CONFINING QUANTUM PARTICLES WITH A PURELY MAGNETIC FIELD

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Summary

- The problem
- The scalar case
- The results
- Agmon-type estimates
- Lower bound for the Dirichlet integral
- Examples

Joint work with Yves Colin de Verdière, Institut Fourier, Grenoble

The problem

- Ω : open set in $\mathbb{R}^d$ ($d \geq 2$)
  - $\partial\Omega$ compact
  - either Ω or $\mathbb{R}^d \setminus \Omega$ is bounded.
- P: a particle in Ω in the presence of a magnetic field B.
- Classically: if $|B(x)| \to +\infty$ as x approaches $\partial\Omega$, we expect that P is confined and never visits the boundary

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- quantically: P never feels the boundary $\Leftrightarrow$ B completely determines the motion $\Leftrightarrow$ no need for (BC)
- mathematically: find conditions on $B(x)$ (as $x \to \partial\Omega$) s.t. the magnetic operator H_A is e.s.a on $C_o^\infty(\Omega)$.
  These conditions will not depend on the gauge A, but only on the field B.

Related result :the scalar case

- Replace B by a scalar potential V
- ? conditions on V (as $x \to \partial\Omega$) s.t. the Schrödinger operator $H = -\Delta + V$ is e.s.a on $C_o^\infty(\Omega)$.

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  Optimal condition on V near $\partial\Omega$, Ω a bounded smooth domain
- In particular If $V(x) \geq (\frac{3}{4})D(x)^{-2}$ where D is the distance to the boundary of Ω.then $H = -\Delta + V$ is e.s.a on $C_o^\infty(\Omega)$.

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- Tools:
  - Agmon-type results on exponential decay of eigenfunctions
  - multidimensional Hardy inequalities.
- the term $\frac{3}{4}$ follows from Hardy inequalities.

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- $\Longrightarrow$ need a lower bound on the magnetic quadratic form h_A associated to the magnetic potential A.
- continuity assumption on the direction of $B(x)$ near $\partial\Omega \Longrightarrow$

$$h_A(u) \geq (1-\epsilon) \int_{\Omega \cap \{x| \ |x| \leq R\}} |B|_{\mathrm{sp}} \ |u|^2 \ |dx| - C_{\epsilon,R} \ \|u\|^2 \ .$$

$\forall u \in C_o^\infty(\Omega)$

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$\forall u \in C_o^\infty(\Omega)$

Theorem If moreover $|B(x)|_{\mathrm{sp}} \geq (1+\eta) D(x)^{-2}$ (MB)

($\eta > 0$, D : the distance to $\partial\Omega$)

Then H_A is essentially self-adjoint

The domain Ω

Ω : open set in $\mathbb{R}^d$ ($d \geq 2$)

- d_R the "Riemannian" distance :

$$d_R(x,y) = \inf_{\gamma \in \Gamma_{x,y}} \mathrm{length}(\gamma)$$

$\Gamma_{x,y}$: smooth curves $\gamma : [0,1] \to \Omega$ s.t. $\gamma(0) = x, \ \gamma(1) = y$.

- $\widehat{\Omega}$ the metric completion of (Ω, d_R)
- $\Omega_\infty = \widehat{\Omega} \setminus \Omega$ the metric boundary of Ω.

