HOLOMORPHIC LINE BUNDLES WITH PARTIALLY VANISHING COHOMOLOGY

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0. Introduction and notation

One of the most fundamental facts of algebraic geometry is the possibility of characterizing ampleness of line bundles by numerical criteria (Nakai-Moishezon, Kleiman-Seshadri, ...), or by cohomology vanishing theorems. Over the complex numbers, ampleness is moreover equivalent to the existence of a metric of positive curvature (Kodaira).

The case of line bundles with curvature of mixed signature is also of a considerable importance. Andreotti and Grauert [AG62] have proved the following result:

Given X a compact complex manifold and L a holomorphic line bundle over X carrying a hermitian metric h whose curvature form $\Theta_h(L)$ is a $(1,1)$-form with at least $n-q$ positive eigenvalues at every point, then for every coherent sheaf $\mathcal{F}$ over X the cohomology groups $H^j(X, \mathcal{F} \otimes \mathcal{O}(mL))$ vanish for $j > q$ and $m \geq m_0(\mathcal{F})$.

The purpose of this paper is to investigate line bundles satisfying partial positivity properties in a systematic way. For this we introduce the following

Definition. — *Let L be a holomorphic line bundle over a projective manifold X. We let $\sigma_+(L)$ be the smallest integer q with the following property: there exists an ample divisor A on X and a constant $C > 0$ such that $H^j(X, mL - pA) = 0$ for all integers $j > q$ and $m, p \geq 0$, $m \geq C(p+1)$.*

One of the reasons of introducing A in the definition is to recover the notion of ampleness: indeed, L is ample precisely if $\sigma_+(L) = 0$. On the other hand $\sigma_+(L) = n = \dim X$ if and only if $c_1(L^\star)$ is in the closure of the cone of effective divisors. Moreover $\sigma_+(L)$ is an upper semicontinuous function of $c_1(L)$ in the Néron-Severi group of X over $\mathbb{Q}$.

The above mentioned vanishing theorem of Andreotti-Grauert takes in this context a slightly more precise form.

Proposition. — *If $\Theta_h(L)$ has at least $n-q$ positive eigenvalues at every point for some integer $q = 0, 1, \ldots, n$, then $\sigma_+(L) \leq q$.*

In view of Kodaira's characterization of ampleness by positive curvature, it would be interesting to know the answer to the following problem.

Problem. — *Let L be a holomorphic line bundle over a projective algebraic* *manifold X and let $q = \sigma_+(L)$. Is there a smooth hermitian metric h on L such that $\Theta_h(L)$ has at least $n - q$ positive eigenvalues at each point ?*

We prove that the problem has a positive answer in case $X = \mathbb{P}(E)$ and $L = \mathcal{O}_{\mathbb{P}(E)}(\pm 1)$, where $E^\star \to Y$ is an ample rank r vector bundle and Y is a curve, or Y is arbitrary and $E^\star$ is generated by sections. This makes use of results of Umemura [Um73] and Sommese [So78] relating the notions of ampleness and Griffiths positivity for vector bundles.

A more algebraic approach leads us to introduce the following definition of a purely numerical nature.

DEFINITION. — *Let X be a projective n-dimensional manifold. A sequence $Y_q \subset Y_{q+1} \subset \ldots \subset Y_{n-1} \subset Y_n = X$ of k-dimensional algebraic subvarieties Y_k of X is called an ample q-flag if for each $k = q, \ldots, n-1$ there exists an ample Cartier divisor Z_k in the normalization $\widetilde{Y}_{k+1}$, such that $Y_k = \nu_{k+1}(\mathrm{Supp} Z_k)$ as a set, where $\nu_{k+1} : \widetilde{Y}_{k+1} \to Y_{k+1}$ is the normalization map.*

We say that a line bundle $L \in \mathrm{Pic}(X)$ is q-flag positive if there exists an ample q-flag $Y_q \subset Y_{q+1} \ldots \subset X$ such that $L_{|Y_q}$ is positive.

The reason for considering normalizations in the definition of ample flags is that we want this notion to be invariant by finite maps. Without taking normalizations, a push forward of a Cartier divisor would not necessarily be a Cartier divisor. Our main result in this direction is that the above numerical criterion implies cohomology vanishing.

THEOREM. — *Let $L \in \mathrm{Pic}(X)$. If L is q-flag positive, then $\sigma_+(L) \le n - q$.*

The example of $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ over $X = \mathbb{P}(E)$ for $E = \Omega^1_S \otimes \mathcal{O}_S(2)$ over a general quartic surface in $\mathbb{P}_3$ shows that the converse to the Theorem is not true when $n = 3$, $q = 2$ (see Example 5.6). However, we do not have counterexamples in the most interesting case $q = 1$. In this case we have a partial positive result, using a recent paper of Campana-Flenner [CF91].

PROPOSITION. — *Let $X = \mathbb{P}(E) \xrightarrow{\pi} C$ be a $\mathbb{P}_{n-1}$-bundle over a smooth curve and let L be a line bundle on X with $\sigma_+(L) \le n - 1$. Then L is 1-flag positive. Moreover there is a base change $F^\star : \widetilde{X} = \mathbb{P}(f^\star E) \to X = \mathbb{P}(E)$ given by a finite map $f : \widetilde{C} \to C$, such that the pull-back $F^\star L$ admits an ample 1-flag*

$$Y_1 \subset \ldots \subset Y_{n-1} \subset X$$

of the form $Y_i = D_i \cap \ldots \cap D_{n-1}$ with $D_i, \ldots, D_{n-1}$ very ample and intersecting transversally.

Our next concern is to study the cone of "ample curves", in relation with effective divisors and ample 1-flags. To this effect, we introduce the following notation, which will be used throughout the paper.

Definition. — (i) *Let X be a nonsingular projective variety, $n = \dim X$. The Neron-Severi group of X is by definition the quotient group*

$$NS(X) = \big(\mathrm{Pic}(X)/{\equiv}\big) \simeq H^2(X,\mathbb{Z}) \cap H^{1,1}(X),$$

where $\equiv$ denotes the numerical equivalence of divisors. We define the real Néron-Severi group to be $NS^1(X) = NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$, and we let its dual be $NS_1(X)$. The Picard number of X is $\rho(X) = \dim_{\mathbb{R}}(NS^1(X)) = \mathrm{rank}_{\mathbb{Z}} NS_{\mathbb{Z}}(X)$.

(i) *We denote by $K_{\rm eff}(X) \subset NS^1(X)$ the cone generated by cohomology classes of effective divisors in X, by $K_{\rm amp}(X) \subset NS^1(X)$ the cone generated by classes of ample divisors, and by $\overline{K}_{\rm eff}(X)$, $\overline{K}_{\rm amp}(X)$ their closures. In a parallel way, we define $N_{\rm eff}(X) \subset NS_1(X)$ to be the cone generated by homology classes of effective curves, and we let $\overline{N}_{\rm amp}(X) \subset NS_1(X)$ be the dual cone of $\overline{K}_{\rm eff}(X)$, i.e. $\xi \in \overline{N}_{\rm amp}(X)$ if and only if $D \cdot \xi \geq 0$ for all $D \in K_{\rm eff}(X)$. The interior $N_{\rm amp}(X)$ of $\overline{N}_{\rm amp}(X)$ will be called the cone of ample curves.*

It is well known that $\overline{K}_{\rm amp}(X)$ is the set of classes of nef divisors, i.e. the dual cone of $\overline{N}_{\rm eff}(X)$ (see [Ha70]). By definition $K_{\rm amp}(X)$ and $N_{\rm amp}(X)$ are open cones. However, $K_{\rm eff}(X)$ and $N_{\rm eff}(X)$ are in general neither closed nor open; the interior $K^\circ_{\rm eff}(X)$ is the cone generated by line bundles of maximum Kodaira-Iitaka dimension $\kappa(L) = \dim X$. The inclusion $K_{\rm amp}(X) \subset K_{\rm eff}(X)$ yields by duality $\overline{N}_{\rm amp}(X) \subset \overline{N}_{\rm eff}(X)$, from which we also deduce $N_{\rm amp}(X) \subset N_{\rm eff}(X)$. Moreover, the equality $\overline{N}_{\rm amp}(X) = \overline{N}_{\rm eff}(X)$ occurs if and only if $\overline{K}_{\rm amp}(X) = \overline{K}_{\rm eff}(X)$, i.e., if and only if every effective divisor of X is nef. Our main results in this direction are:

Theorem. — *For an irreducible curve $C \subset X$, consider the following properties.*

(i) *C is the first member Y_1 of an irreducible ample 1-flag $Y_1 \subset \ldots \subset Y_{n-1} \subset X$. (We say that a flag is irreducible if all subvarieties Y_i are irreducible.)*

(ii) $\{C\} \in N_{\rm amp}(X)$.

(iii) $\{C\} \in \overline{N}_{\rm amp}(X)$.

(iv) *The normal bundle $N_{C/X} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ is ample (i.e., $\mathcal{O}_{\mathbb{P}(N_{C/X})}(1)$ is ample).*

(v) *The normal bundle $N_{C/X}$ is nef (i.e., $\mathcal{O}_{\mathbb{P}(N_{C/X})}(1)$ is nef).*

(vi) *The current of integration $[C]$ is weakly cohomologous to a smooth positive definite form of bidegree $(n-1,n-1)$, i.e. $[C] = u + \partial R + \overline{\partial R} + S$ where u is a smooth positive definite $(n-1,n-1)$-form with $\partial\overline{\partial} u = 0$, R is a current of type $(n-2,n-1)$, and S is a d-closed $(n-1,n-1)$-current whose cohomology class $\{S\} \in H^{n-1,n-1}(X)$ is orthogonal to $NS^1(X) \subset H^{1,1}(X)$.*

(vii) *The current of integration $[C]$ is weakly cohomologous to a smooth semipositive form of bidegree $(n-1,n-1)$ (as in* (vi)*, but with $u \geq 0$ only).*

(viii) *There is a family of generically irreducible curves* (C_t) *covering* X *such that* $C_0 = mC$ *as a cycle.*

Then we have the following implications:

(a) (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii),

(b) (i) $\Rightarrow$ (iv), *if additionally every* Y_i *is Cartier in* Y_{i+1},

(c) (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii), (viii) $\Rightarrow$ (iii)

(d) (ii) $\Leftrightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (iii)

Let $N_\mu(X)$ (or simply N_μ) be the convex cones generated by all classes of curves defined by Property (μ), $1 \leq \mu \leq 8$, and let $\overline{N}_\mu$ be their closures. Then we have:

Theorem. — *There are the following relations between the various cones:*

$$\overline{N}_1 \subset \overline{N}_2 = \overline{N}_3 = \overline{N}_6 = \overline{N}_7, \quad \overline{N}_8 \subset \overline{N}_5 \subset \overline{N}_3, \quad \overline{N}_4 \subset \overline{N}_5.$$

In particular, the inclusions $\overline{N}_1 \subset \overline{N}_4$ *and* $\overline{N}_2 \subset \overline{N}_1$ *would imply the equality of all cones. This is indeed the case if* X *is a surface.*

Finally, we investigate the structure of projective 3-folds with $\sigma_+(-K_X) = 1$. In fact, it is well-known since a long time that the structure of projective varieties should be described in terms of the existence of positive, negative or vanishing "directions" in the canonical bundle K_X. In order to understand the negative directions we put $\sigma(X) = 3 - \sigma_+(-K_X)$. Then the condition $\sigma(X) \geq 1$ holds precisely if $K_X \notin \overline{K}_{\mathrm{eff}}(X)$, which (via deep results of Mori and Miyaoka on the classification of 3-folds) just means that $K_X \notin K_{\mathrm{eff}}(X)$, i.e. $\kappa(X) = -\infty$. Since $\sigma(X) = \dim X$ if and only if X is a Fano manifold, the invariant $\sigma(X)$ measures how far X is from being Fano. First we give various examples of 3-folds with $\sigma(X) = 2$; the most interesting example we have is $X = \mathbb{P}(\Omega^1_S)$ where $S \subset \mathbb{P}_3$ is a general quartic surface. Then we discuss birational properties of $\sigma(X)$. Among other results, we show the following

Proposition. — *Let* X *be a smooth projective* 3*-fold with* $-K_X$ *big and nef but not ample. Let* $\varphi : X \to Y$ *be the map given by the base point free linear system* $|-m_0K_X|$ *with* $m_0 \gg 0$. *Then* $\sigma(X) = 2$ *precisely if all nontrivial fibers of* φ *are of dimension* 1.

An interesting open problem would be to know whether the condition $\sigma(X) \geq 2$ is invariant under birational maps of flipping type for singular X, so that this condition would be invariant by all operations of the Mori program.

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1. Vanishing of top or bottom cohomology groups

Let X be a nonsingular projective variety and let $n = \dim X$. We are interested in line bundles L over X such that all large multiples mL have zero cohomology groups in a certain range of degrees. However, the vanishing properties we want to consider should be stable under small perturbations of L. For this, we make the following definition.

1.1. Definition. — *Let L be a holomorphic line bundle over X.*

(i) *We let $\sigma_+(L)$ be the smallest integer q with the following property: there exists an ample divisor A and a constant $C > 0$ such that $H^j(X, mL - pA) = 0$ for all integers $j > q$ and $m, p \geq 0$, $m \geq C(p+1)$.*

(ii) *Similarly, we let $\sigma_-(L)$ be the largest integer q such that there is an ample divisor A and a constant $C > 0$ for which $H^j(X, mL + pA) = 0$ when $j < q$ and $m, p \geq 0$, $m \geq C(p+1)$.*

Let $\mathcal{F}$ be a coherent sheaf over X and let A be an ample line bundle. Then there exist locally free resolutions of the form

$$(\star) \qquad \cdots \to \bigoplus_{1\leq \ell\leq m_k} \mathcal{O}(-d_{k,\ell}A) \to \cdots \to \bigoplus_{1\leq \ell\leq m_0} \mathcal{O}(-d_{0,\ell}A) \to \mathcal{F} \to 0$$

with suitable integers $m_k, d_{k,\ell} \geq 0$. We define the *height* of $\mathcal{F}$ with respect to A to be the integer

$$\mathrm{ht}_A(\mathcal{F}) = \min_{\{\text{resolutions } (\star) \text{ of } \mathcal{F}\}} \ \max_{0\leq k\leq n,\, 1\leq \ell\leq m_k} d_{k,\ell}\ , \qquad n = \dim X.$$

By taking the tensor product of the resolutions associated with coherent sheaves $\mathcal{F}_1$, $\mathcal{F}_2$, we infer easily $\mathrm{ht}_A(\mathcal{F}_1 \otimes_{\mathcal{O}} \mathcal{F}_2) \leq \mathrm{ht}_A(\mathcal{F}_1) + \mathrm{ht}_A(\mathcal{F}_2)$.

1.2. Proposition. — *With the same constant C as in* 1.1 (i), *we have*

$$H^j(X, \mathcal{O}(mL) \otimes \mathcal{F}) = 0$$

for all integers $j > \sigma_+(L)$ and $m \geq C(\mathrm{ht}_A(\mathcal{F}) + 1)$.

