GROUPES ALGÉBRIQUES TRÈS SPÉCIAUX/VERY SPECIAL ALGEBRAIC GROUPS

par

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Résumé. — We say that a smooth algebraic group G over a field k is very special if for any field extension K/k, every G_K-homogeneous K-variety has a K-rational point. It is known that every split solvable linear algebraic group is very special. In this note, we show that the converse holds, and discuss its relationship with the birational classification of algebraic group actions.

Abstract. — Nous disons qu'un groupe algébrique lisse G sur un corps k est très spécial si pour toute extension de corps K/k, toute K-variété homogène sous G_K a un point K-rationnel. On sait que tout groupe linéaire résoluble scindé est très spécial. Dans cette note, nous obtenons la réciproque et nous discutons ses relations avec la classification birationnelle des actions de groupes algébriques.

1. Introduction

Consider a smooth algebraic group G over a field k, and a G-variety X. By a theorem of Rosenlicht, there exist a dense open G-stable subset $X_0 \subset X$ and a morphism $f : X_0 \to Y$, such that the fiber of f at any point $x \in X_0$ is the orbit of x; moreover, f identifies the function field of Y with the field of G-invariant rational functions on X (see [**14**, Thm. 2], and [**1**, Sec. 7] for a modern proof). We say that the rational map $f : X \dashrightarrow Y$ is the rational quotient of X by G.

From this, one readily derives a birational classification of G-varieties with prescribed invariant function field. To state it, we introduce some notation. Given a finitely generated field extension K/k, we consider pairs (X, ι), where X is a G-variety and $\iota : K \xrightarrow{\simeq} k(X)^G$ is an isomorphism of fields over k. We say that two pairs (X, ι) and (X', ι')

are equivalent, if there exists a G-equivariant birational isomorphism $\varphi : X \dashrightarrow X'$ such that the isomorphism $\varphi^* : k(X')^G \xrightarrow{\cong} k(X)^G$ satisfies $\varphi^* \circ \iota' = \iota$. We may now state :

Proposition 1. — *There is a one-to-one correspondence between equivalence classes of pairs (X, ι) as above, and isomorphism classes of G_K-homogeneous K-varieties.*

This easy result (which is implicitly known, see e.g. [**12**, §2.7]) motivates the consideration of those smooth algebraic groups for which all rational quotients have rational sections. These are described as follows :

Theorem 2. — *The following conditions are equivalent for a smooth algebraic group G :*

(i) *For any G-variety X, the rational quotient $f : X \dashrightarrow Y$ has a rational section.*

(ii) *For any field extension K/k, every G_K-homogeneous K-variety has a K-rational point.*

(iii) *G has a composition series with quotients isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$.*

The equivalence (i)$\Leftrightarrow$(ii) follows readily from Rosenlicht's theorem on rational quotients. The implication (iii)$\Rightarrow$(ii) is also due to Rosenlicht (see [**14**, Thm. 10]). The proof of the converse implication is the main contribution of this note.

The algebraic groups satisfying (iii) are exactly the split solvable linear algebraic groups in the sense of [**10**, Def. 6.33]. On the other hand, (ii) obviously implies that for any field extension K/k, every G-torsor over $\mathrm{Spec}(K)$ is trivial. By [**13**], this is equivalent to G being special as defined by Serre in [**16**], that is, every locally isotrivial G-torsor over a variety is Zariski locally trivial. For this reason, we will call *very special* the algebraic groups satisfying (ii).

In fact, every split solvable linear algebraic group G satisfies a much stronger condition : for any field extension K/k, every G_K-homogeneous variety is rational (as follows from [**15**, Thm. 5]). Equivalently, the field extension $k(X)/k(X)^G$ is purely transcendental for any G-variety X. This yields a further characterization of very special groups.

One may also consider algebraic groups G that are possibly non-smooth, and require that for any field extension K/k, every G_K-homogeneous K-scheme has a K-rational point (where a scheme X equipped with an action a of G is said to be homogeneous if the morphism $\mathrm{id} \times a : G \times X \to X \times X$ is faithfully flat). But the result is unchanged, since every G-torsor over $\mathrm{Spec}(K)$ is trivial, and hence G is smooth in view of [**13**, Prop. 2.3].