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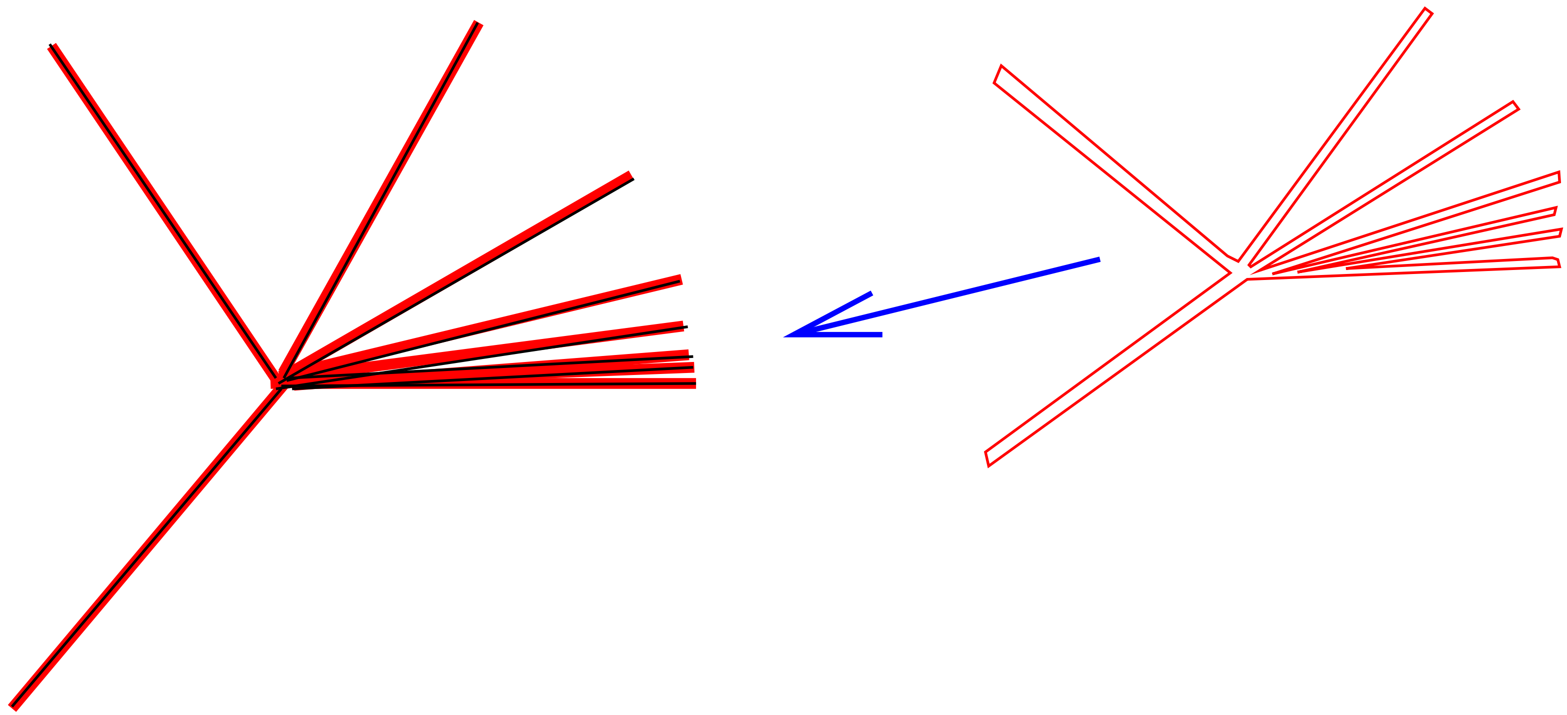
- $\widehat{\Omega}$ the metric completion of (Ω, d_R)
- $\Omega_\infty = \widehat{\Omega} \setminus \Omega$ the metric boundary of Ω.
- Assumptions on Ω
  - Ω_∞ is assumed to be compact. (Ω regular)
  - Either Ω or $\mathbb{R}^d \setminus \Omega$ is bounded.
- If Ω is regular, $\partial\Omega$ is compact. ($\partial\Omega = \bar{\Omega} \setminus \Omega$ the top. boundary of Ω)

An example where $\partial\Omega$ is compact while Ω_∞ is not compact



- e_n a sequence of unit vectors in $\mathbb{R}^2$ converging to e_0
- $X = \cup_{n\in\mathbb{N}}[0,1]e_n$
- $\Omega = \mathbb{R}^2 \setminus X$

- $D(x)$ the distance to the boundary :

$$D(x) = \min_{y \in \Omega_\infty} d_R(x, y) \quad (x \in \Omega)$$

- Property $\quad D$ is 1-Lipschitz and then a. e. differentiable in Ω. At any point x of differentiability of D, $|dD(x)| \leq 1$.

