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VARIATIONAL APPROACH FOR COMPLEX MONGE-AMPÈRE EQUATIONS AND GEOMETRIC APPLICATIONS [after Berman, Berndtsson, Boucksom, Eyssidieux, Guedj, Jonsson, Zeriahi, …]

by **Jean-Pierre DEMAILLY**

INTRODUCTION

Monge-Ampère equations on compact Kähler manifolds can be solved by a variational method that is independent of Yau's theorem. The technique of [BBGZ13] is based on the study of certain functionals (Ding-Tian, Mabuchi) on the space of Kähler metrics, and their geodesic convexity due to [Chen00] and Berman-Berndtsson [BeBe14] in its full generality. Recent applications include the existence and uniqueness of Kähler-Einstein metrics on $\mathbb{Q}$-Fano varieties with log terminal singularities, given in [BBEGZ15], and a new proof by [BBJ15] of a uniform version of the Yau-Tian-Donaldson conjecture [Tian97]. This provides a simpler route to the existence theorem for Kähler-Einstein metrics due to Chen-Donaldson-Sun [CDS15], albeit with a stronger hypothesis. Our goal is to present the main ideas involved in this approach (starting from the basics!)

0.A. Kähler metrics. A *Kähler manifold* (X,ω) is a complex manifold X of dimension $n=\dim_{\mathbb{C}} X$ endowed with a d-closed smooth positive $(1,1)$-form ω. In local holomorphic coordinates $(z_1,\ldots,z_n)$, one can write $\omega = i\sum_{1\le j,k\le n}\omega_{jk}(z)\,dz_j\wedge d\overline{z}_k$, i.e. $(\omega_{jk}(z))$ is a positive definite hermitian matrix at every point, and $d\omega=0$, so that ω is also a (real) symplectic structure on X. The holomorphic tangent bundle T_X is then equipped with the associated hermitian structure $h_\omega=\sum_{1\le j,k\le n}\omega_{jk}(z)\,dz_j\otimes d\overline{z}_k$. There is a unique connection ∇_h on T_X, called the Chern connection, such that h is ∇_h-parallel and $\nabla_h^{0,1}$ coincides with the $\overline{\partial}$ operator given by the complex structure. The Chern curvature tensor, which coincides with the Riemann curvature tensor in the Kähler case, is the $(1,1)$-form form with values in the bundle of endomorphisms of T_X, i.e. a section in $C^\infty(X,\Lambda^{1,1}T_X^*\otimes\mathrm{End}(T_X))$, given by

$$\Theta_{T_X,\omega} := \frac{i}{2\pi}\nabla_h^2 = i\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}dz_j\wedge d\overline{z}_k\otimes \frac{\partial}{\partial z_\lambda}^{*}\otimes\frac{\partial}{\partial z_\mu}. \tag{0.1}$$

Its trace $\mathrm{Tr}(\Theta_{T_X,\omega}) = i\sum_{j,k,\lambda} c_{jk\lambda\lambda}dz_j\wedge d\overline{z}_k$ is also the curvature form of the anti-canonical line bundle $\Lambda^n T_X$ ($=-K_X$ in additive notation), and is by definition the *Ricci curvature* $\mathrm{Ricci}(\omega)$. A standard calculation gives

$$\mathrm{Ricci}(\omega)=\Theta_{\Lambda^nT_X,\Lambda^n\omega} = -dd^c\log\det(\omega_{jk})\quad\text{where } d^c=\tfrac{1}{4i\pi}(\partial-\overline{\partial}),\ dd^c=\tfrac{i}{2\pi}\,\partial\overline{\partial}. \tag{0.2}$$

By definition, $\mathrm{Ricci}(\omega)$ is a closed real $(1,1)$-form, and its De Rham cohomology class is induced by the first Chern class $c_1(X) := c_1(T_X) = -c_1(K_X) \in H^2(X,\mathbb{Z})$.

0.B. Kähler-Einstein metrics and the conjecture of Yau-Tian-Donaldson. A Kähler metric ω is said to be *Kähler-Einstein* if

$$\mathrm{Ricci}(\omega) = \lambda\omega \quad \text{for some } \lambda \in \mathbb{R}. \tag{0.3}$$

This requires $\lambda\omega \in c_1(X)$, hence (0.3) can be solved only when $c_1(X)$ is positive definite, negative definite or zero, and after rescaling ω by a constant, one can always assume that $\lambda \in \{0, 1, -1\}$. Let us fix some reference Kähler metric ω_0. Under the cohomological assumption $c_1(X) = \lambda\{\omega_0\} \in H^2(X,\mathbb{R})$, the $\partial\overline{\partial}$-lemma says that there is a function $f \in C^\infty(X,\mathbb{R})$ such that

$$\mathrm{Ricci}(\omega_0) - \lambda\omega_0 = dd^c f. \tag{0.4}$$

The potential f is defined modulo an additive constant, and we will normalize f so that $\int_X e^f\omega_0^n = \int_X \omega_0^n$. If we look for a solution $\omega = \omega_0 + dd^c\varphi$ of (0.3) in the same cohomology class as ω_0, Formula (0.2) yields $\mathrm{Ricci}(\omega) - \mathrm{Ricci}(\omega_0) = -dd^c \log(\omega_0 + dd^c\varphi)^n/\omega_0^n$, and the Kähler-Einstein condition (0.3) is reduced to solving the Monge-Ampère equation

$$(\omega_0 + dd^c\varphi)^n = e^{-\lambda\varphi+f}\omega_0^n. \tag{0.5}$$

• When $\lambda = -1$ and $c_1(X) < 0$, i.e. $c_1(K_X) > 0$, Aubin [Aub78] has shown that there is always a unique solution, hence a unique Kähler metric $\omega \in c_1(K_X)$ such that

$$\mathrm{Ricci}(\omega) = -\omega.$$

This is a very natural generalization of the existence of constant curvature metrics on complex algebraic curves, implied by Poincaré's uniformization theorem in dimension 1.

• When $\lambda = 0$ and $c_1(X) = 0$, the celebrated result of [Yau78] states that there exists a unique metric $\omega = \omega_0 + dd^c\varphi$ in the given cohomology class $\{\omega_0\}$ such that $\mathrm{Ricci}(\omega) = 0$ (solution of the Calabi conjecture [Cal54], [Cal57]). More generally, without any assumption on $c_1(X)$, [Yau78] showed that the Monge-Ampère equation $(\omega_0 + dd^c\varphi)^n = e^f\omega_0^n$ has a unique solution whenever $\int_X e^f\omega_0^n = \int_X \omega_0^n$, in other words, one can prescribe the volume form $\omega^n = (\omega_0 + dd^c\varphi)^n$ to be any given volume form $e^f\omega_0^n > 0$ under the unique constraint that the volume is preserved. Equivalently, the Ricci curvature form can be prescribed to be equal any given smooth closed $(1,1)$-form

$$\mathrm{Ricci}(\omega) = \rho,$$

provided that $\rho \in c_1(X)$. A synthetic exposition is given in [Bour79], cf. also [Siu87]. Among the numerous posterior geometric applications, let us mention the Bogomolov-Beauville theorem [Beau83] on the structure of Ricci flat manifolds, and the more recent result of [CDP14] on the structure of compact Kähler manifolds with semipositive Ricci class $c_1(X)$.

• A much more difficult problem is to analyze solutions of (0.5) when $\lambda = +1$ and $c_1(X) > 0$, i.e. when $-K_X$ is ample; such manifolds are called *Fano manifolds*. In general, there is neither existence nor uniqueness. However, whenever solutions exist, it is known since [BM87] that they are unique up to the action of the identity component $\mathrm{Aut}^0(X)$ in the complex Lie group of biholomorphisms of X. A necessary and sufficient condition for the existence of Kähler-Einstein metrics had been conjectured by Yau [Yau86], Tian and Donaldson. The necessity was known since [Tian97] (see also [Don02] and the Bourbaki lecture [Bour97]), but the sufficiency, and a solution of the conjecture, has been given only recently, as reported in last year's Bourbaki seminar [Eys15]:

Theorem 0.6 (Chen-Donaldson-Sun [CDS15]; see also [DS15, CSW15] and [Tian15]) *Let X be a Fano manifold. Then X admits a Kähler-Einstein metric if and only if it is K-stable.*

The definition of the K-stability condition will be given in Section 4: the concept is based on a positivity assumption for certain Donaldson-Futaki invariants attached to one parameter degenerations (X_t) of X. In the present paper, we will briefly sketch an alternative variational approach derived from [BBGZ13, BBEGZ15] and [BBJ15]. Together with the usual Kähler geometry functionals which we will describe at some length in Section 1, it also involves non Archimedean counterparts. The following consequence is obtained among many other results:

Theorem 0.7 (Berman-Boucksom-Jonsson [BBJ15]). — *Let X be a Fano manifold with finite automorphism group. Then X admits a Kähler-Einstein metric if and only if it is uniformly K-stable.*

Theorem 0.6 is stronger than Theorem 0.7 since it allows X to have nontrivial vector fields. It also uses K-(poly)stability instead of uniform K-stability. However, the variational proof of 0.7 avoids several of the subtle points in the previous approaches. For example, it uses neither the continuity method, nor partial C^0-estimates, Cheeger-Colding-Tian's theory, or the Kähler-Ricci flow. Moreover, a variant of the proof of Theorem 0.7 gives "directly" the semistable version of the YTD conjecture that was previously deduced from [CDS15] in [Li13]:

Theorem 0.8. — *Let X be a Fano manifold. Then X is K-semistable if and only if its greatest Ricci lower bound $\beta(X)$ is equal to* 1.

Here the value $\beta(X)$ is defined to be the supremum of lower bounds b such that $c_1(X)$ contains a Kähler metric ω_b with $\mathrm{Ricci}(\omega_b) \geq b\,\omega_b$ (this is always possible for $0 < b \ll 1$ by Yau's theorem). By [Szé11], this amounts to the solvability of Aubin's continuity method up to any time $t < \beta(X)$.

