Singular hermitian metrics on positive line bundles

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Abstract. — *The notion of a singular hermitian metric on a holomorphic line bundle is introduced as a tool for the study of various algebraic questions. One of the main interests of such metrics is the corresponding* L^2 *vanishing theorem for* $\overline{\partial}$ *cohomology, which gives a useful criterion for the existence of sections. In this context, numerically effective line bundles and line bundles with maximum Kodaira dimension are characterized by means of positivity properties of the curvature in the sense of currents. The coefficients of isolated logarithmic poles of a plurisubharmonic singular metric are shown to have a simple interpretation in terms of the constant* ε *of Seshadri's ampleness criterion. Finally, we use singular metrics and approximations of the curvature current to prove a new asymptotic estimate for the dimension of cohomology groups with values in high multiples* $\mathcal{O}(kL)$ *of a line bundle* L *with maximum Kodaira dimension.*

1. Introduction

Our purpose is to show that several important concepts of algebraic geometry have a nice interpretation in differential geometric terms, once we admit hermitian metrics with singularities, and especially plurisubharmonic weights with *logarithmic poles.*

A *singular (hermitian) metric* on a line bundle L is simply a hermitian metric which is given in any trivialization by a weight function $e^{-\varphi}$ such that φ is locally integrable. We then have a well-defined *curvature current* $c(L) = \frac{i}{\pi}\partial\overline{\partial}\varphi$ and the case when $c(L) \geq 0$ as a current is especially interesting. One of the main reasons for this is the basic L^2 existence theorem of Hörmander-Andreotti-Vesentini for solutions of $\overline{\partial}$ equations with plurisubharmonic weights. With relatively few efforts, the L^2 theory gives strong vanishing theorems (of Kawamata-Viehweg type) and existence results for sections of the adjoint line bundle $K_X + L$; here K_X denotes the canonical line bundle of the base manifold X, and an additive notation is used for the group $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^\star)$. These techniques can also be applied in combination with the Calabi-Yau theorem to obtain explicit numerical criteria for very ample line bundles; we refer to [De 90] for results in this direction.

On a projective algebraic manifold X, the real vector space generated by the Neron-Severi group $H^2(X,\mathbb{Z})\cap H^{1,1}(X)$ contains two canonical closed convex cones $\Gamma_+\supset\Gamma_a$, which are generated by cohomology classes of effective or ample divisors, respectively. For a line bundle L on X, we show that L has a singular metric with positive curvature current if and only if $c_1(L)\in\Gamma_+$. We also give similar differential geometric descriptions of the line bundles for which $c_1(L)$ belongs to Γ_a (numerically effective line bundles), or to the open cone Γ_+° (line bundles with $\kappa(L)=n$), or else to $\Gamma_+^\circ\cap\Gamma_a$ (big and nef line bundles).

The well-known Seshadri ampleness criterion asserts that a line bundle L on a projective manifold X is ample if and only if there is a constant $\varepsilon>0$ such that $L\cdot C\ge\varepsilon\, m(C)$ for every curve $C\subset X$, where $m(C)$ is the maximum of the multiplicity of the singular points of C. We show that the optimal constant $\varepsilon(L)$ is precisely equal to the supremum of coefficients γ for which a plurisubharmonic weight on L may have an isolated logarithmic pole of slope γ at any point. This result is then refined by introducing "local" Seshadri constants $\varepsilon(L,x)$ which measure ampleness along curves passing though a fixed point x.

Finally we use approximation techniques for singular metrics, combined with the general holomorphic Morse inequalities of [De 85], to obtain an asymptotic upper bound for the dimensions of cohomology groups $H^q(X,kL)$ when L is a line bundle of maximum Kodaira dimension $\kappa(L)=n$ and k tends to $+\infty$. If some multiple of L is written as $mL\simeq\mathcal{O}(A+D)$ where A (resp. D) is an ample (resp. effective) divisor, the upper bound is expressed in a simple way in terms of the first Chern class $c_1(L)$, the multiplicities of the singular points of D and the curvature of the tangent bundle TX.

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2. Notion of singular hermitian metrics

Let L be a holomorphic line bundle over a complex manifold X. We are mostly interested in the case of a compact manifold, but this restriction is irrelevant in the present section.

Definition 2.1. — *A singular (hermitian) metric on L is a metric which is given in any trivialization $\theta: L_{\restriction\Omega}\xrightarrow{\simeq}\Omega\times\mathbb{C}$ by*

$$\|\xi\|=|\theta(\xi)|\,e^{-\varphi(x)},\qquad x\in\Omega,\ \xi\in L_x$$

where $\varphi\in L^1_{\rm loc}(\Omega)$ is an arbitrary function, called the weight of the metric with respect to the trivialization θ.

If $\theta':L_{\restriction\Omega'}\longrightarrow\Omega'\times\mathbb{C}$ is another trivialization, φ' the associated weight and $g\in\mathcal{O}^\star(\Omega\cap\Omega')$ the transition function, then $\theta'(\xi)=g(x)\,\theta(\xi)$ for $\xi\in L_x$, and so

$\varphi' = \varphi + \log|g|$ on $\Omega \cap \Omega'$. The curvature form of L is then given by the closed $(1,1)$-current $c(L) = \frac{i}{\pi}\partial\overline{\partial}\varphi$ on Ω, if we compute formally $c(L) = \frac{i}{2\pi}D^2$ as in the smooth case ; $c(L)$ is, of course, a global current on X which is independent of the choice of trivializations. Our assumption $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ guarantees that $c(L)$ exists in the sense of distribution theory. Then the De Rham cohomology class of $c(L)$ is the image of the first Chern class $c_1(L) \in H^2(X,\mathbb{Z})$ in $H^2_{DR}(X,\mathbb{R})$ (De Rham cohomology can be computed either by means of smooth differential forms or by means of currents). Before going further, we discuss two basic examples.

Example 2.2. — Let $D = \sum \alpha_j D_j$ be a divisor with coefficients $\alpha_j \in \mathbb{Z}$ and let $L = \mathcal{O}(D)$ be the associated invertible sheaf of meromorphic functions f such that $\mathrm{div}(f) + D \geq 0$; the corresponding line bundle can be equipped with the singular metric defined by $||f|| = |f|$. If g_j is a generator of the ideal of D_j on an open set $\Omega \subset X$ then $\theta(f) = f \prod g_j^{\alpha_j}$ defines a trivialization of $\mathcal{O}(D)$ over Ω, thus our singular metric is associated to the weight $\varphi = \sum \alpha_j \log|g_j|$. By the Lelong-Poincaré equation, we find

$$c\big(\mathcal{O}(D)\big) = \frac{i}{\pi}\partial\overline{\partial}\varphi = [D], \tag{2.3}$$

where $[D] = \sum \alpha_j [D_j]$ denotes the current of integration over D (cf. [Le 57] and [Le 69]).

