RATIONAL POINTS AND CURVES ON FLAG VARIETIES

by

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***Abstract.* —** One of the main tool to study the asymptotic behaviour of points of bounded height on projective varieties over a number field K is the height zeta function defined by the series

$$\zeta_{V,H}(s) = \sum_{x \in V(K)} \frac{1}{H(x)^s}$$

where $V(K)$ denotes the set of rational points of V and $H : V(K) \to \mathbf{R}$ is a height on V. If V is a flag variety, Franke, Manin and Tschinkel proved that one may normalize the height so that the height zeta function is an Eisenstein series. One may then apply Langland's work to ascertain the meromorphic properties of this function. This method apply also over global fields of positive characteristic where Eisenstein series have been studied by Morris.

In a joint work with Antoine Chambert-Loir, we are extending this framework to motivic height zeta functions, using results of Kapranov. This generalization makes explicit strong links existing between the asymptotics of points of bounded height and the moduli spaces of curves on the considered varieties.

Résumé. — L'un des meilleur outils pour l'étude du comportement asymptotique des points de hauteur bornée sur les variétés projectives sur un corps de nombres K est la fonction zêta des hauteurs définie par la série complexe

$$\zeta_{V,H}(s) = \sum_{x \in V(K)} \frac{1}{H(x)^s}$$

où $V(K)$ désigne l'ensemble des points rationnels de V et $H : V(K) \to \mathbf{R}$ une hauteur sur V. Si V est une variété de drapeau, Franke, Manin et Tschinkel ont démontré que l'on peut normaliser la hauteur de sorte que la fonction zêta des hauteurs soit une série d'Eisenstein. Il est alors possible d'appliquer les travaux de Langlands pour montrer des propriétés de méromorphie de cette fonction. Cette méthode s'applique aussi aux corps globaux de caractéristique finie pour lesquels les séries d'Eisenstein ont été étudiées par Morris.

Dans un travail en cours avec Antoine Chambert-Loir, nous étendons cette étude aux fonctions zêta des hauteurs motiviques, en utilisant des résultats de Kapranov. Cette généralisation souligne les liens profonds existant entre le comportement symptotique des points de hauteur bornée et les espaces de modules de courbes sur ces variétés.

Joint work in progress with Antoine Chambert-Loir

1. Heights

It is well known that there are many analogies between the rational points on a variety V defined over a number field K and the rational curves on a variety V over $\mathbf{C}$ and that one of the simplest way to make these links more precise is to consider rational points on a global field of finite characteristic.

In this talk we shall consider the three settings simultaneously:

(1) Over $\mathbf{Q}$ we may define several natural heights on the projective space, for example the height $H_N : \mathbf{P}^N(\mathbf{Q}) \to \mathbf{R}$ defined by

$$H_N((x_0 : \ldots : x_N)) = \sqrt{x_0^2 + \cdots + x_N^2},$$

if $x_0, \ldots, x_N$ are coprime integers. The corresponding logarithmic height is $h_N = \log H_N$.

More generally, if K is a number field, let M_K be the set of places of K. For any place v of K, we denote by K_v the completion of K for the topology defined by v and the absolute value $|\cdot|_v$ is normalized by $\mathrm{d}(ax)_v = |a|_v \mathrm{d}x_v$ for any Haar measure $\mathrm{d}x_v$. We then choose v-adic norms $||\cdot||_v : K_v^{N+1} \to \mathbf{R}$, for example we may define the norm $||(x_0, \ldots, x_N)||_v$ as $\sup_{0 \leqslant i \leqslant N} |x_i|_v$ if v is a finite place, as

$\sqrt{\sum_{i=0}^{N} x_i^2}$ if K_v is isormorphic to $\mathbf{R}$, and as $\sum_{i=0}^{N} x_i \bar{x}_i$ if K_v is isomorphic to $\mathbf{C}$. Then $H_N : \mathbf{P}^N(K) \to \mathbf{R}$ is defined by

$$H_N(x_0 : \ldots : x_N) = \prod_v ||(x_0, \ldots, x_N)||_v$$

and $h_N = \log H_N$.