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- Definition Assume Ω regular.
A continuous function $f : \Omega \to \mathbb{C}$ is regular at the boundary if it extends by continuity to $\widehat{\Omega}$.
- Notations
  - $C_o^\infty(\Omega)$: $\{$complex-valued smooth functions with compact support in $\Omega\}$.
  - $|dx|$ the Lebesgue measure
  - $\langle u, v\rangle := \int_\Omega u\bar{v}|dx|$ the L^2 scalar product
  - $\|u\|$ the L^2 norm of u

Magnetic field

- $\wedge^k\mathbb{R}^d$:=$\{$ real-valued k-linear antisymmetric forms on $\mathbb{R}^d\}$.
- $A = \sum_{j=1}^d a_j dx_j$ a smooth real 1-form on Ω (magnetic potential)
- magnetic field associated to A : the two-form $B = dA$.

$$B(x) = \sum_{1\leq j<k\leq d} b_{jk}(x) dx_j \wedge dx_k\,, \quad b_{jk}(x) = \partial_j a_k(x) - \partial_k a_j(x)\ .$$

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- $\nabla_j = \nabla_{\partial/\partial x_j} = \dfrac{\partial}{\partial x_j} - i a_j$ (magnetic connection)
- $H_A = -\sum_{j=1}^d \nabla_j^2$ (magnetic Schrödinger operator)
- $h_A(u) = \int_\Omega \sum_{j=1}^d |\nabla_j u|^2 |dx| \quad u \in C_o^\infty(\Omega)$ (magnetic Dirichlet integral)

The spectral norm

- $B \in \wedge^2\mathbb{R}^d \Longrightarrow \exists$ an orthonormal basis of $\mathbb{R}^d$ so that $B = b_{12}dx_1 \wedge dx_2 + b_{34}dx_3 \wedge dx_4 + \cdots$ with $b_{12} \geq b_{34} \geq \cdots > 0$
- the sequence $b_{12}, b_{34}, \cdots$ is unique
- the non-zero eigenvalues of the antisymmetric endomorphism $\tilde{B}$ of $\mathbb{R}^d$ associated to $B(x)$ are $\pm ib_{12}, \pm ib_{34}, \cdots$.

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- $|B|_{\mathrm{sp}}$ is one half of the trace norm of $\tilde{B} \Longrightarrow$ it is a norm
  - $d = 2 \quad \Longrightarrow \quad |B|_{\mathrm{sp}} = |B|$
  - $d = 3 \quad \Longrightarrow \quad |B|_{\mathrm{sp}}$ the norm of the v. field $\vec{B}$ assoc. to B
- $|B|_{\mathrm{sp}}$ is the infimum of the spectrum of the Schrödinger operator with constant magnetic field B in $\mathbb{R}^d$.

The results

Consider H_A with domain $\mathcal{D}(H_A) = C_o^\infty(\Omega)$.

Theorem ($d = 2$)

If $\partial\Omega$ is compact with a finite number of connected components and

$$|B(x)|_{\mathrm{sp}} \geq (D(x))^{-2}\ , \quad (x \text{ near } \partial\Omega)$$

then the Schrödinger operator H_A is essentially self-adjoint.

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Theorem ($d > 2$)

If Ω is regular, if $\exists \eta > 0$

$$|B(x)|_{\rm sp} \geq (1+\eta)\,(D(x))^{-2} \ , \quad (x \text{ near } \partial\Omega)$$

and if the functions $n_{jk}(x) = \dfrac{b_{jk}(x)}{|B(x)|_{\rm sp}}$ are regular at the boundary Ω_∞

(for any $1 \leq j < k \leq d$), then same conclusion.

Remarks

- If $\Omega := \{x \in \mathbb{R}^d \mid f(x) > 0\}$ with $f : \mathbb{R}^d \to \mathbb{R}$ smooth, $df(y) \neq 0$ for $y \in \partial\Omega$, then $f(x) \sim |df(x)|D(x)$ for x close to $\partial\Omega$. And we can replace $D(x)$ by $f(x)/|df(x)|$.