Example 2.4. — Assume that $\sigma_1, \ldots, \sigma_N$ are non zero holomorphic sections of L. Then we can define a natural (possibly singular) hermitian metric on $L^\star$ by

$$||\xi^\star||^2 = \sum_{1 \leq j \leq n} \big|\xi^\star.\sigma_j(x)\big|^2 \quad \text{for} \quad \xi^\star \in L^\star_x.$$

The dual metric on L is given by

$$||\xi||^2 = \frac{|\theta(\xi)|^2}{|\theta(\sigma_1(x))|^2 + \cdots + |\theta(\sigma_N(x))|^2}$$

with respect to any trivialization θ. The associated weight function is thus given by $\varphi(x) = \log\big(\sum_{1 \leq j \leq N} |\theta(\sigma_j(x))|^2\big)^{1/2}$. In this case φ is a plurisubharmonic function, so $c(L)$ is a (closed) positive current. □

It is worth observing that the weight functions φ have logarithmic poles in both examples. In the second case, the set of poles is the base locus $\bigcap \sigma_j^{-1}(0)$ of the linear system generated by the sections $\sigma_1, \ldots, \sigma_N$. In the sequel, we always suppose that the curvature current $c(L)$ is positive, i.e. that the weight functions φ are plurisubharmonic.

Definition 2.5. — *A singular metric on L with positive curvature current $c(L) \geq 0$ is said to have a logarithmic pole of coefficient γ at a point $x \in X$ if the Lelong number*

$$\nu(\varphi, x) = \liminf_{z \to x} \frac{\varphi(z)}{\log|z - x|}$$

is non zero and if $\nu(\varphi, x) = \gamma$.

For the basic properties of Lelong numbers, we refer to [Le 69], [Siu 74] and [De 87]. It is well known that $\nu(\varphi,x)$ is always equal to the Lelong number of the associated current $T=\frac{i}{\pi}\partial\overline\partial\varphi$, defined by $\nu(T,x)=\lim_{r\to 0_+}\nu(T,x,r)$ with

$$\nu(T,x,r)=\frac{1}{(2\pi r^2)^{n-1}}\int_{B(x,r)}T(z)\wedge(i\partial\overline\partial|z|^2)^{n-1}. \tag{2.6}$$

Finally, for every $c>0$, we consider the sublevel sets

$$E_c(T)=\big\{x\in X\,;\,\nu(T,x)\ge c\big\}. \tag{2.7}$$

By a theorem of [Siu 74], $E_c(T)$ is a (closed) analytic subset of X. If $T=\frac{i}{\pi}\partial\overline\partial\varphi$ on an open set $\Omega\subset X$, we denote accordingly $E_c(\varphi)=E_c(T)\cap\Omega$. The following simple lemma is very useful in this context.

Lemma 2.8. — *If φ is a plurisubharmonic function on X, then $e^{-2\varphi}$ is integrable in a neighborhood of x as soon as $\nu(\varphi,x)<1$ and non integrable as soon as $\nu(\varphi,x)\ge n$.*

Proof. — If $\nu(\varphi,x)=\gamma$, the usual convexity properties of plurisubharmonic functions show that

$$\varphi(z)\le\gamma\log|z-x|+O(1)\quad\text{near }\ x, \tag{2.9}$$

thus $e^{-2\varphi(z)}\ge C|z-x|^{-2\gamma}$ is non integrable as soon as $\gamma\ge n$. For a proof that $e^{-2\varphi}$ is integrable when $\nu(\varphi,x)<1$, we refer to Skoda [Sk 72]. Both bounds are best possible as the examples $\varphi(z)=(n-\varepsilon)\log|z|$ and $\varphi(z)=\log|z_1|$ in $\mathbb{C}^n$ easily show; in the first case $\nu(\varphi,0)=n-\varepsilon$ but $e^{-2\varphi}$ is integrable at 0; in the second case $\nu(\varphi,0)=1$ but $e^{-2\varphi}$ is non integrable at 0. □

3. L^2 vanishing theorem and criterion for the existence of sections

One of the main reasons for which singular metrics are especially interesting is the powerful existence theorem of Hörmander for solutions of equations $\overline\partial u=v$.

Theorem 3.1. — *Suppose that X is a Stein or compact projective manifold equipped with a Kähler metric ω. Let L be a holomorphic line bundle equipped with a singular metric associated to plurisubharmonic weight functions φ such that $c(L)\ge\varepsilon\,\omega$ for some $\varepsilon>0$. For every $q\ge 1$ and every (n,q) form v with values in L such that $\overline\partial v=0$ and $\int_X|v|^2e^{-2\varphi}dV_\omega<+\infty$, there is a $(n,q-1)$-form u with values in L such that $\overline\partial u=v$ and*

$$\int_X|u|^2e^{-2\varphi}dV_\omega\le\frac{1}{2\pi q\varepsilon}\int_X|v|^2e^{-2\varphi}dV_\omega.$$

Here dV_ω stands for the Kähler volume element $\omega^n/n!$ and $|u|^2e^{-2\varphi}$ denotes somewhat abusively the pointwise norm of $u(z)$ at each point $z\in X$, although φ is only defined on an open set in X. The constant 2π comes from the fact that we have included 2π in the definition of $c(L)$.

Proof. — The result is standard when X is Stein and L is the trivial bundle (see [AV 65] and [Hö 66]); the proof can then be reduced to the case of a smooth metric, because any plurisubharmonic function is the limit of a decreasing sequence of smooth plurisubharmonic functions. In general, there exists a hypersurface $H \subset X$ such that $X \setminus H$ is Stein and L is trivial over $X \setminus H$. We then solve the equation $\overline{\partial} u = v$ over $X \setminus H$ and observe that the solution extends to X thanks to the L^2 estimate (cf. [De 82], lemma 6.9). □

From this general theorem, we can easily derive an abstract vanishing theorem for the adjoint line bundle $K_X + L$ and a criterion for the existence of sections of $K_X + L$.

Corollary 3.2. — *Let L be a line bundle with a singular hermitian metric. Assume that $c(L) \geq \varepsilon\,\omega$ and that the metric (i.e. the weight $e^{-2\varphi}$) is integrable near all but finitely many points of X . Then $H^q(X, K_X + L) = 0$ for $q \geq 1$.*

Proof. — Let $x_1, \ldots, x_m$ be the points where the metric is not integrable and let v be a smooth (n,q)–form with values in L such that $\overline{\partial} v = 0$. Let u_j be a smooth solution of $\overline{\partial} u_j = v$ in a neighborhood of V_j of x_j and ψ_j a cut-off function with support in V_j such that $\psi_j(x_j) = 1$. Then $v' = v - \sum \overline{\partial}(\psi_j u_j)$ satisfies $\overline{\partial} v' = 0$ and $\int_X |v'|^2 e^{-2\varphi} d\sigma < +\infty$, because $e^{-2\varphi}$ is integrable except at the x_j's and v' vanishes in a neighborhood of x_j . By theorem 3.1, there is a L^2 solution u' of $\overline{\partial} u' = v'$, hence $u = u' + \sum \psi_j u_j$ is a solution of $\overline{\partial} u = v$. It is well-known that the existence of a L^2_{loc} solution implies the existence of a smooth one, whence the corollary. □

Corollary 3.3. — *Let L be a line bundle with a singular metric such that $c(L) \geq \varepsilon\,\omega$, $\varepsilon > 0$. If the weight function φ is such that $\nu(\varphi, x) \geq n + s$ at some point $x \in X$ which is an isolated point of $E_1(\varphi)$, then $H^0(X, K_X + L)$ generates all s-jets at x.*

