over the collection of irreducible analytic subsets of X and $\{\omega\}$ over the set of Kähler classes of X.*

Our main Theorem 2.1 has an important application to the deformation theory of compact Kähler manifolds.

2.9. Theorem. *Let $\pi : \mathcal{X} \to S$ be a deformation of compact Kähler manifolds over an irreducible base S. Then there exists a countable union $S' = \bigcup S_\nu$ of analytic subsets $S_\nu \subsetneq S$, such that the Kähler cones $\mathcal{K}_t \subset H^{1,1}(X_t, \mathbb{C})$ of the fibers $X_t = \pi^{-1}(t)$ are invariant over $S \smallsetminus S'$ under parallel transport with respect to the $(1,1)$-projection $\nabla^{1,1}$ of the Gauss-Manin connection ∇ in the decomposition of*

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & 0 \\ * & \nabla^{1,1} & * \\ 0 & * & \nabla^{0,2} \end{pmatrix}$$

on the Hodge bundle $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

We moreover conjecture that for an arbitrary deformation $\mathcal{X} \to S$ of compact complex manifolds, the Kähler property is open with respect to the countable Zariski topology on the base S of the deformation.

Sketch of Proof of Theorem 2.9. The result is local on the base, hence we may assume that S is contractible. Then the family is differentiably trivial, the Hodge bundle $t \mapsto H^2(X_t, \mathbb{C})$ is the trivial bundle and $t \mapsto H^2(X_t, \mathbb{Z})$ is a trivial lattice. We use the existence of a relative cycle space $C^p(\mathcal{X}/S) \subset C^p(\mathcal{X})$ which consists of all cycles contained in the fibres of $\pi : X \to S$. It is equipped with a canonical holomorphic projection

$$\pi_p : C^p(\mathcal{X}/S) \to S.$$

We then define the S_ν's to be the images in S of those connected components of $C^p(\mathcal{X}/S)$ which do not project onto S. By the fact that the projection is proper on each component, we infer that S_ν is an analytic subset of S. The definition of the S_ν's imply that the cohomology classes induced by the analytic cycles $\{[Z]\}$, $Z \subset X_t$, remain exactly the same for all $t \in S \smallsetminus S'$. This result implies in its turn that the conditions defining the numerically positive cones $\mathcal{P}_t$ remain the same, except for the fact that the spaces $H^{1,1}_{\mathbb{R}}(X_t) \subset H^2(X_t, \mathbb{R})$ vary along with the Hodge decomposition. At this point, a standard calculation implies that the $\mathcal{P}_t$ are invariant by parallel transport under $\nabla^{1,1}$. Moreover, the connected component $\mathcal{K}_t \subset \mathcal{P}_t$ cannot jump from one component to the other thanks to Kodaira-Spencer theory [KS60] : *every Kähler class in X_{t_0} can be deformed to a nearby Kähler class in nearby fibres X_t* (a result which relies on perturbations of elliptic PDE's). □

As a by-product of our techniques, especially the regularization theorem for currents, we also get the following result for which we refer to [DP03].

2.10. Theorem. *A compact complex manifold carries a Kähler current if and only if it is bimeromorphic to a Kähler manifold (or equivalently, dominated by a Kähler manifold).*

This class of manifolds is called the *Fujiki class* $\mathcal{C}$. If we compare this result with the solution of the Grauert-Riemenschneider conjecture, it is tempting to make the following conjecture which would somehow encompass both results.

2.11 Conjecture. *Let X be a compact complex manifold of dimension n. Assume that X possesses a nef cohomology class $\{\alpha\}$ of type $(1,1)$ such that $\int_X \alpha^n > 0$. Then X*

is in the Fujiki class $\mathcal{C}$. [*Also,* $\{\alpha\}$ *would contain a Kähler current, as it follows from Theorem 2.5 if Conjecture 2.11 is proved*].

We want to mention here that most of the above results were already known in the cases of complex surfaces (i.e. in dimension 2), thanks to the work of N. Buchdahl [Buc99, 00] and A. Lamari [Lam99a, 99b].

Shortly after the original [DP03] manuscript appeared in April 2001, Daniel Huybrechts [Huy01] informed us Theorem 2.1 can be used to calculate the Kähler cone of a very general hyperkähler manifold: the Kähler cone is then equal to a suitable connected component of the positive cone defined by the Beauville-Bogomolov quadratic form. In the case of an arbitrary hyperkähler manifold, S.Boucksom [Bou02] later showed that a $(1,1)$ class $\{\alpha\}$ is Kähler if and only if it lies in the positive part of the Beauville-Bogomolov quadratic cone and moreover $\int_C \alpha > 0$ for all <u>*rational curves*</u> $C \subset X$ (see also [Huy99]).

§3. Cones of curves

In a dual way, we consider in $H^{n-1,n-1}_{\mathbb{R}}(X)$ the cone $\mathcal{N}$ generated by classes of positive currents T of type $(n-1,n-1)$ (i.e., of bidimension $(1,1)$). In the projective case, we also consider the intersection

By extension, we will say that $\overline{\mathcal{K}}$ is the cone of *nef* $(1,1)$-cohomology classes (even though they are not necessarily integral). We now turn ourselves to cones in cohomology of bidegree $(n-1,n-1)$.

3.1. Definition. *Let* X *be a compact Kähler manifold.*

(i) *We define* $\mathcal{N}$ *to be the* (*closed*) *convex cone in* $H^{n-1,n-1}_{\mathbb{R}}(X)$ *generated by classes of positive currents* T *of type* $(n-1,n-1)$ (*i.e., of bidimension* $(1,1)$).

(ii) *We define the cone* $\mathcal{M} \subset H^{n-1,n-1}_{\mathbb{R}}(X)$ *of* <u>*movable classes*</u> *to be the closure of the convex cone generated by classes of currents of the form*

$$\mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})$$

where $\mu : \widetilde{X} \to X$ *is an arbitrary modification* (*one could just restrict oneself to compositions of blow-ups with smooth centers*), *and the* $\widetilde{\omega}_j$ *are Kähler forms on* $\widetilde{X}$. *Clearly* $\mathcal{M} \subset \mathcal{N}$.

(iii) *Correspondingly, we introduce the intersections*

$$\mathcal{N}_{\rm NS} = \mathcal{N} \cap N_1(X), \qquad \mathcal{M}_{\rm NS} = \mathcal{M} \cap N_1(X),$$

in the space of integral bidimension $(1,1)$*-classes*

$$N_1(X) := (H^{n-1,n-1}_{\mathbb{R}}(X) \cap H^{2n-2}(X,\mathbb{Z})/\mathrm{tors}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

(iv) *If* X *is projective, we define* $\mathrm{NE}(X)$ *to be the convex cone generated by all effective curves. Clearly* $\overline{\mathrm{NE}(X)} \subset \mathcal{N}_{\rm NS}$.