0.C. Log Fano manifolds. By definition, a *pair* (X, Δ) is formed by a connected normal compact complex variety X and an effective $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. One then considers the dd^c-cohomology class of $-(K_X + \Delta)$, denoted by $c_1(X, \Delta)$. It is well known, thanks to the Hironaka desingularization theorem, that there exists a *log resolution* $\pi : \tilde{X} \to X$ of (X, Δ), namely a modification of X over the complement of the singular loci of X and Δ, such that the pull-back of Δ and of X_{sing} consists of simple normal crossing (snc) divisors in $\tilde{X}$ and

$$\pi^*(K_X + \Delta) = K_{\tilde{X}} + E, \qquad E = \textstyle\sum_j a_j E_j \tag{0.9}$$

for some $\mathbb{Q}$-divisor E whose push-forward to X is Δ (since X_{sing} has codimension 2, the components E_j that lie over X_{sing} yield $\pi_* E_j = 0$). The coefficient $-a_j \in \mathbb{Q}$ is known as the *discrepancy* of (X, Δ) along E_j. Let r be a positive integer such that $r(K_X + \Delta)$ is Cartier, and σ a local generator of $\mathcal{O}(r(K_X + \Delta))$ on some open set $U \subset X$. Then the (n, n) form

$$|\sigma|^{2/r} := i^{n^2}\, \sigma^{1/r} \wedge \overline{\sigma^{1/r}} \tag{0.10}$$

is a volume form with poles along $S = \operatorname{Supp}\Delta \cup X_{\mathrm{sing}}$. By the change of variable formula, its local integrability can be checked by pulling back σ to $\tilde{X}$, in which case it is easily seen that the integrability occurs if and only if $a_j < 1$ for all j, independently of the log resolution π that has been selected. One then says that the pair (X, Δ) is *klt* (a short-hand for *Kawamata log terminal*). In the special case $\Delta = 0$, one says that X is *log terminal* when the pair $(X, 0)$ is klt (so that K_X is in particular $\mathbb{Q}$-Cartier, i.e. by definition, X is $\mathbb{Q}$-Gorenstein).

Definition 0.11. — *A* log Fano pair *is a klt pair (X, Δ) such that X is projective and the $\mathbb{Q}$-divisor $A = -(K_X + \Delta)$ is ample.*

This assumption implies that the cohomology class $c_1(X, \Delta)$ contains a Kähler form ω_0 (near a singular point, this means that ω_0 can be extended locally as a Kähler form in a smooth ambient space containing the germ of X). Every form $\omega = \omega_0 + dd^c\varphi$ in the same cohomology class can be interpreted as the curvature form of a smooth hermitian metric h on $\mathcal{O}(-(K_X + \Delta))$, whose weight is $\phi = u_0 + \varphi$ where u_0 is a local potential of ω_0. In this setting, we denote

$$\omega = \omega_0 + dd^c\varphi = dd^c\phi \tag{0.12}$$

where ϕ is understood as the weight of a global metric formally denoted $h = e^{-\phi}$ on the $\mathbb{Q}$-line bundle $\mathcal{O}(-(K_X + \Delta))$. Its inverse e^{ϕ} is a hermitian metric on $\mathcal{O}(K_X + \Delta)$, and in our notation, if σ is a local generator of $\mathcal{O}(r(K_X + \Delta))$ as above, the product $|\sigma|^{2/r} e^{\phi} = e^{\varphi + u_0}$ is (locally) a smooth positive function whenever φ is smooth. This implies that

$$e^{-\phi} = |\sigma|^{2/r} e^{-(\varphi + u_0)}$$

should be seen as an integrable volume form on X with poles along $\operatorname{Supp}\Delta$. The Kähler-Einstein condition (0.5) can now be rewritten in a much simpler way

$$(dd^c\phi)^n = c\, e^{-\phi} \quad \text{on } X \smallsetminus S, \tag{0.13}$$

where $c > 0$ is a constant such that $c\int_X e^{-\phi} = \int_X \omega_0^n = A^n$. The corresponding Ricci identity for $\omega = dd^c\phi$, taken in the sense of currents, is

$$\mathrm{Ricci}(\omega) = \omega + [\Delta], \tag{0.14}$$

where $[\Delta]$ is the current of integration over Δ. Of course, it might be desirable to work on a nonsingular variety, and for this, one can try instead to solve the analogous equation $(dd^c\tilde{\phi})^n = e^{-\tilde{\phi}}$ on $\tilde{X} \smallsetminus \operatorname{Supp} E$, putting $\tilde{\phi} = \pi^*\phi = \phi \circ \pi$ for a suitable log resolution π as in (0.9). The expected poles of $e^{-\tilde{\phi}}$ are then given by the snc $\mathbb{Q}$-divisor $E = \sum a_j E_j$ with $a_j < 1$ (notice, however, that the pull-back class $\pi^* c_1(X, \Delta)$ is then merely nef and big, and no longer ample as soon as $\pi \neq \mathrm{Id}_X$).

0.D. Kähler-Einstein metrics and coercivity of the Mabuchi K-energy. Kähler-Einstein metrics can be shown to correspond to critical points of either the Mabuchi K-energy functional M or the Ding functional D, both defined on the space $\mathcal{P}$ of Kähler potentials (see Section 1 for definitions). They are related by an inequality $D \leq M$. Let us denote by $J \geq 0$ the Aubin energy functional, a non-linear higher dimensional version of the classical Dirichlet functional. The results of [Tian97, PSSW08] have established the following fundamental facts:

Theorem 0.15. — *If X is a Fano manifold with finite automorphism group, the following properties are equivalent:*

(i) *X has a Kähler-Einstein metric;*
(ii) *the Ding functional D is coercive, i.e. $D \geq \delta J - C$ on $\mathcal{P}$ for some $\delta, C > 0$;*
(iii) *the Mabuchi functional M is coercive on $\mathcal{P}$.*

The proof that (iii)$\Longrightarrow$(i) will be sketched here via the alternative variational approach of [BBGZ13, BBEGZ15], which moreover also brings an affirmative answer in the log Fano situation. The implication (i)$\Longrightarrow$(ii) has very recently been given a very elegant proof in [DR15], based on new ideas that influenced the strategy of [BBJ15].

0.E. The role of singular potentials. One big issue is that the equations (0.13–0.14) necessarily involve singularities along S, and one has to be able to deal with Monge-Ampère operators of the form $(\omega_0 + dd^c\varphi)^n$ where the potentials φ may exhibit some sort of singularities. At this point, it is not even clear that (0.13–0.14) will make sense. Even in the smooth Fano case where $\Delta = 0$ and $S = \emptyset$, the space of smooth potentials cannot be made compact in any reasonable sense. For this reason, considering more general potentials is needed for proving existence results, even in the absence of singularities in the equations. It is shown here, following [BBGZ13, BBEGZ15], that one appropriate such class is the class $\mathcal{E}^1$ of "finite energy" potentials. The main functionals defined on the space of Kähler potentials can be extended to $\mathcal{E}^1$, and the related convexity and monotonicity properties combined with suitable properness assumptions yield existence and uniqueness results for Kähler-Einstein metrics on general log Fano varieties.

1. FUNCTIONALS ON THE SPACE OF KÄHLER POTENTIALS

1.A. The space of Kähler potentials. Let $A \in H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$ be a Kähler cohomology class, i.e. a class of d-closed $(1,1)$-forms modulo $\partial\overline{\partial}$-exact forms, containing at least one Kähler metric $\alpha > 0$. Let $\omega_0 = \alpha + dd^c\psi_0 = dd^c\phi_0 \in A$ be a Kähler metric on X in the given cohomology class A, where ϕ_0 is thought of as the weight of a hermitian metric $h_0 = e^{-\phi_0}$ on some "virtual" ample line bundle A, although we do not necessarily need A to be an integral or rational class. Later on, we will be mostly interested in the Fano case $A = -K_X$ and the log Fano case $A = -(K_X + \Delta)$. Let $V = \int_X \omega_0^n = A^n$ be the volume of ω_0. One considers the space $\mathcal{P}_A$ of potentials of Kähler metrics $\omega = \omega_0 + dd^c\psi$; again, they are rather thought as hermitian metrics $h = e^{-\phi}$ on A with strictly plurisubharmonic (psh) weight ϕ. They are in $1:1$ correspondance with smooth functions $\psi = \phi - \phi_0 \in C^\infty(X,\mathbb{R})$, so that $h = h_0 e^{-\psi}$. The most basic operator of interest on $\mathcal{P}_A$ is the *Monge-Ampère operator*

$$(1.1) \qquad \mathcal{P}_A \to \mathcal{M}_+, \qquad \mathrm{MA}(\phi) = (dd^c\phi)^n = (\omega_0 + dd^c\psi)^n$$

into the space of measures with positive densities. According to Mabuchi [Mab85], the space $\mathcal{P}_A$ can be seen as some sort of infinite dimensional Riemannian manifold: a "tangent vector" to $\mathcal{P}_A$ is an infinitesimal variation $\delta\phi \in C^\infty(X,\mathbb{R})$ of ϕ (or ψ), and the infinitesimal Riemannian metric at a point $h = e^{-\phi}$ is given by

$$(1.2) \qquad \|\delta\phi\|_2^2 = \frac{1}{V}\int_X (\delta\phi)^2\,\mathrm{MA}(\phi).$$

Observe that the tangent bundle $T_{\mathcal{P}_A} = \mathcal{P}_A \times C^\infty(X,\mathbb{R})$ is trivial here. We let d_2 be the geodesic distance associated with this riemannian metric. In a series of remarkable works [Chen00, CC02, CT08, Chen09, CS09] X.X. Chen and his collaborators have studied the metric and geometric properties of the space $\mathcal{P}_A$, showing in particular that it is a path metric space (a non trivial assertion in this infinite dimensional setting) of nonpositive curvature in the sense of Alexandrov. A key step from [Chen00] has been to produce almost $C^{1,1}$-geodesics which turn out to minimize the intrinsic distance d_2. One can define a similar Finsler metric on $\mathcal{P}_A$ by taking L^p norms instead of L^2 norms

$$(1.2_p) \qquad \|\psi\|_p^p = \frac{1}{V}\int_X |\psi|^p\,\mathrm{MA}(\phi).$$

The associated integrated distance d_p is especially interesting for $p = 1$ as well.