Proof. Choose a resolution $(\star)$ which achieves the actual value of $\mathrm{ht}_A(\mathcal{F})$. Let $\mathcal{F}_k$ be the image sheaf of the differential of degree k in $(\star)$. Then we have short exact sequences

$$0 \to \mathcal{F}_{k+1} \to \bigoplus_{1\leq \ell\leq m_k} \mathcal{O}(-d_{k,\ell}A) \to \mathcal{F}_k \to 0, \qquad 0 \leq k \leq n,$$

and Def. 1.1 (i) yields $H^j(X, mL - d_{k,\ell}A) = 0$ for $j > \sigma_+(L)$ and $m \geq C(d_{k,\ell}+1)$. Since $\mathcal{F}_0 = \mathcal{F}$, this implies inductively

$$H^j(X, \mathcal{O}(mL) \otimes \mathcal{F}) \simeq \ldots H^{j+k}(X, \mathcal{O}(mL) \otimes \mathcal{F}_k) \simeq H^{j+k+1}(X, \mathcal{O}(mL) \otimes \mathcal{F}_{k+1})$$

for $j > q$ and $m \geq C(\mathrm{ht}_A(\mathcal{F}) + 1)$. Taking $k = n = \dim X$, we get the desired conclusion. □

Proposition 1.2 implies immediately that the choice of the ample divisor A in Definition 1.1 is irrelevant: if A' is another ample divisor, then $\mathrm{ht}_A(-pA') \le p\,\mathrm{ht}_A(-A')$, thus the constant C in 1.1 (i) need only be replaced by $C\,\mathrm{ht}_A(-A')$. On the other hand, Serre duality gives:

1.3. Duality formula. — $\sigma_-(L) = n - \sigma_+(L^\star)$.

Proof. Since $H^j(X, mL+pA) = H^{n-j}(X, mL^\star - pA + K_X)$ and $\mathrm{ht}_A(-pA+K_X) \le p + \mathrm{ht}_A(K_X)$, this group vanishes for $m/(p+1)$ large and $n-j > \sigma_+(L)$, hence $\sigma_-(L) \ge n - \sigma_+(L)$. On the other hand, $H^j(X, mL - pA + K_X) = H^{n-j}(X, mL^\star + pA)$ vanishes for $n - j < \sigma_-(L)$ and $m \ge C(p+1)$. Resolving $\mathcal{O}_X$ by direct sums of negative line bundles of the form $\mathcal{O}(-dA+K_X)$ and arguing as in the proof of Prop. 1.2, we conclude that $H^j(X, mL-pA)$ vanishes for $j > n - \sigma_-(L)$ and $m/(p+1)$ large, thus $\sigma_+(L) \le n - \sigma_-(L)$. □

Note also that our definitions imply $\sigma_-(L) \le \sigma_+(L)$, because there must be some degree $j \in \{0, 1, \ldots, n\}$ such that the groups $H^j(X, mL - pA)$ are non zero for arbitrary large values of $m \gg p \gg 1$ (the leading term $(mL - pA)^n/n!$ of the Euler characteristic is a non zero homogeneous polynomial of degree n in m, p). Finally, we have $\sigma_+(L) = \sigma_+(kL)$ for every integer $k > 0$. Hence, for a $\mathbb{Q}$-divisor D, the integer $\sigma_+(D)$ can be defined as $\sigma_+(kD)$ for any common denominator k of the coefficients of D.

1.4. Proposition. — *For $D \in \mathrm{Div}_{\mathbb{Q}}(X)$, the integer $\sigma_+(D)$ depends only on the first Chern class $c_1(D)$. Moreover, the function $c_1(D) \mapsto \sigma_+(D)$ is upper semicontinuous with respect to the vector space topology of the real Néron-Severi group $NS^1(X)$.*

Proof. Fix $D \in \mathrm{Div}_{\mathbb{Z}}(X)$ and $B_i \in \mathrm{Div}_{\mathbb{Z}}(X)$, $1 \le i \le \rho$, such that the Chern classes $c_1(B_i)$ define a basis of $NS^1(X)$. Then for any $D' \in \mathrm{Div}_{\mathbb{Q}}(X)$, the difference $D' - D$ is numerically equivalent to a linear combination $\sum \lambda_i B_i$, $\lambda_i \in \mathbb{Q}$, and $c_1(D')$ is close to $c_1(D)$ if and only if $\sum |\lambda_i|$ is small. Let k be a common denominator of the λ_i's. Then we can write

$$kD' = kD + \sum k\lambda_i B_i + F$$

where F is a numerically trivial integral divisor. Since $\mathrm{Pic}^0(X)$ is compact, there is a uniform bound M of the height of all numerically trivial divisors with respect to the given ample divisor A. Thus for $m, p > 0$ and $j > \sigma_+(D)$, we get

$$H^j(X, mkD' - pA) = H^j\Big(X, mkD + \sum mk\lambda_i B_i + mF - pA\Big) = 0$$

provided that mk is at least equal to

$$C\Big(\mathrm{ht}_A\big(\sum mk\lambda_i B_i + mF - pA\big) + 1\Big) \le C\Big(\sum mk|\lambda_i|\,\mathrm{ht}_A(B_i) + M + p + 1\Big).$$

If the λ_i's are chosen so small that $C\sum |\lambda_i|\,\mathrm{ht}_A(B_i) < 1/2$, it is then enough to take $mk \ge 2C(M + p + 1)$. This implies $\sigma_+(D') \ge \sigma_+(D)$, whence the upper semicontinuity of $\sigma_+(D)$. The fact that $\sigma_+(D)$ depends solely on $c_1(D)$ is proved similarly (and is even easier). $\square$

1.5. General properties. — *All varieties X, X' involved below are supposed to be projective and nonsingular. Let L be a holomorphic line bundle over X.*

(i) *$\sigma_+(L) = 0$ if and only if L is positive (i.e., ample), and $\sigma_-(L) = n$ if and only if L is negative.*

(ii) *$\sigma_-(L) = 0$ if and only if $c_1(L) \in \overline{K}_{\mathrm{eff}}(X)$, and $\sigma_+(L) = n$ if and only if $c_1(L^\star) \in \overline{K}_{\mathrm{eff}}(X)$.*

(iii) *If $f : X' \to X$ is a finite map, then $\sigma_+(f^\star L) = \sigma_+(L)$ and $\sigma_-(f^\star L) = \sigma_-(L)$.*

(iv) *Let $Y \subset X$ be a nonsingular subvariety and let L be a line bundle over X. Then $\sigma_+(L_{|Y}) \le \sigma_+(L)$.*

(v) *Let $L \to X$ and $L' \to X'$ be holomorphic line bundles. Then*
$$\sigma_+(L \boxtimes L') = \sigma_+(L) + \sigma_+(L').$$

(vi) *Let L, L' be holomorphic line bundles over X. Then*
$$\sigma_+(L + L') \le \sigma_+(L) + \sigma_+(L').$$

Proof. (i) If $\sigma_+(L) = 0$, Proposition 1.2 applied to the ideal sheaves $\mathcal{F} = \mathcal{I}_{\{x\}}\mathcal{I}_{\{y\}}$ of pairs of points shows that $H^j(X, \mathcal{O}(mL) \otimes \mathcal{I}_{\{x\}}\mathcal{I}_{\{y\}}) = 0$ for all $m \ge m_0$, $j > 0$ and $x, y \in X$. Hence $\mathcal{O}(mL)$ is very ample for $m \ge m_0$, and L is positive. Conversely, if L is positive, there are integers $k_0, k_1 > 0$ such that $k_0 L - A > 0$ and $k_1 L - K_X > 0$. Then
$$mL - pA - K_X = (m - pk_0 - k_1)L + p(k_0 L - A) + k_1 L - K_X > 0$$
for $m, p \ge 0$ and $m \ge pk_0 + k_1$, and the Kodaira vanishing theorem implies $H^j(X, mL - pA) = 0$ for $j > 0$. Hence $\sigma_+(L) = 0$. The dual case $\sigma_-(L) = n$ follows by Proposition 1.3.

(ii) By Definition, we have $\sigma_-(L) = 0$ if and only if $H^0(X, mL + pA) \ne 0$ for $m, p \ge 0$ and $m/(p+1)$ arbitrary large. The existence of such sections implies $c_1(L) + \frac{p}{m} c_1(A) \in K_{\mathrm{eff}}(X)$, hence $c_1(L) \in \overline{K}_{\mathrm{eff}}(X)$ in the limit. Conversely, the assumption $c_1(L) \in \overline{K}_{\mathrm{eff}}(X)$ implies $\sigma_-(L) = 0$. In fact, for every such L, we claim that $H^0(X, mL + pA) \ne 0$ for all $m \ge 0$ and $p \ge p_0$, where p_0 is chosen so large that $p_0 A - K_X - nH > 0$ for any given very ample line bundle H. To see this, observe that the ample divisors lie in the interior $K^\circ_{\mathrm{eff}}(X)$ of the cone. Hence we have
$$c_1\big(mL + pA - K_X - (n+\varepsilon)H\big) \in K^\circ_{\mathrm{eff}}(X)$$
for all $m \ge 0$, $p \ge p_0$ and $\varepsilon < \varepsilon_0$ sufficiently small. This implies that $mL + pA - K_X - (n+\varepsilon)H$ is linearly equivalent to an effective $\mathbb{Q}$-divisor D plus a numerically trivial line bundle T. Hence
$$\mathcal{O}(mL + pA) \simeq \mathcal{O}(K_X + (n+\varepsilon)H + D + T).$$

Now, fix a point $x_0 \notin D$. The line bundle $G = \mathcal{O}((n+\varepsilon)H + D + T)$ can be equipped with a hermitian metric h with logarithmic poles at x_0 and along D, such that h is nonsingular on $X \setminus (D \cup \{x_0\})$ and has positive definite curvature everywhere on X in the sense of currents. In fact the line bundle $\mathcal{O}(\varepsilon H + D + T)$ possesses a singular metric with curvature form equal to $\varepsilon\omega + [D]$, where ω is a Kähler form in $c_1(H)$ and $[D]$ is the current of integration over D; on the other hand, $\mathcal{O}(nH)$ can be equipped with a hermitian metric of semipositive curvature possessing an isolated pole at x_0, namely a metric of the form

$$\mathcal{O}(nH)_x \ni \xi \longmapsto |\xi|^2/(\sum |h_j(x)|^2)^n$$

where (h_j) is a basis of the space of sections of H vanishing at x_0. Since this last metric is not integrable at x_0, Hörmander's L^2 existence theorem shows that $mL + pA \sim K_X + G$ has a global section which does not vanish at x_0; see e.g. [De90] for details. Therefore $H^0(X, mL + pA) \neq 0$ and $\sigma_-(L) = 0$. The case $\sigma_+(L) = n$ is dual.

(iii) If $f : X' \to X$ is a finite map and A an ample line bundle over X, then $f^\star A$ is ample on X'. The projection formula yields

$$f_\star \mathcal{O}(m\, f^\star L - p\, f^\star A) = \mathcal{O}(mL - pA) \otimes f_\star \mathcal{O}_{X'}.$$

All higher direct images are zero because f is finite. The Leray spectral sequence yields

$$H^j(X', m\, f^\star L - p\, f^\star A) = H^j(X, \mathcal{O}(mL - pA) \otimes f_\star \mathcal{O}_{X'}).$$

Since the height of $\mathcal{O}(-pA) \otimes f_\star \mathcal{O}_{X'}$ is bounded by $p + \mathrm{ht}_A(f_\star \mathcal{O}_{X'})$, we infer from Proposition 1.3 that

$$H^j(X', m\, f^\star L - p\, f^\star A) = 0$$

for $j > \sigma_+(L)$ and $m \geq C(p + \mathrm{Const})$. Hence $\sigma_+(f^\star L) \leq \sigma_+(L)$. Now, the vanishing of $H^j(X', m\, f^\star L - p\, f^\star A)$ implies the vanishing of $H^j(X, mL - pA)$, because $\mathcal{O}_X$ is a direct summand in $f_\star \mathcal{O}_{X'}$. Therefore $\sigma_+(f^\star L) \geq \sigma_+(L)$ and we have equality. The formula for $\sigma_-(f^\star L)$ follows by duality.

(iv) Let A be an ample divisor on X and let $\mathcal{I}_Y$ be the ideal sheaf of Y. Then $A_{|Y}$ is ample and Proposition 1.3 gives

$$H^j(Y, mL_{|Y} - pA_{|Y}) = H^j\big(X, \mathcal{O}(mL - pA) \otimes (\mathcal{O}_X/\mathcal{I}_Y)\big) = 0$$

for $j > \sigma_+(L)$ and $m \geq C(p + \mathrm{ht}_A(\mathcal{O}_X/\mathcal{I}_Y) + 1)$, whence the desired inequality.

(v) Let A, A' be ample divisors on X, X' respectively. Then $A \boxtimes A'$ is ample on $X \times X'$, and the Künneth formula yields

$$H^k(X \times X', m\, L \boxtimes L' - p\, A \boxtimes A') = \bigoplus_{i+j=k} H^i(X, mL - A) \otimes H^j(X', mL' - A').$$

The conclusion follows immediately from this.

(vi) is a straightforward consequence of (iii) and (iv), because the line bundle $L + L'$ is just the restriction of $L \boxtimes L'$ to the diagonal of $X \times X$. $\square$

We now consider the effect of blowing-up on line bundles with partially vanishing cohomology groups.

1.6. Proposition. — *Let $\pi : \widetilde{X} \to X$ be a blow-up with smooth center $Y \subset X$, and let $E \subset \widetilde{X}$ be the exceptional divisor. Then, for every line bundle L over X, we have the following inequalities.*

(i) $\sigma_+(\pi^\star L) \le \max\big\{\sigma_+(L), \sigma_+(L_{|Y}) + \operatorname{codim} Y - 1\big\} \le \sigma_+(L) + \operatorname{codim} Y - 1.$

(ii) $\sigma_-(\pi^\star L) \ge \min\big\{\sigma_-(L), \sigma_-(L_{|Y}) + 1\big\} \ge \sigma_-(L) - (\operatorname{codim} Y - 1).$

(iii) *For $k \ge k_0$ large, $\sigma_+(k\,\pi^\star L - E) \le \sigma_+(L)$.*

Proof. Let A be an ample divisor on X. Then there is an integer $d > 0$ such that $\widetilde{A} = d\,\pi^\star A - E$ is ample on $\widetilde{X}$. We have

$$H^j\big(\widetilde{X}, m\,\pi^\star L - p(d\,\pi^\star A - E)\big) = H^j\big(\widetilde{X}, \pi^\star(L - pdA) + pE\big),$$

and the projection formula implies

$$R^j\pi_\star\mathcal{O}\big(\pi^\star(L - pdA) + pE\big) = \mathcal{O}(L - pdA) \otimes R^j\pi_\star\mathcal{O}(pE).$$

By a well-known formula, the direct images $R^j\pi_\star\mathcal{O}(pE)$ are given by

$$R^j\pi_\star\mathcal{O}(pE) = \begin{cases} \mathcal{I}_Y^{p^-} & \text{for } \ j = 0, \\ 0 & \text{for } \ j \ne 0, r-1, \qquad r = \operatorname{codim} Y, \\ \mathcal{F}_p & \text{for } \ j = r-1, \end{cases}$$

where $\mathcal{F}_p$ is a coherent $\mathcal{O}_X/\mathcal{I}_Y^{(p-r+1)^+}$-module with support on Y (thus equal to 0 for $p < r$), such that

$$\mathcal{F}_p \supset \mathcal{F}_{p-1} \supset \ldots \mathcal{F}_{p-k} = \mathcal{I}_Y^k\mathcal{F}_p \supset \ldots\,, \qquad \mathcal{F}_{p-k}/\mathcal{F}_{p-k-1} \simeq \det N_Y^\star \otimes S^{p-k-r}N_Y^\star$$

for $0 \le k \le p - r$. By the Leray spectral sequence, we infer that the groups $H^j\big(\widetilde{X}, m\,\pi^\star L - p(d\,\pi^\star A - E)\big)$ vanish provided that

$$H^j(X, mL - pdA) = 0, \qquad \text{resp.}$$

$$H^{j-(r-1)}\big(Y, \mathcal{O}(mL_{|Y} - pdA_{|Y}) \otimes \det N_Y^\star \otimes S^{p-k-r}N_Y^\star\big) = 0$$

for $k = r, r+1, \ldots, p$. When $m/(p+1)$ is large enough, these groups actually vanish for $j > \sigma_+(L)$, resp. for $j - (r-1) > \sigma_+(L_{|Y})$; indeed the height of the sheaf $\mathcal{O}(-pdA) \otimes \det N_Y^\star \otimes S^{p-k-r}N_Y^\star$ is bounded by a constant multiple of p. The first inequality in (i) follows from this; the second one is then a consequence of 1.5 (iv). The inequalities in (ii) are equivalent to those in (i) by duality.

