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Introduction

The notion of complex structure was introduced as a local substitute of algebraic structure on algebraic manifolds over complex numbers. It makes possible to apply powerful methods of complex analysis to the problems in algebraic geometry.

All compact complex manifolds of dimension one are projective and the complex structure uniquely defines the adherent algebraic structure. A similar statement is false already in dimension two by the existence of smooth, compact, non-projective complex surfaces. However, the gap between the two structures is not yet very big in dimension two.

From dimension three up, the class of complex compact manifolds becomes substantially larger than the class of projective manifolds. For instance C. Taubes [5] constructed three-dimensional complex manifolds with fundamental group isomorphic to an arbitrarily given finitely presented group, by use of twistor construction due to Penrose.

In this article I want to give a construction of complex structure which retains some algebraic flavor. We produce a complex structure on a real manifold which is transversal to an algebraic foliation on a complex projective manifold. The main result of the article is that any complex structure on a compact manifold can be realized in this way.

§1. Construction of complex structure via algebraic foliation

We begin with the description of our general construction.

Let X be a smooth complex projective manifold and F a holomorphic foliation on X. F is a coherent integrable subsheaf of the holomorphic tangent sheaf Θ_X. F may possess singularities along a closed subset of codimension two or more.

Let S be a real smooth submanifold in X.

Lemma 1. *Assume that S intersects F transversally, i.e.*

$$\Theta_{X|x} \oplus \overline{\Theta_{X|x}} = F_{|x} \oplus \overline{F_{|x}} \oplus (\mathbb{C} \otimes_{\mathbb{R}} T_{S,x})$$

at each point x of S, where $T_{s,x}$ stands for the tangent space at x ot S as a differentiable manifold. Then there is a natural complex structure on S induced by (X, F).

Proof. Put $T_{F,x}$ be the elements $\in F_{|x} \oplus \overline{F_{|x}}$ which is invariant under complex conjugation. The complex structure J on $T_{X,x}$ leaves $T_{F,x}$ stable because F is an analytic foliation. Hence J naturally induces an almost complex structure on $T_{X,x}/T_{F,x}$, which is identified with $T_{S,x}$. This almost complex structure is integrable. In fact, F is a smooth foliation around S and the Frobenius theorem tells us that X is analytically locally a product $\mathbb{C}^{n-r} \times \mathbb{C}^r$ where the fibers of the first projection are the leaves of F of rank r. Thus a local coordinate of S is given by that of $\mathbb{C}^{n-r}$. □

Remark 1. The complex structure stays invariant under small variation of the cycle S since the complex structure is given by the local fibring structure induced by the holomorphic foliation F. Variation of F, on the other hand, will induce non-trivial variation of complex structure of S.

Remark 2. In order to obtain a complex structure on S, we do not have to require S to lie on the smooth locus of F or to be everywhere transversal to F. The right condition is more subtle. By changing F so that the transversality condition fails at a limit, we can obtain a natural degeneration of the complex structure of S. This leads us to a somewhat nonstandard description of deformation and degeneration of complex structure.

We are going to prove that any complex structure on a compact real manifold can be obtained in this way.

This statement means in particular that, given a compact complex manifold, we can choose local coordinate functions such that they are solutions of algebraic differential equations. It is also true for a big class of open complex manifolds.

Let M be a complex manifold of dimension n. The notion of conjugate complex structure on M is always well-defined and the corresponding complex manifold will he denoted by M_c. This structure is obtained from the same covering of M by unit balls, but the holomorphic coordinates are $\overline{z}$ on M_c in place of z on M.

Take the product $M \times M_c$, which is a complex manifold of dimension $2n$. It is a self conjugate manifold with anti-holomorphic involution $i : (x, y) \mapsto (y, x)$. The "diagonal" Δ is defined to be the real points on $M \times M_c$, i.e. the fixed point set of the involution i. Δ is a real submanifold of real dimension $2n$.

By a theorem of Grauert [1], a small tubular neighborhood $U \subset M \times M_c$ of Δ is a Stein manifold. It has a natural holomorphic projection onto M and this fibring defines a holomorphic foliation F_U on U. The diagonal Δ is transversal to F_U and the induced complex structure on Δ coincides with that of M. We would like to embed U into a projective manifold in such a way that F_U is induced by the global algebraic foliation.

Remark 3. The manifold Δ has natural structure of an analytic manifold over real numbers which is obtained from the complex structure on M. A small variation of this structure makes it a component of real algebraic manifold.

We will reduce the problem of extending the holomorphic foliation above to a similar but simpler problem on holomorphic 1-forms. We need the notion of generic holomorphic 1-forms on a complex manifold.

Definition 1. A holomorphic 1-form ω on a complex manifold M is said to be *generic* at $p \in M$ if ω, $(d\omega)^n$, $\omega \wedge (d\omega)^{n-1}$ are non-vanishing at p.

Definition 2. An invertible subsheaf $L \subset \Omega^1_M$ on a compact complex manifold is said to be *generic* if at any point $p \in M$, a local basis of L is generic at p.

Note that the definition does not depend on the choice of local bases of L.

Lemma 2. *Let $L \subset \Omega^1_M$ be a generic invertible subsheaf on a $2n$-dimensional complex manifold X. Then L defines a holomorphic foliation F of rank n.*

Proof. Let ω be a local generator of L at p. The 1-form ω, which is generic at p, has a standard local expression $\sum_{i=1}^n z_{2i-1}dz_{2i}$, by the classical Darboux lemma [3][4]. Then the coordinates with even subscripts define a local product structure on X and hence a holomorphic foliation F near p.