1.B. Some useful functionals. The space $\mathcal{P}_A$ is endowed with several functionals of great geometric significance, which we briefly describe. They a priori depend on the choice of ϕ_0, and not just on $\phi \in \mathcal{P}_A$.

• The *Monge-Ampère functional* is

$$E_{\phi_0}(\phi) = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X (\phi-\phi_0)(dd^c\phi)^j\wedge(dd^c\phi_0)^{n-j} \tag{1.3}$$

$$= \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X \psi(\omega_0+dd^c\psi)^j\wedge\omega_0^{n-j}. \tag{1.3$'$}$$

It is a *primitive* of the Monge-Ampère operator in the sense that $dE_{\phi_0}(\phi) = \frac{1}{V}\,\mathrm{MA}(\phi)$, i.e. for any path in $\mathcal{P}_A$, say $[T,T']\ni t\mapsto\phi_t$, one has

$$\frac{d}{dt}E_{\phi_0}(\phi_t) = \frac{1}{V}\int_X \dot\phi_t\,\mathrm{MA}(\phi_t)\quad\text{where } \dot\phi_t = \frac{d}{dt}\phi_t. \tag{1.4}$$

This is easily checked by a differentiation under the integral sign:

$$\frac{d}{dt}E_{\phi_0}(\phi_t) = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X \big(\dot\phi_t\,dd^c\phi)^j + j(\phi_t-\phi_0)dd^c\dot\phi_t\wedge(dd^c\phi_t)^{j-1}\big)\wedge(dd^c\phi_0)^{n-j},$$

followed by an integration by parts $\int_X(\phi_t-\phi_0)\,dd^c\dot\phi_t\wedge\alpha = \int_X\dot\phi_t\,dd^c(\phi_t-\phi_0)\wedge\alpha$, for suitable d-closed $(n-1,n-1)$-forms α. Identity (1.4) is then obtained by just collecting and cancelling terms together. As a consequence E satisfies the cocycle relation

$$E_{\phi_0}(\phi_1)+E_{\phi_1}(\phi_2) = E_{\phi_0}(\phi_2),$$

so its dependence on ϕ_0 is only up to a constant. Also, $E_{\phi_0}(\phi+c) = E_{\phi_0}(\phi)+c$ if c is a constant. Finally, if ϕ_t depends linearly on t, we have $\ddot\phi_t = \frac{d^2}{dt^2}\phi_t = 0$ and a further differentiation of (1.4) yields

$$\frac{d^2}{dt^2}E_{\phi_0}(\phi_t) = \frac{n}{V}\int_X \dot\phi_t\,dd^c\dot\phi_t\wedge(dd^c\phi_t)^{n-1} = -\frac{n}{V}\int_X d\dot\phi_t\wedge d^c\dot\phi_t\wedge(dd^c\phi_t)^{n-1}\le 0.$$

We conclude from this calculation the fundamental fact that E_{ϕ_0} is *concave* on $\mathcal{P}_A$.

• The concavity of E implies the nonnegativity of $J_{\phi_0}(\phi) := dE_{\phi_0}(\phi_0)\cdot(\phi-\phi_0)-E_{\phi_0}(\phi)$, since the tangent at point ϕ_0 must be above the graph of E. This quantity is called the Aubin J-*energy* functional (cf. [Aub84]):

$$J_{\phi_0}(\phi) = V^{-1}\int_X(\phi-\phi_0)(dd^c\phi_0)^n - E_{\phi_0}(\phi) = V^{-1}\int_X\psi\,\omega_0^n - E_{\phi_0}(\phi)\ge 0. \tag{1.5}$$

Clearly $J_{\phi_0}(\phi+c) = J_{\phi_0}(\phi)$ if c is a constant.

• By exchanging the roles of ϕ, ϕ_0 and putting $J^*_{\phi_0}(\phi) = J_\phi(\phi_0)\ge 0$, the cocycle relation for E yields $E_\phi(-\phi_0) = -E_{\phi_0}(\phi)$. The *transposed* J-*energy functional* is

$$J^*_{\phi_0}(\phi) := E_{\phi_0}(\phi) - V^{-1}\int_X(\phi-\phi_0)(dd^c\phi)^n = E_{\phi_0}(\phi)-V^{-1}\int_X\psi(\omega_0+dd^c\psi)^n\ge 0. \tag{1.6}$$

• The *I-functional* is the symmetric functional defined by

$$I_{\phi_0}(\phi) = I_\phi(\phi_0) := -\frac{1}{V}\int_X (\phi-\phi_0)\big(\mathrm{MA}(\phi)-\mathrm{MA}(\phi_0)\big)$$

$$(1.7)\qquad = \sum_{j=0}^{n-1} V^{-1}\int_X d(\phi-\phi_0)\wedge d^c(\phi-\phi_0)\wedge(dd^c\phi)^j\wedge(dd^c\phi_0)^{n-1-j} \geq 0.$$

One can also write

$$I_{\phi_0}(\phi) = V^{-1}\left(\int_X \psi\,\omega_0^n - \int_X \psi(\omega_0+dd^c\psi)^n\right).$$

From these definitions, one finds immediately

$$(1.8)\qquad I_{\phi_0}(\phi) = J_{\phi_0}(\phi) + J^*_{\phi_0}(\phi).$$

• In the Fano or log Fano setting, the *Ding functional* ([Ding88, DT92]) is defined by

$$(1.9)\qquad D_{\phi_0} = L - L(\phi_0) - E_{\phi_0}, \quad \text{where } \; L(\phi) = -\log\int_X e^{-\phi}.$$

This makes sense, since $e^{-\phi}$ can then be seen as an integrable volume form by the klt condition. By definition, the measure $e^{L(\phi)}e^{-\phi}$ is a probability measure on X. It will be called the *adapted measure* associated with ϕ. Under a change of base metric ϕ_0, the cocycle relation satisfied by E implies

$$(1.9')\qquad D_{\phi_1}(\phi) - D_{\phi_0}(\phi) = \mathrm{Const} = E_{\phi_0}(\phi_1) - \big(L(\phi_1)-L(\phi_0)\big).$$

(Note: in [BBEGZ15], $\mathcal{P}_A$ is denoted $\mathcal{H}$, J^* is denoted E^* ; also, the constant $L(\phi_0)$ in the definition of D is omitted, and the adjustment is made by imposing $L(\phi_0)=0$.)

• Given probability measures μ,ν on a space X, the *relative entropy* $\mathrm{Entr}_\mu(\nu)\in[0,+\infty]$ of ν with respect to μ is defined as the integral

$$\mathrm{Entr}_\mu(\nu) := \int_X \log\left(\frac{d\nu}{d\mu}\right)d\nu,$$

at least when ν is absolutely continuous with respect to μ ; one sets $\mathrm{Entr}_\mu(\nu)=+\infty$ otherwise. The well known Pinsker inequality (see [DZ98, Exercise 6.2.17] for a proof) states that for all μ,ν one has

$$\mathrm{Entr}_\mu(\nu) \geq \frac{1}{2}\|\mu-\nu\|^2 \geq 0.$$

In particular, we must have $\mu=\nu$ whenever $\mathrm{Entr}_\mu(\nu)=0$. In our geometric Fano or log Fano situation, the *entropy functional* $H_{\phi_0}(\phi)$ is defined to be the entropy of the probability measure $\frac{1}{V}(dd^c\phi)^n$ with respect to $e^{L(\phi_0)}e^{-\phi_0}$, namely

$$(1.10)\qquad H_{\phi_0}(\phi) = \int_X \log\left(\frac{(dd^c\phi)^n/V}{e^{L(\phi_0)}e^{-\phi_0}}\right)\frac{(dd^c\phi)^n}{V} \geq 0.$$

• The *Mabuchi functional* (first introduced in [Mab85]) is then defined by

$$(1.11)\qquad M_{\phi_0} = H_{\phi_0} - J^*_{\phi_0}.$$

If we combine (1.6) and (1.11), we get the more explicit expression

$$(1.11') \qquad M_{\phi_0}(\phi) = \int_X \log\left(\frac{e^{\phi}(dd^c\phi)^n}{V}\right)\frac{(dd^c\phi)^n}{V} - E_{\phi_0}(\phi) - L(\phi_0).$$

As a consequence, if the base metric ϕ_0 is changed to ϕ_1, we also have

$$(1.11'') \qquad M_{\phi_1}(\phi) - M_{\phi_0}(\phi) = \text{Const} = E_{\phi_0}(\phi_1) - \big(L(\phi_1) - L(\phi_0)\big).$$

Observation 1.12. — *If c is a constant, then*

$$E_{\phi_0}(\phi + c) = E_{\phi_0}(\phi) + c \quad \textit{and} \quad L(\phi + c) = L(\phi) + c.$$

On the other hand, the functionals $I_{\phi_0}, J_{\phi_0}, J^*_{\phi_0}, D_{\phi_0}, H_{\phi_0}, M_{\phi_0}$ are invariant by $\phi \mapsto \phi + c$ and therefore descend to the quotient space $\mathcal{K}_A = \mathcal{P}_A/\mathbb{R}$ of Kähler metrics $\omega = dd^c\phi \in A$.