(v) *If X is projective, we say that C is a strongly movable curve if*

$$C = \mu_\star(\widetilde{A}_1 \cap \ldots \cap \widetilde{A}_{n-1})$$

for suitable very ample divisors $\widetilde{A}_j$ on $\widetilde{X}$, where $\mu : \widetilde{X} \to X$ is a modification. We let $\mathrm{SME}(X)$ *to be the convex cone generated by all strongly movable (effective) curves. Clearly* $\overline{\mathrm{SME}(X)} \subset \mathcal{M}_{\mathrm{NS}}$.

(vi) *We say that C is a movable curve if $C = C_{t_0}$ is a member of an analytic family $(C_t)_{t\in S}$ such that $\bigcup_{t\in S} C_t = X$ and, as such, is a reduced irreducible 1-cycle. We let* $\mathrm{ME}(X)$ *to be the convex cone generated by all movable (effective) curves.*

The upshot of this definition lies in the following easy observation.

3.2. Proposition. *Let X be a compact Kähler manifold. Consider the Poincaré duality pairing*

$$H^{1,1}_{\mathbb{R}}(X) \times H^{n-1,n-1}_{\mathbb{R}}(X) \longrightarrow \mathbb{R}, \qquad (\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta.$$

Then the duality pairing takes nonnegative values

(i) *for all pairs $(\alpha, \beta) \in \overline{\mathcal{K}} \times \mathcal{N}$;*

(ii) *for all pairs $(\alpha, \beta) \in \mathcal{E} \times \mathcal{M}$.*

(iii) *for all pairs (α, β) where $\alpha \in \mathcal{E}$ and $\beta = [C_t] \in \mathrm{ME}(X)$ is the class of a movable curve.*

Proof. (i) is obvious. In order to prove (ii), we may assume that $\beta = \mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})$ for some modification $\mu : \widetilde{X} \to X$, where $\alpha = \{T\}$ is the class of a positive $(1,1)$-current on X and $\widetilde{\omega}_j$ are Kähler forms on $\widetilde{X}$. Then

$$\int_X \alpha \wedge \beta = \int_X T \wedge \mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1}) = \int_X \mu^* T \wedge \widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1} \geqslant 0.$$

Here, we have used the fact that a closed positive $(1,1)$-current T always has a pull-back $\mu^\star T$, which follows from the fact that if $T = i\partial\overline{\partial}\varphi$ locally for some plurisubharmonic function in X, we can set $\mu^\star T = i\partial\overline{\partial}(\varphi \circ \mu)$. For (iii), we suppose $\alpha = \{T\}$ and $\beta = \{[C_t]\}$. Then we take an open covering (U_j) on X such that $T = i\partial\overline{\partial}\varphi_j$ with suitable plurisubharmonic functions φ_j on U_j. If we select a smooth partition of unity $\sum \theta_j = 1$ subordinate to (U_j), we then get

$$\int_X \alpha \wedge \beta = \int_{C_t} T_{|C_t} = \sum_j \int_{C_t \cap U_j} \theta_j i\partial\overline{\partial}\varphi_{j|C_t} \geqslant 0.$$

For this to make sense, it should be noticed that $T_{|C_t}$ is a well defined closed positive $(1,1)$-current (i.e. measure) on C_t for almost every $t \in S$, in the sense of Lebesgue measure. This is true only because (C_t) covers X, thus $\varphi_{j|C_t}$ is not identically $-\infty$ for almost every $t \in S$. The equality in the last formula is then shown by a regularization argument for T, writing $T = \lim T_k$ with $T_k = \alpha + i\partial\overline{\partial}\psi_k$ and a decreasing sequence of smooth almost plurisubharmonic potentials $\psi_k \downarrow \psi$ such that the Levi forms have a uniform lower bound $i\partial\overline{\partial}\psi_k \geqslant -C\omega$ (such a sequence exists by [Dem92]). Then, writing $\alpha = i\partial\overline{\partial}v_j$ for some smooth potential v_j on U_j, we have $T = i\partial\overline{\partial}\varphi_j$ on U_j with $\varphi_j = v_j + \psi$, and this is the decreasing limit of the smooth approximations $\varphi_{j,k} = v_j + \psi_k$ on U_j. Hence $T_{k|C_t} \to T_{|C_t}$ for the weak topology of measures on C_t. □

If $\mathcal{C}$ is a convex cone in a finite dimensional vector space E, we denote by $\mathcal{C}^\vee$ the dual cone, i.e. the set of linear forms $u \in E^\star$ which take nonnegative values on all elements of $\mathcal{C}$. By the Hahn-Banach theorem, we always have $\mathcal{C}^{\vee\vee} = \overline{\mathcal{C}}$.

Proposition 3.2 leads to the natural question whether the cones $(cK, \mathcal{N})$ and $(\mathcal{E}, \mathcal{M})$ are dual under Poincaré duality. This question is addressed in the next section. Before doing so, we observe that the algebraic and transcendental cones of $(n-1, n-1)$ cohomology classes are related by the following equalities (similar to what we already noticed for $(1,1)$-classes, see Theorem 1.2).

3.3. Theorem. *Let X be a projective manifold. Then*

(i) $\overline{\mathrm{NE}(X)} = \mathcal{N}_{\mathrm{NS}}$.

(ii) $\overline{\mathrm{SME}(X)} = \overline{\mathrm{ME}(X)} = \mathcal{M}_{\mathrm{NS}}$.

Proof. (i) It is a standard result of algebraic geometry (see e.g. [Har70]), that the cone of effective cone $\mathrm{NE}(X)$ is dual to the cone $\overline{\mathcal{K}_{\mathrm{NS}}}$ of nef divisors, hence

$$\mathcal{N}_{\mathrm{NS}} \supset \overline{\mathrm{NE}(X)} = \mathcal{K}^\vee.$$

On the other hand, (3.3) (i) implies that $\mathcal{N}_{\mathrm{NS}} \subset \mathcal{K}^\vee$, so we must have equality and (i) follows.

Similarly, (ii) requires a duality statement which will be established only in the next sections, so we postpone the proof. □

§4. Main results and conjectures

It is very well-known that the cone $\overline{\mathcal{K}_{\mathrm{NS}}}$ of nef divisors is dual to the cone $\mathcal{N}_{\mathrm{NS}}$ of effective curves if X is projective. The transcendental case is Theorem 2.8 which we can restate as follows.

4.1. Theorem (Demailly-Paun, 2001)**.** *If X is Kähler, then the cones $\overline{\mathcal{K}} \subset H^{1,1}_{\mathbb{R}}(X)$ and $\mathcal{N} \subset H^{n-1,n-1}_{\mathbb{R}}(X)$ are dual by Poincaré duality, and $\mathcal{N}$ is the closed convex cone generated by classes $[Y] \wedge \omega^{p-1}$ where $Y \subset X$ ranges over p-dimensional analytic subsets, $p = 1, 2, \ldots, n$, and ω ranges over Kähler forms.*

Proof. Indeed, Prop. 3.4 shows that the dual cone $\mathcal{K}^\vee$ contains $\mathcal{N}$ which itself contains the cone $\mathcal{N}'$ of all classes of the form $\{[Y] \wedge \omega^{p-1}\}$. The main result of [DP03] conversely shows that the dual of $(\mathcal{N}')^\vee$ is equal to $\overline{\mathcal{K}}$, so we must have

$$\mathcal{K}^\vee = \overline{\mathcal{N}'} = \mathcal{N}.$$ □

The other important duality result is the following characterization of pseudo-effective classes, proved in [BDPP03] (the "only if" part already follows from 3.4 (iii)).