Proof. — The proof is a straightforward adaptation of the Hörmander-Bombieri-Skoda technique ([Bo 70], [Sk 75]). We have $e^{-2\varphi(z)} \geq C|z - x|^{-2(n+s)}$ near x by (2.9), and in particular $e^{-2\varphi}$ is non integrable near x. Since x is supposed to be isolated in $E_1(\varphi)$, we infer from lemma 2.8 that $e^{-2\varphi}$ is locally integrable on a small punctured neighborhood $V \setminus \{x\}$. Let $P(z)$ be an arbitrary polynomial of degree $\leq s$ in given analytic coordinates $(z_1, \ldots, z_n)$ on V. Fix a smooth cut-off function χ with compact support in V such that $\chi = 1$ near x and a non vanishing local section $h \in H^0(V, K_X + L)$. Then $v = P\overline{\partial}\chi \otimes h$ can be viewed as a $\overline{\partial}$-closed $(n,1)$-form on X with values in L, such that $\int_X |v|^2 e^{-2\varphi} dV_\omega < +\infty$; indeed v is smooth, has compact support in V and vanishes near x. The solution u of $\overline{\partial} u = v$ is smooth and we have $|u(z)| = o(|z - x|^s)$ near x thanks to the L^2 estimate. Therefore

$$f = \chi P h - u \in H^0(X, K_X + L)$$

has the prescribed s-jet Ph at x. □

4. Numerical cones associated to positive line bundles

We suppose here that X is a projective algebraic manifold and denote $n = \dim X$. It is well known that an integral cohomology class in $H^2(X, \mathbb{Z})$ is the first Chern class of a holomorphic (or algebraic) line bundle if and only if this class is of type $(1,1)$. Hence the so-called *Neron-Severi* group

$$NS(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X) \subset H^2(X, \mathbb{R})$$

is the set of cohomology classes of algebraic line bundles (or of integral divisors).

DEFINITION 4.1. — *A holomorphic line bundle L over X is said to be numerically effective, nef for short, if $L \cdot C = \int_C c_1(L) \geq 0$ for every curve $C \subset X$, and in this case L is said to be big if $L^n = \int_X c_1(L)^n > 0$.*

If L is nef, it is well known that $L^p \cdot Y = \int_Y c_1(L)^p \geq 0$ for any p-dimensional subvariety $Y \subset X$ (see e.g. [Ha 70]). The Nakai-Moishezon ampleness criterion then shows that $L + A$ is ample as soon as A is ample. In fact, if A is ample, it is easy to see that L is nef if and only if $kL + A$ is ample for all integers $k \geq 1$. We are going to describe a simple dictionary relating these concepts to similar concepts in the context of differential geometry.

Let $NS_{\mathbb{R}}(X)$ be the real vector space $NS(X) \otimes \mathbb{R} \subset H^2(X, \mathbb{R})$ and let $\Gamma_+ \subset NS_{\mathbb{R}}(X)$, resp. $\Gamma_a \subset \Gamma_+$, be the closed convex cone generated by cohomology classes of effective (resp. ample) divisors D; denote by Γ_+° (resp. Γ_a°) the interior of Γ_+ (resp. Γ_a). We will call Γ_+ and Γ_a° respectively the *effective cone* and the *ample cone* of X. Finally, recall that the *Kodaira dimension* $\kappa(L)$ is the supremum of the rank of the canonical maps

$$\Phi_m : X \setminus Z_m \longrightarrow P(V_m^\star), \qquad x \longmapsto H_x = \{\sigma \in V_m \,;\, \sigma(x) = 0\}, \qquad m \geq 1$$

with $V_m = H^0(X, mL)$ and $Z_m = \bigcap_{\sigma \in V_m} \sigma^{-1}(0) =$ base locus of V_m. If $V_m = \{0\}$ for all $m \geq 1$, we set $\kappa(L) = -\infty$. Then we have $h^0(X, mL) \leq O(m^{\kappa(L)})$ for $m \geq 1$, and $\kappa(L)$ is the smallest constant for which this estimate holds.

PROPOSITION 4.2. — *If L is a holomorphic line bundle on X, ω a Kähler metric and $\varepsilon > 0$, we have the following equivalent properties:*

(a) $c_1(L) \in \Gamma_+ \Longleftrightarrow L$ has a singular metric with $c(L) \geq 0$;

(b) $c_1(L) \in \Gamma_+^\circ \Longleftrightarrow \exists\varepsilon,\ L$ has a singular metric with $c(L) \geq \varepsilon\,\omega \Longleftrightarrow \kappa(L) = n$;

(c) $c_1(L) \in \Gamma_a \Longleftrightarrow \forall\varepsilon,\ L$ has a smooth metric with $c(L) \geq -\varepsilon\,\omega \Longleftrightarrow L$ *is nef*;

(d) $c_1(L) \in \Gamma_a^\circ \Longleftrightarrow \exists\varepsilon,\ L$ has a smooth metric with $c(L) \geq \varepsilon\,\omega \Longleftrightarrow L$ *is ample.*

Proof. — It is well known that L is ample if and only if L has a smooth metric with positive definite curvature, and this gives the last equivalence in (4.2d).

(4.2a) Suppose that $c_1(L) \in \Gamma_+$. By definition $c_1(L)$ is a limit of cohomology classes of effective real divisors $D_k = \sum \lambda_{j,k} D_{j,k}$. Then $[D_k]$ is a sequence of closed positive currents, with a uniform bound of the mass $\int_X [D_k]\wedge\omega^{n-1}$, for this integral converges to $\int_X c_1(L)\wedge\omega^{n-1}$. By weak compactness, there is a subsequence $[D_{k_\nu}]$ converging to some closed positive current T of bidegree $(1,1)$, such that the cohomology class of T is equal to $c_1(L)$. Therefore, if L is equipped with an arbitrary smooth metric, we find $T = c(L) + \frac{i}{\pi}\partial\overline{\partial}\psi$ for some function $\psi \in L^1(X)$, and T is the curvature current of the singular metric obtained by multiplication of the original smooth metric by $e^{-\psi}$. Conversely, if L has a singular metric with $c(L) \geq 0$, we fix a point $x_0 \in X$ such that the associated weight satisfies $\nu(\varphi_L, x_0) = 0$. Let ψ_0 be a smooth function on $X \smallsetminus \{x_0\}$ which is equal to $n\log|z-x_0|$ (in some coordinates) near x_0, and let A be a fixed ample line bundle, equipped with a smooth metric of positive curvature. For m_0 large enough, we have $m_0 c(A) + \frac{i}{\pi}\partial\overline{\partial}\psi_0 \geq \omega$ and the tensor product metric on $kL + m_0A$ multiplied by $e^{-2\psi_0}$ is associated to a weight $\varphi_k = k\varphi_L + m_0\varphi_A + \psi_0$ such that

$$\frac{i}{\pi}\partial\overline{\partial}\varphi_k = k\,c(L) + m_0 c(A) + \frac{i}{\pi}\partial\overline{\partial}\psi_0 \geq k\,c(L) + \omega \geq \omega \qquad \forall k \geq 1.$$

Moreover $\nu(\varphi_k, x_0) = n$, whereas $\nu(\varphi_k, x) = k\,\nu(\varphi_L, x) < 1$ for $x \neq x_0$ near x_0. We infer from corollary 3.3 that $kL + m_0A + K_X$ has non zero sections for all $k \geq 1$. Let D_k be the divisor of any such section and $\{D_k\}$ its cohomology class. Then

$$c_1(L) = \frac{1}{k}\Big(\{D_k\} - m_0 c_1(A) - c_1(K_X)\Big) = \lim_{k\to+\infty}\frac{1}{k}\{D_k\}$$

and therefore $c_1(L) \in \Gamma_+$.