1.C. Comparison estimates between these functionals. Let us first note the following sequence of elementary inequalities (see for instance [BBGZ13, Lemma 2.2]):

$$(1.13) \qquad \frac{1}{n}J_{\phi}(\phi_0) \le J_{\phi_0}(\phi) \le \frac{n+1}{n}J_{\phi_0}(\phi) \le I_{\phi_0}(\phi) \le (n+1)J_{\phi_0}(\phi).$$

For the proof, notice that in (1.3) we have for $j = 1, 2, \ldots, n$

$$\begin{aligned}\delta_j &:= \int_X (\phi - \phi_0)(dd^c\phi)^{j-1} \wedge (dd^c\phi_0)^{n-j+1} - \int_X (\phi - \phi_0)(dd^c\phi)^{j} \wedge (dd^c\phi_0)^{n-j} \\ &= \int_X (\phi - \phi_0)dd^c(\phi_0 - \phi) \wedge (dd^c\phi)^{j-1} \wedge (dd^c\phi_0)^{n-j-1} \ge 0\end{aligned}$$

thanks to an integration by parts. Hence $E_{\phi_0}(\phi)$ is an average of $(n+1)$ terms that may only decrease when j increases, and from there we get an estimate

$$\frac{1}{V}\int_X (\phi - \phi_0)(dd^c\phi)^n \le E_{\phi_0}(\phi) \le \frac{1}{V}\int_X (\phi - \phi_0)(dd^c\phi_0)^n$$

in the interval between the $j = n$ and $j = 0$ terms. By definition, $I_{\phi_0}(\phi)$ is the difference of the two extreme terms and $J_{\phi_0}(\phi)$ is the difference of the last two terms, namely

$$I_{\phi_0}(\phi) = \delta_1 + \ldots + \delta_n, \quad J_{\phi_0}(\phi) = \frac{\delta_1 + 2\delta_2 + \ldots + n\delta_n}{n+1}.$$

All inequalities of (1.13) are an immediate consequence, except possibly the first one. For the latter, we exploit the symmetry of I to infer from what we already proved that

$$\frac{1}{n}J_{\phi}(\phi_0) \le \frac{1}{n+1}I_{\phi}(\phi_0) = \frac{1}{n+1}I_{\phi_0}(\phi) \le J_{\phi_0}(\phi). \qquad \square$$

By using (1.13), the equality $J^* = I - J$ (cf. (1.8)) implies

$$(1.14) \qquad \frac{1}{n}J \le J^* \le nJ,$$

hence all three functionals I, J, J^* have the same growth “at infinity” on $\mathcal{P}_A$. A further and more important fact is a comparison of the Ding and Mabuchi functionals for log Fano varieties (X, Δ). It leads to a formal characterization of Kähler-Einstein metrics (assuming unrealistically that we only deal with smooth Kähler metrics, including the Kähler-Einstein one – in fact, this may only happen when $\Delta = 0$!).

PROPOSITION 1.15. — *Let* (X, Δ) *be a log Fano manifold. Then* $M_{\phi_0}(\phi) \geq D_{\phi_0}(\phi)$ *and, in case of equality,* ϕ *must be Kähler-Einstein.*

Proof. — Unravelling the definitions we get $M - D = (H - J^*) - (L - L(\phi_0) - E)$ and $E_{\phi_0}(\phi) - J^*_{\phi_0}(\phi) = V^{-1}\int_X (\phi - \phi_0)(dd^c\phi)^n$ by (1.6), hence

$$\begin{aligned}M_{\phi_0}(\phi) - D_{\phi_0}(\phi) &= \int_X \left(\log\left(\frac{(dd^c\phi)^n/V}{e^{L(\phi_0)}e^{-\phi_0}}\right) + (\phi - \phi_0)\right)\frac{(dd^c\phi)^n}{V} + L(\phi_0) - L(\phi)\\ &= \int_X \log\left(\frac{(dd^c\phi)^n/V}{e^{L(\phi)}e^{-\phi}}\right)\frac{(dd^c\phi)^n}{V} \geq 0.\end{aligned}$$

In case of equality, the Pinsker inequality implies $\frac{(dd^c\phi)^n}{V} = e^{L(\phi)}e^{-\phi}$, hence $\omega = dd^c\phi$ is Kähler-Einstein. □

As hinted above, it will be absolutely necessary to extend the functionals to suitable spaces of non necessarily smooth metrics if we wish to use Proposition 1.15. It will also be needed to achieve compactness properties to ensure that the equality is reached.

1.D. A quasi-triangle inequality for *I*. We refer to [BBEGZ15] for the proof of the following inequality. It is based on a combination of the Cauchy-Schwarz inequality and an iteration of integration by parts.

PROPOSITION 1.16. — *There exists a constant* $c_n > 0$*, only depending on the dimension* n*, such that*

$$I_{\phi_0}(\phi) \leq c_n\big(I_{\phi_0}(\phi_1) + I_{\phi_1}(\phi)\big).$$

for all ϕ_0, ϕ_1, $\phi \in \mathcal{P}_A$.

2. MONGE-AMPÈRE OPERATORS WITH SINGULAR POTENTIALS

We sketch here a number of preliminary facts about functions and measures with finite energy on a normal compact Kähler space, which rely on a combination of the main results from [BEGZ10, BBGZ13, EGZ09].

2.A. Monge-Ampère operators in the sense of Bedford-Taylor. Consider locally bounded plurisubharmonic (psh) functions $u_1, \ldots, u_n \in L^\infty_{\mathrm{loc}}$ be on a complex space X. Then, following [BT76, BT82], one can define inductively any Monge-Ampère product as a closed positive current by putting

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k := dd^c(u_1\, dd^c u_2 \wedge \ldots \wedge dd^c u_k) \tag{2.1}$$

in the sense of distributions. In fact, by induction, the coefficients $dd^c u_2 \wedge \ldots \wedge dd^c u_k$ are complex Radon measures, their product by the locally bounded Borel function u_1 is thus well defined, and one can take the $dd^c(...)$ in the sense of distributions (currents on a complex space X being defined as the dual space to the space of forms on the regular locus X_{reg} that extend to a nonsingular ambient space). One needs to check that $dd^c u_1 \wedge \ldots \wedge dd^c u_k$ is again a closed positive current. For this, one expresses locally $u_1 = \lim \downarrow u_{1,\nu}$ as a decreasing limit of smooth functions; this can be done e.g. by locally extending u_1 to a nonsingular ambient open chart $\Omega \subset \mathbb{C}^N$ and using convolution. Then one gets a weak limit

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k = \lim_{\nu \to +\infty} dd^c u_{1,\nu} \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k \geq 0. \tag{2.2}$$

Such products can be shown to be continuous by taking monotone limits of bounded psh functions $u_{j,\nu}$. However, there is no such continuity for arbitrary weak limits $u_{j,\nu} \to u_j$. The next step is to deal with non necessarily bounded potentials.

2.B. Non pluripolar Monge-Ampère products. Let X be a normal compact complex space endowed with a fixed Kähler form $\omega_0 = dd^c\phi_0$ and let $V := \int_X \omega_0^n$. We denote by $\mathcal{P}(X, \omega_0)$ be the set of ω_0-psh potentials, namely $\phi = \phi_0 + \psi$ such that $dd^c\phi = \omega_0 + dd^c\psi \geq 0$. The functions $\psi_\nu := \max\{\psi, -\nu\}$ are again ω_0-psh and bounded for all $\nu \in \mathbb{N}$. The Monge-Ampère measures $(\omega_0 + dd^c\psi_\nu)^n$ are therefore well-defined in the sense of Bedford-Taylor, with

$$\int_X (\omega_0 + dd^c\psi_\nu)^n = V = \int_X \omega_0^n.$$

By [BT87], the positive measures $\mu_\nu := \mathbf{1}_{\{\psi > -\nu\}}(\omega_0 + dd^c\psi_\nu)^n$ satisfy

$$\mathbf{1}_{\{\psi > -\nu\}}\mu_{\nu+1} = \mu_\nu,$$

and in particular $\mu_\nu \leq \mu_{\nu+1}$. As in [BEGZ10], we will say that ψ has *full Monge-Ampère mass* if the total mass of μ_ν converges to V, i.e.

$$\lim_{\nu \to \infty} \mu_\nu(X) = \lim_{\nu \to \infty} \int_{\{\psi > -\nu\}} (\omega_0 + dd^c \max\{\psi, -\nu\})^n = V.$$

In that case one sets $(\omega_0 + dd^c\psi)^n := \lim_{\nu \to +\infty} \mu_\nu$, which is thus a positive measure on X with mass V.

More generally, according to [GZ07], for given Kähler classes $\{\omega_1\}, \ldots, \{\omega_p\}$ (say $\omega_j = dd^c\phi_{0,j}$), and arbitrary $\phi_j = \phi_{0,j} + \psi_j \in \mathcal{P}(X, \omega_j)$, $1 \leq j \leq p$, the positive current

$$T = \langle(\omega_1 + dd^c\psi_1) \wedge ... \wedge (\omega_p + dd^c\psi_p)\rangle \tag{2.3}$$

is also well-defined as the monotone limit of

$$T_\nu = \mathbf{1}_{\bigcap\{\psi_j > -\nu\}}(\omega_1 + dd^c \max\{\psi_1, -\nu\}) \wedge ... \wedge (\omega_p + dd^c \max\{\psi_p, -\nu\})$$

as $\nu \to +\infty$. It depends continuously on the ψ_j's when the latter converge monotonically. By [BT82], the coefficients of $T = \langle(\omega_1 + dd^c\psi_1) \wedge ... \wedge (\omega_p + dd^c\psi_p)\rangle$ carry zero mass on all pluripolar sets, and by [BEGZ10, Théorème 1.8], $T = \lim T_\nu$ is a closed current. This is not a priori trivial since the T_ν's are not closed; the idea is similar to the technique introduced in Skoda [Sko82], El Mir [ElM84] and Sibony [Sib85]. This limit T is called the *non pluripolar product* of the currents $\omega_j + dd^c\psi_j$. If $\psi'_j \in \mathcal{P}(X, \omega_j)$ is less singular than ψ_j in the sense that $\psi'_j \geq \psi_j + \mathrm{Const}$, it is easy to show that

$$\int_X \langle(\omega_1 + dd^c\psi_1) \wedge ... \wedge (\omega_p + dd^c\psi_p)\rangle \wedge \alpha \leq \int_x \langle(\omega_1 + dd^c\psi'_1) \wedge ... \wedge (\omega_p + dd^c\psi'_p)\rangle \wedge \alpha$$

whenever $\alpha \geq 0$ is a smooth closed $(n-p, n-p)$-form on X, and one could say that the p-tuple $(\psi_1, \ldots, \psi_p)$ has full Monge-Ampère mass if the closed positive current $\langle(\omega_1 + dd^c\psi_1) \wedge ... \wedge (\omega_p + dd^c\psi_p)\rangle$ actually represents the cup-product cohomology class $\{\omega_1\} \ldots \{\omega_p\}$ in $H^{p,p}_{\partial\bar\partial}(X)$. One denotes by

$$\mathcal{P}_{\mathrm{full}}(X, \omega_0) \subset \mathcal{P}(X, \omega_0)$$

the set of ω_0-potentials ϕ with full Monge-Ampère mass $(\omega_0 + dd^c\psi)^n$. In a related way, one can introduce the spaces

$$\mathcal{T}(X, \omega_0) = \mathcal{P}(X, \omega_0)/\mathbb{R}, \quad \mathcal{T}_{\mathrm{full}}(X, \omega_0) = \mathcal{P}_{\mathrm{full}}(X, \omega_0)/\mathbb{R} \tag{2.4}$$

of currents $T = \omega_0 + dd^c\psi$ in the coholomogy class $\{\omega_0\} \in H^{1,1}_{\partial\bar\partial}(X)$ (resp. the subspace of currents with full Monge-Ampère measure). One can then define a Monge-Ampère operator with values in the space of probability measures of X

$$\mathcal{T}_{\mathrm{full}}(X, \omega_0) \longrightarrow \mathcal{M}(X), \quad T \longmapsto \mathrm{MA}(T) := V^{-1}\langle T^n\rangle. \tag{2.5}$$

It should be strongly emphasized that for $n \geq 2$ this operator is *not continuous* in the weak topology of $\mathcal{T}_{\mathrm{full}}(X, \omega_0)$ (and the corresponding weak topology of $\mathcal{M}(X)$). Another important fact is that potentials with full Monge-Ampère mass must have zero Lelong numbers (essentially, the argument is that otherwise these Lelong numbers would create mass on analytic sets, which are pluripolar).