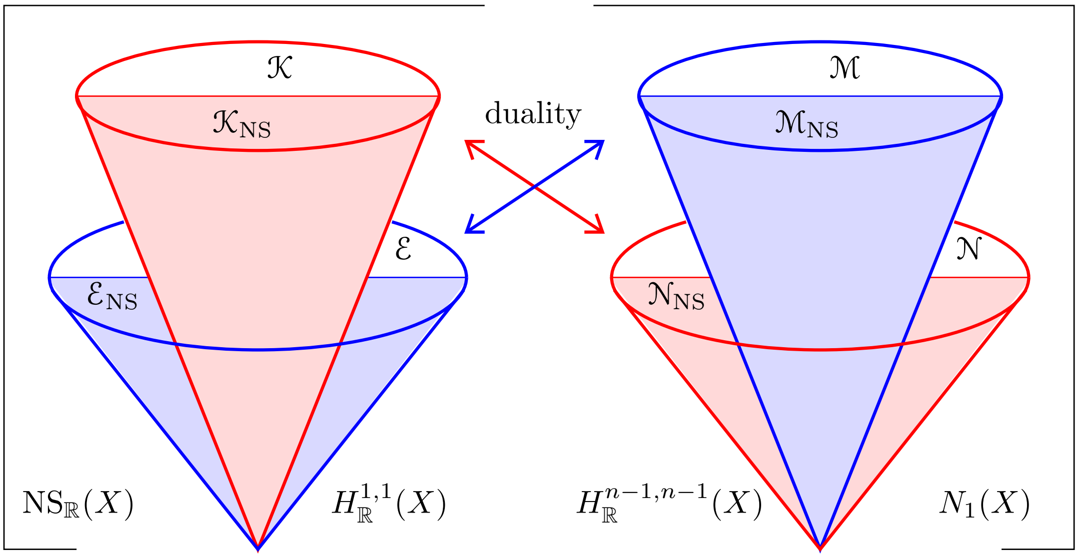
4.2. Theorem. *If X is projective, then a class $\alpha \in \mathrm{NS}_{\mathbb{R}}(X)$ is pseudo-effective if (and only if) it is in the dual cone of the cone* $\mathrm{SME}(X)$ *of strongly movable curves.*

In other words, a line bundle L is pseudo-effective if (and only if) $L \cdot C \geqslant 0$ for all *movable curves*, i.e., $L \cdot C \geqslant 0$ for every very generic curve C (not contained in a countable union of algebraic subvarieties). In fact, by definition of $\mathrm{SME}(X)$, it is enough to consider only those curves C which are images of generic complete intersection of very ample divisors on some variety $\widetilde{X}$, under a modification $\mu : \widetilde{X} \to X$.

By a standard blowing-up argument, it also follows that a line bundle L on a normal Moishezon variety is pseudo-effective if and only if $L \cdot C \geq 0$ for every movable curve C.

The Kähler analogue should be :

4.3. Conjecture. *For an arbitrary compact Kähler manifold X, the cones $\mathcal{E}$ and $\mathcal{M}$ are dual.*



The relation between the various cones of movable curves and currents in (3.5) is now a rather direct consequence of Theorem 4.2. In fact, using ideas hinted in [DPS96], we can say a little bit more. Given an irreducible curve $C \subset X$, we consider its normal "bundle" $N_C = \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C)$, where $\mathcal{I}$ is the ideal sheaf of C. If C is a general member of a covering family (C_t), then N_C is nef. Now [DPS96] says that the dual cone of the pseudo-effective cone of X contains the closed cone spanned by curves with nef normal bundle, which in turn contains the cone of movable curves. In this way we get :

4.4. Theorem. *Let X be a projective manifold. Then the following cones coincide.*

(i) *the cone* $\mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X)$;

(ii) *the closed cone* $\overline{\mathrm{SME}(X)}$ *of strongly movable curves* ;

(iii) *the closed cone* $\overline{\mathrm{ME}(X)}$ *of movable curves* ;

(iv) *the closed cone* $\overline{\mathrm{ME}_{\mathrm{nef}}(X)}$ *of curves with nef normal bundle.*

Proof. We have already seen that

$$\mathrm{SME}(X) \subset \mathrm{ME}(X) \subset \mathrm{ME}_{\mathrm{nef}}(X) \subset (\mathcal{E}_{\mathrm{NS}})^{\vee}$$

and

$$\mathrm{SME}(X) \subset \mathrm{ME}(X) \subset \mathcal{M}_{\mathrm{NS}} \subset (\mathcal{E}_{\mathrm{NS}})^{\vee}$$

by 3.4 (iii). Now Theorem 4.2 implies $(\mathcal{M}_{\mathrm{NS}})^{\vee} = \overline{\mathrm{SME}(X)}$, and 4.4 follows. □

4.5. Corollary. *Let X be a projective manifold and L a line bundle on X.*

(i) *L is pseudo-effective if and only if $L \cdot C \geq 0$ for all curves C with nef normal sheaf N_C.*

(ii) *If L is big, then $L \cdot C > 0$ for all curves C with nef normal sheaf N_C.*

4.5 (i) strenghtens results from [PSS99]. It is however not yet clear whether $\mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X)$ is equal to the closed cone of curves with *ample* normal bundle (although we certainly expect this to be true).

The most important special case of Theorem 4.2 is

4.6. Theorem. *If X is a projective manifold and is not uniruled, then K_X is pseudo-effective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$.*

Proof. If $K_X \notin \mathcal{E}_{\mathrm{NS}}$, Theorem 3.2 shows that there is a moving curve C_t such that $K_X \cdot C_t < 0$. The "bend-and-break" lemma then implies that there is family Γ_t of rational curves with $K_X \cdot \Gamma_t < 0$, so X is uniruled. □

A stronger result is expected to be true, namely :

4.7. Conjecture (special case of the "abundance conjecture")**.** *If K_X is pseudo-effective, then $\kappa(X) \geqslant 0$.*

§5 Zariski decomposition and movable intersections

Let X be compact Kähler and let $\alpha \in \mathcal{E}^{\circ}$ be in the *interior* of the pseudo–effective cone. In analogy with the algebraic context such a class α is called "big", and it can then be represented by a *Kähler current* T, i.e. a closed positive $(1,1)$-current T such that $T \geqslant \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.