(2) If $K = \mathbf{F}_q(\mathscr{C})$ where $\mathscr{C}$ is a smooth projective curve of genus g over $\mathbf{F}_q$, then there is a bijection from the set of points in the projective space $\mathbf{P}^N(K)$ to the set $\mathrm{Mor}(\mathscr{C}, \mathbf{P}^N_{\mathbf{F}_q})$. Let us denote by $\tilde{x}$ the image of a point x. Then

$$h_N(x) = \deg(\tilde{x}^*(\mathscr{O}(1)))$$

where $\tilde{x}^*(\mathscr{O}(1))$ belongs to the Picard group of the curve $\mathscr{C}$. We also put $H_N = q^{h_N}$.

(3) Similarly, if $K = k(\mathscr{C})$ where $\mathscr{C}$ is a smooth projective curve over a field k, we define

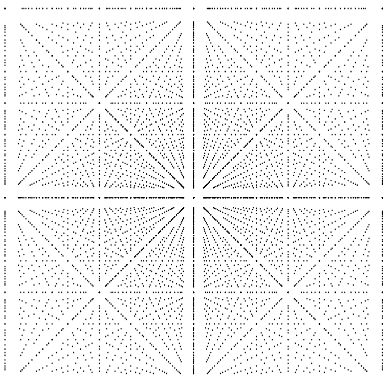
$$h_N(x) = \deg(\tilde{x}^*(\mathscr{O}(1)))$$

where $\tilde{x}^*(\mathscr{O}(1))$ belongs to the Picard group of the curve $\mathscr{C}$.

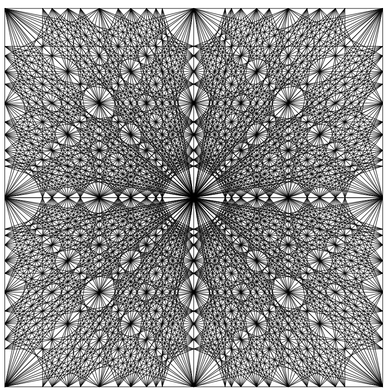
In all settings, if V is a variety over K, any morphism $\phi : V \to \mathbf{P}^N_K$ induces a map $h : V(K) \to \mathbf{R}$ defined by $h = h_N \circ \phi$. We want to study asymptotically the set

$$\{ x \in V(K) \mid h(x) < \log(B) \}$$

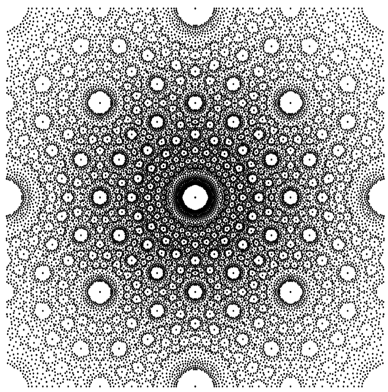
as B goes to $+\infty$. To illustrate this, I represented such sets as points on the projective plane, as lines on the plane and as points in $\mathbf{P}^1_{\mathbf{Q}(i)}$.



$\mathbf{P}^2_{\mathbf{Q}}$



Lines in $\mathbf{P}^2_{\mathbf{Q}}$



$\mathbf{P}^1_{\mathbf{Q}(i)}$

2. Height zeta functions

One of the main tool to study the asymptotic behaviour of the number of points of bounded height is the height zeta function.

(1) Over a number field, it is defined for any open subset U of V by

$$\zeta_{U,H}(s) = \sum_{x \in U(K)} \frac{1}{H(x)^s}$$

where this series converges.

(2) Similarly, over $\mathbf{F}_q(T)$, for any open subset U of V

$$Z_{U,h}(T) = \sum_{x \in U(K)} T^{h(x)} \quad \text{and} \quad \zeta_{U,H}(s) = Z_{U,h}(q^{-s}).$$

(3) In the functional setting, we are in fact interested in moduli spaces of morphisms from the curve $\mathscr{C}$ to the variety V. Let $\mathscr{M}_k$ be the group generated by symbols $[V]$ for V variety over k with the relations $[V] = [V']$ if V and V' are isomorphic and

$$[V] = [F] + [V - F]$$

for any closed subset F of V.