(4.2b) Without loss of generality, we may suppose that the cohomology class $\{\omega\}$ is integral, i.e. $\{\omega\} \in NS(X)$. The first equivalence in (4.2b) is then an immediate consequence of (4.2a), since a class $\{\alpha\}$ is in the interior Γ_+° if and only if $\{\alpha - \varepsilon\,\omega\} \in \Gamma_+$ for ε small enough ($\{\omega\}$ is obviously an interior point). If $c(L) \geq \varepsilon\,\omega$, we can construct as above a singular metric φ_k on $kL - K_X$ such that $\frac{i}{\pi}\partial\overline{\partial}\varphi_k \geq \omega$ for $k \geq k_0$, $\nu(\varphi_k, x_0) = n+1$ and $\nu(\varphi_k, x) < 1$ for $x \neq x_0$ near x_0. Then corollary 3.3 shows that kL has sections with arbitrary 1-jets at x_0, hence $\kappa(L) = n$. Conversely, if $\kappa(L) = n$, then $h^0(X, kL) \geq ck^n$ for $k \geq k_0$ and $c > 0$. Let A be a smooth ample divisor. The exact cohomology sequence

$$0 \longrightarrow H^0(X, kL - A) \longrightarrow H^0(X, kL) \longrightarrow H^0(A, kL_{\restriction A})$$

where $h^0(A, kL_{\restriction A}) = O(k^{n-1})$ shows that $kL - A$ has non zero sections for k large. If D is the divisor of such a section, then $kL \simeq \mathcal{O}(A + D)$. If we select a smooth metric on A such that $c(A) \geq \varepsilon_0\omega$ and the singular metric on $\mathcal{O}(D)$ described in example 2.2, then $c(L) = \frac{1}{k}\big(c(A) + [D]\big) \geq (\varepsilon_0/k)\,\omega$, as desired.

(4.2c) If $c_1(L) \in \Gamma_a$, then L is nef, because the condition that a cohomology class of type $(1,1)$ has nonnegative integrals over curves is preserved through convex combinations and limits. Conversely, when L is nef, $kL + A$ is ample as soon as A is ample; thus $c_1(kL + A)$ is the cohomology class of an ample rational divisor D_k and

$c_1(L) = \lim_{k\to+\infty} k^{-1}\{D_k\} \in \Gamma_a$. Moreover, arbitrary choices of smooth metrics with positive curvature on A and $kL + A$ yield a smooth metric on L such that

$$c(L) = \frac{1}{k}\big(c(kL + A) - c(A)\big) \geq -\frac{1}{k}c(A)\,;$$

in this way the negative part can be made smaller than $\varepsilon\,\omega$ for any $\varepsilon > 0$ if we take k large enough. Finally, if $c(L) \geq -\varepsilon\,\omega$ for every $\varepsilon > 0$, then $L \cdot C \geq -\varepsilon \int_C \omega$ and we conclude that $L \cdot C \geq 0$, thus L is nef.

(4.2d) Only the first equivalence remains to be checked: this is an immediate consequence of (4.2c) and the fact that $\{\omega\} \in \Gamma_a^\circ$. $\square$

Corollary 4.3. — *If L is nef, then $\kappa(L) = n$ if and only if $L^n > 0$. Moreover, the following properties are equivalent:*

(a) *L is nef and big;*

(b) $c_1(L) \in \Gamma_a \cap \Gamma_+^\circ$;

(c) *for every $\delta > 0$, L has a singular metric such that $c(L) \geq \varepsilon\,\omega$ for some $\varepsilon > 0$ and such that* $\max_{x\in X} \nu(\varphi, x) \leq \delta$.

The metrics obtained in (c) *can be chosen to be smooth on the complement of a fixed divisor D, with logarithmic poles along D.*

Proof. — The first statement is well-known and obtained as follows: if L is nef, the Hilbert polynomial of $\chi(X, kL)$ has leading coefficient $L^n/n! \geq 0$; by the Kodaira-Nakano vanishing theorem, we have an exact sequence

$$H^{j-1}\big(A, (kL + A)_{\restriction A}\big) \longrightarrow H^j(X, kL) \longrightarrow H^j(X, kL + A) = 0, \qquad j \geq 1$$

whenever A is a smooth divisor chosen sufficiently ample so that $A - K_X$ is ample; thus $h^j(X, kL) = O(k^{n-1})$ for $j \geq 1$ and

$$h^0(X, kL) = (L^n/n!)\, k^n + O(k^{n-1}),$$

therefore L is big if and only if $\kappa(L) = n$. The equivalence of (a) and (b) follows immediately from this and from (4.2b), (4.2c).

We also observe that $H^0(X, kL - A)$ is the kernel of the restriction morphism $H^0(X, kL) \to H^0(A, kL_{\restriction A})$ in which the target has dimension $O(k^{n-1})$. If $L^n > 0$, we infer $H^0(X, kL - A) \neq 0$ for k large, so there is an effective divisor D such that $kL \simeq \mathcal{O}(A + D)$. Now, $pL + A$ is ample for every $p \geq 0$, so $pL + A$ has a smooth metric with $c(pL + A) \geq \varepsilon_p\omega$ and the isomorphism $(k + p)L \simeq pL + A + D$ gives a metric on L such that

$$c(L) = (k + p)^{-1}c(pL + A) + (k + p)^{-1}[D] \geq (k + p)^{-1}\varepsilon_p\,\omega.$$

Observe that the singular part $(p + k)^{-1}[D]$ can be chosen as small as desired by taking p large, so $\max_X \nu(\varphi, x) \leq (k+p)^{-1} \max_D \nu(D, x)$ can be made arbitrarily small. Hence (a) implies (c).

Finally, if property (c) holds, the regularization theorem of [De 91] applied to $T = c(L)$ shows that L has smooth metrics such that the regularized curvature form T_ε has arbitrary small negative part. Hence L is nef by (4.2c) and $\kappa(L) = n$ by (4.2b). Therefore (c) implies (a). $\square$

5. The Kawamata-Viehweg vanishing theorem

To illustrate the strength of theorem 3.1, we give below a very simple derivation of the Kawamata-Viehweg vanishing theorem [Ka 82], [Vi 82]. Only the case of maximum Kodaira dimension will be treated here (the general case can be easily deduced by a slicing argument and an induction on $\dim X$, cf. [De 89]).