- About optimality
  Proposition: For any $0 < \alpha < \sqrt{3}/2$, there exists a magnetic field B for which H_A (with $dA = B$) is not e.s.a. and such that

$$|B(x)|_{\mathrm{sp}} \geq \frac{\alpha}{(D(x))^2} \quad (x \text{ near } \partial\Omega).$$

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Consequence

The optimal constant in front of the leading term $(D(x))^{-2}$ is in $[\sqrt{3}/2, 1]$.
$\Longrightarrow$ the situation for magnetic fields is different from the situation for scalar potentials (for which the optimal constant is $3/4$)

Sketch of the proof

- Technical lemmas
  - Magnetic inequality
  - Local lower bound
- Lower bounds for the magnetic Dirichlet integrals
  - $d = 2$
  - $d > 2$ (if the direction of B is regular)
- Agmon-type estimates

Technical lemma

Magnetic inequality

For any $u \in C_o^\infty(\Omega)$, we have

$$h_A(u) \geq |\langle b_{12} u | u \rangle| + |\langle b_{34} u | u \rangle| + \cdots \ .$$

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Proof

Commutator formula : $[\nabla_j, \nabla_k] = -i \ b_{jk} \quad \Longrightarrow$

$$|\langle b_{12}u|u\rangle| = |\langle[\nabla_1, \nabla_2]u|u\rangle| \leq 2|\langle\nabla_1 u|\nabla_2 u\rangle| \leq \int_\Omega (|\nabla_1 u|^2 + |\nabla_2 u|^2)|dx| \ .$$

Then take the sum of similar inequalities replacing the indices 12 by $34, \ 56, \cdots$.

Consequence: Local lower bound

Ω : regular open set in $\mathbb{R}^d$, $x_0 \in \Omega_\infty$

If $B(x) \neq 0$ near x_0 and the direction of B is regular near x_0 then, $\forall \epsilon > 0$,

$\exists\, U$ nbhd of x_0 in $\mathbb{R}^d$ so that, for any $\phi \in C_o^\infty(U \cap \Omega)$,

$$h_A(\phi) \geq (1-\epsilon) \int_U |B(x)|_{\mathrm{sp}} |\phi(x)|^2 |dx| \ ,$$

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Proof

- $n(x)$ reg. at the point $x_0 \Longrightarrow$ we can choose U s.t., $\forall x \in U \cap \Omega$, $|n(x) - n(x_0)|_{\mathrm{Eucl}} \leq \epsilon \sqrt{\frac{2}{d(d-1)}}$
- We choose orth. coord. so that $n(x_0) = n_{12} dx_1 \wedge dx_2$ $+ n_{34} dx_3 \wedge dx_4 + \cdots$ with $n_{2k-1,2k} \geq 0$ and $\sum_k n_{2k-1,2k} = 1$.
- Magnetic inequality $\Longrightarrow$ for $\phi \in C_o^\infty(\Omega \cap U)$,
$$h_A(\phi) \geq \int_U |B(x)|_{\mathrm{sp}} (n_{12}(x) + n_{34}(x) + \cdots) |\phi(x)|^2 |dx|$$

and $\quad n_{12}(x) + n_{34}(x) + \cdots \geq 1 - \epsilon$.

Lower bounds for the magnetic Dirichlet integrals

Theorem $d = 2$ Assume that $\partial\Omega \subset B(O,R)$. If $B \neq 0$ near $\partial\Omega$, then there exists $c_R \in \mathbb{R}$ so that, $\forall u \in C_o^\infty(\Omega)$,

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Proof

- The sign of B is constant near each conn. component of $\partial\Omega$.
- Write $\bar\Omega \subset \cup_{l=1}^3 \Omega_l$, $\Omega_1 \cap \partial\Omega = \emptyset$, $B > 0$ on Ω_2 and $B < 0$ on Ω_3.
- Take a partition of unity $\phi_j,\ j = 1, 2, 3$,
- Use IMS formula

$$h_A(u) = \sum_{l=0}^{2} h_A(\phi_l u) \, - \int_\Omega \left(\sum_{l=0}^{2} |d\phi_l|^2\right) |u|^2 \, |dx| \, ,$$

apply (MI) in $\Omega_l \cap \Omega$ for $l = 2, 3$ and the lower bound 0 for Ω_1.