If U is an open subset of V, for any integer n, there exists a variety U_n over k such that for any extension k' of k, there is a functorial bijection from $U_n(k')$ to the set of points of $U(k'(\mathscr{C}))$ of height n. The motivic height zeta function is the formal series in $\mathscr{M}_k[[T]]$ defined by

$$Z^{\mathrm{mot}}_{U,h}(T) = \sum_{n \in \mathbf{N}} [U_n] T^n.$$

If k is a finite field, one may go from the functional setting to the classical one by using the map

$$\begin{array}{rcl} \mathscr{M}_k & \to & \mathbf{Z} \\ [V] & \mapsto & \#V(\mathbf{F}_q). \end{array}$$

This map sends $Z^{\mathrm{mot}}_{U,h}$ to the classical zeta function $Z_{U,h}$.

3. The case of flag varieties

For flag varieties, one may use the fact, first discovered by Franke, Manin and Tschinkel [**FMT**] that, in that case, the height zeta function coincides with an Eisenstein series. One may then apply the difficult and deep results obtained for Eisenstein series by Langlands over number fields [**Lan**], by Harder [**Harder**] and Morris ([**Mo1**] and [**Mo2**]) over global fields of finite characteristic and by Kapranov [**Ka**] in the functional setting.

Notations 3.0.1. — Let G be a split semi-simple simply-connected algebraic group over K, let P be a smooth parabolic subgroup of G, let B be a Borel subgroup of G contained in P and let T be a split maximal torus of G contained in B. We denote by Φ the root system of T in G, by Φ^+ the positive roots corresponding to B and by Δ the corresponding basis of the root system. Let Φ_P be the roots of T in the Lie algebra $\mathrm{Lie}(R_u(P))$ of the unipotent radical of P. The set Φ_P is contained in the set of positive roots. We also put $\Delta_P = \Phi_P \cap \Delta$.

Let $V = G/P$. There exists a canonical isomorphism from the character group $X^*(P)$ of P to $\mathrm{Pic}(V)$ sending the character χ to the line bundle $\mathscr{L}_\chi = G \times^P \mathbf{A}^1_K$ where P acts on the affine line via χ. There is also an injective restriction map $\mathrm{res} : X^*(P) \to X^*(T)$. Let ρ_P (resp. ρ_B) be the half-sum of the roots in Φ_P (resp. Φ^+) then $2\rho_P$ belongs to the image of $X^*(P)$ and we denote also by $2\rho_P$ its inverse image in $X^*(P)$. The line bundle $\mathscr{L}_{2\rho_P}$ is isomorphic to the anticanonical line bundle ω_V^{-1} and is very ample. From now on, all the heights used will be relative to ω_V^{-1}.

(1) In the number field case, it is possible to choose the height on V so that the height zeta function coincides with the value of an Eisenstein series:

$$\zeta_{V,H}(s) = \sum_{x \in G/P(K)} H(x)^{-s} = E_P^G((2s-1)\rho_P, e).$$

Franke, Manin and Tschinkel then applied the work of Langlands and have proven the following results:

- $\zeta_{V,H}(s)$ converges for $\mathrm{Re}(s) > 1$,
- It extends to a meromorphic function on the projective plane,
- It has a pole of order $t = \mathrm{rk}\,\mathrm{Pic}(V)$ at $s = 1$,
- There is a explicit formula for the leading term of the development of $\zeta_{V,H}(s)$ in Laurent series at $s = 1$:

$$\lim_{s \to 1} (s-1)^t \zeta_{V,H}(s) = \prod_{\alpha \in \Phi_P - \Delta_P} \frac{\xi_K(\langle \check{\alpha}, \rho_B \rangle)}{\xi_K(\langle \check{\alpha}, \rho_B \rangle + 1)} \prod_{\alpha \in \Delta_P} \frac{\mathrm{res}_{s=1}\, \xi}{\xi_K(2)\langle \check{\alpha}, 2\rho_P \rangle},$$

where

$$\xi_K(s) = d_K^{s/2} (\pi^{-s/2}\Gamma(s/2))^{r_1} ((2\pi)^{-s}\Gamma(s))^{r_2} \zeta_K(s),$$

r_1 being the number of real places of K and r_2 the number of complex places. This limit may be reinterpreted as

$$\lim_{s\to 1}(s-1)^t\zeta_{V,H}(s) = \prod_{\alpha\in\Delta_P}\frac{1}{\langle\check{\alpha},2\rho_P\rangle}\boldsymbol{\omega}_H(V(\boldsymbol{A}_K))$$

where $\boldsymbol{\omega}_H$ is a Tamagawa measure on $V(\boldsymbol{A}_K)$.