This note is organized as follows. The proof of Proposition 1 is presented in Section 2. The implications (i)$\Leftrightarrow$(ii)$\Leftarrow$(iii) are proved in Section 3, which also makes the first steps in the proof of (ii)$\Rightarrow$(iii). In Section 4, we show that any very special torus is split. Together with the fact that any special unipotent group is split (see [**11**, Thm. 1.1]), this enables us to complete the proof of (ii)$\Rightarrow$(iii) in Section 5.

Notation and conventions. We fix a ground field k and choose an algebraic closure $\bar{k}$. We denote by k_s the separable closure of k in $\bar{k}$, and by Γ_k the Galois group of k_s/k. Given a field extension K/k and a k-scheme X, we denote by X_K the K-scheme $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$.

A *variety* is an integral separated k-scheme of finite type. An *algebraic group G* is a k-group scheme of finite type. We say that G is *linear* if it is smooth and affine.

Given an algebraic group G, a *G-variety* is a variety X equipped with a G-action,

$$a : G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x.$$

We say that a G-variety X is *G-homogeneous* if G is smooth, X is geometrically reduced, and the morphism

$$\mathrm{id} \times a : G \times X \longrightarrow X \times X, \quad (g, x) \longmapsto (x, g \cdot x)$$

is surjective. If in addition X is equipped with a k-rational point x, then we say that X is a *G-homogeneous space*; then $X \simeq G/\mathrm{Stab}_G(x)$, where $\mathrm{Stab}_G(x)$ denotes the stabilizer.

Every homogeneous space is smooth and quasi-projective; thus, so is every homogeneous variety.

2. Proof of Proposition 1

Consider a pair (X, ι) and choose a dense open G-stable subset $X_0 \subset X$ with quotient $f : X_0 \to Y$ as in Rosenlicht's theorem. Identifying $k(Y)$ with K via ι, the generic fiber of f is a G_K-homogeneous K-variety, say Z_0. If we replace X_0 with an open subset X_1 satisfying the same properties, then Z_0 is replaced with another G_K-homogeneous K-variety Z_1, which is G_K-equivariantly birationally isomorphic to Z_0. But every G_K-equivariant birational isomorphism $Z_0 \dashrightarrow Z_1$ is an isomorphism : this is proved in [**5**, Lem. 4] for homogeneous spaces, and the general case follows by Galois descent. So we obtain a G_K-homogeneous K-variety Z; moreover, replacing (X, ι) with an equivalent pair replaces Z with a G_K-equivariantly isomorphic variety.

Conversely, consider a G_K-homogeneous K-variety Z. We may then choose an immersion of Z in some projective space $\mathbb{P}^n_K$. Also, choose a k-variety Y with function field K and consider the closure W of Z in $\mathbb{P}^n_Y$. Then W is a k-variety equipped with a k-morphism $f : W \to Y$. The action map $a : G_K \times_{\mathrm{Spec}(K)} Z \to Z$ is identified with a morphism $G \times_{\mathrm{Spec}(k)} Z \to Z$, which yields a rational action of G on W (since W and Z have the same function field). By construction, the field of invariant rational functions on W is identified with K. We now use Weil's regularization theorem (see the main result of [**18**] and [**14**, Thm. 1]) : W is G-birationally isomorphic to a G-variety X. This associates with Z a pair (X, ι), unique up to equivalence.

One may readily check that the two constructions above are mutually inverse, by using again the fact that every equivariant birational isomorphism between homogeneous varieties is an isomorphism.

3. Proof of Theorem 2 : first steps

We first show the equivalence (i)$\Longleftrightarrow$(ii). Under the correspondence described in the proof of Proposition 1, the rational sections of $f : X \dashrightarrow Y$ correspond to the K-points of the associated G_K-homogeneous variety. Thus, (i) is equivalent to the assertion that (ii) holds for any finitely generated field extension of k. Given an arbitrary field extension K/k and a G_K-homogeneous variety Z, there exist a finitely generated subextension L/k and a G_L-homogeneous variety W such that $W_L \simeq Z$. Then every L-rational point of W yields a K-rational point of Z; this completes the proof.