Theorem $d > 2$

Assume that $\partial\Omega \subset B(O,R)$. If $B = dA \neq 0$ near $\partial\Omega$ and if the $n_{jk}(x)$ are regular at $\partial\Omega$, then, for any $\epsilon > 0$, there exists $C_{\epsilon,R} > 0$ so that, $\forall u \in C_o^\infty(\Omega)$,

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Proof

- Choose a finite covering of Ω_∞ by open sets $U_l,\ l = 1, \cdots N$ of $\mathbb{R}^d$ s.t (LLB) are satisfied.
- Choose a partition of unity $\phi_l, l = 0, \cdots, N$ with
  - For $l \geq 1$, $\phi_l \in C_o^\infty(U_l)$, $\quad \phi_0$ is $C_o^\infty(\Omega)$
  - $\sum_l \phi_l^2 \equiv 1$ in Ω, $\quad \sup \sum_l |d\phi_l|^2 = M \quad (*)$
- Use (LLB) for $l \geq 1$, the bound $(*)$ and IMS identity

Agmon-type estimates

Theorem

Assume that $\partial\Omega$ is compact, and that $\exists c \in \mathbb{R}$ such that, $\forall u \in C_o^\infty(\Omega)$,

$$h_A(u) - \int_{\{x\in\Omega \mid D(x)\leq 1\}} D(x)^{-2}|u(x)|^2|dx| \geq c\|u\|^2 \ . \quad (**)$$

Then, for $E << 0$, if v is a weak $L^2(\Omega)$-solution of $(H_A - E)v = 0$, v vanishes identically and H_A is essentially self-adjoint.

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Proof

- Lemma Let v be a weak solution of $(H_A - E)v = 0$, and let f be a real-valued Lipschitz function with compact support. Then

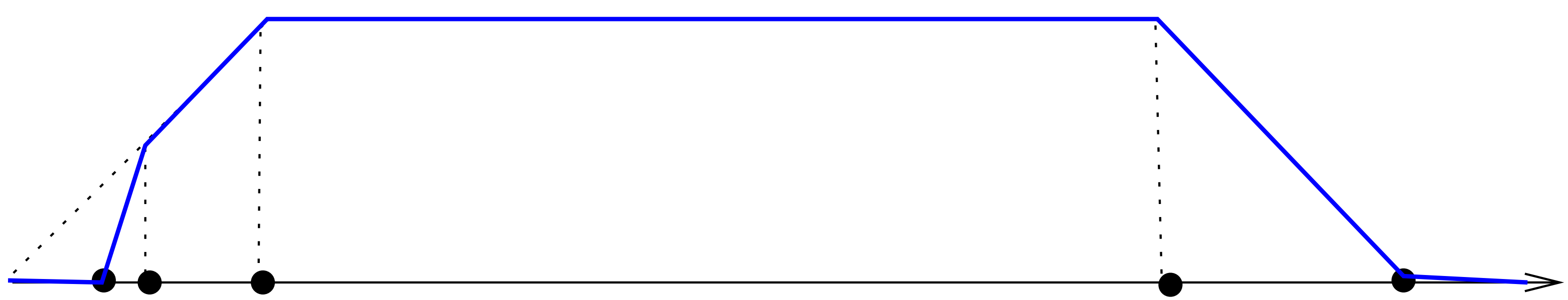
$$\langle fv|(H_A - E)(fv)\rangle = \langle v \mid |df(x)|^2 v\rangle \,. \quad (***)$$

- Apply the formula $(***)$ to an appropriate f.