5.1. Theorem (Demailly [Dem92], [Bou02, 3.1.24]**.** *If T is a Kähler current, then one can write $T = \lim T_m$ for a sequence of Kähler currents T_m which have logarithmic poles with coefficients in $\frac{1}{m}\mathbb{Z}$, i.e. there are modifications $\mu_m : X_m \to X$ such that*

$$\mu_m^{\star} T_m = [E_m] + \beta_m$$

where E_m is an effective $\mathbb{Q}$-divisor on X_m with coefficients in $\frac{1}{m}\mathbb{Z}$ (the "fixed part") and β_m is a closed semi-positive form (the "movable part").

Proof. Since this result has already been studied extensively, we just recall the main idea. Locally we can write $T = i\partial\overline{\partial}\varphi$ for some strictly plurisubharmonic potential φ. By a Bergman kernel trick and the Ohsawa-Takegoshi L^2 extension theorem, we get local approximations

$$\varphi = \lim \varphi_m, \qquad \varphi_m(z) = \frac{1}{2m} \log \sum_{\ell} |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the set of holomorphic functions which are L^2 with respect to the weight $e^{-2m\varphi}$. This Hilbert basis is also a family of local generators of the globally defined multiplier ideal sheaf $\mathcal{I}(mT) = \mathcal{I}(m\varphi)$. Then $\mu_m : X_m \to X$ is obtained by blowing-up this ideal sheaf, so that

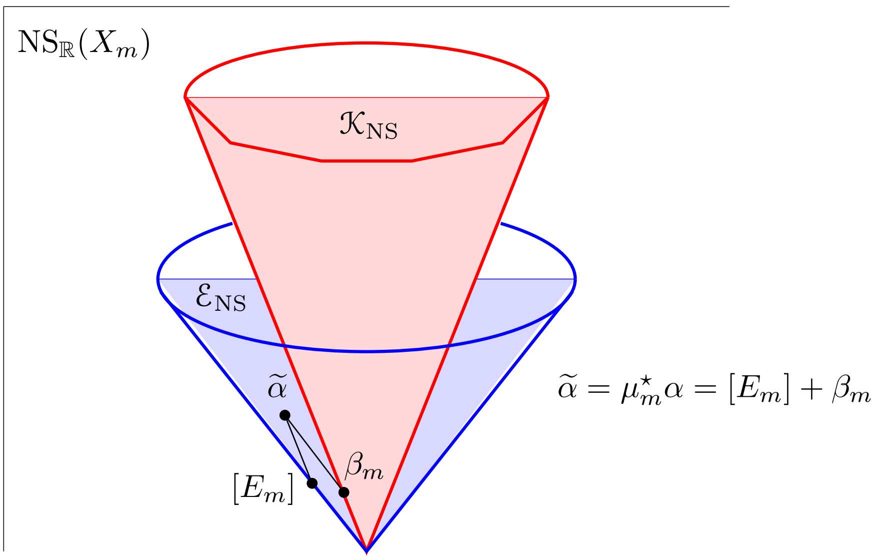
$$\mu_m^\star \mathcal{I}(mT) = \mathcal{O}(-mE_m).$$

We should notice that by approximating $T - \frac{1}{m}\omega$ instead of T, we can replace β_m by $\beta_m + \frac{1}{m}\mu^\star\omega$ which is a big class on X_m; by playing with the multiplicities of the components of the exceptional divisor, we could even achieve that β_m is a Kähler class on X_m, but this will not be needed here. □

The more familiar algebraic analogue would be to take $\alpha = c_1(L)$ with a big line bundle L and to blow-up the base locus of $|mL|$, $m \gg 1$, to get a $\mathbb{Q}$-divisor decomposition

$$\mu_m^\star L \sim E_m + D_m, \qquad E_m \ \text{ effective}, \quad D_m \ \text{ free}.$$

Such a blow-up is usually referred to as a "log resolution" of the linear system $|mL|$, and we say that $E_m + D_m$ is an approximate Zariski decomposition of L. We will also use this terminology for Kähler currents with logarithmic poles.



5.2. Definition. *We define the* **volume**, *or* **movable self-intersection** *of a big class* $\alpha \in \mathcal{E}^\circ$ *to be*

$$\mathrm{Vol}(\alpha) = \sup_{T \in \alpha} \int_{\widetilde{X}} \beta^n > 0$$

where the supremum is taken over all Kähler currents $T \in \alpha$ with logarithmic poles, and $\mu^\star T = [E] + \beta$ with respect to some modification $\mu : \widetilde{X} \to X$.

By Fujita [Fuj94] and Demailly-Ein-Lazarsfeld [DEL00], if L is a big line bundle, we have

$$\mathrm{Vol}(c_1(L)) = \lim_{m\to+\infty} D_m^n = \lim_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL),$$

and in these terms, we get the following statement.

5.3. Proposition. *Let L be a big line bundle on the projective manifold X. Let $\epsilon > 0$. Then there exists a modification $\mu : X_\epsilon \to X$ and a decomposition $\mu^*(L) = E + \beta$ with E an effective $\mathbb{Q}$-divisor and β a big and nef $\mathbb{Q}$-divisor such that*

$$\mathrm{Vol}(L) - \varepsilon \leqslant \mathrm{Vol}(\beta) \leqslant \mathrm{Vol}(L).$$

It is very useful to observe that the supremum in Definition 5.2 is actually achieved by a collection of currents whose singularities satisfy a filtering property. Namely, if $T_1 = \alpha + i\partial\overline{\partial}\varphi_1$ and $T_2 = \alpha + i\partial\overline{\partial}\varphi_2$ are two Kähler currents with logarithmic poles in the class of α, then

$$T = \alpha + i\partial\overline{\partial}\varphi, \qquad \varphi = \max(\varphi_1, \varphi_2) \tag{5.4}$$

is again a Kähler current with weaker singularities than T_1 and T_2. One could define as well

$$T = \alpha + i\partial\overline{\partial}\varphi, \qquad \varphi = \frac{1}{2m}\log(e^{2m\varphi_1} + e^{2m\varphi_2}), \tag{5.4'}$$

where $m = \mathrm{lcm}(m_1, m_2)$ is the lowest common multiple of the denominators occuring in T_1, T_2. Now, take a simultaneous log-resolution $\mu_m : X_m \to X$ for which the singularities of T_1 and T_2 are resolved as $\mathbb{Q}$-divisors E_1 and E_2. Then clearly the associated divisor in the decomposition $\mu_m^\star T = [E] + \beta$ is given by $E = \min(E_1, E_2)$. By doing so, the volume $\int_{X_m} \beta^n$ gets increased, as we shall see in the proof of Theorem 5.5 below.

5.5. Theorem (Boucksom [Bou02]). *Let X be a compact Kähler manifold. We denote here by $H^{k,k}_{\geqslant 0}(X)$ the cone of cohomology classes of type (k,k) which have non-negative intersection with all closed semi-positive smooth forms of bidegree $(n-k, n-k)$.*

(i) *For each integer $k = 1, 2, \ldots, n$, there exists a canonical "movable intersection product"*

$$\mathcal{E} \times \cdots \times \mathcal{E} \to H^{k,k}_{\geqslant 0}(X), \quad (\alpha_1, \ldots, \alpha_k) \mapsto \langle \alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1} \cdot \alpha_k \rangle$$

such that $\mathrm{Vol}(\alpha) = \langle \alpha^n \rangle$ whenever α is a big class.