(iii) Quite similarly, we find

$$\begin{aligned} H^j\big(\widetilde{X}, m(k\,\pi^\star L - E) - p(d\,\pi^\star A - E)\big) &= H^j\big(\widetilde{X}, \pi^\star(mkL - pdA) - (m-p)E\big) \\ &\simeq H^j\big(X, \mathcal{O}(mkL - pdA) \otimes \mathcal{I}_Y^{m-p}\big) \end{aligned}$$

for $m \ge p$ (in that case, $\mathcal{O}(-(m-p)E)$ has no higher direct images). The height of $\mathcal{O}(-pdA) \otimes \mathcal{I}_Y^{m-p}$ is bounded by $pd + (m-p)\mathrm{ht}_A(\mathcal{I}_Y)$, so we get vanishing for $j > \sigma_+(L)$ and $mk \ge C(pd + (m-p)C' + 1)$. The last condition is satisfied for $k \ge CC' + 1$ and $m \ge C(pd + 1)$, whence the inequality $\sigma_+(k\,\pi^\star L - E) \le \sigma_+(L)$ for $k \ge CC' + 1$.

2. Cohomology vanishing and signature properties of the curvature

Suppose that the line bundle L over X is equipped with a hermitian metric L and let $\Theta_h(L) = \frac{i}{2\pi} D_{L,h}^2$ be the Chern curvature form of h. The Andreotti-Grauert theorem implies that one can deduce coarse vanishing theorems from a knowledge of upper and lower bounds on the signature of $\Theta_h(L)$. Here we obtain a slightly more precise statement by means of the Bochner-Kodaira formula.

2.1. Proposition. — *Suppose that L has a smooth hermitian metric h such that the curvature form $\Theta_h(L)$ has at least q' negative eigenvalues and at least $n - q''$ positive eigenvalues at each point of X, for some integers $0 \le q' \le q'' \le n$. Then $q' \le \sigma_-(L) \le \sigma_+(L) \le q''$.*

Proof. This is essentially well-known, but we reproduce briefly the argument for self-containedness. By duality, it is enough to show the inequality $\sigma_+(L) \le q''$, i.e., $H^j(X, mL - pA) = 0$ for $j > q''$ and $m \ge C(p+1)$. For this, we apply the Bochner-Kodaira formula in the case of a nonnecessarily Kähler metric ω on X (see e.g. [De84]). Let u be a smooth $(0, j)$-form with values in a hermitian line bundle G. The Bochner-Kodaira formula implies an a priori estimate

$$\int_X \big(\|\overline{\partial} u\|^2 + \|\overline{\partial}^\star u\|^2\big) dV \ge \int_X (\gamma_1 + \ldots + \gamma_j - C_\omega) \|u\|^2 dV,$$

where $\gamma_1 \le \ldots \le \gamma_n$ are the eigenvalues of the curvature form of G at any point of X, and C_ω is a constant depending only on the torsion and Ricci curvature of ω. In particular, if $\gamma_1 + \ldots + \gamma_j > C_\omega$ everywhere, then $H^j(X, G) = 0$. We select the hermitian metric ω such that the eigenvalues $\lambda_1 \le \ldots \le \lambda_n$ of L satisfy $\lambda_{q''+1} = \ldots = \lambda_n = 1$ and $\lambda_1 \ge -\varepsilon$. (For this it is enough to take ω large enough on the negative eigenspaces of the curvature form $\Theta_h(L)$ and equal to $\Theta_h(L)$ on the positive eigenspaces.) Then, if α is a bound for the eigenvalues of $\Theta(A)$, the eigenvalues of $G = mL - pA$ satisfy $\gamma_j \ge m\lambda_j - p\alpha$, hence

$$\gamma_1 + \ldots + \gamma_j \ge m\big((j - q'') - q''\varepsilon\big) - pj\alpha.$$

If we choose $j > q''$ and $\varepsilon < 1/2q''$, we get $\gamma_1 + \ldots + \gamma_j - C_\omega > m/2 - pn\alpha - C_\omega$, hence $H^j(X, mL - pA) = 0$ for $m \ge 2pn\alpha + 2C_\omega$. □

In view of the Kodaira ampleness criterion, it is natural to ask whether the following "converse" to Proposition 2.1 holds.

2.2. Problem. — *Let L be a holomorphic line bundle over a projective algebraic manifold X and let $q' = \sigma_-(L)$, $q'' = \sigma_+(L)$. Is there a smooth hermitian metric h' (resp. h'') on L such that $\Theta_{h'}(L)$ has at least q' negative eigenvalues and $\Theta_{h''}(L)$ has at least $n - q''$ positive eigenvalues at each point?*

We now study the particular case of line bundles such that the nonzero cohomology groups arise only in one degree, in relation with the signature of the curvature form.

2.3. Definition. — *A line bundle L has cohomology concentrated in degree q if $\sigma_+(L) = \sigma_-(L) = q$.*

2.4. Definition. — *A line bundle L has constant signature (p, q) with $p+q \le n$, if there is a smooth hermitian metric h on L such that $\Theta_h(L)$ has signature (p, q) at every point.*

By Proposition 2.1, if L has signature $(n-q, q)$, then L has cohomology concentrated in degree q. The converse is not true, even though we expect Problem 2.2 to have a positive answer. The trouble is that the metrics h' and h'' will in general not coincide even if we assume that $q' = q''$.

2.5. Example. — An obvious necessary condition for the existence of a line bundle of constant signature $(n-q, q)$ is that the tangent bundle T_X splits topologically into a direct sum of complex subbundles of rank $(n-q)$ and q (the positive and negative eigenspaces with respect to any fixed hermitian metric on the base). Take for instance a surface X which is the blow-up of some other surface X' at a point, and let $L = \pi^\star L' - kE$ where E is the exceptional divisor, L' is ample on X' and $k \gg 0$; it is indeed enough to take k such that $L \cdot A = \pi^\star L' \cdot A - kE \cdot A < 0$ for some ample divisor A on X. Then neither $mL + pA$ nor $-mL + pA$ have sections for $m \gg p$ (take e.g. $A = \mu\pi^\star L' - E$ with μ large enough to see this easily), hence $\sigma_+(L) = \sigma_-(L) = 1$. However T_X does not split topologically into a sum of two line bundles if we take for instance X to be the blow-up of $\mathbb{P}_2$ at two points, hence L cannot have constant signature $(1, 1)$. (This simple example has been communicated to us by A. Beauville and F.A. Bogomolov.) □

By 1.5 (i) and the Kodaira ampleness criterion, Problem 2.2 has an affirmative answer in the positive or negative definite cases $q = 0$, $q = n$. The nondefinite case, however, seems to be a very hard problem, and we have very little evidence for it. The only general indication we have is that the sign of the determinant is correct in the average, i.e. that $(-1)^q c_1(L)^n = (-1)^q \int_X \Theta_h(L)^n \ge 0$. (We can even obtain this integral to be > 0 when $\dim X \le 3$.)

2.6. Proposition. — *Suppose that L has cohomology concentrated in degree q. Let k be the maximal integer with $c_1(L)^k \ne 0$. Then $n-k$ is even and $(-1)^q c_1(L)^k \cdot c_1(G)^{n-k} \ge 0$ for all $G \in \mathrm{Pic}(X)$, in particular $(-1)^q c_1(L)^n \ge 0$. Moreover, we have $(-1)^q c_1(L)^n > 0$ if $n = \dim X \le 3$.*

Proof. For m large, the Riemann-Roch formula gives

$$0 \le h^q(X, mL) = (-1)^q \chi(X, mL) \sim (-1)^q c_1(L)^n m^n/n!,$$

thus $(-1)^q c_1(L)^n \ge 0$. Since $\sigma_+(L)$ and $\sigma_-(L)$ are left unchanged by small perturbations of L, we conclude that $(-1)^q c_1(L + \varepsilon G)^n \ge 0$ for any $G \in \mathrm{Pic}(X)$ and $\varepsilon \in \mathbb{Q}$ small. Expanding the n-th power and letting ε tend to 0, we get $(-1)^q c_1(L)^k \cdot (\pm c_1(G))^{n-k} \ge 0$ for all $G \in \mathrm{Pic}(X)$. Since the product is not always zero when G runs over $\mathrm{Pic}(X)$, $n-k$ must be even. Observe that $k > 1$, otherwise L is numerically trivial and $\sigma_-(L) = 0 < \sigma_+(L) = n$. If $n = \dim X \le 3$, the only remaining possibility is $k = n$, whence $(-1)^q c_1(L)^n > 0$. $\square$

2.7. Examples. —

(i) If $L \to X$ is positive, $L' \to X'$ is negative and $n = \dim X$, $n' = \dim X'$, then $L \boxtimes L'$ has constant signature (n, n').

(ii) Let E be a holomorphic vector bundle of rank r over a n-dimensional variety Y, and let L be the tautological line bundle $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ over $X = \mathbb{P}(E)$. Then $\sigma_+(L) \le n$. Moreover, L has cohomology concentrated in degree n, i.e. $\sigma_+(L^\star) \le r-1$, if and only if $E^\star$ is ample.

(iii) Let $E \to Y$ be as in (ii). If E is negative in the sense of Griffiths, then L has signature $(r-1, n)$ over X. In particular, Problem 2.2 has a positive answer for $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ when $E^\star \to Y$ is an ample vector bundle over a curve.

Proof. (i) is obvious.

(ii) Let $\pi : X \to Y$ be the natural projection. There is an ample line bundle H on Y such that $A = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^\star H$ is ample on X. Then, for $m, p \ge 0$ and $\varepsilon = \pm 1$, the line bundle

$$mL + \varepsilon pA = \mathcal{O}_{\mathbb{P}(E)}(m + \varepsilon p) \otimes \pi^\star \mathcal{O}(\varepsilon pH)$$

has direct image $\pi_\star \mathcal{O}(mL + \varepsilon pA) = S^{m+\varepsilon p}E \otimes \mathcal{O}(\varepsilon pH)$ and zero higher direct images when $m \ge -\varepsilon p$. The Leray spectral sequence implies, together with Serre duality,

$$\begin{aligned} H^j(X, mL + \varepsilon pA) &\simeq H^j\big(Y, S^{m+\varepsilon p}E \otimes \mathcal{O}(\varepsilon pH)\big) \\ &\simeq H^{n-j}\big(Y, S^{m+\varepsilon p}E^\star \otimes \mathcal{O}(-\varepsilon pH) \otimes K_Y\big). \end{aligned}$$

By taking $\varepsilon = -1$, it follows immediately that $\sigma_+(L) \le n$. Moreover, if $E^\star$ is ample and $\varepsilon = +1$, the last group vanishes for $j < n$ and $m/(p+1)$ large, thus $\sigma_-(L) = n$. Conversely, if $\sigma_-(L) = n$, we have $H^j(Y, S^{m+p}E^\star \otimes \mathcal{O}(-pH) \otimes K_Y) = 0$ for $j > 0$ and $m/(p+1)$ large; resolving arbitrary coherent sheaves $\mathcal{F}$ by line bundles of the form $\mathcal{O}(-pH) \otimes K_Y$, we conclude that $H^j(Y, S^m E^\star \otimes \mathcal{F}) = 0$ for $j > 0$ and $m \ge m_0(\mathcal{F})$, hence $E^\star$ is ample.

(iii) Suppose that E carries a hermitian metric h with negative curvature tensor

$$\Theta_h(E) = \sum_{\substack{1 \le j,k \le n \\ 1 \le \lambda,\mu \le n}} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e_\lambda^\star \otimes e_\mu$$

in the sense of Griffiths. We suppose here that (e_λ) is a local holomorphic frame of E near a point $x \in X$, which is orthonormal at x and satisfies $\nabla e_\lambda(x) = 0$. The Griffiths negativity assumption means that $\sum c_{jk\lambda\mu} t_j \overline{t}_k v_\lambda \overline{v}_\mu < 0$ for all non zero vectors $t = \sum t_j \partial/\partial z_j \in T_{X,x}$ and $v = \sum v_\lambda e_\lambda \in E_x$. Let $(z_1, \ldots, z_n, w_1, \ldots, w_n)$

be the normal coordinates on $E^\star$ given by the dual frame $(e^\star_\lambda)$. A standard calculation shows that the curvature of $L=\mathcal{O}_{\mathbb{P}(E)}(1)$ at $[e^\star_\lambda]\in\mathbb{P}(E)$ is

$$\Theta_h(L)_{[e^\star_\lambda]}=\sum_{1\le j,k\le n}c_{jk\lambda\lambda}dz_j\wedge d\overline{z}_k+\sum_{1\le\mu\le r,\ \mu\ne\lambda}dw_\mu\wedge d\overline{w}_\mu$$

in terms of the coordinates $(z_1,\ldots,z_n,w_1,\ldots,w_\lambda{=}1,\ldots,w_n)$ on $\mathbb{P}(E)$. Hence the Griffiths negativity of E implies that $\Theta_h(L)$ has signature $(r-1,n)$, with positive eigenvalues in the fibre directions and negative eigenvalues in the horizontal directions. Finally, every ample vector bundle over a curve is positive in the sense of Griffiths (Umemura), so Conjecture 2.5 holds in that case. $\square$

There are only very few known cases of Conjecture 2.5, even in the case of line bundles $L=\mathcal{O}_{\mathbb{P}(E)}(1)$ with $E^\star$ ample. The following results are due to A. Sommese.

2.8. Proposition (A. Sommese). — *Conjecture 2.5 holds for $L=\mathcal{O}_{\mathbb{P}(E)}(1)$ if $E^\star\to Y$ is ample and satisfies one of the following additional properties:*

(i) *E has a strictly pseudoconvex neighborhood U of the zero section such that all slices $E_x\cap U$ are linearly strictly convex (U is supposed to be relatively compact with smooth boundary in E).*

(ii) *E is seminegative in the sense of Griffiths.*

(iii) *$E^\star$ is generated by sections.*

Sketch of the proof (see [So78]). First observe that the hypotheses are related by the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). The implication (iii) $\Rightarrow$ (ii) is standard, while (ii) $\Rightarrow$ (i) is checked as follows. Since $E^\star$ is ample, there is a stricly plurisubharmonic Finsler metric on E, i.e., a smooth strictly plurisubharmonic function F on $E\smallsetminus\{0\}$ such that $F(\lambda\xi)=|\lambda|\,F(\xi)$. On the other hand, the Griffiths seminegativity means that there is a weakly plurisubharmonic hermitian norm $\|\xi\|$. Then, for $\varepsilon>0$ small, the Finsler metric $\|\xi\|+\varepsilon F(\xi)$ is strictly plurisubharmonic and fibrewise strictly convex; thus $U=\{\xi\in E\,;\,\|\xi\|+\varepsilon F(\xi)<1\}$ satisfies (i).