DEFINITION 5.1. — *We say that a divisor $D = \sum \alpha_j D_j$ with rational coefficients $\alpha_j \in \mathbb{Q}$ is integrable at a point $x_0 \in X$ if the function $\prod |g_j|^{-2\alpha_j}$ associated to local generators g_j of the ideal of D_j at x_0 is integrable on a neighborhood of x_0 .*

Observe that $\prod |g_j|^{-2\alpha_j} = e^{-2\varphi}$ where φ is the weight function of the natural singular metric on $\mathcal{O}(D)$ described in example 2.2. When D has normal crossings, the g_j's can be taken to be coordinates at x_0 ; thus D is integrable if and only if $\alpha_j < 1$ for all j . When D is effective and has arbitrary singularities, lemma 2.8 shows that a sufficient condition for the integrability of D at x_0 is that the multiplicity (or Lelong number)

$$\nu(D, x_0) = \sum \alpha_j \, \nu(D_j, x_0)$$

be < 1. If neither D has normal crossings nor $\nu(D, x_0) < 1$, the integrability condition can be checked by means of a sequence of blowing-ups which lift D into a divisor with normal crossings (this is always possible by [Hi 64]). Taking into account the jacobian divisor J of the blow-up morphism π, we get at the end a divisor $D' = \pi^\star D - J$ with normal crossings which is integrable if and only if D is integrable. A consequence of this is that integrability is an open condition : if E is an arbitrary effective divisor and if D is integrable at x_0, then $D + p^{-1}E$ is again integrable at x_0 for p large enough. With these definitions, we have:

THEOREM 5.2 (Kawamata-Viehweg). — *Let L be a line bundle over a projective manifold X with $\kappa(L) = n$. Assume that some positive multiple mL can be written $mL = \mathcal{O}(F + D)$ where F is a nef line bundle and D an effective divisor such that $m^{-1}D$ is integrable on $X \setminus \{$finite set$\}$. Then*

$$H^q(X, K_X + L) = 0 \quad \text{for} \quad q \geq 1.$$

Proof. — By the proof of (4.2b), there is an ample divisor A and an effective divisor E such that $kL \simeq \mathcal{O}(A + E)$. Then $(pm + k)L \simeq \mathcal{O}(pF + A + pD + E)$ where $pF + A$ is ample and

$$(pm + k)^{-1}(pD + E) \leq m^{-1}D + (pm + k)^{-1}E$$

is integrable on $X \setminus \{\text{finite set}\}$ for $p \geq 1$ large enough (integrability is an open condition). If we select a smooth metric on $pF+A$ with positive curvature $\omega_p = c(pF+A)$ and take the singular metric on $\mathcal{O}(pD+E)$ described in example 2.2, we find a singular metric on L such that

$$c(L) = (pk+m)^{-1}\omega_p + (pk+m)^{-1}(p[D]+[E]),$$

and the associated weight $e^{-2\varphi}$ is locally integrable on $X \setminus \{\text{finite set}\}$. Hence we can apply corollary 3.2 to conclude that $H^q(X, K_X+L) = 0$ for $q \geq 1$. $\square$

6. Seshadri constants of nef line bundles

Let L be a nef line bundle over a projective algebraic manifold X. To every point $x \in X$, we attach the number

$$\varepsilon(L,x) = \inf_{C \ni x} \frac{L \cdot C}{\nu(C,x)} \tag{6.1}$$

where the infimum is taken over all irreducible curves C passing through x and $\nu(C,x)$ is the multiplicity of C at x. The infimum

$$\varepsilon(L) = \inf_{x \in X} \varepsilon(L,x) = \inf_C \frac{L \cdot C}{\nu(C)} \quad \text{where} \quad \nu(C) = \max_{x \in C} \nu(C,x) \tag{6.1'}$$

will be called the *Seshadri constant* of L. It is well known that L is ample if and only if $\varepsilon(L) > 0$ (Seshadri's criterion [Ha 70]). For two nef line bundles L_1, L_2 we have $\varepsilon(L_1+L_2) \geq \varepsilon(L_1)+\varepsilon(L_2)$, in particular this shows again that L_1+L_2 is ample if L_1 or L_2 is ample.

If L is a nef line bundle, we are especially interested in singular metrics with isolated logarithmic poles: we say that a logarithmic pole x of the weight φ is *isolated* if φ is finite and continuous on $V \setminus \{x\}$ for some neighborhood V of x and we define

$$\gamma(L,x) = \sup \left\{ \begin{array}{l} \gamma \in \mathbb{R}_+ \text{ such that } L \text{ has a singular metric with } ic(L) \geq 0 \\ \text{and with an isolated log pole of coefficient } \gamma \text{ at } x \end{array} \right\}; \tag{6.2}$$

if there are no such metrics, we set $\gamma(L,x) = 0$.

The numbers $\varepsilon(L,x)$ and $\gamma(L,x)$ will be seen to carry a lot of useful information about the global sections of L and its multiples kL. To make this precise, we first introduce some further definitions. Let $s(L,x)$ be the largest integer $s \in \mathbb{N}$ such that the global sections in $H^0(X,L)$ generate all s-jets $J^s_x L = \mathcal{O}_x(L)/\mathcal{M}^{s+1}_x\mathcal{O}_x(L)$. If L_x is not generated, i.e. if all sections of L vanish at x, we set $s(L,x) = -\infty$. We also introduce the limit value

$$\sigma(L,x) = \limsup_{k \to +\infty} \frac{1}{k} s(kL,x) = \sup_{k \in \mathbb{N}^\star} \frac{1}{k} s(kL,x) \tag{6.3}$$

if $s(kL,x)\neq -\infty$ for some k, and $\sigma(L,x)=0$ otherwise. The limsup is actually equal to the sup thanks to the superadditivity property

$$s(L_1+L_2,x)\geq s(L_1,x)+s(L_2,x).$$

The limsup is in fact a limit as soon as kL spans at x for $k\geq k_0$, e.g. when L is ample.

THEOREM 6.4. — *Let L be a nef line bundle over X. For every point $x\in X$ we have*

$$\varepsilon(L,x)\geq\gamma(L,x)\geq\sigma(L,x).$$

If L is ample, the equality holds for every $x\in X$. If L is nef and big, the equality holds outside any divisor D prescribed by corollary 4.3.

Proof. — Fix a point $x\in X$ and a coordinate system $(z_1,\ldots,z_n)$ centered at x. If $s=s(kL,x)$, then $H^0(X,kL)$ generates all s-jets at x and we can find holomorphic sections $f_1,\ldots,f_N$ whose s-jets are all monomials z^α, $|\alpha|=s$. We define a global singular metric on L by

$$|\xi|=\Big(\sum_{1\leq j\leq N}|f_j(z)\cdot\xi^{-k}|^2\Big)^{-1/2k},\quad \xi\in L_z \tag{6.5}$$

associated to the weight function $\varphi(z)=\frac{1}{2k}\log\sum\big|\theta(f_j(z))\big|^2$ in any trivialization $L_{\restriction\Omega}\simeq\Omega\times\mathbb{C}$. Then φ has an isolated logarithmic pole of coefficient s/k at x, thus

$$\gamma(L,x)\geq\frac{1}{k}s(kL,x)$$

and in the limit we get $\gamma(L,x)\geq\sigma(L,x)$.