(ii) *The product is increasing, homogeneous of degree 1 and superadditive in each argument, i.e.*

$$\langle \alpha_1 \cdots (\alpha_j' + \alpha_j'') \cdots \alpha_k \rangle \geqslant \langle \alpha_1 \cdots \alpha_j' \cdots \alpha_k \rangle + \langle \alpha_1 \cdots \alpha_j'' \cdots \alpha_k \rangle.$$

It coincides with the ordinary intersection product when the $\alpha_j \in \overline{\mathcal{K}}$ are nef classes.

(iii) *The movable intersection product satisfies the Teissier-Hovanskii inequalities*

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_n \rangle \geqslant (\langle \alpha_1^n \rangle)^{1/n} \ldots (\langle \alpha_n^n \rangle)^{1/n} \qquad (\textit{with } \langle \alpha_j^n \rangle = \mathrm{Vol}(\alpha_j)\,).$$

(iv) *For $k = 1$, the above "product" reduces to a (non linear) projection operator*

$$\mathcal{E} \to \mathcal{E}_1, \qquad \alpha \to \langle \alpha \rangle$$

onto a certain convex subcone $\mathcal{E}_1$ of $\mathcal{E}$ such that $\overline{\mathcal{K}} \subset \mathcal{E}_1 \subset \mathcal{E}$. Moreover, there is a "divisorial Zariski decomposition"

$$\alpha = \{N(\alpha)\} + \langle \alpha \rangle$$

where $N(\alpha)$ is a uniquely defined effective divisor which is called the "negative divisorial part" of α. The map $\alpha \mapsto N(\alpha)$ is homogeneous and subadditive, and $N(\alpha) = 0$ if and only if $\alpha \in \mathcal{E}_1$.

(v) *The components of $N(\alpha)$ always consist of divisors whose cohomology classes are linearly independent, especially $N(\alpha)$ has at most $\rho = \mathrm{rank}_{\mathbb{Z}}\,\mathrm{NS}(X)$ components.*

Proof. We essentially repeat the arguments developped in [Bou02], with some simplifications arising from the fact that X is supposed to be Kähler from the start.

(i) First assume that all classes α_j are big, i.e. $\alpha_j \in \mathcal{E}^\circ$. Fix a smooth closed $(n-k, n-k)$ *semi-positive* form u on X. We select Kähler currents $T_j \in \alpha_j$ with logarithmic poles, and a simultaneous log-resolution $\mu : \widetilde{X} \to X$ such that

$$\mu^\star T_j = [E_j] + \beta_j.$$

We consider the direct image current $\mu_\star(\beta_1 \wedge \ldots \wedge \beta_k)$ (which is a closed positive current of bidegree (k,k) on X) and the corresponding integrals

$$\int_{\widetilde{X}} \beta_1 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \geqslant 0.$$

If we change the representative T_j with another current T'_j, we may always take a simultaneous log-resolution such that $\mu^\star T'_j = [E'_j] + \beta'_j$, and by using (5.4′) we can always assume that $E'_j \leqslant E_j$. Then $D_j = E_j - E'_j$ is an effective divisor and we find $[E_j] + \beta_j \equiv [E'_j] + \beta'_j$, hence $\beta'_j \equiv \beta_j + [D_j]$. A substitution in the integral implies

$$\begin{aligned}
&\int_{\widetilde{X}} \beta'_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \\
&\qquad = \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u + \int_{\widetilde{X}} [D_1] \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \\
&\qquad \geqslant \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u.
\end{aligned}$$

Similarly, we can replace successively all forms β_j by the β'_j, and by doing so, we find

$$\int_{\widetilde X} \beta'_1 \wedge \beta'_2 \wedge \ldots \wedge \beta'_k \wedge \mu^\star u \geqslant \int_{\widetilde X} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u.$$

We claim that the closed positive currents $\mu_\star(\beta_1 \wedge \ldots \wedge \beta_k)$ are uniformly bounded in mass. In fact, if ω is a Kähler metric in X, there exists a constant $C_j \geqslant 0$ such that $C_j\{\omega\} - \alpha_j$ is a Kähler class. Hence $C_j\omega - T_j \equiv \gamma_j$ for some Kähler form γ_j on X. By pulling back with μ, we find $C_j\mu^\star\omega - ([E_j] + \beta_j) \equiv \mu^\star\gamma_j$, hence

$$\beta_j \equiv C_j\mu^\star\omega - ([E_j] + \mu^\star\gamma_j).$$

By performing again a substitution in the integrals, we find

$$\int_{\widetilde X} \beta_1 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \leqslant C_1 \ldots C_k \int_{\widetilde X} \mu^\star\omega^k \wedge \mu^\star u = C_1 \ldots C_k \int_X \omega^k \wedge u$$

and this is true especially for $u = \omega^{n-k}$. We can now arrange that for each of the integrals associated with a countable dense family of forms u, the supremum is achieved by a sequence of currents $(\mu_m)_\star(\beta_{1,m}\wedge\ldots\wedge\beta_{k,m})$ obtained as direct images by a suitable sequence of modifications $\mu_m : \widetilde X_m \to X$. By extracting a subsequence, we can achieve that this sequence is weakly convergent and we set

$$\langle\alpha_1 \cdot \alpha_2 \cdots \alpha_k\rangle = \lim_{m\to+\infty}\uparrow \{(\mu_m)_\star(\beta_{1,m} \wedge \beta_{2,m} \wedge \ldots \wedge \beta_{k,m})\}$$

(the monotonicity is not in terms of the currents themselves, but in terms of the integrals obtained when we evaluate against a smooth closed semi-positive form u). By evaluating against a basis of positive classes $\{u\} \in H^{n-k,n-k}(X)$, we infer by Poincaré duality that the class of $\langle\alpha_1 \cdot \alpha_2 \cdots \alpha_k\rangle$ is uniquely defined (although, in general, the representing current is not unique).

(ii) It is indeed clear from the definition that the movable intersection product is homogeneous, increasing and superadditive in each argument, at least when the α_j's are in $\mathcal{E}^\circ$. However, we can extend the product to the closed cone $\mathcal{E}$ by monotonicity, by setting

$$\langle\alpha_1 \cdot \alpha_2 \cdots \alpha_k\rangle = \lim_{\delta\downarrow 0}\downarrow\langle(\alpha_1 + \delta\omega) \cdot (\alpha_2 + \delta\omega) \cdots (\alpha_k + \delta\omega)\rangle$$

for arbitrary classes $\alpha_j \in \mathcal{E}$ (again, monotonicity occurs only where we evaluate against closed semi-positive forms u). By weak compactness, the movable intersection product can always be represented by a closed positive current of bidegree (k, k).