The function f

- Consider $0 < \rho < \frac{1}{2}$ and $1 < R < +\infty$.
- Define $f = F(D)$ with

$$F(u) = \begin{cases} 0 \text{ for } u \leq \rho \text{ and for } u \geq R+1 \\ 2(u-\rho) \text{ for } \rho \leq u \leq 2\rho \\ u \text{ for } 2\rho \leq u \leq 1 \\ 1 \text{ for } 1 \leq u \leq R \\ R+1-u \text{ for } R \leq u \leq R+1 \end{cases}$$



- $|df|^2 = F'(D)^2$ almost everywhere.
- Apply inequality $(**)$ to $u = fv \Longrightarrow$

$$\langle (H_A - E)(fv) \mid fv \rangle \geq \int_{2\rho \leq D(x) \leq 1} |v|^2 |dx| + c\|fv\|^2 \ .$$

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- On the other hand, use the explicit values of df and the equality $(***)$ to get:

$$\begin{aligned} \langle (H_A - E)(fv) \mid fv \rangle \leq \quad & 4 \int_{\rho \leq D(x) \leq 2\rho} |v|^2 |dx| + \cdots \\ \cdots \int_{2\rho \leq D(x) \leq 1} |v|^2 |dx| + \quad & \int_{R \leq D(x) \leq R+1} |v|^2 |dx| \ . \end{aligned}$$

- $\Longrightarrow c\|fv\|^2 \leq 4 \displaystyle\int_{\rho \leq D(x) \leq 2\rho} |v|^2 |dx| + \int_{R \leq D(x) \leq R+1} |v|^2 |dx| \ .$
- Take $\rho \to 0$ and $R \to +\infty$

End of proof, remarks

- It is enough to show that $\exists c \in \mathbb{R}$ such that, $\forall u \in C_o^\infty(\Omega)$,

$$h_A(u) \geq \int_{\Omega \cap B(O,R)} |D(x)|^{-2} |u(x)|^2 |dx| - c\|u\|^2,$$

under the assumpt of Thms $d = 2$ and $d > 2$. This is a consequence of lower bounds for MDI

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- Essential self-adjointness depends only on the boundary behavior
X a smooth manifold with a smooth density $|dx|$. If $L_j,\ j = 1, 2$ are symm. ell. diff. op. of degree m on $L^2(X, |dx|)$ so that L_1 e.s.a. and $L_2 - L_1 = M$ compactly supported, then L_2 is e.s.a..

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$\Longrightarrow$ To prove self-adjointness in Ω, nothing to prove at ∞ in $\mathbb{R}^d$

- Essential self-adjointness is independent of the choice of a gauge
If $A_2 = A_1 + dF$, $F \in C^\infty(X, \mathbb{R})$, then H_{A_1} e.s.a. $\Longrightarrow H_{A_2}$ e.s.a.

About optimality

A non e.s.a. Schrödinger operator with large magnetic field near $\partial\Omega$

- Theorem

  Consider

  - $\Omega = \{(x,y) \in \mathbb{R}^2 |\, x^2 + y^2 = r^2 < 1\}$
  - $A = \alpha(xdy - ydx)/(r-1)$, $0 < \alpha < \sqrt{3}/2$.

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Proof

- The corresponding magnetic field B writes $B(x, y) = \frac{\alpha(r-2)}{(r-1)^2} dx \wedge dy$,
- $|B(x)| \sim \; \alpha/(D(x))^2$ (near the boundary)
- $H_A = -\dfrac{\partial^2}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial}{\partial r} - \dfrac{2i\alpha r}{r-1}\dfrac{\partial}{\partial \theta} + \dfrac{\alpha^2 r^2}{(r-1)^2}$.
- $\Longrightarrow H_A$ splits as a sum $\sum_{m \in \mathbb{Z}} H_{A,m}$ where $H_{A,m}$ acts on functions $e^{im\theta} f(r)$.

- Look at the $m = 0$ component
- Reduce the measure $|rdrd\theta|$ to $2\pi dr$ by a change of function:
  $\Longrightarrow$ for any function $u(r) = r^{-1/2}v(r)$,
  $H_A u = r^{-1/2}\left(-\frac{d^2}{dr^2} + V(r)\right)v$, where $V(r) = -\frac{1}{4r^2} + \frac{\alpha^2 r^2}{(r-1)^2}$.