Now, assuming that (i) holds, Sommese [So78] has made an explicit calculation of the Levi form of the dual neighborhood

$$U^\circ=\big\{\xi^\star\in E^\star_x\,;\,x\in X,\ F^\star(\xi^\star):=\sup_{\xi\in E_x\cap U}|\xi^\star(\xi)|<1\big\}.$$

It follows that the Finsler metric $F^\star(\xi^\star)$ on $E^\star$ has a Levi form $i\partial\overline{\partial}F^\star$ of signature (r,n), where $r=\mathrm{rank}(E)$ and $n=\dim Y$ (in particular, U° is a $(n+1)$-convex neighborhood of the zero section in $E^\star$). Then $F^\star$ can be also seen as a hermitian metric on $L^\star=\mathcal{O}_{\mathbb{P}(E)}(-1)$ whose curvature form has signature $(n,r-1)$ on $\mathbb{P}(E)$. $\square$

It is interesting to observe that Conjecture 2.5 is related to several natural questions in analytic geometry. In particular, it would yield a considerably simpler proof of the regularization theorem for closed positive $(1,1)$-currents established in [De92]. More importantly, it would yield a positive answer to the following conjecture made by M. Schneider [Sch73].

2.9. Conjecture (M. Schneider). — *Let $Y \subset X$ be a nonsingular subvariety with ample normal bundle N_Y. Then, denoting $k = \operatorname{codim} Y$, the complement $X \setminus Y$ is k-convex in the sense of Andreotti-Grauert.*

2.10. Proposition. — *Conjecture 2.9 holds provided that Problem 2.2 has an affirmative solution in the case of line bundles of the form $L = \mathcal{O}_{\mathbb{P}(N_Y^\star)}(1)$. Especially, $X \setminus Y$ is k-convex if N_Y is ample and generated by sections.*

Proof. Let $\widetilde{X} \to X$ be the blow-up of X with center Y, let E be the exceptional divisor and $n = \dim X$. Then $X \setminus Y \simeq \widetilde{X} \setminus E$ and $\mathcal{O}(E)_{|E} \simeq N_{E/\widetilde{X}} \simeq \mathcal{O}_{\mathbb{P}(N_Y^\star)}(-1)$ has a metric of signature $(n-k, k-1)$ by Conjecture 2.5. Therefore, $\mathcal{O}(E)$ can be equipped with a metric of signature $(n-k+1, k-1)$ in a neighborhood of E. (Take a smooth extension of the metric to $\widetilde{X}$ and multiply if necessary by a factor of the form $\exp(-Cd(z,E)^2)$ in order to produce a positive curvature eigenvalue in the normal directions to E.) If σ is the canonical section of $\mathcal{O}(E)$, then $-\log ||\sigma(z)||$ is an exhaustion function on $\widetilde{X} \setminus E$, and its Levi form has signature $(n-k, k-1)$ in a neighborhood of E. It follows that $\widetilde{X} \setminus E$ is k-convex. $\square$

3. Ample flags and partial cohomology vanishing

Proposition 1.4 shows that $\sigma_+(L)$ is a numerical invariant of L. For instance, the ampleness of L (i.e., the vanishing of $\sigma_+(L)$) is characterized by the well-known Nakai-Moishezon criterion: L is ample if and only if $L^p \cdot Y > 0$ for every p-dimensional subvariety $Y \subset X$. We expect that there is such a nice numerical criterion characterizing the invariant $\sigma_+(L)$. First, we introduce the notion of an ample flag of algebraic subvarieties.

3.1. Definition. — *Let X be a projective n-dimensional manifold. A sequence $Y_q \subset Y_{q+1} \subset \ldots \subset Y_{n-1} \subset Y_n = X$ of k-dimensional algebraic subvarieties of X is called an ample q-flag if for each $k = q, \ldots, n-1$ there exists an ample Cartier divisor Z_k in the normalization $\widetilde{Y}_{k+1}$, such that $Y_k = \nu_{k+1}(\mathrm{Supp} Z_k)$ as a set, where $\nu_{k+1} : \widetilde{Y}_{k+1} \to Y_{k+1}$ is the normalization map.*

3.2. Definition. — *We say that $L \in \mathrm{Pic}(X)$ is q-flag positive (resp. negative) if there exists an ample q-flag $Y_q \subset Y_{q+1} \ldots \subset X$ such that $L_{|Y_q}$ is positive (negative).*

The reason for considering normalizations in the definition of ample flags is that we want this notion to be invariant by finite maps. Without taking normalizations, a push forward of a Cartier divisor would not necessarily be a

Cartier divisor. The invariance by finite maps can be stated as follows.

3.3. Proposition. — *Let $f : X' \to X$ be a finite map and let $L \in \mathrm{Pic}(X)$. Then L is q-flag positive if and only if $f^\star L$ is q-flag positive. Moreover, if $(Y'_k)_{q\le k\le n}$ is an ample q-flag for $f^\star L$, then $(f(Y'_k))_{q\le k\le n}$ is an ample q-flag for L, and if $(Y_k)_{q\le k\le n}$ is an ample q-flag for L, then there exists an ample q-flag $(Y'_k)_{q\le k\le n}$ for $f^\star L$ such that $Y_k = f(Y'_k)$ (however Y'_k may be smaller than $f^{-1}(Y_k)$).*

Proof. First suppose that $f^\star L$ is q-flag positive and that $(Y'_k)_{q\le k\le n}$ is an ample q-flag with $L_{|Y'_q}$ ample. Then we set $Y_k = f(Y'_k)$. This gives a commutative diagram of normalizations

$$(\star)\qquad \begin{array}{ccc} \widetilde{Y}'_{k+1} & \xrightarrow{\widetilde{f}} & \widetilde{Y}_{k+1} \\ \nu'_{k+1}\downarrow & & \downarrow\nu_{k+1} \\ Y'_{k+1} & \xrightarrow{f} & Y_{k+1}. \end{array}$$

Let Z'_k be a Cartier divisor in $\widetilde{Y}'_{k+1}$ such that $Y'_k = \nu'_{k+1}(\mathrm{Supp} Z'_k)$ set theoretically. We define Z_k to be the push forward $Z_k = \widetilde{f}_\star Z'_k$. Recall that if U is a small open set in $\widetilde{Y}_{k+1}$ and g' a generator of $\mathcal{O}(-Z'_k)$ on $\widetilde{Y}'_{k+1} \cap \widetilde{f}^{-1}(U)$, then $g = \widetilde{f}_\star g'$ defined by $g(x) := \prod_{x'\in\widetilde{Y}'_{k+1}} g'(x')$ is a generator of $\mathcal{O}(-Z_k)$ on U; note that g is actually holomorphic on $\widetilde{Y}_{k+1}$ because $\widetilde{Y}_{k+1}$ is a normal space. Hence Z_k is a Cartier divisor, and we have

$$\nu_{k+1}(Z_k) = \nu_{k+1}(\widetilde{f}(Z'_k)) = f(\nu'_{k+1}(Z'_k)) = f(Y'_k) = Y_k.$$

Finally, $\mathcal{O}(Z_k)$ is ample on Y_{k+1}. Indeed, for m large enough and for all $x_1 \ne x_2$ in $\widetilde{Y}_{k+1}$, there are sections $h' \in H^0(\widetilde{Y}'_{k+1}, \mathcal{O}(mZ'_k))$ which vanish at a given point of a fibre $\widetilde{f}^{-1}(x_1)$, but do not vanish at the other points nor at the points of $\widetilde{f}^{-1}(x_2)$. Hence $h = \widetilde{f}_\star h' \in H^0(\widetilde{Y}_{k+1}, \mathcal{O}(mZ_k))$ vanishes at x_1 but does not vanish at x_2. This implies that $\mathcal{O}(Z_k)$ is ample on $\widetilde{Y}_{k+1}$.

Now, suppose that L is q-flag positive and that $(Y_k)_{q\le k\le n}$ is an ample q-flag with $L_{|Y_q}$ ample. Let Z_k be a Cartier divisor in $\widetilde{Y}_{k+1}$ with $Y_k = \nu_{k+1}(\mathrm{Supp} Z_k)$. We construct an ample flag $(Y'_k)_{q\le k\le n}$ in X' such that $f(Y'_k) = Y_k$, defining Y'_k by backward induction on k. We set of course $Y'_n = X'$. If Y'_{k+1} has already been constructed, we get a commutative diagram of normalizations $(\star)$ as above. We then define $Z'_k := \widetilde{f}^\star(Z_k)$ and $Y'_k := \nu'_{k+1}(\mathrm{Supp} Z'_k)$ as a set. Since the pull-back of an ample Cartier divisor is an ample Cartier divisor, it is clear that $(Y'_k)_{q\le k\le n}$ is an ample q-flag. Moreover, $(f^\star L)_{|Y'_q}$ is the pull-back of the ample line bundle $L_{|Y_q}$, hence $(f^\star L)_{|Y'_q}$ is ample. □

3.4. Theorem. — *Let $L \in \mathrm{Pic}(X)$. If L is q-flag positive, then $\sigma_+(L) \le n - q$, and thus $\sigma_-(L^\star) \ge q$.*

Proof. Let $(Y_k)_{q\le k\le n}$ be an ample q-flag such that $L_{|Y_q}$ is ample, and let Z_k be an ample Cartier divisor in the normalization $\widetilde Y_{k+1}$ of Y_k with $Y_k=\nu_{k+1}(\mathrm{Supp}Z_k)$. The inclusions $j_{k,\ell}:Y_k\hookrightarrow Y_\ell$ can be lifted as finite maps $\widetilde j_{k,\ell}:\widetilde Y_k\to Y_\ell$. For each multi-index $\alpha=(\alpha_k,\,\ldots\,,\alpha_{n-1})$ of nonnegative integers, we set

$$G^\alpha=\widetilde j_{k,k+1}^\star\mathcal{O}(\alpha_kZ_k)\otimes\ldots\otimes\widetilde j_{k,n}^\star\mathcal{O}(\alpha_{n-1}Z_{n-1}).$$

Since $\mathcal{O}(Z_\ell)$is ample on $\widetilde Y_{\ell+1}$, we conclude that G^α is nef on $\widetilde Y_k$ and ample if $\alpha\ne0$. Let A be an ample line bundle on X, and let $\mathcal{F}$ be an arbitrary coherent sheaf on $\widetilde Y_k$. We prove by induction on $k=q,q+1,\,\ldots\,,n$ the following property:

$$(\mathcal{P}_k)\qquad\begin{cases}H^i(\widetilde Y_k,\mathcal{O}(mL-pA)_{|\widetilde Y_k}\otimes G^\alpha\otimes\mathcal{F})=0\\[1ex] \text{for all }\ i>k-q,\ \alpha\in\mathbb{N}^{n-k},\ m\ge C(\mathcal{F})(p+1).\end{cases}$$

Here we write $\mathcal{O}(mL-pA)_{|\widetilde Y_k}=\widetilde j_{k,n}^\star\mathcal{O}(mL-pA)$ for the simplicity of notations. The desired conclusion is just statement $(\mathcal{P}_n)$ in the case $G^\alpha=\mathcal{F}=\mathcal{O}_X$. Observe that it is sufficient to prove $(\mathcal{P}_k)$ in the special case $\mathcal{F}=\mathcal{O}_{\widetilde Y_k}$, since we may otherwise take resolutions of $\mathcal{F}$ by locally free sheaves of the form $\mathcal{O}(-dA)^{\oplus N}$, and argue by backward induction on i as in the proof of Proposition 1.2.

First step: $k=q$. — By our assumption, $\mathcal{O}(L)_{|\widetilde Y_q}=\widetilde j_{q,n}^\star\mathcal{O}(L)$ is ample. Using a resolution of $\mathcal{O}(-A)_{|\widetilde Y_q}$ of the form

$$\mathcal{O}(-d_sL)^{\oplus N_s}_{|\widetilde Y_q}\to\cdots\to\mathcal{O}(-d_1L)^{\oplus N_1}_{|\widetilde Y_q}\to\mathcal{O}(-d_0L)^{\oplus N_0}_{|\widetilde Y_q}\to\mathcal{O}(-A)_{|\widetilde Y_q}\to0$$

and raising it to the power p to get a resolution of $\mathcal{O}(-pA)_{|\widetilde Y_q}$, we are led to prove the vanishing property

$$H^i(\widetilde Y_q,\mathcal{O}((m-d_{s_1}-\cdots-d_{s_p})L)_{|\widetilde Y_k}\otimes G^\alpha)=0,\qquad\forall i>0,\ s_j\le n,$$

for $m'=m-d_{s_1}-\cdots-d_{s_p}\ge m-pd_n\ge m_0$ independent of α. Hence we merely have to consider the groups $H^i(\widetilde Y_q,\mathcal{O}(mL)\otimes G^\alpha)$. By the Kodaira-Serre vanishing theorem and the ampleness of L on $\widetilde Y_q$, we know that these groups vanish for $m\ge m_0(\alpha)$. We have to check that $m_0(\alpha)$ can be taken independent of α. (This would be of course a straightforward consequence of the precise vanishing theorem if $\widetilde Y_q$ were smooth.) For this, we use the ampleness of $\mathcal{O}(Z_\ell)$ to obtain resolutions of $\mathcal{O}_{\widetilde Y_q}$ of the form

$$\mathcal{O}(-\delta_{n,\ell}Z_\ell)^{\oplus N_{n,\ell}}_{|\widetilde Y_q}\to\cdots\to\mathcal{O}(-\delta_{1,\ell}Z_\ell)^{\oplus N_{1,\ell}}_{|\widetilde Y_q}\to\mathcal{O}(-\delta_{0,\ell}Z_\ell)^{\oplus N_{0,\ell}}_{|\widetilde Y_q}\to\mathcal{O}_{\widetilde Y_q}\to0$$

with $\delta_{n,\ell}>\ldots>\delta_{1,\ell}>\delta_{0,\ell}>0$, for each $\ell=k,\,\ldots\,,n-1$. If $\alpha_\ell\ge\delta_{n,\ell}$, we take the tensor product of this resolution with $\mathcal{O}(mL)\otimes G^\alpha$. This reduces the vanishing of $H^i(\widetilde Y_q,\mathcal{O}(mL)\otimes G^\alpha)$, $i>0$, to the vanishing of all analogous groups with $\alpha'=(\alpha_q,\,\ldots\,,\alpha_\ell-d_{s,\ell},\,\ldots\,,\alpha_{n-1})$ in place of α. Hence we are reduced inductively to the case where $\alpha_\ell<\delta_{n,\ell}$ for all ℓ, and we may take $m_0=\max_{\{\alpha;\,\alpha_\ell<\delta_{n,\ell}\}}m_0(\alpha)$.

Second step. — Suppose that property $(\mathcal{P}_k)$ holds for some index k in the range $\{q,\,\ldots\,,n-1\}$. Observe that G^α is the restriction to $\widetilde Y_k$ of an invertible sheaf defined on $\widetilde Y_{k+1}$, namely

$$\mathcal{O}(\alpha_kZ_k)\otimes G^\beta\quad\text{ with }\ \alpha=(\alpha_k,\beta),\ \ \beta=(\alpha_{k+1},\,\ldots\,,\alpha_{n-1}).$$

The exact sequence

$$0 \longrightarrow \mathcal{O}(-Z_q) \longrightarrow \mathcal{O}_{\widetilde{Y}_{k+1}} \longrightarrow \mathcal{O}_{Z_k} \longrightarrow 0$$

defines a (possibly non reduced) scheme structure on Z_k. Taking the tensor product with $\mathcal{O}(\alpha_k Z_k) \otimes G^\beta$ and the associated cohomology sequence, we get the exact sequence

$$\begin{aligned}
&H^{i-1}\big(Z_k, \mathcal{O}(mL-pA)_{|Z_k} \otimes \mathcal{O}(\alpha_k Z_k)_{|Z_k} \otimes G^\beta_{|Z_k}\big)\\
(\dagger)\qquad &\longrightarrow H^i\big(\widetilde{Y}_{k+1}, \mathcal{O}(mL-pA)_{|\widetilde{Y}_{k+1}} \otimes \mathcal{O}((\alpha_k-1)Z_k) \otimes G^\beta\big)\\
&\longrightarrow H^i\big(\widetilde{Y}_{k+1}, \mathcal{O}(mL-pA)_{|\widetilde{Y}_{k+1}} \otimes \mathcal{O}(\alpha_k Z_k) \otimes G^\beta\big)\\
&\longrightarrow H^i\big(Z_k, \mathcal{O}(mL-pA)_{|Z_k} \otimes \mathcal{O}(\alpha_k Z_k)_{|Z_k} \otimes G^\beta_{|Z_k}\big).
\end{aligned}$$

Claim 3.5. — *For every subscheme $S \subset Z_k \subset \widetilde{Y}_{k+1}$ and every coherent sheaf $\mathcal{F}$ on S, we have*

$$H^i\big(S, \mathcal{O}(mL-pA)_{|S} \otimes \mathcal{O}(\alpha_k Z_k)_{|S} \otimes G^\beta_{|S} \otimes \mathcal{F}\big) = 0$$

for all $i > k-q \geq 0$, $\alpha \in \mathbb{N}^{n-k}$ and $m \geq C(\mathcal{F})(p+1)$.