Proposition 2.6 ([BBEGZ15]). — *Let $\phi \in \mathcal{P}_{\mathrm{full}}(X, \omega_0)$ and let $\pi : \tilde{X} \to X$ be any resolution of singularities of X. Then $\tilde\phi := \phi \circ \pi$ has zero Lelong numbers. Equivalently, $e^{-\tilde\psi} \in L^p(\tilde{X})$ for all $p < +\infty$.*

By using analytic Zariski decomposition (cf. [Dem92, Bouc04]), non pluripolar products can be extended to the case of big cohomology classes, i.e. classes $A \in H^{1,1}_{\partial\bar\partial}(X)$ containing a Kähler current $T_0 = \theta_0 + dd^c\psi \geq \varepsilon\omega_0$. In this context, the main results on non pluripolar Monge-Ampère operators can be summarized as follows (cf. [BEGZ10]).

Theorem 2.7. — *Let $A \in H^{1,1}(X, \mathbb{R})$ be a big class on a compact Kähler manifold X. If μ is a positive measure on X that puts no mass on pluripolar subsets and satisfies the compatibility condition $\mu(X) = \mathrm{Vol}(A)$, then there exists a unique closed positive $(1,1)$-current $T \in A$ such that*

$$\langle T^n\rangle = \mu.$$

The proof of Theorem 2.7 consists in reducing the situation to the Kähler case *via* approximate Zariski decomposition. Uniqueness is obtained by adapting the proof of S. Dinew [Din09] (which also deals with the Kähler class case).

When the measure μ satisfies some additional regularity condition, the authors show how to adapt Kołodziej's pluripotential theoretic approach to the sup-norm *a priori* estimates [Kol05] to get *global* information on the singularities of T.

PROPOSITION 2.8. — *Assume that the measure μ in Theorem 2.7 furthermore has $L^{1+\varepsilon}$ density with respect to Lebesgue measure for some $\varepsilon > 0$. Then the solution $T \in A$ to $\langle T^n \rangle = \mu$ has* minimal singularities.

Currents with minimal singularities were introduced in [DPS01]. For any pseudoeffective class $A \in H^{1,1}_{\partial\bar\partial}(X)$ (i.e. any class $\{\theta_0\}$ containing at least one positive current), one can obtain them by considering an upper regularized envelope:

$$T_{\min} = \theta_0 + dd^c\psi_{\min}, \qquad \psi_{\min}(x) := \Big(\sup_\psi \big\{\psi(x)\,;\ \psi \le 0 \text{ and } \theta_0 + dd^c\psi \ge 0\big\}\Big)^*.$$

When A is a Kähler class, the positive currents $T \in A$ with minimal singularities are exactly those with *locally bounded* potentials. When A is merely big all positive currents $T \in A$ will have poles in general, and the minimal singularity condition on T essentially says that T has the least possible poles among all positive currents in A. Currents with minimal singularities have in particular locally bounded potentials on the *ample locus* $\mathrm{Amp}(A)$ of A, which is roughly speaking the largest Zariski open subset where A locally looks like a Kähler class. Regarding local regularity properties, the following result can be obtained.

PROPOSITION 2.9. — *In the setting of Theorem 2.7, assume that μ is a smooth strictly positive volume form. Assume also that A is* nef. *Then the solution $T \in A$ to the equation $\langle T^n \rangle = \mu$ is C^∞ on $\mathrm{Amp}(A)$.*

One can likewise consider Monge-Ampère equations of the form

$$\langle (\theta_0 + dd^c\psi)^n \rangle = e^\psi dV \tag{2.10}$$

where θ_0 is a smooth representative of a big cohomology class A, ψ is a θ_0-psh function and dV is a smooth positive volume form. One can show that (2.10) admits a unique solution ψ such that $\int_X e^\psi dV = \mathrm{Vol}(A)$. Theorem 2.8 then shows that ψ has minimal singularities, and in the easier case of varieties of general type, one obtains as a special case:

THEOREM 2.11. — *Let X be a smooth projective variety of general type. Then X admits a unique singular Kähler-Einstein volume form of total mass equal to $\mathrm{Vol}(K_X)$. In other words the canonical bundle K_X can be endowed with a unique non-negatively curved metric $e^{-\phi_{KE}}$ whose curvature current $dd^c\phi_{KE}$ satisfies*

$$\langle (dd^c\phi_{KE})^n \rangle = e^{\phi_{KE}} \tag{i}$$

and such that

$$\int_X e^{\phi_{KE}} = \mathrm{Vol}(K_X). \tag{ii}$$

The weight ϕ_{KE} furthermore has minimal singularities.

Remark 2.12. — In [Cao14, Dem15], a slightly more elaborate concept of positive Monge Ampère products $\langle(\theta_1+dd^c\psi_1)\wedge\ldots\wedge(\theta_p+dd^c\psi_p)\rangle$ is introduced for arbitrary pseudoeffective classes. It is defined by means of the Bergman kernel approximation technique of [Dem92, Bouc04], and has the property of neglecting Monge-Ampère masses only on the analytic sets associated with the positive Lelong numbers of the potentials ψ_j. Therefore, this product is cohomologically "more comprehensive" and larger than the non pluripolar product (which a priori neglects all pluripolar sets). The general definition of the numerical dimension of a current and the study of the abundance conjecture seem to require such a generalization, although it is not needed here.

3. RESULTS INVOLVING FINITE ENERGY CURRENTS

3.A. Functions and currents of finite energy. Let $A = \{\omega_0\}$ be a Kähler class, $\omega_0 = dd^c\phi_0$. Following [BBEGZ15], one introduces for any $p \in [1, +\infty[$ the space

$$\mathcal{E}^p(X,\omega_0) := \left\{\phi = \phi_0 + \psi \in \mathcal{P}_{\mathrm{full}}(X,\omega_0)\,;\ \int_X |\psi|^p\,\mathrm{MA}(\omega_0 + dd^c\psi) < +\infty\right\}, \tag{3.1}$$

and say that functions $\psi \in \mathcal{E}^p(X,\omega_0)$ have *finite $\mathcal{E}^p$-energy.* The class $\mathcal{E}^1(X,\omega_0)$ ($p = 1$) is the most important in this context. One denotes by

$$\mathcal{T}^p(X,\omega_0) \subset \mathcal{T}^p_{\mathrm{full}}(X,\omega_0)$$

the corresponding set of *currents with finite $\mathcal{E}^p$-energy*, which can be identified with the quotient space

$$\mathcal{T}^p(X,\omega_0) = \mathcal{E}^p(X,\omega_0)/\mathbb{R} \quad \text{via} \quad \phi \mapsto dd^c\phi = \omega_0 + dd^c\psi.$$

(It is important to note that $\mathcal{T}^p(X,\omega_0)$ is *not* a closed subset of $\mathcal{T}(X,\omega_0)$ for the weak topology). From these definitions, the following fact is not very hard to check.

Theorem 3.2. — *All functionals E, L, I, J, J^*, D, H, M have a natural extension to arguments $\phi, \phi_0 \in \mathcal{E}^1(X,\omega_0)$, and I, J, J^*, D, H, M descend to $\mathcal{T}^1(X,\omega_0) = \mathcal{E}^1(X,\omega_0)/\mathbb{R}$.*

3.B. Measures of finite energy. As in [BBGZ13], one defines the *energy* of a probability measure μ on X (with respect to $\omega_0 = dd^c\phi_0$) as the Legendre-Fenchel transform

$$E_0^*(\mu) := \sup_{\phi=\phi_0+\psi\in\mathcal{E}^1(X,\omega_0)} \left(E_0(\psi) - \int_X \psi\,\mu\right) \in [0,+\infty] \tag{3.3}$$

where $E_0(\psi)$ means here $E_{\phi_0}(\phi_0+\psi)$ in the notation of Section 1. This defines a convex lower semicontinuous function $E_0^* : \mathcal{M}(X) \to [0,+\infty]$, and a probability measure μ is said to have *finite energy* if $E_0^*(\mu) < +\infty$. We denote the set of probability measures with finite energy by

$$\mathcal{M}^1(X,\omega_0) := \{\mu \in \mathcal{M}(X) \mid E_0^*(\mu) < +\infty\}\,. \tag{3.4}$$

It follows from well known facts of pluripotential theory (see e.g. [BBGZ13, Corollaire 2.11]) that every pluripolar set S is contained in the poles of a potential in $\mathcal{E}^1(X,\omega_0)$, hence every measure $\mu \in \mathcal{M}^1(X,\omega_0)$ has mass $\mu(S)=0$ on pluripolar sets.