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- $0 < \alpha < \sqrt{3/2} \Longrightarrow \exists\epsilon > 0$ with
  $$V(r) \leq \left(\frac{3}{4} - \epsilon\right)(r-1)^{-2} \quad \text{near} \quad r = 1$$
  $\Longrightarrow$ the operator $H = -\frac{d^2}{dr^2} + V(r)$ is in the limit circle case at $r = 1$ (Theorem X.10 (Reed-Simon))
- Let $v(r)$ be an L^2 solution of $(H - E)v = 0$
  then $u(r) = r^{-1/2}v(r)$ is an L^2 solution of $(H_A - E)u = 0$ in Ω.

Example 1 :Polytopes

- Ω : a polytope given by $\Omega = \cap_{i=1}^{N}\{x \mid L_i(x) < 0\}$,
- L_i's are the affine real-valued functions $L_i(x) = \displaystyle\sum_{j=1}^{d} n_{ij}x_j + a_i$.
- Assume that, for $i = 1, \cdots, d,\ \sum_{j=1}^{d} n_{ij}^2 = 1$ and $n_{i1} \neq 0$

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- Assume that, for $i = 1, \cdots, d$, $\sum_{j=1}^{d} n_{ij}^2 = 1$ and $n_{i1} \neq 0$
- Theorem The operator H_A in Ω with
  $A = \left(\dfrac{1}{n_{11} L_1} + \dfrac{1}{n_{21} L_2} + \cdots \right) dx_2$ is essentially self-adjoint.
- Proof
  $B = \left(\dfrac{1}{L_1^2} + \dfrac{1}{L_2^2} + \cdots \right) dx_1 \wedge dx_2 + \displaystyle\sum_{j=3}^{d} b_j dx_j \wedge dx_2$, and
  $D = \min_{1 \leq i \leq N} |L_i|$.
  $\Longrightarrow B = b_1 dx_1 \wedge dx_2 + \sum_{j=3}^{d} b_j dx_j \wedge dx_2$ with $b_1 \geq D^{-2}$.
  $\Longrightarrow$ apply (MI) and Agmon-type estimates.

Example 2 :“toroidal domains”

- $\partial\Omega$: a smooth compact manifold of co-dimension 1
- $j : \partial\Omega \to \mathbb{R}^d$ the injection of $\partial\Omega$ into $\mathbb{R}^d$. (H. Hopf) :
  there exists a nowhere vanishing tangent vector field to $\partial\Omega$ (or 1-form) $\Leftrightarrow$ the Euler characteristic of $\partial\Omega$ vanishes (Ω is toroidal).

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- Theorem
  - Assume Ω is toroidal.
  - A_0 : a smooth 1-form on $\bar{\Omega}$ s. t. the 1-form on $\partial\Omega$: $\omega = j^\star(A_0)$ does not vanish
  - Define near $\partial\Omega$: $A = A_0/D^\alpha$, $\alpha > 1$,
    (or $\alpha = 1$ and $\forall y \in \partial\Omega$, $|\omega(y)| > 1$.)
  - Then H_A is essentially self-adjoint.
- Remark The assumption on $\partial\Omega$ is fullfilled if $\Omega \subset \mathbb{R}^3$ is bounded by a 2-torus. It is the case for tokamacs.

Proof

Check:

- The regularity of the magnetic field (to be checked locally near $\partial\Omega$.) In local coordinates
  - $A_0 = a_1 dx_1 + \beta$ with $\beta = a_2 dx_2 + \cdots$
  - $\omega == j^\star(A_0) = a_2(0,x')dx_2 + \cdots$
  - so $B = d\left(\frac{A_0}{x_1^\alpha}\right) = \frac{x_1 dA_0 - \alpha dx_1 \wedge \beta}{x_1^{\alpha+1}}$.
  - $\Longrightarrow$ the direction of B is equivalent as $x_1 \to 0^+$ to that of $dx_1 \wedge \omega$ which is non vanishing and continuous on $\bar{\Omega}$.
- The lower bound $|B|_{\mathrm{sp}} \geq (1+\eta)D^{-2}$ near $\partial\Omega$. The norm of B near the boundary is given, as $x \to y$ by $|B(x)|_{\mathrm{sp}} \sim |\omega(y)|/D^{\alpha+1}$.

$\Longrightarrow$ Apply the Theorem $d > 2$.