We apply this to $S = Z_k$ itself and $\mathcal{F} = \mathcal{O}_S = \mathcal{O}_{Z_k}$. Then $(\dagger)$ implies that the groups

$$H^i\big(\widetilde{Y}_{k+1}, \mathcal{O}(mL-pA)_{|\widetilde{Y}_{k+1}} \otimes \mathcal{O}(\alpha_k Z_k) \otimes G^\beta\big)$$

are independent of α_k for all $i > k+1-q$, $\beta \in \mathbb{N}^{n-k-1}$ and $m \geq C(p+1)$. However, these groups vanish for α_k large since $\mathcal{O}(Z_k)$ is ample on $\widetilde{Y}_{k+1}$. Hence they vanish for $\alpha_k = 0$ and for all $i > k+1-q$, $\beta \in \mathbb{N}^{n-k-1}$ and $m \geq C(p+1)$. Therefore property $(\mathcal{P}_{k+1})$ holds.

Third step: $(\mathcal{P}_k)$ implies Claim 3.5. — We prove the claim by induction on $\dim S$, the result being obvious if $\dim S = 0$. In fact, it is sufficient to prove the claim when S is reduced. Otherwise, let $\mathcal{N}$ be the sheaf of nilpotent elements of the structure sheaf $\mathcal{O}_S$; then $(\mathcal{N}^j\mathcal{F})$ defines a finite filtration of $\mathcal{F}$ and the quotients $\mathcal{N}^j\mathcal{F}/\mathcal{N}^{j+1}\mathcal{F}$ are coherent sheaves over S_{red}. Therefore, if the vanishing property 3.5 holds on S_{red}, it also holds on S with $C(\mathcal{F}) = \max_j C(\mathcal{N}^j\mathcal{F}/\mathcal{N}^{j+1}\mathcal{F})$. We can also suppose $\mathcal{F} = \mathcal{O}_S$, otherwise we are reduced to this case by taking resolutions of $\mathcal{F}$ by locally free sheaves of the form $\mathcal{O}(-dA)^{\oplus N}_{|S}$).

Now, suppose that S is reduced and $\mathcal{F} = \mathcal{O}_S$. The map $\mu = \nu_{k+1|S} : S \to Y_k$ can be lifted to a map $\widetilde{\mu} : \widetilde{S} \to \widetilde{Y}_k$. This gives a commutative diagram

$$\begin{array}{ccccc}
\widetilde{S} & \xrightarrow{\widetilde{\mu}} & \widetilde{Y}_k & \xrightarrow{\widetilde{j}_{k,k+1}} & \widetilde{Y}_{k+1}\\
\nu\downarrow & & \downarrow\nu_k & & \downarrow\nu_{k+1}\\
S & \xrightarrow{\mu} & Y_k & \overset{j_{k,k+1}}{\hookrightarrow} & Y_{k+1}
\end{array}$$

where the vertical arrows are the normalization maps. Moreover, the diagram commutes with the diagonal injection $S \hookrightarrow \widetilde{Y}_{k+1}$.

Since we have vanishing on $\widetilde{Y}_k$ by hypothesis $(\mathcal{P}_k)$ and since $\widetilde{\mu}$ is a finite map, the Leray spectral sequence implies

$$\begin{aligned} H^i\big(\widetilde{S}, \mathcal{O}(mL-pA)_{|\widetilde{S}} \otimes \mathcal{O}(\alpha_k Z_k)_{|\widetilde{S}} \otimes G^\beta_{|\widetilde{S}}\big) \\ = H^i\big(\widetilde{Y}_k, \mathcal{O}(mL-pA)_{|\widetilde{Y}_k} \otimes \mathcal{O}(\alpha_k Z_k)_{|\widetilde{Y}_k} \otimes G^\beta_{|\widetilde{Y}_k} \otimes \widetilde{\mu}_\star \mathcal{O}_{\widetilde{S}}\big) = 0 \end{aligned}$$

for $i > k-q$, $\alpha \in \mathbb{N}^{n-k}$ and $m \geq C(p+1)$. Taking the direct image by ν, this gives

$$(\dagger\dagger) \qquad H^i\big(S, \mathcal{O}(mL-pA)_{|S} \otimes \mathcal{O}(\alpha_k Z_k)_{|S} \otimes G^\beta_{|S} \otimes \nu_\star \mathcal{O}_{\widetilde{S}}\big) = 0.$$

Let $\mathcal{I} \subset \mathcal{O}_S$ be the sheaf of universal denominators of $\nu_\star \mathcal{O}_{\widetilde{S}}$, i.e., the largest ideal $\mathcal{I} \subset \mathcal{O}_S$ such that $\mathcal{I} \cdot \nu_\star \mathcal{O}_{\widetilde{S}} \subset \mathcal{O}_S$. The support of $\mathcal{O}_S/\mathcal{I}$ coincides with the subvariety of non normal points in S. Hence, for $d > 0$ large enough, $\mathcal{O}(dA)_{|S} \otimes \mathcal{I}$ has a global section u whose set of zeros is a subvariety $S' \subset S$ with $\dim S' < \dim S$ at every point. We get an exact sequence

$$0 \longrightarrow \mathcal{O}(-dA)_{|S} \otimes \nu_\star \mathcal{O}_{\widetilde{S}} \xrightarrow{u\times} \mathcal{O}_S \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{F}$ is supported on S'. By the induction hypothesis on $\dim S$ we have

$$H^i\big(S', \mathcal{O}(mL-pA)_{|S'} \otimes \mathcal{O}(\alpha_k Z_k)_{|S'} \otimes G^\beta_{|S'} \otimes \mathcal{F}\big) = 0$$

for $i > k-q$ and $m \geq C(\mathcal{F})(p+1)$, while $(\dagger\dagger)$ implies

$$H^i\big(S, \mathcal{O}(mL-pA)_{|S} \otimes \mathcal{O}(\alpha_k Z_k)_{|S} \otimes G^\beta_{|S} \otimes \mathcal{O}(-dA)_{|S} \otimes \nu_\star \mathcal{O}_{\widetilde{S}}\big) = 0$$

for $i > k-q$ and $m \geq C(p+d+1)$. The above exact sequence shows that the cohomology groups of the central term also vanish in degrees $i > k-q$, as asserted by Claim 3.5. □

Theorem 3.4 leads in a natural way to the following problem.

3.6. Problem. — *Given a line bundle $L \in \mathrm{Pic}(X)$ such that $\sigma_+(L) \leq n-q$, under which conditions can one conclude that L is q-flag positive ?*

If no further assumptions are made, the answer is in general negative by Example 5.6 below (in which case we have $n = 3$ and $q = 2$). However, we have no counterexample in the most interesting case $q = 1$. Since the problem seems very difficult to attack in full generality, we will content ourselves with rather special cases.

3.7. Lemma. — *Let X be a projective manifold and let $D_1, \ldots, D_{n-1} \subset X$ be irreducible divisors, such that the $\mathcal{O}(D_i)$ are nef and $Y_i = D_i \cap \ldots \cap D_{n-1}$ is i-dimensional. Let L be a line bundle with $L \cdot Y_1 > 0$. Then there is an ample 1-flag for L of the form $Y'_i = D'_i \cap \ldots \cap D'_{n-1}$ with all D'_j being very ample, nonsingular and intersecting transversally.*

Proof. Fix an ample divisor H on X. Choosing smooth generic members $D'_i \in |k(mD_i + H)|$, $k \gg 0$, $m \gg 0$, we get the desired 1-ample flag for L. (Note that $L \cdot Y'_i \sim (km)^{n-1} L \cdot Y_1$ as $k, m \to +\infty$.) $\square$

3.8. PROPOSITION. — *Let $X = \mathbb{P}(E) \xrightarrow{\pi} C$ be a $\mathbb{P}_{n-1}$-bundle over a smooth curve and let L be a line bundle on X with $\sigma_+(L) \le n-1$. Then L is 1-flag positive. Moreover there is a base change $F^\star : \widetilde{X} = \mathbb{P}(f^\star E) \to X = \mathbb{P}(E)$ given by a finite map $f : \widetilde{C} \to C$, such that the pull-back $F^\star L$ admits an ample 1-flag*

$$Y_1 \subset \ldots \subset Y_{n-1} \subset X$$

of the form $Y_i = D_i \cap \ldots \cap D_{n-1}$ with $D_i, \ldots, D_{n-1}$ very ample and intersecting transversally.

Proof. All line bundles over $\mathbb{P}(E)$ are of the form $L = \mathcal{O}_{\mathbb{P}(E)}(k) \otimes \pi^\star G$ for some $G \in \mathrm{Pic}(C)$. If $k = 0$, then G must have positive degree and the conclusion is obtained by taking an arbitrary ample flag such that Y_1 is mapped onto C by π. Hence suppose $k \ne 0$. After taking a finite covering $f : \widehat{C} \to C$, we may assume that $G = \mathcal{O}_C$, replacing E by $f^\star(E \otimes G^{1/k})$. By Proposition 3.3, the conclusion is invariant by finite maps, hence we may replace C by $\widehat{C}$ and assume $L = \mathcal{O}_{\mathbb{P}(E)}(\pm 1)$.

In the case $L = \mathcal{O}_{\mathbb{P}(E)}(1)$, we take $Y_1 \subset \ldots \subset Y_{n-1}$ to be a flag of linear subspaces in a fibre $Y_{n-1} = \mathbb{P}(E_x)$. Then $D_{n-1} = \mathbb{P}(E_x)$ and $D_2, \ldots, D_{n-2}$ can be taken to be suitable $\mathbb{P}_{n-2}$-subbundles of $\mathbb{P}(E)$ defined by sections of an ample line bundle $\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^\star H$. Note that D_{n-1} is just nef, but Lemma 3.7 can be applied to get an ample 1-flag.

The only nontrivial case is $L = \mathcal{O}_{\mathbb{P}(E)}(-1)$. Our assumption $\sigma_+(L) \le n-1$ implies that $E^\star$ is ample (cf. 2.7 (ii)). By a result of Campana and Flenner [CF91], there is a finite map $f : \widetilde{C} \to C$, a very ample line bundle A on $\widetilde{C}$ and a surjective morphism

$$A^{\oplus N} \longrightarrow f^\star E^\star \longrightarrow 0.$$

Replacing C by $\widetilde{C}$ and E by $f^\star E$, we may assume that we have $f = \mathrm{Id}$. Then $E^\star \otimes A^\star$ is generated by sections and we get bundle-monomorphisms

$$\varphi_j : A^{\oplus(n-1)} \hookrightarrow E^\star,$$

sending the sum of factors A of indices $m \in \{1, \ldots, n\} \setminus \{j\}$ in $A^{\oplus N}$ into $E^\star$. (Otherwise, change $A^{\oplus N} \to E$ by a constant linear transformation acting on $A^{\oplus N}$ to achieve this property.) These monomorphisms define exact sequences

$$(\star\star) \qquad 0 \longrightarrow K \longrightarrow E \xrightarrow{\varphi_j^\star} (A^\star)^{\oplus(n-1)} \longrightarrow 0$$

and divisors

$$D_j := \mathbb{P}\big((A^\star)^{\oplus(n-1)}\big) \subset \mathbb{P}(E).$$

These divisors intersect transversally. We set

$$Y_j := D_j \cap \ldots \cap D_{n-1}.$$

(a) First, we check that $\mathcal{O}(D_j)$ is nef in $\mathbb{P}(E)$. Since $\mathcal{O}(D_j) = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^\star(K^\star)$, we have to show that $E \otimes K^\star$ is nef. From $(\star\star)$, we get

$$0 \longrightarrow \mathcal{O} \longrightarrow E \otimes K^\star \longrightarrow (A^\star \otimes K^\star)^{\oplus(n-1)} \longrightarrow 0.$$

Hence it is enough to show that $A^\star \otimes K^\star$ is nef. However, this is clear because $K^\star$ is a quotient of $E^\star$ and thus of $A^{\oplus N}$.

(b) Now, we have to check that $L \cdot Y_1 > 0$. First suppose that $n = 2$, thus $Y_1 = D_1$. Then

$$\begin{aligned} L \cdot D_1 &= c_1\big(\mathcal{O}_{\mathbb{P}(E)}(-1)\big) \cdot \Big(c_1\big(\mathcal{O}_{\mathbb{P}(E)}(1)\big) + \pi^\star c_1(K^\star)\Big) \\ &= -c_1(E) + c_1(K) = c_1(A) > 0 \end{aligned}$$

thanks to $(\star\star)$ and to the relations $h^2 - \pi^\star c_1(E)h + \pi^\star c_2(E) = 0$, $\pi_\star h = 1$ satisfied by $h = c_1\big(\mathcal{O}_{\mathbb{P}(E)}(1)\big)$.

(c) In the general case $n > 2$, we have

$$Y_1 = \mathbb{P}(A^\star) \subset Y_2 = \mathbb{P}(A^{\star\oplus 2}) \subset \mathbb{P}(E), \qquad L_{|Y_2} = \mathcal{O}_{\mathbb{P}(A^{\star\oplus 2})}(-1),$$

where the inclusions are defined by the projections $E \to A^\star$ (resp. $E \to A^{\star\oplus 2}$) on the n-th factor (resp. on the first and n-th factor). This reduces the problem to the case of the rank 2 bundle $E = (A^\star)^{\oplus 2}$. By applying case (b), we thus get $L \cdot Y_1 > 0$, as desired. □

4. The cone of ample curves

In this section, we concentrate ourselves on the study of the cone of "ample curves", in relation with effective divisors and ample 1-flags.

4.1. Theorem. — *For an irreducible curve $C \subset X$, consider the following properties.*

(i) *C is the first member Y_1 of an irreducible ample 1-flag $Y_1 \subset \ldots \subset Y_{n-1} \subset X$. (We say that a flag is irreducible if all subvarieties Y_i are irreducible.)*

(ii) $\{C\} \in N_{\mathrm{amp}}(X)$.

(iii) $\{C\} \in \overline{N}_{\mathrm{amp}}(X)$.

(iv) *The normal bundle $N_{C/X} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ is ample (i.e., $\mathcal{O}_{\mathbb{P}(N_{C/X})}(1)$ is ample).*

(v) *The normal bundle $N_{C/X}$ is nef (i.e., $\mathcal{O}_{\mathbb{P}(N_{C/X})}(1)$ is nef).*

(vi) *The current of integration $[C]$ is weakly cohomologous to a smooth positive definite form of bidegree $(n-1, n-1)$, i.e. $[C] = u + \partial R + \overline{\partial R} + S$ where u is a smooth positive definite $(n-1, n-1)$-form with $\partial\overline{\partial} u = 0$, R is a current of type $(n-2, n-1)$, and S is a d-closed $(n-1, n-1)$-current whose cohomology class $\{S\} \in H^{n-1,n-1}(X)$ is orthogonal to $NS^1(X) \subset H^{1,1}(X)$.*

(vii) *The current of integration $[C]$ is weakly cohomologous to a smooth semipositive form of bidegree $(n-1,n-1)$ (as in* (vi)*, but with $u \geq 0$ only).*

(viii) *There is a family of generically irreducible curves (C_t) covering X such that $C_0 = mC$ as a cycle.*

Then we have the following implications:

(a) (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii),

(b) (i) $\Rightarrow$ (iv), *if additionally every Y_i is Cartier in Y_{i+1},*

(c) (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii), (viii) $\Rightarrow$ (iii)

(d) (ii) $\Leftrightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (iii)

Proof. The implications (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (vii) are obvious.