Theorem 3.5 ([BBGZ13, Theorem 4.7]). — *The map $T=\omega_0+dd^c\psi \mapsto V^{-1}\langle T^n\rangle$ is a bijection between $\mathcal{T}^1(X,\omega_0)$ and $\mathcal{M}^1(X,\omega_0)$ (but it is not continuous with respect to weak convergence).*

3.C. The strong topology of currents with finite energy. With respect to the weak topology of currents, compactness in $\mathcal{T}^1(X,\omega_0)$ is easy to obtain: any set of currents with uniformly bounded energy is weakly compact. The drawback of weak topology is that the Monge-Ampère operator is *not* weakly continuous as soon as $n \geq 2$. In order to overcome this difficulty, [BBEGZ15] has introduced the following "strong topologies" on $\mathcal{T}^1X,\omega_0)$ and $\mathcal{M}^1(X,\omega_0)$. This topology has been studied further in [Dar15].

Definition 3.6. — *The* strong topology *on $\mathcal{T}^1(X,\omega_0)$ and $\mathcal{M}^1(X,\omega_0)$ are respectively defined as the coarsest refinement of the weak topology such that the functionals J and E_0^* become continuous.*

With this ad hoc strong topology, as could be expected, one gets

Proposition 3.7 ([BBEGZ15, Proposition 2.6]). — *The map*

$$T=\omega_0+dd^c\psi \mapsto V^{-1}\langle T^n\rangle$$

is a homeomorphism between $\mathcal{T}^1(X,\omega_0)$ and $\mathcal{M}^1(X,\omega_0)$.

3.D. Weak geodesics and convexity. Guedj conjectured that the completion of the space $\mathcal{P}(X,\omega_0)$ of smooth potentials equipped with the Mabuchi metric (1.2) precisely consists of the space $\mathcal{E}^2(X,\omega_0)$ of potentials of finite $\mathcal{E}^2$-energy (cf. [Gue14]). This has been shown by Darvas [Dar14, Dar15].

Let $\omega(0)=\omega_0+dd^c\psi^0$, $\omega(1)=\omega_0+dd^c\psi^1 \in \mathcal{T}^2(X,\omega_0)$ be currents with continuous potentials (so they even lie in $\mathcal{T}^\infty(X,\omega_0)$). Let $S\subset\mathbb{C}$ be the open strip $0<\mathrm{Re}\,t<1$ and let $\psi(z,t)$ be the upper semicontinuous regularization of the envelope of the family of all continuous functions $\varphi(z,t)$ that are $\mathrm{pr}_1^*\omega_0$-psh on $X\times\overline{S}$ and such that $\varphi(z,t)\leq\psi^0(z)$ for $\mathrm{Re}\,t=0$ and $\varphi(z,t)\leq\psi^1(z)$ for $\mathrm{Re}\,t=1$. Setting $\psi^t(z):=\psi(z,t)$ and $\omega(t):=\omega_0+dd^c\psi^t$ we have by [BD12] and [Bern15, §2.2] the following statement.

Lemma 3.8. — *Let ψ be the ω_0-psh envelope defined above. Then:*

(i) *ψ is $\mathrm{pr}_1^*\omega_0$-psh and bounded on $X\times S$.*

(ii) *$(\mathrm{pr}_1^*\omega_0+dd^c\psi)^{n+1}=0$ on $X\times S$.*

(iii) *$t\mapsto\psi^t$ is Lipschitz continuous, and converges uniformly on X to ψ^0 (resp. ψ^1) as $\mathrm{Re}\,t\to 0$ (resp. $\mathrm{Re}\,t\to 1$).*

When dealing with Kähler forms on a non-singular variety X, (ii) gives the geodesic equation for the Mabuchi metric defined on the space of Kähler metrics, as was observed by Donaldson and Semmes. Therefore, we will call $(\omega(t))_{t\in[0,1]}$ the *weak geodesic* joining $\omega(0)$ to $\omega(1)$ (and will also call the function ψ the “weak geodesic” joining ψ^0 to ψ^1).

Lemma 3.9. — *Let ψ be a $\mathrm{pr}_1^*\omega_0$-psh function on $X \times S$, and set $\psi^t(z) := \psi(z,t)$, which is an ω_0-psh function unless $\psi^t \equiv -\infty$. Let us also set $\phi^t = \phi_0 + \psi^t$. Then*

(i) *$t \mapsto E_0(\psi^t) = E_{\phi_0}(\phi^t)$ is subharmonic on S, and so is $t \mapsto L(\phi^t)$ if $\omega_0 \in c_1(X,\Delta)$.*
(ii) *If ψ further satisfies* (i) *and* (ii) *of Lemma* 3.8*, then $t \mapsto E_0(\psi^t)$ is even harmonic on S.*

Proof. — The assertions for E are well-known in the smooth case, and the proof in the present context reduces to [BBGZ13, Proposition 6.2] by passing to a log resolution otherwise. The subharmonicity of $L(\phi^t)$ is deeper, and is a special case of Berndtsson's theorem on the plurisubharmonic variation of Bergman kernels [Bern06]. □

Combining these results we get the following crucial convexity property of the Ding functional along weak geodesics:

Lemma 3.10. — *Let $\omega(t) = dd^c\phi^t$, $t \in [0,1]$, be the weak geodesic joining two currents $\omega(0) = dd^c\phi^0$, $\omega(1) = dd^c\phi^1 \in \mathcal{T}^2(X,\omega_0)$ with continuous potentials and $\omega_0 \in c_1(X,\Delta)$. Then $t \mapsto D_{\phi_0}(\phi^t)$ is convex and continuous on* $[0,1]$.

Another fundamental result proved by Berndtsson and Berman [BeBe14] is the convexity of the Mabuchi functional on weak geodesics. The key ingredient is a local positivity property of weak solutions to the homogeneous Monge-Ampère equation on a product domain, whose proof again uses the plurisubharmonic variation of Bergman kernels.

Theorem 3.11 ([BeBe14]). — *With the same notation as in Lemma* 3.10*, the Mabuchi functional $t \mapsto M_{\phi_0}(\phi^t)$ is convex and continuous on* $[0,1]$.

3.E. Variational characterization of Kähler-Einstein metrics. In this section, we give after [BBEGZ15] a proof of the following generalization to log Fano pairs (X,Δ) of a result of Ding and Tian for Fano manifolds, assuming the absence of holomorphic vector fields. Here the Ding and Mabuchi functionals are taken relatively to a given Kähler metric $\omega_0 = dd^c\phi_0 \in A = c_1(X,\Delta)$, and we assume for simplicity that ϕ_0 is normalized so that $L(\phi_0) = 0$.

Proposition 3.12. — *For a current $\omega = dd^c\phi \in \mathcal{T}^1(X,A)$, the following conditions are equivalent.*

(i) *ω is a Kähler-Einstein metric for (X,Δ).*
(ii) *The Ding functional reaches its infimum at ϕ : $D_{\phi_0}(\phi) = \inf_{\mathcal{E}^1(X,A)/\mathbb{R}} D_{\phi_0}$.*
(iii) *The Mabuchi functional reaches its infimum at ϕ : $M_{\phi_0}(\phi) = \inf_{\mathcal{E}^1(X,A)/\mathbb{R}} M_{\phi_0}$.*

Proof. — The equivalence betwen (i) and (ii) is proved as in [BBGZ13, Theorem 6.6], which we summarize for completeness. To prove (ii)⇒(i), one introduces the ω_0-psh envelope Pu of a function u on X as the upper semicontinuous upper envelope of the family of all ω_0-psh functions ψ such that $\psi \le u$ on X (or $Pu \equiv -\infty$ if this family is empty). Given $v \in C^0(X)$ one sets for all $t \in \mathbb{R}$

$$\psi(t) := P(\psi + tv).$$

One has of course $\psi^0 = \psi$. On the one hand, $t \mapsto L(\phi + tv) = -\log \int_X e^{-(\phi+tv)}$ is concave by Hölder's inequality, and its right-hand derivative at $t = 0$ is easily seen to be given by

$$\int_X v\, e^{-\phi} \Big(\int_X e^{-\phi} \Big)^{-1} = e^{L(\phi)} \int_X v\, e^{-\phi},$$

see [BBGZ13, Lemma 6.1]. On the other hand, the differentiability theorem of [BeBo10] (applied in the present case to a resolution of singularities of X) shows that $t \mapsto E_0(\psi(t))$ is differentiable, with derivative at $t = 0$ given by

$$\frac{1}{V} \int_X v\,(\omega_0 + dd^c\psi)^n = \frac{1}{V} \int_X v\,(dd^c\phi)^n.$$

Since $\psi(t)$ belongs to $\mathcal{E}^1(X, \omega_0)$, (ii) shows that $L(\psi+tv)-E_0(\psi(t))$ achieves is minimum for $t = 0$, and hence

$$\frac{d}{dt}_{|t=0+} \left(L(\psi + tv) - E_0(\psi(t)) \right) \ge 0,$$

i.e.

$$e^{L(\phi)} \int_X v\, e^{-\phi} \ge V^{-1} \int_X v\,(dd^c\phi)^n.$$

Applying this to both v and $-v$ shows that $e^{L(\phi)}e^{-\phi} = V^{-1}(dd^c\phi)^n$, which means that $\omega = dd^c\phi$ is a Kähler-Einstein metric.

To prove (i)⇒(ii), we rely on the convexity of the Ding functional along weak geodesics. Let ω be any Kähler-Einstein metric. Since every ω_0-psh function on X is the decreasing limit of a sequence of continous ω_0-psh functions thanks to [EGZ15], it is enough to show that $D_{\phi_0}(\phi) \le D_{\phi_0}(\phi')$ for all $\phi' \in \mathcal{E}^1(X, A)$ with continuous potentials. Let $\omega(t) = dd^c\phi^t$, $t \in [0, 1]$, be the weak geodesic between $\omega(0) = \omega = dd^c\phi$ and $\omega(1) = \omega' = dd^c\phi'$. By Lemma 3.10, $t \mapsto D_{\phi_0}(\phi^t)$ is convex and continuous on $[0, 1]$. To get as desired that $D_{\phi_0}(\phi) \le D_{\phi_0}(\phi')$, it is thus enough to show that

$$\frac{d}{dt}_{|t=0_+} D_{\phi_0}(\phi^t) \ge 0, \tag{3.13}$$

which is proved exactly as in the last part of the proof of [BBGZ13, Theorem 6.6]. More specifically, by convexity with respect to $t \mapsto \phi^t$, the function $u_t := (\phi^t - \phi)/t$ decreases to a bounded function v, and the concavity of E yields

$$\frac{d}{dt}_{|t=0_+} E_0(\psi_t) \le V^{-1} \int_X v\,(dd^c\phi)^n.$$

On the other hand, monotone convergence shows that

$$\frac{d}{dt}_{|t=0+} L(\phi^t) = \int_X v\, e^{-\phi} = V^{-1} \int_X v\, (dd^c\phi)^n,$$

hence (3.13).