Now, suppose that L has a singular metric with an isolated log pole of coefficient $\geq\gamma$ at x. Set $c(L)=\frac{i}{\pi}\partial\overline{\partial}\varphi$ on a neighborhood Ω of x and let C be an irreducible curve passing through x. Then all weight functions associated to the metric of L must be locally integrable along C (since φ has an isolated pole at x). We infer

$$L\cdot C=\int_C c(L)\geq\int_{C\cap\Omega}\frac{i}{\pi}\partial\overline{\partial}\varphi\geq\gamma\,\nu(C,x)$$

because the last integral is larger than the Lelong number of the current $[C]$ with respect to the weight φ (cf. [De 87]) and we may apply the comparison theorem with the ordinary Lelong number associated to the weight $\log|z-x|$. Therefore

$$\varepsilon(L,x)=\inf\frac{L\cdot C}{\nu(C,x)}\geq\sup\gamma=\gamma(L,x).$$

Finally, we show that $\sigma(L,x)\geq\varepsilon(L,x)$ when L is ample. This is done essentially by same arguments as in the proof of Seshadri's criterion, as explained in [Ha 70]. Consider the blow-up $\pi:\widetilde{X}\to X$ at point x, the exceptional divisor $E=\pi^{-1}(x)$ and the line bundles $F_{p,q}=\mathcal{O}(p\,\pi^\star L-q\,E)$ over $\widetilde{X}$, where $p,q>0$. Recall that

$\mathcal{O}(-E)_{\restriction E}$ is the canonical line bundle $\mathcal{O}_E(1)$ over $E \simeq \mathbb{P}^{n-1}$, in particular we have $E^n = \mathcal{O}_E(-1)^{n-1} = (-1)^{n-1}$. For any irreducible curve $\widetilde{C} \subset \widetilde{X}$, either $\widetilde{C} \subset E$ and

$$F_{p,q} \cdot \widetilde{C} = \mathcal{O}(-q\,E) \cdot \widetilde{C} = q\,\mathcal{O}_E(1) \cdot \widetilde{C} = q \deg \widetilde{C}$$

or $\pi(\widetilde{C}) = C$ is a curve and

$$F_{p,q} \cdot \widetilde{C} = p\,L \cdot C - q\,\nu(C,x) \geq \big(p - q/\varepsilon(L,x)\big) L \cdot C.$$

Thus $F_{p,q}$ is nef provided that $p \geq q/\varepsilon(L,x)$. Since $F_{p,q}$ is ample when p/q is large, a simple interpolation argument shows that $F_{p,q}$ is ample for $p > q/\varepsilon(L,x)$. In that case, the Kodaira-Serre vanishing theorem gives

$$H^1(\widetilde{X}, k\,F_{p,q}) = H^1\big(\widetilde{X}, \mathcal{O}(kp\,\pi^\star L - kq\,E)\big) = 0$$

for k large. Hence we get a surjective map

$$H^0(\widetilde{X}, kp\,\pi^\star L) \longrightarrow\!\!\!\!\!\rightarrow H^0\Big(\widetilde{X}, \mathcal{O}(kp\,\pi^\star L) \otimes (\mathcal{O}/\mathcal{O}(-kq\,E))\Big) \simeq J_x^{kq-1}(kp\,L),$$

that is, $H^0(X, kp\,L)$ generates all $(kq-1)$ jets at x. Therefore $p > q/\varepsilon(L,x)$ implies $s(kp\,L, x) \geq kq - 1$ for k large, so $\sigma(L,x) \geq q/p$. At the limit we get $\sigma(L,x) \geq \varepsilon(L,x)$.

Assume now that L is nef and big and that $\varepsilon(L,x) > 0$. By the proof of lemma 4.3, there exist an integer $k_0 \geq 1$ and effective divisors A, D such that $k_0 L \simeq A + D$ where A is ample. Then $a\,\pi^\star A - E$ is ample for a large. Hence there are integers $a, b > 0$ such that $a\,\pi^\star A - b\,E - K_{\widetilde{X}}$ is ample. When $F_{p,q}$ is nef, the sum with any positive multiple $k\,F_{p,q}$ is still ample and the Akizuki-Nakano vanishing theorem gives

$$H^1(\widetilde{X}, k\,F_{p,q} + a\,\pi^\star A - b\,E) = H^1\big(\widetilde{X}, (kp + k_0 a)\,\pi^\star L - a\,\pi^\star D - (kq+b)E\big) = 0$$

when we substitute $A = k_0 L - D$. As above, this implies that we have a surjective map

$$H^0\big(X, (kp + k_0 a)\,L - a\,D\big) \longrightarrow\!\!\!\!\!\rightarrow J_x^{kq+b-1}\big((kp + k_0 a)\,L - a\,D\big)$$

when $p \geq q/\varepsilon(L,x)$. Since $\mathcal{O}(-aD) \subset \mathcal{O}$, we infer $s\big((kp+k_0 a)L, x\big) \geq kq + b - 1$ at every point $x \in X \smallsetminus D$ and at the limit $\sigma(L,x) \geq \varepsilon(L,x)$. $\square$

Remark 6.6. — Suppose that L is ample. The same arguments show that if $\pi : \widetilde{X} \to X$ is the blow-up at two points x, y and if $E_x + E_y$ is the exceptional divisor, then $F_{p,q} = p\,\pi^\star L - q\,E_x - E_y$ is ample for $p > q/\varepsilon(L,x) + 1/\varepsilon(L,y)$. In that case, $H^0(X, kp\,L)$ generates $J_x^{kq-1}(kp\,L) \oplus J_y^{k-1}(kp\,L)$ for k large. Take $p > q/\varepsilon(L,x) + 1/\varepsilon(L)$ and let y run over $X \smallsetminus \{x\}$. For k large, we obtain sections $f_j \in H^0(X, kp\,L)$ whose jets at x are all monomials z^α, $|\alpha| = kq - 1$, and with no other common zeros. Moreover, formula (6.5) produces a metric on L which is smooth and has positive definite curvature on $X \smallsetminus \{x\}$, and which has a log pole of coefficient $(kq-1)/kp$ at x. Therefore the supremum $\gamma(L,x) = \sup\{\gamma\}$ is always achieved by metrics that are smooth and have positive definite curvature on $X \smallsetminus \{x\}$.

Remark 6.7. — If Y is a p-dimensional algebraic subset of X passing through x, then

$$L^p \cdot Y \geq \varepsilon(L,x)^p \nu(Y,x)$$

where $L^p \cdot Y = \int_Y c_1(L)^p$ and $\nu(Y,x)$ is the multiplicity of Y at x (equal by Thie's theorem [Th 67] to the Lelong number of the integration current $[Y]$). If L is ample, we can take a metric on L which is smooth on $X \setminus \{x\}$ and defined on a neighborhood Ω of x by a weight function φ with a log pole of coefficient γ at x. By the comparison theorem for Lelong numbers, we get

$$L^p \cdot Y \geq \int_{Y\cap\Omega} \Big(\frac{i}{\pi}\partial\overline{\partial}\varphi\Big)^p \geq \gamma^p \nu(Y,x)$$

and γ can be chosen arbitrarily close to $\varepsilon(L,x)$. If L is nef, we apply the inequality to $k\,L + M$ with M ample and take the limit as $k \to +\infty$. $\square$

The numbers $\varepsilon(L,x)$ and Seshadri's constant $\varepsilon(L) = \inf \varepsilon(L,x)$ are especially interesting because they provide effective results concerning the existence of sections of $K_X + L$. The following proposition illustrates this observation.