To show the equivalence (ii)$\Longleftrightarrow$(iii), we begin with some easy observations. First, if G is very special, then G_K is a very special K-group for any field extension K/k. Further properties are gathered in the following :

Lemma 3. — *Consider an exact sequence of algebraic groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1.$$

(i) *If G is very special, then so is Q.*

(ii) *If N and Q are very special, then so is G.*

Démonstration. — (i) Just note that Q is smooth and every Q_K-homogeneous variety is homogeneous under the induced action of G_K.

(ii) Since N and Q are smooth, G is smooth as well. Let K/k be a field extension, and X a G_K-homogeneous K-variety. Then there is a quotient $f : X \to Y = X/N_K$, where Y is a Q_K-homogeneous variety : indeed, if X has a K-rational point x, then $X \simeq G_K/H$ where $H := \mathrm{Stab}_{G_K}(x)$ and we may take for f the canonical morphism $G/H \to G/N_K \cdot H$. The case of an arbitrary G_K-homogeneous variety X follows from this by using Galois descent together with the smoothness and quasi-projectivity of X.

Since Q_K is very special, Y has a K-rational point y. The fiber of f at y is a K-variety, homogeneous under N_K. As the latter is very special, it follows that this fiber has a K-rational point. $\qquad\square$

Finally, note that a smooth commutative algebraic group G is very special if and only if for any field extension K/k, every quotient of G_K is special.

These observations yield a quick proof of the implication (iii)$\Rightarrow$(ii) : by Lemma 3 (ii), it suffices to show that $\mathbb{G}_a$ and $\mathbb{G}_m$ are very special. Since these groups are commutative, it suffices in turn to show that for any field extension K/k, every quotient of $\mathbb{G}_{a,K}$ or

$\mathbb{G}_{m,K}$ is special. But every quotient of $\mathbb{G}_{a,K}$ is isomorphic to $\mathbb{G}_{a,K}$ (see [**6**, IV.2.1.1]), and likewise for $\mathbb{G}_{m,K}$; moreover, $\mathbb{G}_a$ and $\mathbb{G}_m$ are special. This yields the assertion.

One may show similarly that every G_K-homogeneous K-variety is rational, for any split solvable linear algebraic group G and any field extension K/k.

We now start the proof of the implication (ii)$\Rightarrow$(iii) with the following :

Lemma 4. — *Let G be a very special algebraic group. Then G is connected, linear and solvable.*

Démonstration. — As the assertions are invariant under field extensions, we may assume k algebraically closed. Since G is special, it is connected and linear by [**16**, Thm. 1]. Moreover, every quotient group of G is special in view of Lemma 3 (i). In particular, the largest semisimple quotient H of G is special, as well as the largest adjoint semisimple quotient $H/Z(H)$, where $Z(H)$ denotes the (scheme-theoretic) center. By a result of Grothendieck (see [**7**, Thm. 3]), the special semisimple groups are exactly the products of special linear groups and symplectic groups. In particular, every special adjoint semisimple group is trivial. Thus, so is $H/Z(H)$, and G is solvable. □

4. Very special tori

Let T be a torus. We denote by $M = \mathrm{Hom}_{k_s\text{--gp}}(T_{k_s}, \mathbb{G}_{m,k_s})$ its character group; this is a free abelian group of finite rank equipped with a continuous action of the absolute Galois group $\Gamma = \Gamma_k$. By an unpublished result of Colliot-Thélène (see [**8**, Thm. 18]; this result is implicitly contained in [**4**]), T is special if and only if the Γ-module M is *invertible*, i.e., a direct factor of a permutation Γ-module. From this, we derive a criterion for T to be very special :

Lemma 5. — *The following conditions are equivalent :*

   (i) *T is very special.*

   (ii) *Every quotient group of T is special.*

   (iii) *For any subgroup of finite index $\Gamma' \subset \Gamma$, every Γ'-submodule $M' \subset M$ is invertible.*

Démonstration. — (i)$\Rightarrow$(ii) This follows from Lemma 3.