Finally, the equivalence between (ii) and (iii) is a consequence of Proposition 1.15. □

4. FURTHER RESULTS OBTAINED BY THE VARIATIONAL TECHNIQUE

4.A. Existence and uniqueness of Kähler-Einstein metrics. One says that the Mabuchi functional is *proper* if $M_{\phi_0}(\phi) \to +\infty$ as ϕ approaches the boundary of $\mathcal{P}_A/\mathbb{R}$, i.e. $J_{\phi_0}(\phi) \to +\infty$ (one could omit the dependence on ϕ_0 here by (1.11″), (1.13) and Prop. 1.16). This is usually called properness in the sense of Tian. The first main result of [BBEGZ15] is:

Theorem 4.1. — *Let X be a $\mathbb{Q}$-Fano variety with log terminal singularities.*

(i) *The identity component* $\mathrm{Aut}^0(X)$ *of the automorphism group of X acts transitively on the set of Kähler-Einstein metrics on X,*

(ii) *If the Mabuchi functional of X is proper, then* $\mathrm{Aut}^0(X) = \{1\}$ *and X admits a unique Kähler-Einstein metric, which is the unique minimizer of the Mabuchi functional in an appropriate space of finite energy metrics* (*cf. section* 3).

When X is non-singular, (i) is a classical result of S. Bando and T. Mabuchi [BM87]. The present variational proof is built in part on the work of B. Berndtsson [Bern15]. Point (ii) generalizes a result of W.Y. Ding and G. Tian (see [Tian00]), and relies (in the same way as in [Berm13]) on Proposition 3.12, plus a compactness argument.

It should be recalled that, when X is non-singular and $\mathrm{Aut}^0(X) = \{1\}$, a deep result of G. Tian [Tian97], strengthened in [PSSW08], conversely shows that the existence of a Kähler-Einstein metric implies the properness of the Mabuchi functional. A similar result is expected for singular varieties – this should be a consequence of [BBEGZ15] and of the recent work of Darvas-Rubinstein [DR15].

4.B. Ricci iteration. In their independent works [Kel09] and [Rub08], J. Keller and Y. Rubinstein investigated the dynamical system known as *Ricci iteration*, defined by iterating the inverse Ricci operator. The idea of considering Ricci iteration had been considered earlier by Nadel in [Nad95]. The second main result of [BBEGZ15] deals with the existence and convergence of Ricci iteration in the more general context of $\mathbb{Q}$-Fano varieties.

Theorem 4.2. — *Let X be a $\mathbb{Q}$-Fano variety with log terminal singularities.*

(i) *Given a smooth form $\omega_0 \in c_1(X)$, there exists a unique sequence of closed positive currents $\omega_j \in c_1(X)$ with continuous potentials on X, smooth on X_{reg}, and such that*
$$\mathrm{Ricci}(\omega_{j+1}) = \omega_j$$
on X_{reg} for all $j \in \mathbb{N}$.

(ii) *If we further assume that the Mabuchi functional of X is proper and let ω_{KE} be the unique Kähler-Einstein metric provided by Theorem 4.1, then $\lim_{j\to+\infty} \omega_j = \omega_{\mathrm{KE}}$ in the C^∞ topology on X_{reg}, and uniformly in $C^0(X)$ at the level of potentials.*

When X is non-singular, this result settles [Rub08, Conjecture 3.2], which was obtained in [Rub08, Theorem 3.3] under the more restrictive assumption that Tian's α-invariant satisfies $\alpha(X) > 1$ (an assumption that implies the properness of the Mabuchi functional). Building on a preliminary version of [BBEGZ15], a more precise version of Theorem 4.2 was obtained in [JMR16, Theorem 2.5] for Kähler-Einstein metrics with cone singularities along a smooth hypersurface of a non-singular variety.

4.C. Convergence of the Kähler-Ricci flow. When X is a $\mathbb{Q}$-Fano variety with log terminal singularities, the work of J. Song and G. Tian [ST09] shows that given an initial closed positive current $\omega_0 \in c_1(X)$ with continuous potentials, there exists a unique solution $(\omega_t)_{t>0}$ to the normalized Kähler-Ricci flow, in the following sense:

(i) For each $t > 0$, ω_t is a closed positive current in $c_1(X)$ with continuous potentials;

(ii) On $X_{\mathrm{reg}} \times]0, +\infty[$, ω_t is smooth and satisfies $\dot{\omega}_t = -\mathrm{Ricci}(\omega_t) + \omega_t$;

(iii) $\lim_{t\to 0_+} \omega_t = \omega_0$, in the sense that their local potentials converge in $C^0(X_{\mathrm{reg}})$.

The third main result of [BBEGZ15] studies the long time behavior of this normalized Kähler-Ricci flow, and provides a weak analogue for singular Fano varieties of G. Perelman's result on the convergence of the Kähler-Ricci flow on Kähler-Einstein Fano manifolds:

Theorem 4.3. — *Assume that the Mabuchi functional of X is proper, and denote by ω_{KE} its unique Kähler-Einstein metric. Then $\lim\limits_{t\to+\infty} \omega_t = \omega_{\mathrm{KE}}$ and $\lim\limits_{t\to+\infty} \omega_t^n = \omega_{\mathrm{KE}}^n$, both in the weak topology.*

When X is non-singular, the above result is weaker than Perelman's theorem, which yields convergence in C^∞-topology (see [SeT08]). On the other hand, the variational approach of [BBGZ13, BBEGZ15] is completely independent of Perelman's deep estimates, which seem at the moment out of reach on singular varieties.

5. A VARIATIONAL APPROACH TO THE YAU-TIAN-DONALDSON CONJECTURE

We describe here the main ideas of [BHJ15a, BBJ15] towards the solution of the Yau-Tian-Donaldson conjecture. The technique involves the variational approach and non-Archimedean counterparts of the functionals of Kähler geometry that were introduced in Section 1.

5.A. Test configurations. Let (X, A) be a ($\mathbb{Q}$-)polarized projective variety. Following Li-Xu [LX14], one usually assumes the total space of the test configuration is normal. Also, as in Donaldson's original definition, it is needed to consider the case where A may be an ample $\mathbb{Q}$-line bundle (one takes suitable powers when genuine line bundles have to be considered, e.g. to deal with $\mathbb{C}^*$ actions).

Definition 5.1. — *A test configuration $(\mathcal{X}, \mathcal{A})$ for (X, A) consists of the following data:*

(i) *a flat and proper morphism $\pi : \mathcal{X} \to \mathbb{C}$ of algebraic varieties; one denotes by $X_t = \pi^{-1}(t)$ the fiber over $t \in \mathbb{C}$.*
(ii) *a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{C}$;*
(iii) *an isomorphism $X_1 \simeq X$.*
(iv) *a $\mathbb{C}^*$-linearized ample line bundle $\mathcal{A}$ on $\mathcal{X}$; one puts $A_t = \mathcal{A}_{|X_t}$.*
(v) *an isomorphism $(X_1, A_1) \simeq (X, A)$ extending the one in* (iii).

Every $\mathbb{C}^*$-equivariant action on X induces a diagonal $\mathbb{C}^*$-action on $\mathcal{X} = X \times \mathbb{C}$, and hence a test configuration $(\mathcal{X}, \mathcal{A})$ with $\mathcal{A} = \mathrm{pr}_1^* L$. Such test configurations are called *product* test configurations. A product test configuration is *trivial* if the $\mathbb{C}^*$-action on (X, A) is trivial.

Since A is assumed to be very ample, there is an embedding $X \hookrightarrow \mathbb{P}(V)$ where $V := H^0(X, A)$ and $\mathbb{P}(V)$ denotes the projective space of hyperplanes of V. Every 1-parameter subgroup $\rho : \mathbb{C}^* \to \mathrm{GL}(V)$ induces a test configuration $(\mathcal{X}_\rho, \mathcal{A}_\rho)$ for (X, A). By definition, $\mathcal{X}_\rho$ is the Zariski closure in $\mathbb{P}(V) \times \mathbb{C}$ of the image of the closed embedding $X \times \mathbb{C}^* \hookrightarrow \mathbb{P}(V) \times \mathbb{C}^*$ mapping (x, t) to $(\rho(t)x, t)$. Note that ρ is trivial if and only if $(\mathcal{X}_\rho, \mathcal{A}_\rho)$ is, while $(\mathcal{X}_\rho, \mathcal{A}_\rho)$ is a product if and only if ρ preserves X. Conversely, it is easy to check that every ample test configuration $(\mathcal{X}, \mathcal{A})$ may be obtained as above.

5.B. Donaldson-Futaki invariants and K-stability. The exposition follows here essentially [Don05]. Write $N_m = h^0(X, mA)$ for $m \geq 1$. The Donaldson-Futaki invariant of an ample test configuration $(\mathcal{X}, \mathcal{A})$ for (X, A) describes the subdominant term in the asymptotic expansion of w_m / mN_m as $m \to \infty$, where $w_m \in \mathbb{Z}$ is the weight of the $\mathbb{C}^*$-action on the determinant $\det H^0(\mathcal{X}_0, m\mathcal{A}_0)$. A Riemann-Roch argument (cf. [BHJ15a, Lemma 3.1]) then yields:

Lemma 5.2. — *Let $\pi : (\mathcal{X}, \mathcal{A}) \to \mathbb{C}$ be a test configuration for (X, A), with compactification $\bar{\pi} : (\bar{\mathcal{X}}, \bar{\mathcal{A}}) \to \mathbb{P}^1$. For every $m \in \mathbb{N}$ large enough, one has*

$$w_m = \chi(\bar{\mathcal{X}}, m\bar{\mathcal{A}}) - N_m,$$

where χ stands for the Euler characteristic. In particular, w_m is a polynomial of m of degree at most $n+1$.