(iii) The Teissier-Hovanskii inequalities are a direct consequence of the fact that they hold true for nef classes, so we just have to apply them to the classes $\beta_{j,m}$ on $\widetilde X_m$ and pass to the limit.

(iv) When $k = 1$ and $\alpha \in \mathcal{E}^0$, we have

$$\alpha = \lim_{m\to+\infty}\{(\mu_m)_\star T_m\} = \lim_{m\to+\infty}(\mu_m)_\star[E_m] + \{(\mu_m)_\star\beta_m\}$$

and $\langle\alpha\rangle = \lim_{m\to+\infty}\{(\mu_m)_\star\beta_m\}$ by definition. However, the images $F_m = (\mu_m)_\star F_m$ are effective $\mathbb{Q}$-divisors in X, and the filtering property implies that F_m is a decreasing sequence. It must therefore converge to a (uniquely defined) limit $F = \lim F_m := N(\alpha)$ which is an effective $\mathbb{R}$-divisor, and we get the asserted decomposition in the limit.

Since $N(\alpha) = \alpha - \langle\alpha\rangle$ we easily see that $N(\alpha)$ is subadditive and that $N(\alpha) = 0$ if α is the class of a smooth semi-positive form. When α is no longer a big class, we define

$$\langle\alpha\rangle = \lim_{\delta\downarrow 0}\downarrow\langle\alpha+\delta\omega\rangle,\qquad N(\alpha) = \lim_{\delta\downarrow 0}\uparrow N(\alpha+\delta\omega)$$

(the subadditivity of N implies $N(\alpha+(\delta+\varepsilon)\omega)\leqslant N(\alpha+\delta\omega)$). The divisorial Zariski decomposition follows except maybe for the fact that $N(\alpha)$ might be a convergent countable sum of divisors. However, this will be ruled out when (v) is proved. As $N(\bullet)$ is subadditive and homogeneous, the set $\mathcal{E}_1 = \{\alpha\in\mathcal{E}\ ;\ N(\alpha)=0\}$ is a closed convex conne, and we find that $\alpha\mapsto\langle\alpha\rangle$ is a projection of $\mathcal{E}$ onto $\mathcal{E}_1$ (according to [Bou02], $\mathcal{E}_1$ consists of those pseudo-effective classes which are "nef in codimension 1").

(v) Let $\alpha\in\mathcal{E}^\circ$, and assume that $N(\alpha)$ contains linearly dependent components F_j. Then already all currents $T\in\alpha$ should be such that $\mu^\star T = [E]+\beta$ where $F = \mu_\star E$ contains those linearly dependent components. Write $F = \sum\lambda_j F_j$, $\lambda_j>0$ and assume that

$$\sum_{j\in J} c_j F_j \equiv 0$$

for a certain non trivial linear combination. Then some of the coefficients c_j must be negative (and some other positive). Then E is numerically equivalent to

$$E' \equiv E + t\mu^\star\Big(\sum\lambda_j F_j\Big),$$

and by choosing $t>0$ appropriate, we obtain an effective divisor E' which has a zero coefficient on one of the components $\mu^\star F_{j_0}$. By replacing E with $\min(E,E')$ via (5.4′), we eliminate the component $\mu^\star F_{j_0}$. This is a contradiction since $N(\alpha)$ was supposed to contain F_{j_0}. □

5.6. Definition. *For a class $\alpha\in H^{1,1}_{\mathbb{R}}(X)$, we define the* <u>*numerical dimension*</u> *$\nu(\alpha)$ to be $\nu(\alpha) = -\infty$ if α is not pseudo-effective, and*

$$\nu(\alpha) = \max\{p\in\mathbb{N}\ ;\ \langle\alpha^p\rangle\neq 0\},\qquad \nu(\alpha)\in\{0,1,\ldots,n\}$$

if α is pseudo-effective.

By the results of [DP03], a class is big ($\alpha\in\mathcal{E}^\circ$) if and only if $\nu(\alpha)=n$. Classes of numerical dimension 0 can be described much more precisely, again following Boucksom [Bou02].

5.7. Theorem. *Let X be a compact Kähler manifold. Then the subset $\mathcal{D}_0$ of irreducible divisors D in X such that $\nu(D)=0$ is countable, and these divisors are rigid as well as their multiples. If $\alpha\in\mathcal{E}$ is a pseudo-effective class of numerical dimension 0, then α is numerically equivalent to an effective $\mathbb{R}$-divisor $D=\sum_{j\in J}\lambda_j D_j$, for some finite* *subset $(D_j)_{j\in J} \subset \mathcal{D}_0$ such that the cohomology classes $\{D_j\}$ are linearly independent and some $\lambda_j > 0$. If such a linear combination is of numerical dimension 0, then so is any other linear combination of the same divisors.*

Proof. It is immediate from the definition that a pseudo-effective class is of numerical dimension 0 if and only if $\langle\alpha\rangle = 0$, in other words if $\alpha = N(\alpha)$. Thus $\alpha \equiv \sum \lambda_j D_j$ as described in 5.7, and since $\lambda_j\langle D_j\rangle \leqslant \langle\alpha\rangle$, the divisors D_j must themselves have numerical dimension 0. There is at most one such divisor D in any given cohomology class in $NS(X) \cap \mathcal{E} \subset H^2(X,\mathbb{Z})$, otherwise two such divisors $D \equiv D'$ would yield a blow-up $\mu : \widetilde{X} \to X$ resolving the intersection, and by taking $\min(\mu^\star D, \mu^\star D')$ via (5.4′), we would find $\mu^\star D \equiv E + \beta$, $\beta \neq 0$, so that $\{D\}$ would not be of numerical dimension 0. This implies that there are at most countably many divisors of numerical dimension 0, and that these divisors are rigid as well as their multiples. □

The above general concept of numerical dimension leads to a very natural formulation of the abundance conjecture for non-minimal (Kähler) varieties.

5.8. Generalized abundance conjecture. *For an arbitrary compact Kähler manifold X, the Kodaira dimension should be equal to the numerical dimension :*

$$\kappa(X) = \nu(X) := \nu(c_1(K_X)).$$

This appears to be a fairly strong statement. In fact, it is not difficult to show that the generalized abundance conjecture would contain the $C_{n,m}$ conjectures.

5.9. Remark. Using the Iitaka fibration, it is immediate to see that $\kappa(X) \leq \nu(X)$.

5.10. Remark. It is known that abundance holds in case $\nu(X) = -\infty$ (if K_X is not pseudo-effective, no multiple of K_X can have sections), or in case $\nu(X) = n$. The latter follows from the solution of the Grauert-Riemenschneider conjecture in the form proven in [Dem85] (see also [DP03]).