Proposition 6.8. — *Let L be a big nef line bundle over X.*

(a) *If $\varepsilon(L,x) > n+s$, then $H^0(X, K_X+L)$ generates all s-jets at x.*

(b) *If $\varepsilon(L) > 2n$, then $K_X + L$ is very ample.*

Proof. — By the proof of theorem 6.4, the line bundle $\pi^\star L - q\,E$ is nef for $q \leq \varepsilon(L,x)$. Moreover, its n-th self intersection is equal to $L^n + (-q)^n E^n = L^n - q^n$ and as $L^n \geq \varepsilon(L,x)^n$ by remark 3.5, we see that $\pi^\star L - q\,E$ is big for $q < \varepsilon(L,x)$. The Kawamata-Viehweg vanishing theorem 5.2 then gives

$$H^1(\widetilde{X}, K_{\widetilde{X}} + \pi^\star L - q\,E) = H^1\big(\widetilde{X}, \pi^\star K_X + \pi^\star L - (q-n+1)E\big) = 0,$$

since $K_{\widetilde{X}} = \pi^\star K_X + (n-1)E$. Thus we get a surjective map

$$\begin{array}{ccc} H^0\big(\widetilde{X}, \pi^\star K_X + \pi^\star L) & \twoheadrightarrow & H^0\big(\widetilde{X}, \pi^\star \mathcal{O}(K_X+L) \otimes \mathcal{O}/\mathcal{O}(-(q-n+1)E)\big) \\ \| & & \| \\ H^0(X, K_X+L) & \twoheadrightarrow & J_x^{q-n}(K_X+L) \end{array}$$

provided that $\varepsilon(L,x) > q$. The first statement is proved. To show that $K_X + L$ is very ample, we blow up at two points x, y. The line bundle $\pi^\star L - n\,E_x - n\,E_y$ is ample for $1/\varepsilon(L,x) + 1/\varepsilon(L,y) < 1/n$, a sufficient condition for this is $\varepsilon(L) > 2n$. Then we see that

$$H^0(X, K_X+L) \longrightarrow (K_X+L)_x \oplus (K_X+L)_y$$

is also surjective. $\square$

These results are related to a conjecture of Fujita [Fu 88], asserting that if L is an ample line bundle, then $K_X + mL$ is spanned for $m \geq n+1$ and very ample for

$m \geq n+2$. The answer is positive for surfaces, thanks to I. Reider's numerical criterion (a deep extension of Bombieri's work [Bo 73] on pluricanonical embeddings of surfaces of general type). Our paper [De 90] describes a new method which gives partial results in the higher dimensional case. By proposition 6.8 above, we know that $K_X + mL$ generates s-jets for $m > (n+s)/\varepsilon(L)$ and is very ample for $m > 2n/\varepsilon(L)$. It is easy to see for example that $\varepsilon(L) \geq 1$ for any flag manifold, in which the ample cone has a very simple structure. In general, unfortunately, it seems to be a rather hard problem to compute the Seshadri constant $\varepsilon(L)$, even in the case of surfaces. An answer to the following question would be urgently needed.

Question 6.9. — *Given a projective algebraic manifold X, is there always a universal lower bound for $\varepsilon(L)$ when L runs over all ample line bundles of X ? In this case, is it possible to compute explicitly such a lower bound in terms of geometric invariants of X ?*

7. Asymptotic estimates of cohomology groups

Let X be a compact Kähler manifold, E a holomorphic vector bundle of rank r and L a line bundle over X. If L is equipped with a smooth metric of curvature form $c(L)$, we define the q-index set of L to be the open subset

$$(7.1)\qquad X(q,L) = \left\{ x \in X \;;\; c(L)_x \text{ has } \begin{matrix} q & \text{negative eigenvalues} \\ n-q & \text{positive eigenvalues} \end{matrix} \right\}, \qquad 0 \leq q \leq n.$$

It is shown in [De 85] that the cohomology groups $H^q\big(X, E \otimes \mathcal{O}(kL)\big)$ satisfy the asymptotic "Morse inequalities"

$$(7.2)\qquad h^q\big(X, E\otimes \mathcal{O}(kL)\big) \leq r\frac{k^n}{n!}\int_{X(q,L)} (-1)^q \big(c(L)\big)^n + o(k^n) \quad \text{ as } \; k \to +\infty.$$

One difficulty in the application of this result is that the curvature integral is in general quite uneasy to compute, since it is neither a topological nor a holomorphic invariant. However, when $c_1(L) \in \Gamma_+$, the results of [De 91] allow us to measure the distance of L to the ample cone Γ_a. In that case, a use of singular metrics combined with the approximation theorem of [De 91] produces smooth metrics on L for which an explicit bound of the negative part of the curvature is known. It follows that (7.2) gives an explicit upper bound of the cohomology groups of $E \otimes \mathcal{O}(kL)$ in terms of a polynomial in the first Chern class $c_1(L)$ (related techniques have already been used in [Su] in a slightly different context). To state the precise result, we need the notion of nefness for a real $(1,1)$ cohomology class which does not necessarily belong to $NS_{\mathbb{R}}(X)$: we say that $\{u\} \in H^{1,1}(X)$ is nef if $\{u\}$ belongs to the closure of the convex cone generated by classes of Kähler forms $\big($the so-called *Kähler cone* of $H^{1,1}(X)\big)$.

Theorem 7.3. — *Suppose that there is a nef cohomology class $\{u\}$ in $H^{1,1}(X)$ such that $c_1\big(\mathcal{O}_{TX}(1)\big) + \pi^\star\{u\}$ is nef over the hyperplane bundle $P(T^\star X)$. Suppose* *moreover that L is equipped with a singular metric such that $T = c(L) \geq 0$. For $p = 1, 2, \ldots, n, n+1$ set*

$$b_p = \inf\{c > 0\,;\, \operatorname{codim} E_c(T) \geq p\},$$

with $b_{n+1} = \max_{x \in X} \nu(T, x)$. Then for any holomorphic vector bundle E of rank r over X we have

$$h^q\big(X, E \otimes \mathcal{O}(kL)\big) \leq A_q r\, k^n + o(k^n)$$

where A_q is the cup product

$$A_q = \frac{1}{q!\,(n-q)!}\big(b_{n-q+1}\{u\}\big)^q \cdot \big(c_1(L) + b_{n-q+1}\{u\}\big)^{n-q}$$

in $H^{2n}(X, \mathbb{R})$, identified to a positive number.