The arguments of the proof and more explicit calculations actually give the following consequence (see [Wang12] and [LX14, Example 3]).

Proposition 5.3. — *Let $\pi : (\mathcal{X}, \mathcal{A}) \to \mathbb{C}$ be a test configuration for (X, A),*

(i) *There is an asymptotic expansion*

$$\frac{w_m}{mN_m} = F_0 + m^{-1}F_1 + m^{-2}F_2 + \dots \ .$$

(ii) *The coefficient F_0 is given by*

$$F_0(\mathcal{X}, \mathcal{A}) = \frac{\left(\bar{\mathcal{A}}^{n+1}\right)}{(n+1)(A^n)}.$$

(iii) *If $\mathcal{X}$ is normal, the coefficient F_1 is given by*

$$-2F_1 = V^{-1}\left(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{A}}^n\right) + \bar{S}\, F_0(\mathcal{X}, \mathcal{A})$$

where $V := (A^n)$ and

$$\bar{S} := -n\,\frac{(K_X \cdot A^{n-1})}{(A^n)}.$$

coincides with the mean value of the scalar curvature $S(\omega)$ of any Kähler form $\omega \in c_1(A)$ (hence the chosen notation).

Definition 5.4. — *The* Donaldson-Futaki invariant *of the test configuration $(\mathcal{X}, \mathcal{A})$ is*

$$\mathrm{DF}(\mathcal{X}, \mathcal{A}) := -2F_1.$$

Definition 5.5. — *The polarized variety (X, A) is said to be K-stable if* $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq 0$ *for all normal test configurations, with equality if and only if $(\mathcal{X}, \mathcal{A})$ is trivial.*

The main motivation behind these definitions is the following

Generalized Yau-Tian-Donaldson conjecture 5.6.
Let (X, A) be a polarized variety. Then X admits a cscK metric (short hand for Kähler metric with constant scalar curvature) $\omega \in c_1(A)$ if and only if (X, A) is K-stable.

By elaborating further [Don01, AP06], it was proved by Stoppa [Sto09] that K-stability indeed follows from the existence of a cscK metric: [Sto09] deals with the case when X admits no non-trivial holomorphic vector fields; the general case has been considered by Mabuchi and an alternative general proof can be found in [Berm15]. In [BDL16], it is further proved that, for X smooth, the existence of a cscK metric implies a generalized form of properness (taking vector fields into account). At about the same time the K-stability was also obtained using an algebro-geometric argument in Codogni-Stoppa [CS16].

The main result of [CDS15] (see also [Tian15]) is a solution of the conjecture in the special case $A = -K_X$; in this case a cscK metric is the same as a Kähler-Einstein metric.

5.C. Duistermaat-Heckman measures and uniform K-stability. The Duistermaat-Heckman measure $\mathrm{DH}_{(X,A)}$ is the probability measure on $\mathbb{R}$ describing the asymptotic distribution as $m \to \infty$ of the (scaled) weights of the $\mathbb{C}^*$-action on $H^0(X, mA)$, counted with multiplicity, namely

$$\mathrm{DH}_{(X,A)} = \lim_{m\to\infty} \sum_{\lambda\in\mathbb{Z}} \frac{\dim H^0(X,mA)_\lambda}{\dim H^0(X,mA)}\, \delta_{\lambda/m}, \quad \delta_p := \text{Dirac measure at } p, \tag{5.7}$$

where $H^0(X, mA) = \bigoplus_{\lambda\in\mathbb{Z}} H^0(X, mA)_\lambda$ is the weight space decomposition. For each $p \in [1, \infty]$, the L^p-norm $\|(\mathcal{X}, \mathcal{A})\|_p$ of an ample test configuration $(\mathcal{X}, \mathcal{A})$ is defined as the L^p norm

$$\|(\mathcal{X}, \mathcal{A})\|_p = \left(\int_{\mathbb{R}} |\lambda - b(\mu)|^p \, d\mu(\lambda)\right)^{1/p} \quad \text{where} \quad b(\mu) = \int_{\mathbb{R}} \lambda \, d\mu(\lambda) \tag{5.8}$$

is the barycenter of $\mu = \mathrm{DH}_{(X,A)}$. Then (iii) asserts in particular that $\|(\mathcal{X}, \mathcal{A})\|_p = 0$ if and only if $(\tilde{\mathcal{X}}, \tilde{\mathcal{A}})$ is trivial. Following ideas originating in G. Székelyhidi's thesis (see also [Szé15]), and according to [Der14, BHJ15a], one introduces:

Definition 5.9. — (Székelyhidi [Szé15]) *The polarized variety (X, A) is said to be* L^p-uniformly K-stable *if there exists $\delta > 0$ such that* $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq \delta \, \|(\mathcal{X}, \mathcal{A})\|_p$ *for all normal test configurations.*

One can show that L^p-uniform K-stability can only hold for $p \leq \frac{n}{n-1}$. Theorem 0.7 together with the results of [CDS15] shows *in fine* that uniform K-stability is equivalent to K-stability, at least in the case of Fano manifolds with finite automorphism group.

5.D. The non-Archimedean approach. This subsection is essentially borrowed from the introduction of [BBJ15] and relies on the foundational material developed in [BHJ15a]. One assumes here that X is a Fano manifold and $A = -K_X$. A ray $(\phi_t)_{t\geq 0}$ in $\mathcal{P}_A$ corresponds to an S^1-invariant metric Φ on the pull-back of $-K_X$ to the product of X with the punctured unit disc $\mathbb{D}^*$. The ray is called *subgeodesic* when Φ is plurisubharmonic (psh for short). Denoting by F any of the functionals M, D or J, the asymptotic behavior of $F(\phi_t)$ as $t \to +\infty$ is well-understood whenever the corresponding metric Φ extends to a smooth metric on a test configuration $(\mathcal{X}, \mathcal{A})$ of (X, A). Indeed, one has

$$\lim_{t\to+\infty} \frac{F(\phi_t)}{t} = F^{\mathrm{NA}}(\mathcal{X}, \mathcal{A}), \tag{5.10}$$

where F^{NA} is the corresponding non-Archimedean functional introduced in [BHJ15a]. For $F = D$, this is a reformulation of a key technical step in [Berm15]. For $F = M$ or

J, this is established in [BHJ15b], but less precise formulations have appeared several times in the literature over the last two decades.

Denoting by $\mathrm{DF}(\mathcal{X}, \mathcal{A})$ the Donaldson-Futaki invariant of a normal test configuration $(\mathcal{X}, \mathcal{A})$, one has $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq M^{\mathrm{NA}}(\mathcal{X}, \mathcal{A}) \geq D^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})$. In this context, uniform K-stability means the existence of $\delta > 0$ such that $\mathrm{DF} \geq \delta J^{\mathrm{NA}}$, and this condition turns out to be equivalent to a lower bound $M^{\mathrm{NA}} \geq \delta J^{\mathrm{NA}}$ [BHJ15a]. The approach to Theorem 0.7 consists in establishing equivalences between Archimedean estimates and their non Archimedean counterparts:

(5.11) the Ding functional D is coercive, i.e. $D \geq \delta J - C$ on $\mathcal{P}_A$ for some $\delta, C > 0$;

(5.11^{NA}) $D^{\mathrm{NA}} \geq \delta J^{\mathrm{NA}}$ for some $\delta > 0$;

(5.12) the Mabuchi functional M is coercive, i.e. $M \geq \delta J - C$ on $\mathcal{P}_A$ for some $\delta, C > 0$;

(5.12^{NA}) $M^{\mathrm{NA}} \geq \delta J^{\mathrm{NA}}$ for some $\delta > 0$.

The implications (5.11) $\Longrightarrow$ (5.11^{NA}) and (5.12) $\Longrightarrow$ (5.12^{NA}) are immediate consequences of (5.10).

In a first purely algebro-geometric step, one establishes (5.12^{NA}) $\Longrightarrow$ (5.11^{NA}), the converse implication being trivial since $M^{\mathrm{NA}} \geq D^{\mathrm{NA}}$. This is accomplished by using the Minimal Model Program, very much in the same way as in [LX14].

The heart of the proof is the implication (5.11^{NA}) $\Longrightarrow$ (5.12). For this, one argues by contradiction, assuming that M is not coercive. Using a compactness argument inspired by Darvas and He [DH14] (itself relying on the energy-entropy compactness theorem in [BBEGZ15]), one produces a subgeodesic ray along which M has slow growth. As in [DH14], this ray does not lie in $\mathcal{P}_A$, but in the space $\mathcal{E}^1$ of metrics of finite energy, a space whose structure was recently clarified by Darvas [Dar15]. As in [DR15], to control the Mabuchi functional along the ray, one also uses a recent result by Berman and Berndtsson (see [BeBe14, CLP14]) to the effect that M is convex along geodesic segments (cf. Theorem 3.11).

Since the Ding functional D is dominated by the Mabuchi functional, D also has slow growth along the geodesic ray. If Φ happens to extend to a bounded metric on some test configuration $(\mathcal{X}, \mathcal{A})$ of $(X, -K_X)$, the slope of D at infinity is given by $D^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})$, and ($5.11^{\mathrm{NA}}$) yields a contradiction. In the general case, one can assume that Φ extends to a psh metric on the pullback of $-K_X$ to $X \times \Delta$, but the singularities along the central fiber may be quite complicated. Nevertheless, the slope of D at infinity can be analyzed using the multiplier ideals of $m\Phi$, $m \in \mathbb{N}$; these give rise to a sequence of test configurations to which one can apply the assumption (5.11^{NA}) and derive a contradiction. This step is quite subtle and involves some non-Archimedean analysis in the spirit of [BFJ08, BFJ12] in order to calculate the slope at infinity of the Ding functional.

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Jean-Pierre DEMAILLY

Université Grenoble Alpes
Institut Fourier
Laboratoire de Mathématiques
UMR 5582 du CNRS, CS 40700
100 rue des Maths
F–38610 Gières
E-mail : jean-pierre.demailly@univ-grenoble-alpes.fr