(a) We only have to show (i) $\Rightarrow$ (ii). Let $C = Y_1$ be the first member of the ample irreducible flag

$$Y_1 \subset Y_2 \subset \ldots \subset Y_{n-1} \subset X.$$

Let $D \in \overline{K}_{\text{eff}}(X)$. We first check $D \cdot C \geq 0$, so as to conclude that $C \in \overline{N}_{\text{amp}}(X)$. Otherwise $L = \mathcal{O}(-D)$ satisfies $L \cdot C > 0$, i.e. $L_{|C}$ is ample. By Theorem 3.4 we conclude that $\sigma_+(L) \leq n-1$, hence $D \notin \overline{K}_{\text{eff}}(X)$ by 1.5 (ii), contradiction. This shows that the cone N_1 generated by all curves $C = Y_1$ is contained in $\overline{N}_{\text{amp}}(X)$.

Now, we check that C cannot be a boundary point of $\overline{N}_{\text{amp}}(X)$. Let H be a very ample line bundle on X and let $(Z_k)_{1\leq k\leq n-1}$ be Cartier divisors in $\widetilde{Y}_{k+1}$ such that $Y_k = \nu_{k+1}(\text{Supp} Z_k)$ (see Def. 3.1). There is an integer $m_1 > 0$ such that $m_1 Z_1 - H_{|\widetilde{Y}_2}$ is very ample on $\widetilde{Y}_2$. Select an irreducible divisor $Z_1' \in |m_1 Z_1 - H_{|\widetilde{Y}_2}|$. Since the 1-cycle $(\nu_2)_\star(Z_1)$ is a multiple $p_1 Y_1$, we conclude that $Y_1' = (\nu_2)_\star(Z_1') \equiv m_1 p_1 Y_1 - (H \cdot Y_2) \in N_1$. Quite similarly, we have irreducible very ample divisors $Z_k' \sim m_k Z_k - H_{|\widetilde{Y}_{k+1}}$ and k-dimensional cycles $Y_k' = (\nu_{k+1})_\star(Z_k') \equiv m_k p_k Y_k - H \cdot Y_{k+1}$. Moreover

$$H^{k-1} \cdot Y_k' \equiv m_k p_k H^{k-1} \cdot Y_k - H^k \cdot Y_{k+1} \in \overline{N}_1$$

because H is very ample and Y_k' is a member of a k-flag. By multiplying the above line by $m_{k+1}p_{k+1} \ldots m_{n-1}p_{n-1}$ and adding everything together for $k = 1, \ldots, n-1$, we find

$$m_1 p_1 \ldots m_{n-1} p_{n-1} Y_1 - H^{n-1} \in \overline{N}_1 \subset \overline{N}_{\text{amp}}(X).$$

However H^{n-1} is in the open cone $N_{\text{amp}}(X)$. Therefore $\{Y_1\} \in N_{\text{amp}}(X)$ and $N_1 \subset N_{\text{amp}}(X)$.

(b) Next we prove (i) $\Rightarrow$ (iv) in case every Y_i is Cartier in Y_{i+1}. This is done easily by induction using the exact sequences of vector bundles

$$0 \longrightarrow N_{C/Y_i} \longrightarrow N_{C/Yi+1} \longrightarrow N_{Y_i/Y_{i+1}|C} \longrightarrow 0.$$

(c) The implication (v) $\Rightarrow$ (iii) is more complicated. Let $C \subset X$ be an irreducible curve such that $N_{C/X}$ is nef. Let $D \subset X$ be an irreducible hypersurface. It is sufficient to prove $D \cdot C \geq 0$ if $C \subset D$. Let $\mathcal{I}$ be the ideal defining C in X, $\mathcal{J}$ the ideal defining C in D. Then there is an exact sequence

$$\mathcal{O}_X(-D)_{|C} = N^\star_{D/X|C} \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow 0.$$

If $C \not\subset \mathrm{Sing}(D)$, the map $\alpha : N^\star_{D/X|C} \longrightarrow \mathcal{I}/\mathcal{I}^2$ is non-zero, hence injective. Thus

$$\alpha^\star : N_{C/X} \longrightarrow N_{D/X|C}$$

is generically surjective. Since $\mathrm{Im}\alpha^\star$ is again nef (observe that $\mathbb{P}(\mathrm{Im}\alpha^\star) \subset \mathbb{P}((\mathcal{I}/\mathcal{I}^2)^\star)$ and $\mathcal{O}_{\mathbb{P}(\mathrm{Im}\alpha^\star)}(1) = \mathcal{O}_{\mathbb{P}((\mathcal{I}/\mathcal{I}^2)^\star)}(1)_{|\mathbb{P}(\mathrm{Im}\alpha^\star)})$, it follows clearly that $\deg(N_{D/X|C}) = D \cdot C \geq 0$. If however $\alpha = 0$ we have slightly to modify the above arguments. We want to "generalize" the diagram

$$\begin{array}{ccc} & \mathcal{I}/\mathcal{I}^2 & \\ \alpha \nearrow & & \searrow \\ N^\star_{D/X|C} & \xrightarrow{\gamma} & \Omega^1_{X|C} \end{array}$$

where γ was given just by taking the linear part of the power series expansion along D of a local equation f for D. Suppose now that $C \subset \mathrm{Sing}(D)$, so that $\alpha = 0$. Let m be the vanishing order of f along C. By taking the homogeneous part of degree m in f, we obtain a map

$$\widetilde{\gamma} : N^\star_{D/X|C} \longrightarrow S^m(\Omega^1_{X|C}).$$

As before $\widetilde{\gamma}$ factors over

$$\widetilde{\alpha} : N^\star_{D/X|C} \longrightarrow S^m(\mathcal{I}/\mathcal{I}^2).$$

Since $S^m((\mathcal{I}/\mathcal{I}^2)^\star) = S^m(\mathcal{I}/\mathcal{I}^2)^\star$ is again nef, we conclude in the same way as before that $D \cdot C \geq 0$.

The implication (viii) $\Rightarrow$ (iii) is again easy. Let (C_t) be a family of curves covering X such that $C_0 = mC$ as cycles. It is sufficient to prove that for every effective divisor D we have $D \cdot C \geq 0$. But this is obvious: we choose t general, such that $C_t \not\subset \mathrm{Supp}(D)$.

(d) Finally, the implications (vi) $\Rightarrow$ (ii) and (vii) $\Rightarrow$ (iii) follow from the following fact: if $D \in \overline{K}_{\mathrm{eff}}(X)$ and $[C] \sim u$ weakly, $u \geq 0$, then

$$D \cdot C = \int_D u \geq 0,$$

with strict inequality if $D \neq 0$ and $u > 0$. We now prove the converse implication (ii) $\Rightarrow$ (vi) as a consequence of the Hahn-Banach theorem.

Let $\mathcal{D}'_{1,1}$ be the space of currents of bidegree $(1,1)$ equipped with the weak topology. In $\mathcal{D}'_{1,1}$ we consider the closed convex cone of positive currents $\mathcal{D}^+_{1,1}$. Let ω be a Kähler metric on X. It is well-known that the intersection of $\mathcal{D}^+_{1,1}$ with the hyperplane of currents T such that $\int_X T \wedge \omega^{n-1} = 1$ is weakly compact, so $\mathcal{D}^+_{1,1}$ has a compact base. On the other hand, we pick a class $\{C\} \in N_{\mathrm{amp}}(X)$ and consider the vector space $\mathcal{E}_C$ of d-closed currents T such that the cohomology class $\{T\}$ belongs to $NS^1(X)$ and satisfies $\{T\}\cdot\{C\}=0$. We claim that $\mathcal{D}^+_{1,1}\cap\mathcal{E}_C=\{0\}$. In fact the set of classes in $NS^1(X)$ represented by positive currents is precisely $\overline{K}_{\rm eff}(X)$ (see e.g. [De92], Prop. 6.1 (vi)), hence a class $\mathcal{D}^+_{1,1}\cap\mathcal{E}_C$ is the set of classes $\{T\}\in\overline{K}_{\rm eff}(X)$ such that $\{T\}\cdot\{C\}=0$, but this set reduces to $\{0\}$ by our assumption $\{C\}\in N_{\rm amp}(X)$. The Hahn-Banach theorem implies that there is a linear form u on $\mathcal{D}'_{1,1}$ such that u is positive on $\mathcal{D}^+_{1,1}\setminus\{0\}$ and vanishes on $\mathcal{E}$. From the positivity property, we conclude that u is a positive definite smooth $(n-1,n-1)$-form on X. On the other hand, since $\mathcal{E}_C$ contains $\partial\overline{\partial}\mathcal{D}'_{0,0}$, we have $\partial\overline{\partial}u=0$. In fact $\mathcal{E}_C$ is the set of d-closed $(1,1)$-currents whose classes are orthogonal to $\mathbb{R}\{C\}\oplus NS^1(X)^\perp\subset H^{n-1,n-1}(X)$. Hence there is a class $\lambda\{C\}+\{S\}$ with a constant $\lambda\in\mathbb{R}$ and $\{S\}\in NS^1(X)^\perp$ such that $\{u-\lambda[C]-S\}$ vanishes on all d-closed $(1,1)$-currents. But this means precisely that there is a $(n-2,n-1)$-current R such that $u-\lambda[C]-S=\partial R+\overline{\partial R}$. Now λ must be positive, otherwise $u>0$ would have nonpositive intersection product with effective divisors. Hence we get the desired conclusion (vi) by multiplying u, S, R with λ^{-1}. □

4.2. Remark. — In general neither (vi) nor (ii) implies (v). For example take a Fano 3-fold X with $b_2=1$ and index 1. Then $\{C\}\in N_{\rm amp}(X)$ (resp. $[C]$ is cohomologous to a smooth positive definite form of bidegree $(2,2)$ for every curve $C\subset X$, however $N_{C/X}$ might not be nef. Take e.g. a line in X, this is a smooth rational curve C with $N_{C/X}=\mathcal{O}\oplus\mathcal{O}(-1)$ or $\mathcal{O}(1)\oplus\mathcal{O}(-2)$.

However we may hope that (vi) or (ii) imply (v) on the level of cones, e.g., given $\alpha\in N_{\rm amp}(X)$ which is represented by curves, then one can find C with $\alpha=\lambda\{C\}$ with $\lambda\in\mathbb{Q}^+$ such that $N_{C/X}$ is nef (or even ample). An implication (viii) $\Rightarrow$ (v) is true generically as we show in Theorem 4.3, however there seems to be no hope to prove (iv) $\Rightarrow$ (viii) or (v) $\Rightarrow$ (viii) despite a concrete counterexample seems still not to exist, at least to our knowledge.

Let $N_\mu(X)$ (or simply N_μ) be the convex cones generated by all classes of curves in 4.1(μ), $1\le\mu\le 8$, and let $\overline{N}_\mu$ be their closures. Then we have:

4.3. Theorem. — *There are the following relations between the various cones:* $\overline{N}_1\subset\overline{N}_2=\overline{N}_3=\overline{N}_6=\overline{N}_7,\quad \overline{N}_8\subset\overline{N}_5\subset\overline{N}_3,\quad \overline{N}_4\subset\overline{N}_5.$

Proof. The equalities $\overline{N}_2=\overline{N}_3$ and $\overline{N}_6=\overline{N}_7$ are obvious. Moreover $\overline{N}_2=\overline{N}_6$, $\overline{N}_1\subset\overline{N}_2$ and $\overline{N}_5\subset\overline{N}_3$ follow from 4.1.

The only remaining case is $\overline{N}_8\subset\overline{N}_5$. Let (C_t) be a covering family of irreducible curves. We show that for general t, $N_{C_t/X}$ is nef. Consider the graph of the family

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ q\downarrow & & \\ T & & \end{array}$$

We may assume $\dim T = n-1$, hence p is generically finite. Let $\mathcal{I}$ be the ideal of C_t in X, $\mathcal{J}$ the ideal of C_t in Z. As C_t is a fiber of q, $(\mathcal{J}/\mathcal{J}^2)^\star$ is nef; it is even trivial for generic t. Now the canonical map

$$p^\star(\Omega^1_{X|C_t}) \longrightarrow \Omega^1_{Z|C_t}$$

is generically surjective. Thus for generic t, $N_{C_t/Z}$ embeds into $p^\star(N_{C_t/X})$ and consequently –both sheaves being of same rank– $N_{C_t/X}$ is nef. □

In this context, the special case $q = 1$ of Problem 3.6 can be restated in a slightly stronger form:

4.4. Proposition. — *The following three statements are equivalent:*

(i) *A line bundle* $L \in \mathrm{Pic}(X)$ *satisfies* $\sigma_+(L) \le n-1$ *if and only if there is an irreducible ample* 1*-flag* $Y_1 \subset \ldots \subset Y_{n-1} \subset X$ *such that* $L_{|Y_1}$ *is ample.*

(ii) $\overline{N}_1(X) = \overline{N}_2(X)$;

(iii) $N_1(X) = N_2(X)$.

Proof. Thanks to 1.5 (ii), Statement 4.4(i) can be restated: $\{L^\star\} \notin \overline{K}_{\mathrm{eff}}(X)$ if and only if there is an irreducible ample 1-flag $Y_1 \subset \ldots \subset Y_{n-1} \subset X$ with $L_{|Y_1}$ ample. Changing L into $L^\star$ and taking the negation, this becomes: $\{L\} \in \overline{K}_{\mathrm{eff}}(X)$ if and only if for every irreducible ample 1-flag $Y_1 \subset \ldots \subset Y_{n-1} \subset X$, then $L \cdot Y_1 \ge 0$. By duality, this means precisely that the dual cone $\overline{N}_{\mathrm{amp}}(X)$ of $\overline{K}_{\mathrm{eff}}(X)$ is the closure $\overline{N}_1(X)$ of the convex cone generated by all curves Y_1. This is actually equivalent to $N_{\mathrm{amp}}(X) = N_1(X)$, because we have shown in the Proof of 4.1(a) that $N_1(X)$ does not contain boundary points of $\overline{N}_{\mathrm{amp}}(X)$. □

4.5. Proposition. — *Let X be a projective surface. Then all cones* $\overline{N}_\mu$ *coincide, in particular Statements* 4.4 (i,ii,iii) *hold on X.*

Proof. By definition, we have $K_{\mathrm{eff}}(X) = N_{\mathrm{eff}}(X)$ in dimension 2. Thus by duality, $N_{\mathrm{amp}}(X) = K_{\mathrm{amp}}(X)$. This means that $N_{\mathrm{amp}}(X)$ is generated by the irreducible curves Y_1 which are ample divisors in X, i.e. $N_1 = N_2$. Now, an irreducible ample curve has an ample normal bundle $\mathcal{O}(C)_{|C}$, hence $N_1 \subset N_4$. By Theorem 4.3, this implies the equality of all cones $\overline{N}_\mu$. □

4.6. Remarks and open problems. —

(1) Observe that the inclusions $\overline{N}_2 \subset \overline{N}_1 \subset \overline{N}_4$ and $\overline{N}_1 \subset \overline{N}_8$ would imply the equality of all cones $\overline{N}_\mu$. The first inclusion seems to be the most difficult step.