(ii)$\Rightarrow$(iii) The invariant subfield $K := k_s^{\Gamma'} \subset k_s$ is a finite separable extension of k with absolute Galois group Γ', and T_K is the K-torus with character group M viewed as a Γ'-module. Moreover, the Γ'-submodule M' of M corresponds to a quotient torus T' of T_K. By asumption, T' is special; thus, M' is invertible.

(iii)⇒(i) Let K/k be a field extension. Then again, the character group of T_K is M equipped with its action of the absolute Galois group Γ_K. We claim that this action factors through that of Γ.

To show this, note that the action of Γ_K on K_s stabilizes the subfields k_s and Kk_s. This yields an exact sequence

$$1 \longrightarrow \mathrm{Gal}(K_s/Kk_s) \longrightarrow \Gamma_K = \mathrm{Gal}(K_s/K) \longrightarrow \mathrm{Gal}(Kk_s/K) \longrightarrow 1.$$

Since T_{k_s} is split, so is T_{Kk_s} and hence the action of Γ_K on M factors through an action of $\mathrm{Gal}(Kk_s/K)$. The latter group may be identified with a subgroup of $\mathrm{Gal}(k_s/k) = \Gamma$, proving the claim.

Every T_K-homogeneous K-variety X is a torsor under a quotient T' of T_K, which in turn corresponds to a Γ_K-stable submodule of M. By the claim and the assumption, it follows that T' is special, i.e., X has a K-rational point. □

Lemma 6. **—** *Let T be a very special torus. Then T is split.*

Démonstration. — It suffices to show that Γ acts trivially on M. Equivalently, for any subgroup $\Gamma' \subset \Gamma$ acting on M via a quotient of prime order $p \geqslant 2$, the Γ'-action on M is trivial.

Denote by C_p the cyclic group of order p. By Lemma 5, the C_p-module M is invertible, as well as any submodule M'. In particular, there exist a C_p-module N and two integers $a, b \geqslant 0$ such that

$$M' \oplus N \simeq \mathbb{Z}^a \oplus (\mathbb{Z}C_p)^b$$

as C_p-modules. By localizing at the prime ideal $(p) \subset \mathbb{Z}$, we obtain an isomorphism of $\mathbb{Z}_{(p)}C_p$-modules

$$M'_{(p)} \oplus N_{(p)} \simeq \mathbb{Z}^a_{(p)} \oplus (\mathbb{Z}_{(p)}C_p)^b.$$

Moreover, $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)}C_p$ are indecomposable $\mathbb{Z}_{(p)}C_p$-modules (this is obvious for $\mathbb{Z}_{(p)}$; for $\mathbb{Z}_{(p)}C_p$, one uses the isomorphism of $\mathbb{Q}C_p$-modules

$$\mathbb{Q}C_p \simeq \mathbb{Q} \oplus V,$$

where $\mathbb{Q}$ is a trivial module and V is an irreducible non-trivial module of dimension $p-1$; moreover, this isomorphism is not defined over $\mathbb{Z}_{(p)}$). As the Krull-Schmidt theorem holds for $\mathbb{Z}_{(p)}C_p$-modules (see [**9**, Thm. 2]), there exist integers $c, d \geqslant 0$ such that

$$M'_{(p)} \simeq \mathbb{Z}^c_{(p)} \oplus (\mathbb{Z}_{(p)}C_p)^d.$$

In particular, if M' is not fixed pointwise by C_p, then its rank (as a $\mathbb{Z}$-module) is at least p.

Consider the $\mathbb{Q}C_p$-module $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$. If M is not fixed pointwise by C_p, then $M_{\mathbb{Q}}$ contains a $\mathbb{Q}C_p$-module W isomorphic to V (since every simple $\mathbb{Q}C_p$-module is isomorphic to $\mathbb{Q}$ or V). Thus, $M' := M \cap W$ is a C_p-submodule of M, not fixed pointwise by C_p and of rank $p-1$; a contradiction. So Γ acts trivially on M as desired. $\qquad\square$

Remark 7. — The localization argument in the above proof cannot be avoided, since the Krull-Schmidt theorem generally fails for C_p-modules. More specifically, we may choose p so that the ring R has nontrivial class group, and choose a non-principal ideal $A \subset R$. Then A is a summand of a free R-module, but is not isomorphic to R.