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- Remark 2 If $d = 3$, magnetic field $\Leftrightarrow$ vector field in Ω.
  The assumptions $\Longrightarrow$ this vector field is asymptotic to $-\alpha V^{\perp}/D^{\alpha+1}$
  - V : the vector field associated to ω
  - $V^{\perp}$: deduced from V by a rotation of $\pm\pi/2$

  $\Longrightarrow$ B is very large near $\partial\Omega$ and parallel to $\partial\Omega$.

Point of view of classical mechanics: the trajectories of the charged particle are spiraling around the field lines and do not cross the boundary.
Open problem : precise statement.

Exemple 3 : Non toroidal domains

- Remark 1

  Now any 1-form on $X = \partial\Omega$ may have some zeroes.

- Definition

  A 1-form ω on a compact manifold X is generic if ω has a finite number of zeroes and $d\omega$ does not vanish at the zeroes of ω.

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  A 1-form ω on a compact manifold X is generic if ω has a finite number of zeroes and $d\omega$ does not vanish at the zeroes of ω.

- Theorem
  - $\Omega \subset \mathbb{R}^d$ with a smooth compact boundary $X = \partial\Omega$.
  - A_0 a smooth 1-form in $\mathbb{R}^d$ s. t. $\omega = j_X^\star(A_0)$ is generic. and at each zero m of ω, $|d\omega(m)|_{\mathrm{sp}} > 1$.

    Then, if A is a 1-form in Ω such that near X, $A = A_0/D^2$, $B = dA$ is confining in Ω.

- Remark 2

  The field B is more singular than in the toroidal case.

Exemple 4. Singular points : monopoles

$\Omega = \mathbb{R}^3 \setminus 0 \quad m \in \mathbb{Z} \setminus 0$

- Monopole of degree m: the magnetic field $B_m = (m/2)p^\star(\sigma)$
  - $p : \mathbb{R}^3 \setminus 0 \to S^2$:the radial projection
  - σ : the area form on S^2.

    In coordinates $B_m = \dfrac{m}{2} \dfrac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$.

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- Remark

  $|B_m| \geq \frac{|m|}{2} r^{-2}$ where the constant is sharp.
- The flux of B_m through S^2 is equal to $2\pi m$ quantization condition
- L_m: complex line bundle, ∇_m : Herm. connexion on Ω with curv. B_m.
- $H_m = -\dfrac{\partial^2}{\partial r^2} - \dfrac{2}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}K_m$, (K_m the angular Schr. op. on S^2)
- Theorem H_m (the monopole of degree m) is e.s.a. $\Leftrightarrow |m| \geq 2$.
- Proof λ_1^m the lowest eigenvalue of K_m.
  - H_m is e.s.a. $\Leftrightarrow \lambda_1^m \geq 3/4$.(Weyl's theory for Sturm-Liouville equ.)
  - $\lambda_1^m = |m|/2$

Exemple 4. Singular points : multipoles

- Denote, for $x \in \mathbb{R}^3$, $\mathcal{B}_x$ the monopole of degree 2 with center x.
- $P\left(\frac{\partial}{\partial x}\right)$: a homogeneous linear diff. op. of degree n on $\mathbb{R}^3$ with constant coefficients.
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  A multipole of degree 1 is called a dipole.
- Theorem If $B_V = dA_V$ is a dipole , H_{A_V} is essentially self-adjoint.
- Proof
  - B_V homogeneous of degree -3
    $\Longrightarrow$ enough to show that B_V does not vanish.
  - we can take $V = \partial/\partial z$.
  - $B_{\partial/\partial z} = \dfrac{d}{dt}_{|t=0} \dfrac{xdy \wedge dz + ydz \wedge dx + (z-t)dx \wedge dy}{\left(x^2 + y^2 + (z-t)^2\right)^{3/2}}$,
  - $\Longrightarrow B_{\partial/\partial z} = \frac{3xzdy \wedge dz + 3yzdz \wedge dx + (2z^2 - x^2 - y^2)dx \wedge dy}{(x^2+y^2+z^2)^{5/2}}$.
  - The form $B_{\partial/\partial z}$ does not vanish in Ω.