(2) The comparison between $\overline{N}_1$ and $\overline{N}_4$ is not quite clear, however the (a priori) slightly smaller cone $\overline{N}'_1$ consisting of classes of those C which are the first members of an ample irreducible flag

$$Y_1 \subset \ldots \subset Y_{n-1} \subset X$$

such that every Y_i is Cartier in Y_i, is contained in $\overline{N}_4$.

Problem: Is $\overline{N}_1 = \overline{N}'_1$? Is at least $\overline{N}_1 \subset \overline{N}_4$?

(3) Is always $\overline{N}_4 = \overline{N}_5$?

(4) What is the relation between $\overline{N}_8$ and $\overline{N}_1$? Given a covering family (C_t) of irreducible curves in X, it is at least true that we obtain a flag of irreducible varieties

$$Y_1 \subset \ldots \subset Y_{n-1} \subset X$$

such that the deformations of Y_i fill up Y_{i+1}.

4.7. Remark. — Exceptional curves are far from being ample. This is made precise by the following statement. Let X be a projective manifold, $f : X \longrightarrow Y$ be a modification such that Y carries a big line bundle (e.g. Y is smooth or normal projective). Let $C \subset X$ be an irreducible curve with $\dim f(C) = 0$. Then $\{C\} \notin \overline{N}_{\mathrm{amp}}(X)$.

At this point, one should note that $N_{\mathrm{amp}}(X)$ is in general not generated by complete intersection curves of the type $H_1 \cdot \ldots \cdot H_{n-1}$ with divisors H_i that are ample. This is shown by the following example:

4.8. Example. — Let $X = \mathbb{P}(E)$ over $\mathbb{P}_1$, with $E = \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. Let p be the projection. Assume that $N_{\mathrm{amp}}(X)$ is generated by curves $H_1 \cdot H_2$ with H_i ample on X. Let $L = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes p^\star(\mathcal{O}(\lambda))$ with $\lambda \in \mathbb{Q}$. Then for every $\lambda < 0$, we have $\{L\} \notin \overline{K}_{\mathrm{eff}}(X)$. We will show that for $\lambda \geq -1$ there cannot exist H_1, H_2 ample such that $L \cdot H_1 \cdot H_2 < 0$.

Write $H_i = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes p^\star(\mathcal{O}(a_i))$ as $\mathbb{Q}$-divisors, after possibly rescaling H_i by a positive factor. Since $H_i = \mathcal{O}_{\mathbb{P}(E \otimes \mathcal{O}(a_i))}(1)$, H_i is ample if and only if $a_i > 2$. Denoting $h = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$, we have $c_1(E) = -3$, thus $h^3 + 3h^2 = 0$ and $p_\star h^2 = 1$. This implies

$$L \cdot H_1 \cdot H_2 = (h+\lambda)(h+a_1)(h+a_2) = p_\star\big(h^3 + (a_1+a_2+\lambda)h^2\big) = a_1+a_2+\lambda-3 > \lambda+1.$$

Hence we must have $L \cdot H_1 \cdot H_2 > 0$ for $\lambda \geq -1$, although $\{L\} \notin \overline{K}_{\mathrm{eff}}(X)$ for $\lambda < 0$. This example also shows that f cannot be taken to be the identity map in the conclusion of Theorem 3.8. □

We now discuss Problem 3.6 in the case $\sigma_+(L) \leq n-1$ for 3-folds and only for the most important line bundle, namely the anti-canonical bundle $-K_X$. The condition $\sigma_+(-K_X) \leq 2$ is equivalent to $K_X \notin \overline{N}_{\mathrm{eff}}(X)$ which in turn just means $\kappa(X) = -\infty$, i.e. X is uniruled ([Mo88]). Instead of X being smooth, we deal even with normal projective varieties having at most terminal singularities ([KMM87]).

4.9. Proposition. — *Let X be a normal projective $\mathbb{Q}$-Gorenstein 3-fold, $\varphi : X \to Y$ a surjective holomorphic map to a normal projective variety Y of dimension at most 2, such that $-K_{X|\varphi^{-1}(y)}$ is ample for general $y \in Y$. Then* *there exists H ample on X with $K_X \cdot H^2 < 0$. In particular there exists an irreducible ample 1-flag $Y_1 \subset Y_2 \subset X$ such that $-K_{X|Y_1}$ is ample.*

Proof. If $\dim Y = 0$, then $-K_X$ is ample and there is nothing to prove. So assume $\dim Y \geq 0$ and fix an ample divisor H_0 on X. If $K_X \cdot H_0^2 < 0$, we are done. Thus we assume $K_X \cdot H_0^2 \geq 0$. Choose $L \in \mathrm{Pic}(Y)$ ample and define

$$H = H_0 + \varphi^\star(mL)$$

with $m \gg 0$ to be specified in a moment. Clearly H is ample. We have

$$K_X \cdot H^2 = K_X \cdot H_0^2 + 2K_X \cdot H_0 \cdot \varphi^\star(mL) + K_X \cdot (\varphi^\star(mL))^2.$$

(a) If $\dim Y = 2$, then $K_X \cdot \varphi^\star(L)^2 < 0$, since $L \cdot L$ is cohomologous to a multiple of a generic fibre and $-K_{X|\varphi^{-1}(y)}$ is ample on such fibres. Hence for $m \gg 0$, $K_X \cdot H^2 < 0$.

(b) If $\dim Y = 1$, we obtain $\varphi^\star(mL)^2 = 0$. But now

$$K_X \cdot H_0 \cdot \varphi^\star(mL) < 0,$$

hence again $K_X \cdot H^2 < 0$ for $m \gg 0$. This finishes the proof of 4.9, the existence of the ample 1-flag being an obvious consequence of the main statement.

4.10. Corollary. — *Let X be a normal projective $\mathbb{Q}$-Gorenstein 3-fold with at most terminal singularities. Assume $\kappa(X) = -\infty$. If one of the following conditions is satisfied, there exists an ample divisor H on X with $K_X \cdot H^2 < 0$:*

(a) *The dimension of the image of the Albanese map α of X is 2 (or, more generally, there exists a surjective map of X to an irreducible reduced surface whose desingularisation has Kodaira dimension at least 0).*

(b) *There exists a modification $\varphi : X \to X'$ onto a normal projective $\mathbb{Q}$-Gorenstein 3-fold X' such that X' admits a map onto Y as in 4.9.*

(c) *$\kappa(-K_X) \geq 0$.*

Proof. (a) The general fiber of α is $\mathbb{P}_1$, by Iitaka's conjecture $C_{3,2}$ or the uniruledness of X, hence we can apply 4.9.

(b) By 4.9 there exists H' ample on X' such that $K_{X'} \cdot H'^2$. We conclude $K_X \cdot \varphi^\star(H')^2 < 0$, hence, approximating the nef and big divisor $\varphi^\star(H')$ by ample $\mathbb{Q}$-divisors, we obtain our claim.

c) Take $Z \in |-mK_X|$ and any ample divisor H. Then clearly $Z \cdot H^2 > 0$, hence our claim follows.

4.11. Remark. — By the minimal model theory [Mo88] every smooth projective 3-fold X with $\kappa(X) = -\infty$ is birational to a normal projective variety X' having at most terminal singularities which admits a map $\varphi : X' \longrightarrow Y$ with $-K_X'$ being φ-ample. Hence there exists H' ample on X' such that $K_X' \cdot H'^2 < 0$. However we cannot easily conclude the same for X. The difficulty is the following: X'

arises from X by a sequence of contractions of prime divisors and by flips. For contractions of prime divisors everything is settled by 4.10 (b). What is totally unclear and maybe even false is the following:

4.12. PROBLEM. — Assume X and X' are normal projective varieties having at most terminal singularities. Let $\varphi : X \longrightarrow X'$ be a birational rational map of flipping type (in particular isomorphic in codimension 1). Assume that there exists H' ample on X' such that $K_{X'} \cdot H'^2 < 0$. Does there exist an ample divisor H on X with the analogous property ? Slightly more modest would be to ask for an ample 1-flag for $-K_X$ if there is one for $-K_{X'}$.

5. Threefolds with $\sigma_-(K_X) = 2$

Our goal here is to study 3-folds which have a noneffective canonical divisor K_X. To this end, we introduce the following definition.

5.1. DEFINITION. — *Let X be a normal projective variety, assume X to be $\mathbb{Q}$-Gorenstein. Choose $m \in \mathbb{N}$ such that mK_X is Cartier. Then we define:*

$$\sigma(X) = \sigma_-(mK_X).$$

Note that $\sigma(X)$ does not depend on the choice of m (we could also take the minimal m such that mK_X is Cartier). We have $0 \leq \sigma(X) \leq n = \dim X$; and $\sigma(X) \geq 1$ is equivalent to saying that $K_X \notin \overline{N}_{\mathrm{eff}}(X)$. Moreover $\sigma(X) = n$ iff $-K_X$ is ample, i.e. X is $\mathbb{Q}$-Fano. Thus $\sigma(X)$ measures how far X is from being Fano. We now concentrate on 3-folds. Then for X smooth, $\sigma(X) \geq 1$ iff $\kappa(X) = -\infty$ (see [Mi87], [Mo88]), thus it is interesting to investigate 3-folds with $\sigma(X) = 2$.

5.2. EXAMPLE. — Let X be a smooth projective 3-fold, $\varphi : X \to Y$ a surjective holomorphic map onto a compact Riemann surface Y such that $-K_X$ is φ-ample. Then $\sigma(X) \geq 2$ (in particular, if $g(Y) \geq 1$, then $\sigma(X) = 2$).

Proof. Fix A ample on X. Since $-K_X$ is φ-ample, we have $R^i\varphi_\star(-mK_X - pA) = 0$, for $i \geq 1$ and $m \geq C(p+1)$. Hence

$$H^2(X, -mK_X - pA) = H^2(Y, \varphi_\star(-mK_X - pA)) = 0$$

for those m, p, i.e. $\sigma_+(-K_X) \leq 1$. (Of course here we do not need $\dim X = 3$.) □

Examples of maps φ arising in Example 5.2 are contractions of extremal rays; note also that the proof still works if X is only normal and $\mathbb{Q}$-Gorenstein.

Now we turn to fibrations over surfaces. Since this seems to be rather complicated in general, we restrict ourselves to the case of $\mathbb{P}_1$-bundles. Recall that for a $\mathbb{P}_m$-bundle $X = \mathbb{P}(E) \xrightarrow{\pi} Y$ over a projective manifold Y we have the formula

$$K_X = \pi^\star(K_Y + \det E) \otimes \mathcal{O}_{\mathbb{P}(E)}(-m-1).$$

5.3. PROBLEM. — Let $X = \mathbb{P}(E) \xrightarrow{\pi} S$ be a $\mathbb{P}_1$-bundle over a smooth projective surface S. Under which conditions is $\sigma(X) = 2$?

This question will be considered in (5.5), (5.6). Note that in the Fano case $\sigma(X)=3$, the situation is not so difficult to understand: S has automatically to be a del Pezzo surface and the condition on E can be expressed in the following way: $E\otimes(\det E\otimes K_S)^{-1/2}$ has to be ample. At least for $S=\mathbb{P}_2$ all those E are classified (see [SW90]).

5.4. Examples. —

(1) Let $X=\mathbb{P}_1\times S$ with a projective surface S. Then $\sigma(X)=\sigma(S)+1$, hence $\sigma(X)\geq 2$ iff $\kappa(S)=-\infty$, with equality iff S is not Fano.

(2) Now assume $X=C\times S$ with $g(C)\geq 1$. Then $\sigma(X)=\sigma(S)$, hence $\sigma(X)=2$ iff S is Fano. In fact, let $A=A_1\boxtimes A_2$ with A_1 ample on C, A_2 ample on S. Then
$$\begin{aligned}H^q(X,mK_X+pA)&=H^q\big(X,\mathcal{O}(mK_C+pA_1)\boxtimes\mathcal{O}(mK_S+pA_2)\big)\\&=H^0(X,\mathcal{O}(mK_C+pA_1))\otimes H^q(S,\mathcal{O}(mK_S+pA_2))\end{aligned}$$
by Künneth, hence $\sigma_-(K_X)=\sigma_-(K_S)$.

(3) Let $X=\mathbb{P}\big(\mathcal{O}_{\mathbb{P}_2}(1)\oplus\mathcal{O}_{\mathbb{P}_2}(n)\big)$, $n\in\mathbb{N}$. Then
$$\begin{aligned}\sigma(X)=3\quad&\text{iff}\quad n=1\text{ or }2,\\ \sigma(X)=2\quad&\text{iff}\quad n=3.\end{aligned}$$
(See 5.5 below for more general calculations).

5.5. Remarks. — Let $X=\mathbb{P}(E)\to S$ be a $\mathbb{P}_1$-bundle over a projective surface S. We may assume E to be ample. In order to compute $\sigma(X)$ we have to consider $H^1(X,mK_X+pA)$ for some ample A. We choose $A=\mathcal{O}_{\mathbb{P}(E)}(1)$. Then
$$H^1(X,mK_X+pA)\simeq H^0(S,S^{2m-p-2}(E^\star)\otimes(\det E)^{m-1}\otimes K_S^m).$$
Thus $\sigma(X)=2$ implies (and, if X is not Fano, is in fact equivalent to)

$$(\star)\qquad H^0(S,S^{2m-p-2}(E^\star)\otimes(\det E)^{m-1}\otimes K_S^m)=0,\quad m\geq C(p+1).$$

Consider the $\mathbb{Q}$-vector bundle $E_0=E\otimes(\det E)^{-1/2}$, so that $E_0^\star\simeq E_0$ and $c_1(E_0)=0$. Then $(\star)$ is equivalent to

$$(\star\star)\qquad H^0(S,S^{2m-p-2}E_0\otimes(\det E)^{p/2}\otimes K_S^m)=0,\quad m\geq C(p+1).$$

This should be considered as a kind of stability property of E. For instance, if $\kappa(S)\geq 0$, then $(\det E)^{p/2}\otimes K_S^m$ has sections for all $p\geq p_0$ and $m\geq 0$, so $(\star\star)$ immediately implies that
$$H^0(S,S^kE_0)=0,\quad\forall k>0.$$
(Note that E_0 is not necessarily a "real" vector bundle if $\det E$ is not divisible by 2 in $\mathrm{Pic}(S)$; in that case we only consider the case of even integers k so that S^kE_0 is actually a vector bundle.)

(a) Assume that $c_1(E)^2>4\,c_2(E)$, S minimal and $\kappa(S)\geq 0$. Then $\sigma(X)=1$.

Proof. If $\sigma(X) = 2$, then by $(\star\star)$ with $m = (k + p + 2)/2$ and p chosen such that $(\det E)^{p/2} \otimes K_S^{(p+2)/2}$ has sections, we conclude (at least for even k) that

$$(+) \qquad H^0\big(S, S^k(E_0 \otimes K_S^{1/2})\big) = 0, \forall k \geq 0.$$

By Riemann-Roch, $\chi\big(S^k(E_0 \otimes K_S^{1/2})\big) \sim (c_1^2 - c_2)k^3/6$ where $c_i = c_i(E_0 \otimes K_S^{1/2})$. By assumption $c_1^2(E) > 4\,c_2(E)$ and $K_S^2 \geq 0$, hence

$$c_1^2 - c_2 = \frac{3}{4}K_S^2 - c_2(E_0) = \frac{3}{4}K_S^2 + \frac{1}{4}c_1(E)^2 - c_2(E) > 0.$$

Thus Riemann-Roch implies

$$H^2\big(S, S^k(E_0 \otimes K_S^{1/2})\big) \neq 0$$

for k large. By duality we obtain

$$H^0\big(S, S^k(E_0 \otimes K_S^{-1/2}) \otimes K_S\big) \neq 0.$$

This contradicts $(+)$.