This foliation is uniquely determined by the conformal class of ω, or equivalently by L. Indeed the leaves of the foliation is defined as the completely integrable holonomic system given by the holomorphic vector fields satisfying $\omega(X) = 0$, $(\omega \wedge d\omega)(X) = 0$, so that we get the same equation after replacing ω by $f\omega$, see [3][4]. □

Let M be a compact complex manifold. Let $\Delta \subset U \subset M \times M_c$ and F_U be as above. Suppose that our initial manifold M is of dimension ≥ 2. Take a general homomorphism $\omega : \Theta_U/F_U \to \mathcal{O}_U$, ω is naturally a holomorphic 1-form on U, and $L = \mathcal{O}_U\omega$ is generic in the sense in Definition 2, with F_U being the associated foliation on U.

Realize the real algebraic manifold Δ as the real point set of an algebraic variety P. U is then viewed as an open subset in P. Let $j : U \hookrightarrow P$ be the embedding.

We may assume that the restriction map $\operatorname{Pic} P \to \operatorname{Pic} U$ is surjective and the sheaf F_U extends to a coherent sheaf on P. Furthermore, the embedding $F_U \hookrightarrow T_U$ is approximated by global algebraic maps.

Lemma 3. *Let D be a very ample divisor on P. Then ω is approximated by a sequence $\omega_i \in H^0(P, \Omega^1_P(iD))$, $i = 1, 2, \ldots$ which generate generic invertible subsheaves L_i of Ω^1_P.*

In particular, the complex structure of M is approximated by a sequence of complex structure induced from algebraic foliations on P.

Proof. The totally real submanifold Δ is away from D, and so is U. Hence any ω is approximated by a sequence of algebraic 1-forms with poles along D. On the other hand, the property of being generic is an open condition, and we can assume that $\mathcal{O}_P\omega_i$ is generic. □

Let $\mathcal{M}$ denote the set of the complex structure on M, and $\mathcal{M}_0$ the connected component which contains the original complex structure. The variation of the complex structures attached to the choice of ω_i is connected, since $\bigcup H^0(P, \Omega^1_P(iD))$ is a vector space and the genericity condition on ω_i is an open, algebraic condition. Hence the set of the complex structures $\mathcal{M}'_0$ which is obtained by foliations F' attached to generic algebraic invertible subsheaves $L \subset \Omega^1_P$ is dense in $\mathcal{M}_0$ which is finite-dimensional since M is compact. On the other hand, it is easy to see that a limit of algebraic foliations is again algebraic. Thus we have that $\mathcal{M}_0 = \mathcal{M}'_0$.

Thus we have arrived at the conclusion:

Theorem 4. *The complex structure on M is induced by an algebraic foliation on P.*

§2 Conclusion

The above result confirms once more that the structure of algebraic foliation is too rich to be described in purely algebraic terms. In particular, the result of Taubes may transfer some of the statement which says that some group cannot be realized as the fundamental group of algebraic manifolds into similar results on the leaves of algebraic foliations. It potentially provides us with a series of applications in analytic geometry and algebraic geometry which are comparable to the ones in the topology of 4-manifolds provided by gauge theory.

It is difficult to predict right now how useful this approach could be in the theory of complex manifolds, but at least it may help settle some fundamental problems, including, for example, the following question which I believe is of basic importance:

Problem. *Does any compact complex manifold contain an open Stein subset U, such that the complement is a CW-complex of a strictly smaller dimension?*

The answer is clearly affirmative for projective manifolds, complex tori and some non-algebraic complex surfaces. But even in the case of surfaces, we need precise knowledge on the structure of the surfaces to get the "Stein cell" U above.

I have raised this question more than twenty-five years ago and have not received any serious answer yet. I am now much less sure about it now than when I was a student (Taubes threefolds may actually give a counterexample), but I still view it as a basic problem in complex analytic geometry.

Our realization of a given complex manifold as a real cycle M on an algebraic manifold X may also have potential application to algebraic geometry, I have in mind a construction of counterexamples to the Hodge conjecture and similar problems of general character. For instance, if M is not algebraic and if we know that the cohomology class of the cycle M is of type (p, p), it might be a potential counterexample to the Hodge conjecture.

The missing ingredient here is a natural cohomological correspondence between the cycles transversal to a given holomorphic foliation. Such a correspondence exists on the level of subrings generated by the Chern classes of the normal bundle, thanks to a modified version of the Bott-Baum theorems on the holomorphic foliations with singularities.

Many examples give evidence, for a general hypothesis that an algebraic foliation will induce general correspondence between the cohomology groups of generically transversal cycles.

The class of three-dimensional complex manifolds contains lot of manifolds with non-trivial cohomological operations. We can easily produce simply-connected complex manifolds of dimension six or greater with non-trivial cohomological operations, by using, say, symmetric power construction and desingularization. It would be natural to expect that the properties of simply connected manifolds will better survive the above correspondence, and that such cohomological operation will provide an obstruction for a (p, p)-cycle to be algebraic. This speculation gives a hint that the first natural codimension for the simplest counterexample to Hodge conjecture would be six.

The idea of foliations with transversal structure seems to have been known for a long time, but it was developed mostly in the work of A. Haefliger [2].

The construction above was discovered by the author in the late 70's, but it lacked the final step which reduces the extension of foliations to the extension of 1-forms, and therefore was incomplete at the time.

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