In the remaining cases, the most tractable situation is probably the case when $\nu(X) = 0$. In fact Theorem 5.7 then gives $K_X \equiv \sum \lambda_j D_j$ for some effective divisor with numerically independent components, $\nu(D_j) = 0$. It follows that the λ_j are rational and therefore

$$(*) \qquad K_X \sim \sum \lambda_j D_j + F \qquad \text{where } \lambda_j \in \mathbb{Q}^+,\ \nu(D_j) = 0 \text{ and } F \in \mathrm{Pic}^0(X).$$

Especially, if we assume additionally that $q(X) = h^{0,1}(X)$ is zero, then mK_X is linearly equivalent to an integral divisor for some multiple m, and it follows immediately that $\kappa(X) = 0$. The case of a general projective (or compact Kähler) manifold with $\nu(X) = 0$ and positive irregularity $q(X) > 0$ would be interesting to understand.

§6 The orthogonality estimate

The goal of this section is to show that, in an appropriate sense, approximate Zariski decompositions are almost orthogonal.

6.1. Theorem. *Let X be a projective manifold, and let $\alpha = \{T\} \in \mathcal{E}^\circ_{\rm NS}$ be a big class represented by a Kähler current T. Consider an approximate Zariski decomposition*

$$\mu_m^\star T_m = [E_m] + [D_m]$$

Then

$$(D_m^{n-1} \cdot E_m)^2 \leqslant 20\,(C\omega)^n \big(\mathrm{Vol}(\alpha) - D_m^n\big)$$

where $\omega = c_1(H)$ is a Kähler form and $C \geqslant 0$ is a constant such that $\pm\alpha$ is dominated by $C\omega$ (i.e., $C\omega \pm \alpha$ is nef).

Proof. For every $t \in [0,1]$, we have

$$\mathrm{Vol}(\alpha) = \mathrm{Vol}(E_m + D_m) \geqslant \mathrm{Vol}(tE_m + D_m).$$

Now, by our choice of C, we can write E_m as a difference of two nef divisors

$$E_m = \mu^\star\alpha - D_m = \mu_m^\star(\alpha + C\omega) - (D_m + C\mu_m^\star\omega).$$

6.2. Lemma. *For all nef $\mathbb{R}$-divisors A, B we have*

$$\mathrm{Vol}(A - B) \geqslant A^n - nA^{n-1} \cdot B$$

as soon as the right hand side is positive.

Proof. In case A and B are integral (Cartier) divisors, this is a consequence of the holomorphic Morse inequalities, [Dem01, 8.4]. If A and B are $\mathbb{Q}$-Cartier, we conclude by the homogeneity of the volume. The general case of $\mathbb{R}$-divisors follows by approximation using the upper semi-continuity of the volume [Bou02, 3.1.26]. □

6.3. Remark. We hope that Lemma 6.2 also holds true on an arbitrary Kähler manifold for arbitrary nef (non necessarily integral) classes. This would follow from a generalization of holomorphic Morse inequalities to non integral classes. However the proof of such a result seems technically much more involved than in the case of integral classes.

6.4. Lemma. *Let $\beta_1, \ldots, \beta_n$ and $\beta'_1, \ldots, \beta'_n$ be nef classes on a compact Kähler manifold $\widetilde{X}$ such that each difference $\beta'_j - \beta_j$ is pseudo-effective. Then the n-th intersection products satisfy*

$$\beta_1 \cdots \beta_n \leqslant \beta'_1 \cdots \beta'_n.$$

Proof. We can proceed step by step and replace just one β_j by $\beta' j \equiv \beta_j + T_j$ where T_j is a closed positive $(1,1)$-current and the other classes $\beta'_k = \beta_k$, $k \neq j$ are limits of Kähler forms. The inequality is then obvious. □

End of proof of Theorem 6.1. In order to exploit the lower bound of the volume, we write

$$tE_m + D_m = A - B, \qquad A = D_m + t\mu_m^\star(\alpha + C\omega), \quad B = t(D_m + C\mu_m^\star\omega).$$

By our choice of the constant C, both A and B are nef. Lemma 6.2 and the binomial formula imply

$$\begin{aligned}\mathrm{Vol}(tE_m + D_m) &\geqslant A^n - nA^{n-1}\cdot B\\ &= D_m^n + nt\,D_m^{n-1}\cdot \mu_m^\star(\alpha + C\omega) + \sum_{k=2}^{n} t^k\binom{n}{k} D_m^{n-k}\cdot \mu_m^\star(\alpha+C\omega)^k\\ &\quad - nt\,D_m^{n-1}\cdot(D_m + C\mu_m^\star\omega)\\ &\quad - nt^2\sum_{k=1}^{n-1} t^{k-1}\binom{n-1}{k} D_m^{n-1-k}\cdot \mu_m^\star(\alpha+C\omega)^k\cdot(D_m + C\mu_m^\star\omega).\end{aligned}$$

Now, we use the obvious inequalities

$$D_m \leqslant \mu_m^\star(C\omega), \qquad \mu_m^\star(\alpha+C\omega)\leqslant 2\mu_m^\star(C\omega), \qquad D_m + C\mu_m^\star\omega \leqslant 2\mu_m^\star(C\omega)$$

in which all members are nef (and where the inequality $\leqslant$ means that the difference of classes is pseudo-effective). We use Lemma 6.4 to bound the last summation in the estimate of the volume, and in this way we get

$$\mathrm{Vol}(tE_m + D_m) \geqslant D_m^n + ntD_m^{n-1}\cdot E_m - nt^2\sum_{k=1}^{n-1} 2^{k+1}t^{k-1}\binom{n-1}{k}(C\omega)^n.$$

We will always take t smaller than $1/10n$ so that the last summation is bounded by $4(n-1)(1+1/5n)^{n-2} < 4ne^{1/5} < 5n$. This implies

$$\mathrm{Vol}(tE_m + D_m) \geqslant D_m^n + nt\,D_m^{n-1}\cdot E_m - 5n^2t^2(C\omega)^n.$$

Now, the choice $t = \frac{1}{10n}(D_m^{n-1}\cdot E_m)((C\omega)^n)^{-1}$ gives by substituting

$$\frac{1}{20}\frac{(D_m^{n-1}\cdot E_m)^2}{(C\omega)^n} \leqslant \mathrm{Vol}(E_m + D_m) - D_m^n \leqslant \mathrm{Vol}(\alpha) - D_m^n$$

(and we have indeed $t \leqslant \frac{1}{10n}$ by Lemma 6.4), whence Theorem 6.1. Of course, the constant 20 is certainly not optimal. □

6.5. Corollary. *If $\alpha \in \mathcal{E}_{\mathrm{NS}}$, then the divisorial Zariski decomposition $\alpha = N(\alpha) + \langle\alpha\rangle$ is such that*

$$\langle\alpha^{n-1}\rangle\cdot N(\alpha) = 0.$$

Proof. By replacing α by $\alpha + \delta c_1(H)$, one sees that it is sufficient to consider the case where α is big. Then the orthogonality estimate implies

$$(\mu_m)_\star(D_m^{n-1})\cdot(\mu_m)_\star E_m = D_m^{n-1}\cdot(\mu_m)^\star(\mu_m)_\star E_m \leqslant D_m^{n-1}\cdot E_m \leqslant C(\mathrm{Vol}(\alpha) - D_m^n)^{1/2}.$$