(b) We observe also that $\sigma(X) = 2$ implies $\kappa(K_S + \det E^\star) = -\infty$, and even that $K_S + \det E^\star \notin \overline{N}_{\mathrm{eff}}(S)$. Furthermore the proof of a) yields $\frac{3}{4}K_S^2 \leq c_2(E_0)$ if $\sigma(X) = 2$. Our conclusion is that the condition "$\sigma(X) = 2$" becomes more and more difficult to achieve when K_S becomes more positive. Hence the question arises whether there exists any $\mathbb{P}_1$-bundle X over a surface S with $\kappa(S) \geq 0$ having $\sigma(X) = 2$.

5.6. Example. — Let $S \subset \mathbb{P}_3$ be a general quartic surface. Then S is a $K3$ surface and has Picard number $\rho(S) = 1$. Let $X = \mathbb{P}(\Omega_S^1) \to S$. We are going to show that $\sigma(X) = 2$.

By (5.5) it is sufficient to prove that

$$(\star) \qquad H^0(S, S^{2m-p-2}(\Omega_S^1) \otimes A^p) = 0, \quad m \geq C(p+1)$$

where A is a sufficiently ample line bundle (so that $E = \Omega_S^1 \otimes A$ in the context of 5.5; note that $\det \Omega_S^1 = \mathcal{O}_S$, hence $E_0 = \Omega_S^1$). We may take for instance $A = \mathcal{O}_S(2) = \mathcal{O}_{\mathbb{P}_3}(2)_{|S}$. Now $(\star)$ is easily verified by considering the standard exact sequences

$$\begin{aligned}
&0 \longrightarrow \mathcal{O}_S(-4) \longrightarrow \Omega^1_{\mathbb{P}_3|S} \longrightarrow \Omega_S^1 \longrightarrow 0,\\
&0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(1)^{\oplus 4} \longrightarrow T_{\mathbb{P}_3|S} \longrightarrow 0 \quad \text{(Euler sequence)},\\
&0 \longrightarrow \Omega^1_{\mathbb{P}_3|S} \longrightarrow \mathcal{O}_S(-1)^{\oplus 4} \longrightarrow \mathcal{O}_S \longrightarrow 0 \quad \text{(dual sequence)},
\end{aligned}$$

and their k-th symmetric powers

$$\begin{aligned}
&0 \longrightarrow \mathcal{O}_S(-4) \otimes S^{k-1}\Omega^1_{\mathbb{P}_3|S} \longrightarrow S^k\Omega^1_{\mathbb{P}_3|S} \longrightarrow S^k\Omega_S^1 \longrightarrow 0,\\
&0 \longrightarrow S^k\Omega^1_{\mathbb{P}_3|S} \longrightarrow \bigoplus \mathcal{O}_S(-k) \longrightarrow \bigoplus \mathcal{O}_S(1-k) \longrightarrow 0.
\end{aligned}$$

In fact we find for every $k \in \mathbb{N}$ a constant C_k such that

$$H^0\big(S, S^m\Omega_X^1 \otimes \mathcal{O}_X(k)\big) = 0, \quad m \geq C_k.$$

Note that property $(\star)$ for an arbitrary 2-bundle E with $c_1(E) = 0$ on a $K3$ surface with $\rho(S) = 1$ is just slightly stronger than the condition that all $S^m E$ are stable. Namely, $(\star)$ can be rephrased by saying that $\mathcal{O}_{\mathbb{P}(E)}(1) \notin \overline{N}_{\text{eff}}(\mathbb{P}(E))$, while stability of $S^m E$ means that $\mathcal{O}_{\mathbb{P}(E)}(m)$ is never effective. So it seems that 5.6 should work for most Hermite-Einstein bundles for which $S^m E$ never splits off a direct summand.

Now we use this example to show that Problem 3.6 has not always a positive answer. In fact we prove

$$(\star\star) \qquad \text{If } Y \subset X \text{ is ample, then } K_X^2 \cdot Y < 0,$$

whereas $\sigma_+(K_X) = 1$. We can write

$$\mathcal{O}_X(Y) = \mathcal{O}_{\mathbb{P}(\Omega_S^1)}(\alpha) \otimes \pi^\star \mathcal{O}_S(\beta),$$

with $\alpha, \beta \in \mathbb{Z}$. Then certainly $\alpha > 0$, and also $\beta > 0$ since Ω_S^1 is not ample. Replacing Y by some multiple, we may assume the existence of a smooth curve $C \in |\mathcal{O}_S(\beta)|$. Since

$$-K_X = \mathcal{O}_{\mathbb{P}(\Omega_S^1)}(2),$$

we have (abbreviating $\mathcal{O}_{\mathbb{P}(\Omega_S^1)}(2)$ by $\mathcal{O}_{\mathbb{P}}(2)$):

$$\begin{aligned} K_X^2 \cdot Y &= 4\, c_1\big(\mathcal{O}_{\mathbb{P}}(1)\big)^2 \big(\alpha\, c_1(\mathcal{O}_{\mathbb{P}}(1)) + \pi^\star C\big) \\ &= 4\alpha\, c_1\big(\mathcal{O}_{\mathbb{P}}(1)\big)^3 + 4\, c_1\big(\mathcal{O}_{\mathbb{P}(\Omega^1_{S|C})}(1)\big)^2. \end{aligned}$$

Since

$$c_1\big(\mathcal{O}_{\mathbb{P}}(1)\big)^3 = c_1(\Omega_S^1)^2 - c_2(\Omega_S^1) = -c_2(S) = -24$$

and

$$c_1\big(\mathcal{O}_{\mathbb{P}(\Omega^1_{S|C})}(1)\big)^2 = c_1(\Omega^1_{S|C}) = 0,$$

our claim $(\star\star)$ follows.

Now we discuss birational properties of $\sigma(X)$.

5.7. Proposition. — *Let X and Y be normal projective varieties of dimension 3 with at most terminal singularities. Let $\varphi : X \to Y$ be a birational map with exceptional set E such that $-K_X$ is φ-ample. Then:*

(a) $\sigma(Y) = 2 \quad \Longrightarrow \quad \sigma(X) = 2 \quad$ *if* $\dim \varphi(E) = 0$;

(b) $\sigma(X) = 2 \quad \Longrightarrow \quad \sigma(Y) \geq 2.$

Proof. Fix an ample divisor A on Y. Let E_i be the (automatically 1-codimensional) components of E. For simplicity of notations we may assume X and Y Gorenstein (otherwise substitute K by mK). Write

$$K_X = \varphi^\star(K_Y) + \sum \lambda_i E_i.$$

Because Y has terminal singularities, all λ_i are > 0. Since $-K_X$ is φ-ample, so is $-\sum \lambda_i E_i$. Hence we find p_0 such that

$$B = \varphi^\star(p_0 A) - \sum \lambda_i E_i$$

is ample on X. Substituting A by p_0A, we may assume $p_0 = 1$.

Let us first prove (b). So assume $\sigma(X) = 2$. Then

$$H^q(X, -mK_X - pB) = 0, \quad m \geq C(p+1), \quad q \geq 2.$$

Since

$$\begin{aligned} &-mK_X - pB = -\varphi^\star(mK_Y + pA) - \sum(m-p)\lambda_i E_i \quad \text{and} \\ &R^q\varphi_\star\big(\mathcal{O}\big(\sum(p-m)\lambda_i E_i\big)\big) = 0 \end{aligned}$$

for $p < m$ and $q > 0$ ($-\sum \lambda_i E_i$ being φ-ample), we conclude via the Leray spectral sequence that

$$H^q(Y, -mK_Y - pA) = 0, \quad m \geq C(p+1), \quad q \geq 2,$$

hence $\sigma(Y) \geq 2$. For (a) assume $\sigma(Y) = 2$, so

$$H^q(Y, -mK_Y - pA) = H^{3-q}(Y, (m+1)K_Y + pA) = 0, \quad m \geq C(p+1), \quad q \geq 2.$$

We observe that

$$\varphi_\star(-mK_X - pB) = \mathcal{O}(-mK_Y - pA) \otimes \varphi_\star\Big(\mathcal{O}\big(\sum(p-m)\lambda_i E_i\big)\Big).$$

Since $p < m$, we see that $\varphi_\star\big(\mathcal{O}\big(\sum(p-m)\lambda_i E_i\big)\big)$ is an ideal sheaf $\mathcal{I}$ with $\mathrm{Supp}(\mathcal{O}/\mathcal{I}) = \varphi(E)$, hence with finite support. Thus

$$H^q\big(Y, \varphi_\star(-mK_X - pB)\big) = H^q(Y, -mK_Y - pA), \quad q \geq 2.$$

By the Leray spectral sequence, we conclude as before that

$$H^q(X, -mK_X - pB) = H^q(Y, -mK_Y - pA) = 0, \quad m \geq C'(p+1), \quad q \geq 2.$$

Hence $\sigma(X) \geq 2$. If finally $\sigma(X) = 3$, we have

$$H^1\big(Y, \varphi_\star(-mK_X - pB)\big) = 0,$$

hence $H^1(Y, -mK_Y - pA) = 0$ and consequently $\sigma(Y) = 3$. □

5.8. Remark. — If in particular X is a projective 3-fold with at most terminal singularities and $\varphi : X \to Y$ is a birational contraction of an extremal ray whose exceptional set is not of codimension ≥ 2, then Proposition 5.7 applies.

5.9. Problem. — Let X, Y be projective 3-folds with only terminal singularities and let $\varphi : X \dashrightarrow Y$ be a birational morphism of flipping type. Is $\sigma(X) \geq 2$ iff $\sigma(Y) \geq 2$?

If this is true, then by Mori [Mo88] we conclude the following: if X and Y are smooth projective 3-folds (or $\mathbb{Q}$-factorial with at most terminal singularities) and if $f : X \to Y$ is a birational morphism, then $\sigma(X) = 2$ implies $\sigma(Y) \geq 2$.

5.10. Example. — We will show that in Prop. 5.7 (a) the conclusion $\sigma(X) = 2$ no longer holds if $\dim \varphi(E) = 1$. Let $Y = \mathbb{P}_3$, let D be a smooth complete intersection curve of type (a, a) with $a \geq 7$. Let $\varphi : X \to Y$ be the blow-up of Y along D.

Claim. — *We have $\sigma(X) = 1$.*

Proof. Let A and B as in the proof of Prop. 5.7, so $A = \mathcal{O}_{\mathbb{P}_3}(p_0)$ and $B = \varphi^\star A - E$ for some integer p_0 (In this situation, the exceptional divisor E has just one component with $\lambda = 1$.) We will show that

$$H^2(X, -mK_X) \neq 0$$

for $m \gg 0$. In fact we have an exact sequence

$$\begin{array}{l} 0 \to \mathcal{I}_D^m \otimes \mathcal{O}(-mK_Y) \to \mathcal{O}(-mK_Y) \to \mathcal{O}(-mK_Y)_{|D_m} \to 0, \\ \qquad\qquad || \\ \varphi_\star(-mK_X) \end{array}$$

where D_m is the $(m-1)$-th infinitesimal neighborhood of D defined by $\mathcal{I}_D^m$. Since $H^q(Y, -mK_Y) = 0$ for $q > 0$ and all $m > 0$, it is sufficient to prove

$$H^1(D_m, -mK_Y) \neq 0, \quad m \gg 0.$$

Since

$$\chi(D_m, \mathcal{O}(4m)) = \sum_{0 \le \nu \le m-1} \chi\big(D, S^\nu N_C^\star \otimes \mathcal{O}_D(4m)\big)$$

it is sufficient to prove that this last sum is negative for $m \gg 0$. By Riemann-Roch we have

$$\begin{aligned} &\sum_{0 \le \nu \le m-1} \chi\big(D, S^\nu N_C^\star \otimes \mathcal{O}_D(4m)\big) = \\ &\qquad = \sum_{0 \le \nu \le m-1} \big((1-g)(\nu+1) + (\nu+1)a^2(4m - \nu a)\big) \\ &\qquad = (1 - g + 4ma^2)m(m+1)/2 \;-\; a^3 m(m^2-1)/3. \end{aligned}$$

The leading term of this cubic polynomial in m is

$$a^2(2 - a/3)m^3$$

which is negative by our assumption $a \geq 7$. $\square$

5.11. Proposition. — *Let X be a smooth projective 3-fold. Assume $-K_X$ to be big and nef but not ample. Let $\varphi : X \to Y$ be the map given by the base point free linear system $|-m_0K_X|$ with $m_0 \gg 0$. Then:*

$$\sigma(X) = 2 \ \text{ iff all nontrivial fibers of } \varphi \text{ are 1-dimensional.}$$

Proof. By the base point free theorem, some multiple $-m_0K_X$ is globally generated and since $-K_X$ is big, the associated map $\varphi : X \to Y$ is a modification onto a normal projective variety Y.

(1) Assume that φ has at least one 2-dimensional fiber F.

In this case, we will show that $H^2(X, -mK_X - pA) \neq 0$, $p \gg 0$, where A is an arbitrary fixed ample divisor on X. For this we use the Leray spectral sequence

$$E_2^{p,q} = H^p\big(Y, R^q\varphi_\star\mathcal{O}(-mK_X - pA)\big).$$

First consider $E_2^{0,2} = H^0\big(Y, R^2\varphi_\star\mathcal{O}(-mK_X - pA)\big)$. Since $\sigma_-(mK_X) = \sigma_-(K_X)$ we may assume $m_0 = 1$ in our considerations, hence $K_X = \varphi^\star(K_Y)$. So

$$E_2^{0,2} = H^0\big(Y, R^2\varphi_\star\mathcal{O}(-pA) \otimes \mathcal{O}(-mK_Y)\big).$$

We show that $R^2\varphi_\star\mathcal{O}(-pA) \neq 0$ in order to conclude $E_2^{0,2} \neq 0$. Let $\{x\} = \varphi(F)$. Then $\mathrm{rk}(R^2\varphi_\star\mathcal{O}(-pA)_x) \geq h^2(F, -pA_{|F})$. Here we equip F with the reduced structure. Since F is Cohen-Macaulay, we have

$$H^2(F, -pA_{|F}) \simeq H^0(F, K_F \otimes pA_{|F}),$$

which is nonzero for all $p \gg 0$. Hence $E_2^{0,2} \neq 0$ for $p \gg 0$ and all m. Next consider $E_0^{2,1} = H^2\big(Y, R^1\varphi_\star\mathcal{O}(-pA) \otimes \mathcal{O}(-mK_Y)\big)$. Since $\dim \mathrm{Supp} R^1\varphi_\star\mathcal{O}(-pA) = 0$, $E_2^{2,1} = 0$. Hence $E_2^{0,2} \simeq E_3^{0,2} \simeq E_\infty^{0,2}$ and thus

$$H^2(X, -mK_X - pA) \neq 0, \quad p \gg 0.$$

(2) Now assume that all fibers of φ have dimension 1.

Then take a general smooth ample (hyper)surface $H \subset X$. It will meet every fiber in a finite set. Hence $-K_X \cdot C > 0$ for every curve $C \subset H$, otherwise C would be contracted to a point by φ. Furthermore $K_X^2 \cdot H > 0$ since $-K_X$ is big and nef on the general member of the linear system $|mH|$, $m \gg 0$. Hence $-K_{X|H}$ is ample by Nakai-Moishezon. Theorem 3.4 implies $\sigma_-(K_X) \geq 2$, and a strict inequality cannot occur because $-K_X$ is not ample. $\square$

5.12. Remark. — The proof of 5.11 can be easily modified to make it work for all $\mathbb{Q}$-Gorenstein normal projective varieties X. In that case F might not be Cohen-Macaulay, so we possibly have to pass to the normalization which is Cohen-Macaulay.

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