(2) The connection with Eisenstein series is also valid for global fields of finite characteristic and we may apply the work of Harder and Morris to get the following results:

- $Z_{V,h}(z)$ converges for $|z| < q^{-1}$,
- $Z_{V,h}(T)$ is a rational function,
- $Z_{V,h}(z)$ has a pole of order t at $z = q^{-1}$
- the leading term at $z = q^{-1}$, that is $\lim_{z\to q^{-1}}(z-q^{-1})^t Z_{V,h}(z)$ is given by

$$q^{\dim(V)(1-g)}\prod_{\alpha\in\Phi_P-\Delta_P}\frac{Z_K(q^{-\langle\check{\alpha},\rho_B\rangle})}{Z_K(q^{-\langle\check{\alpha},\rho_B\rangle-1})}\prod_{\alpha\in\Delta_P}\frac{\operatorname{res}_{z=q^{-1}}Z_K}{Z_K(q^{-2})\langle\check{\alpha},2\rho_P\rangle}$$

which may be reinterpreted as in the number field case.

(3) In the functional setting, we need to use an extension of $\mathscr{M}_k$ constructed by Denef et Loeser to give an analog to the last assertion. Let $\mathscr{M}_k^{\mathrm{loc}}$ be the ring $\mathscr{M}_k[\mathbf{L}^{-1}]$ and, for any integer n, let $F^n\mathscr{M}_k^{\mathrm{loc}}$ be the subgroup of $\mathscr{M}_k^{\mathrm{loc}}$ generated by the elements of the form $\mathbf{L}^{-i}[V]$ where $i - dim(V) \geqslant m$. Then $\widehat{\mathscr{M}}_k$ is the completion of $\mathscr{M}_k^{\mathrm{loc}}$ for this filtration.

- The varieties V_n verify:

$$\overline{\lim_{n\to+\infty}}\frac{\dim(V_n)}{n}\leqslant 1,$$

- $Z_{V,h}^{\mathrm{mot}}(T)$ is a rational function,
- the formal series

$$\Big(\prod_{\alpha\in\Delta_P}\Big(1-(\mathbf{L}T)^{\langle\check{\alpha},2\rho_P\rangle}\Big)\Big)Z_{V,h}^{\mathrm{mot}}(T)$$

  converges in $\widehat{\mathscr{M}}_k$ at $T = \mathbf{L}^{-1}$,
- at this point it takes the value

$$\mathbf{L}^{\dim(V)(1-g)}\prod_{\alpha\in\Phi_P-\Delta_P}\frac{Z_K^{\mathrm{mot}}(\mathbf{L}^{-\langle\check{\alpha},\rho_B\rangle})}{Z_K^{\mathrm{mot}}(\mathbf{L}^{-\langle\check{\alpha},\rho_B\rangle-1})}\prod_{\alpha\in\Delta_P}\frac{Z_K^{\mathrm{mot}}(T)(1-\mathbf{L}T)(\mathbf{L}^{-1})}{Z_K^{\mathrm{mot}}(\mathbf{L}^{-2})}$$

where Z_K^{mot} is the zeta function of the field defined by

$$Z_K^{\mathrm{mot}}(T) = \sum_{n\in\mathbf{N}} [\mathscr{C}^{(n)}]T^n,$$

$\mathscr{C}^{(n)}$ being the n-th symmetric power of $\mathscr{C}$. Kapranov proved that Z_K^{mot} verifies

$$Z_K^{\mathrm{mot}}(T) = \frac{P(T)}{(1-T)(1-\mathbf{L}T)}$$

for a polynomial P in $\mathscr{M}_k[T]$ of degree $2g$ which satisfies a functional equation. Once again this may be interpreted in terms of a Tamagawa number in a motivic setting.

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