5. Completion of the proof of Theorem 2

It remains to show the implication (ii)$\Rightarrow$(iii). Let G be a very special algebraic group, and recall from Lemma 4 that G is connected, linear and solvable. Choose a maximal torus T of G; then T_K is a maximal torus of G_K for any field extension K/k (see [**10**, Thm. 17.82]).

If k is perfect, then $G = U \rtimes T$, where U denotes the unipotent radical of G (indeed, $G_{\bar{k}} = U_{\bar{k}} \rtimes T_{\bar{k}}$, see e.g. [**10**, Thm. 16.33]). As a consequence, T is a quotient group of G, and hence is very special by Lemma 3 (i). In view of Lemma 6, it follows that T is split. But the smooth connected unipotent group U is split as well (see e.g. [**10**, Cor. 16.23]), and hence so is G.

For an arbitrary field k, consider the derived subgroup $\mathscr{D}(G)$; this is a smooth connected unipotent normal subgroup of G (see e.g. [**10**, Cor. 6.19, Prop. 16.34]). The quotient group $G/\mathscr{D}(G)$ lies in a unique exact sequence of commutative algebraic groups

$$(5.1) \qquad\qquad 0 \longrightarrow M \longrightarrow G/\mathscr{D}(G) \longrightarrow V \longrightarrow 0,$$

where M is of multiplicative type and V is unipotent; moreover, (5.1) splits uniquely over the perfect closure k_i of k (see e.g. [**10**, Thm. 16.3]). As a consequence, the natural morphism $T \to G/\mathscr{D}(G)$ induces an isomorphism $T \xrightarrow{\simeq} M$. Also, V is a quotient group of G, and hence is special. Since V is unipotent, it is split by [**11**, Thm. 1.1]. In view of [**2**, Lem. 5.7], it follows that the exact sequence (5.1) has a unique splitting. Thus, we may identify $G/\mathscr{D}(G)$ with $T \times V$. In particular, T is a quotient group of G. Using Lemmas 3 and 6, it follows that T is split.

Denote by U the pull-back of V in G. Then U is a smooth connected unipotent group, and $G \simeq U \rtimes T$. By [**2**, Thm. 3.7], U has a largest split subgroup U_{split}; moreover, the formation of U_{split} is compatible with separable field extensions. As a consequence, U_{split} is normal in G. Also, T acts trivially on U/U_{split} in view of [**2**, Cor. 4.4]. Thus,

$G/U_{\mathrm{split}} \simeq U/U_{\mathrm{split}} \times T$. In particular, U/U_{split} is a quotient group of G, and hence is special. Using again [**11**, Thm. 1.1], it follows that U is split, and hence so is G.

Remark 8. — By inspecting the proof of Theorem 2, one may check that the conditions (i), (ii), (iii) are equivalent to

(iv) *Every quotient group of G is special.*

If G is linear, they are also equivalent to

(i)' *The rational quotient map $V \dashrightarrow V/G$ has a rational section for any finite-dimensional representation $G \to \mathrm{GL}(V)$.*

Indeed, the implication (i)⇒(i') is obvious. We show that (i)'⇒(iv) : let Q be a quotient group of G, and choose a faithful finite-dimensional representation $\rho : Q \to \mathrm{GL}(W)$. Then G acts on $\mathrm{GL}(W)$ by right multiplication via ρ. Moreover, $\mathrm{GL}(W)$ may be viewed as an open subset of $V := \mathrm{End}(W)$, stable by the linear representation of G in V by right multiplication via ρ again. Thus, the rational quotient map $V \dashrightarrow V/G$ may be viewed as the Q-torsor $\mathrm{GL}(W) \to \mathrm{GL}(W)/Q$. By assumption, this torsor has a rational section, and hence is locally trivial for the Zariski topology. It follows that Q is special by using [**16**, Thm. 2] (which is obtained over an algebraically closed field, but whose proof holds unchanged over an arbitrary field).

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