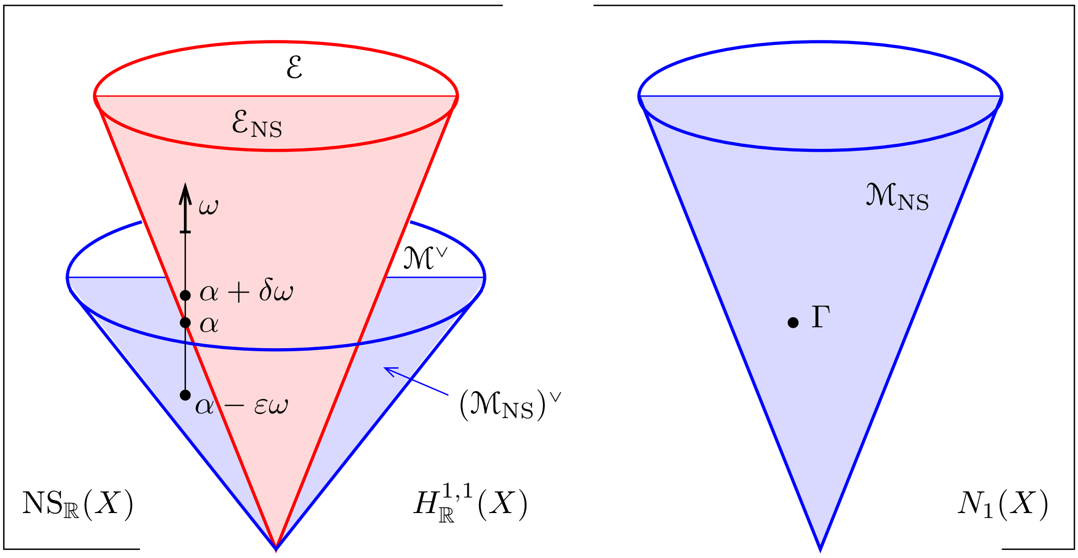
Since $\langle\alpha^{n-1}\rangle = \lim(\mu_m)_\star(D_m^{n-1})$, $N(\alpha) = \lim(\mu_m)_\star E_m$ and $\lim D_m^n = \mathrm{Vol}(\alpha)$, we get the desired conclusion in the limit. □

§7 Proof of the duality theorem

The proof is reproduced from [BDPP03]. We want to show that $\mathcal{E}_{\mathrm{NS}}$ and $\mathrm{SME}(X)$ are dual (Theorem 4.2). By 3.4 (iii) we have in any case

$$\mathcal{E}_{\mathrm{NS}} \subset (\mathrm{SME}(X))^\vee.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$ on the boundary of $\mathcal{E}_{\mathrm{NS}}$ which is in the interior of $\mathrm{SME}(X)^\vee$.



Let $\omega = c_1(H)$ be an ample class. Since $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$, the class $\alpha + \delta\omega$ is big for every $\delta > 0$, and since $\alpha \in ((\mathrm{SME}(X))^\vee)^\circ$ we still have $\alpha - \varepsilon\omega \in (\mathrm{SME}(X))^\vee$ for $\varepsilon > 0$ small. Therefore

$$\alpha \cdot \Gamma \geqslant \varepsilon\omega \cdot \Gamma \tag{7.1}$$

for every movable curve Γ. We are going to contradict (7.1). Since $\alpha + \delta\omega$ is big, we have an approximate Zariski decomposition

$$\mu_\delta^\star(\alpha + \delta\omega) = E_\delta + D_\delta.$$

We pick $\Gamma = (\mu_\delta)_\star(D_\delta^{n-1})$. By the Hovanskii-Teissier concavity inequality

$$\omega \cdot \Gamma \geqslant (\omega^n)^{1/n}(D_\delta^n)^{(n-1)/n}.$$

On the other hand

$$\begin{aligned}\alpha \cdot \Gamma &= \alpha \cdot (\mu_\delta)_\star(D_\delta^{n-1}) \\ &= \mu_\delta^\star\alpha \cdot D_\delta^{n-1} \leqslant \mu_\delta^\star(\alpha + \delta\omega) \cdot D_\delta^{n-1} \\ &= (E_\delta + D_\delta) \cdot D_\delta^{n-1} = D_\delta^n + D_\delta^{n-1} \cdot E_\delta.\end{aligned}$$

By the orthogonality estimate, we find

$$\frac{\alpha \cdot \Gamma}{\omega \cdot \Gamma} \leqslant \frac{D_\delta^n + \big(20(C\omega)^n(\mathrm{Vol}(\alpha+\delta\omega) - D_\delta^n)\big)^{1/2}}{(\omega^n)^{1/n}(D_\delta^n)^{(n-1)/n}}$$
$$\leqslant C'(D_\delta^n)^{1/n} + C''\frac{(\mathrm{Vol}(\alpha+\delta\omega) - D_\delta^n)^{1/2}}{(D_\delta^n)^{(n-1)/n}}.$$

However, since $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$, the class α cannot be big so

$$\lim_{\delta\to 0} D_\delta^n = \mathrm{Vol}(\alpha) = 0.$$

We can also take D_δ to approximate $\mathrm{Vol}(\alpha+\delta\omega)$ in such a way that $(\mathrm{Vol}(\alpha+\delta\omega)-D_\delta^n)^{1/2}$ tends to 0 much faster than D_δ^n. Notice that $D_\delta^n \geqslant \delta^n\omega^n$, so in fact it is enough to take

$$\mathrm{Vol}(\alpha+\delta\omega) - D_\delta^n \leqslant \delta^{2n}.$$

This is the desired contradiction by (7.1). □

7.2. Remark. If holomorphic Morse inequalities were known also in the Kähler case, we would infer by the same proof that "α not pseudo-effective" implies the existence of a blow-up $\mu : \widetilde{X} \to X$ and a Kähler metric $\widetilde{\omega}$ on $\widetilde{X}$ such that $\alpha \cdot \mu_\star(\widetilde{\omega})^{n-1} < 0$. In the special case when $\alpha = K_X$ is not pseudo-effective, we would expect the Kähler manifold X to be covered by rational curves. The main trouble is that characteristic p techniques are no longer available. On the other hand it is tempting to approach the question via techniques of symplectic geometry :

7.3. Question. *Let (M,ω) be a compact real symplectic manifold. Fix an almost complex structure J compatible with ω, and for this structure, assume that $c_1(M)\cdot\omega^{n-1} > 0$. Does it follow that M is covered by rational J-pseudoholomorphic curves ?*

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Jean-Pierre Demailly
Université de Grenoble I, Institut Fourier, UMR 5582 du CNRS
BP74, 100 rue des Maths, 38402 Saint-Martin d'Hères Cedex, France
E-mail: demailly@fourier.ujf-grenoble.fr

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