Remark 7.4. — When X is projective algebraic and $\kappa(L) = n$, the proof of (4.2b) shows that $mL \simeq \mathcal{O}(A + D)$ with A ample and D effective, for some $m \geq 1$. Then we can choose a singular metric on L such that $T = c(L) = \omega + m^{-1}[D]$, where $\omega = m^{-1}c(A)$ is a Kähler metric. As $\nu(T, x) = m^{-1}\nu(D, x)$ at each point, the constants b_j of theorem 7.3 are obtained by counting the multiplicities of the singular points of D; for example, if D only has isolated singularities, then $b_1 = 0$, $b_2 = \ldots = b_n = 1/m$. Observe moreover that the nefness assumption on $\mathcal{O}_{TX}(1)$ is satisfied with $\{u\} = c_1(G)$ if G is a nef $\mathbb{Q}$-divisor such that $\mathcal{O}(TX) \otimes \mathcal{O}(G)$ is nef, e.g. if $\mathcal{O}(S^m TX) \otimes \mathcal{O}(mG)$ is spanned by sections for some $m \geq 1$.

Proof of theorem 7.3. — By definition, we have $0 = b_1 \leq b_2 \leq \ldots \leq b_n \leq b_{n+1}$, and for $c \in\,]b_p, b_{p+1}]$, $E_c(T)$ has codimension $\geq p$ with some component(s) of codimension p exactly. Let ω be a fixed Kähler metric on X. By adding $\varepsilon\omega$ to u if necessary, we may assume that $u \geq 0$ and that $\mathcal{O}_{TX}(1)$ has a smooth hermitian metric such that $c\big(\mathcal{O}_{TX}(1)\big) + \pi^\star u \geq 0$.

Under this assumption, the main approximation theorem of [De 91] shows that the metric of L can be approximated by a sequence of smooth metrics such that the associated curvature forms T_j satisfy the uniform lower bound

$$T_j \geq -\lambda_j(x)\, u(x) - \varepsilon_j\, \omega(x) \tag{7.5}$$

where $\lim\!\downarrow_{j\to+\infty} \varepsilon_j = 0$ and $(\lambda_j)_{j>0}$ is a decreasing sequence of continuous functions on X such that $\lim_{j\to+\infty} \lambda_j(x) = \nu(T, x)$ at each point.

Estimate (7.2) cannot be used directly with $T = c(L)$ because wedge products of currents do not make sense in general. Therefore, we replace $c(L)$ by its approximations T_j and try to find an upper bound for the limit.

LEMMA 7.6. — *Let $U_j = X(q, T_j)$ be the q-index set associated to T_j and let c be a positive number. On the open set $\Omega_{c,j} = \{x \in X\,;\, \lambda_j(x) < c\}$ we have*

$$(-1)^q \mathbb{1}_{U_j} T_j^n \leq \frac{n!}{q!\,(n-q)!}\big(c\,u + \varepsilon_j\,\omega\big)^q \wedge \big(T_j + c\,u + \varepsilon_j\,\omega\big)^{n-q}.$$

Proof. — Write $v = c\,u + \varepsilon_j\,\omega > 0$ and let $\alpha_{1,j} \le \ldots \le \alpha_{n,j}$ be the eigenvalues of T_j with respect to v at each point. Then $T_j^n = \alpha_{1,j} \ldots \alpha_{n,j}\, v^n$ and

$$v^q \wedge (T_j + v)^{n-q} = \frac{q!\,(n-q)!}{n!} \sum_{1 \le i_1 < \ldots < i_{n-q} \le n} (1 + \alpha_{i_1,j}) \ldots (1 + \alpha_{i_{n-q},j})\, v^n.$$

On $\Omega_{c,j}$ we get $T_j \ge -v$ by inequality (7.5), thus $\alpha_{i,j} \ge -1$; moreover, we have $\alpha_1 \le \ldots \le \alpha_q < 0$ and $0 < \alpha_{q+1} \le \ldots \le \alpha_n$ on U_j. On $\Omega_{c,j}$ we thus find

$$0 \le (-1)^q \mathbb{1}_{U_j} \alpha_{1,j} \ldots \alpha_{n,j} \le \mathbb{1}_{U_j} \alpha_{q+1,j} \ldots \alpha_{n,j} \le (1 + \alpha_{q+1,j}) \ldots (1 + \alpha_{n,j}),$$

therefore $(-1)^q \mathbb{1}_{U_j} T_j^n \le \big(n!/q!\,(n-q)!\big)\, v^q \wedge (T_j + v)^{n-q}$. $\square$

End of the proof of theorem 7.3. — Set $\Lambda = \max_X \lambda_1(x)$. By lemma 7.6 applied with an arbitrary $c > \Lambda$ we have

$$(-1)^q \mathbb{1}_{U_j} T_j^n \le \frac{n!}{q!(n-q)!} (\Lambda\, u + \varepsilon_1 \omega)^q \wedge (T_j + \Lambda\, u + \varepsilon_1 \omega)^{n-q} \quad \text{on } X.$$

Then estimate (7.2) and lemma 7.6 again imply

$$\begin{aligned} h^q\big(X, E \otimes \mathcal{O}(kL)\big) &\le r \frac{k^n}{n!} \int_X (-1)^q \mathbb{1}_{U_j} T_j^n + o(k^n) \\ &\le \frac{r\,k^n}{q!\,(n-q)!} \Big(\int_{\Omega_{c,j}} (c\,u + \varepsilon_j\,\omega)^q \wedge (T_j + c\,u + \varepsilon_j\,\omega)^{n-q} \\ &\quad + \int_{X \setminus \Omega_{c,j}} (\Lambda\, u + \varepsilon_1 \omega)^q \wedge (T_j + \Lambda\, u + \varepsilon_1 \omega)^{n-q} \Big) + o(k^n). \end{aligned} \tag{7.7}$$

Since $\lambda_j(x)$ decreases to $\nu(T, x)$ as $j \to +\infty$, the set $X \setminus \Omega_{c,j}$ decreases to $E_c(T)$. Now, $T_j + \Lambda\, u + \varepsilon_1 \omega$ is a closed positive $(1,1)$-form belonging to a fixed cohomology class, so the mass of any wedge power $(T_j + \Lambda\, u + \varepsilon_1 \omega)^p$ with respect to ω is constant. By weak compactness, there is a subsequence (j_ν) such that $(T_{j_\nu} + \Lambda\, u + \varepsilon_1 \omega)^p$ converges weakly to a closed positive current Θ_p of bidegree (p,p), for each $p = 1, \ldots, n$. For $c > b_{p+1}$, we have $\operatorname{codim} E_c(T) \ge p + 1$, hence $\mathbb{1}_{E_c(T)} \Theta_p = 0$. It follows that the integral over $X \setminus \Omega_{c,j}$ in (7.7) converges to 0 when $c > b_{n-q+1}$. For the same reason the integral over $\Omega_{c,j}$ converges to the same limit as its value over X: observe that $(T_j + c\,u + \varepsilon_j\,\omega)^{n-q}$ can be expressed in terms of powers of u, ω and of the positive forms $(T_j + \Lambda\, u + \varepsilon_1 \omega)^p$ with $p \le n - q$; thus the limit is a linear combination with smooth coefficients of the currents Θ_p, which carry no mass on $E_c(T)$. In the limit, we obtain

$$h^q\big(X, E \otimes \mathcal{O}(kL)\big) \le \frac{r\,k^n}{q!\,(n-q)!} (c\{u\})^q \cdot \big(c_1(L) + c\{u\}\big)^{n-q} + o(k^n),$$

and since this is true for every $c > b_{n-q+1}$, theorem 7.3 follows. $\square$

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