APPLICATION OF MOTIVIC COMPLEXES TO NEGLIGIBLE CLASSES*

by

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Abstract. — Lichtenbaum's complex enables one to relate Galois cohomology to $\mathscr{K}$-cohomology groups. In this paper, we consider the first terms of the Hochschild-Serre spectral sequence for the cohomology of these complexes, which was developed by Kahn, in the case of quotients of "big" open sets in cellular varieties. In the particular case of a faithful representation W of a finite group G over an algebraically closed field k, this yields that the group of negligible classes in the cohomology group $H^3(G, \mathbf{Q}/\mathbf{Z}(2))$ is canonically isomorphic to the second equivariant Chow group of a point. It also implies that the unramified classes in the cohomology group $H^3(k(W)^G, (\mathbf{Q}/\mathbf{Z})'(2))$ come from the cohomology of G, which had been proved by Saltman when k is the field of complex numbers.

Using the motivic complexes of Voevodsky, we then prove similar results in degrees four and five.

Résumé. — Le complexe de Lichtenbaum fournit un lien entre la cohomologie galoisienne et les groupes de $\mathscr{K}$-cohomologie Dans ce texte nous considérons les premiers termes de la suite spectrale de Hochschild-Serre pour l'hypercohomologie de ces complexes, qui a été développée par Bruno Kahn, dans le cas du quotient d'un ouvert d'une variété cellulaire dont le complémentaire est de codimension assez grande. Dans le cas particulier d'une représentation fidèle W d'un groupe G sur un corps algébriquement clos k, cela implique que le groupe des classes négligeables dans le groupe de cohomologie $H^3(G, \mathbf{Q}/\mathbf{Z}(2))$ est canoniquement isomorphe au second groupe de Chow équivariant du point. Cela implique également que les classes non ramifiées dans le groupe de cohomologie $H^3(k(W)^G, (\mathbf{Q}/\mathbf{Z})'(2))$ proviennent de la cohomologie de G, ce qui avait été démontré par Saltman quand k est le corps des complexes.

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Contents

1. Introduction

The unramified cohomology groups were first developed by Colliot-Thélène and Ojanguren as invariants for stable rationality which generalize the unramified Brauer group. It has been used in [**CTO**] and [**Pe1**] to give new examples of unirational varieties which are not stably rational.

Unirational fields of special interest are given by Noether's problem: if G is a finite group and W a faithful representation of G over a field k, then the field of invariant functions $k(W)^G$ does not depend, up to stable equivalence, on W. The problem is to determine for which fields k and groups G the field $k(W)^G$ is stably rational. The first counter-example over $\mathbf{C}$ was constructed by Saltman in [**Sa1**] using the unramified Brauer group. Bogomolov [**Bo**] gave a complete description of the unramified Brauer group of the field $\mathbf{C}(W)^G$ in terms of the cohomology of the group G.

The study of the higher unramified cohomology groups for these fields is made more complicated by the existence of negligible classes in the cohomology of finite groups which vanish when lifted to Galois groups. The first interesting results about the third unramified cohomology group for such fields have been obtained by Saltman in [**Sa2**].

More precisely, he proved that this cohomology group for $k = \mathbf{C}$ is contained in the image of the inflation map

$$H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^3(k(W)^G, \mathbf{Q}/\mathbf{Z})$$

and that, if $H^3(G, \mathbf{Q}/\mathbf{Z})_{\mathrm{n}}$ is the kernel of this map and if G is a p-group, then there is a natural isomorphism

$$H^3(G, \mathbf{Q}/\mathbf{Z})_{\mathrm{n}}/H^3(G, \mathbf{Q}/\mathbf{Z})_{\mathrm{p}} + H^3(G, \mathbf{Q}/\mathbf{Z})_{\mathrm{c}} \xrightarrow{\sim} N^3(G)$$

where

$$H^3(G, \mathbf{Q}/\mathbf{Z})_{\mathrm{p}} = \mathrm{Ker}\Big(H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^3(G, \mathbf{C}(W)^*)\Big)$$

which may be computed in terms of the cohomology of G,

$$H^3(G,\mathbf{Q}/\mathbf{Z})_{\mathrm{c}} = \sum_{H\subsetneq G} \mathrm{Cores}_H^G H^3(H,\mathbf{Q}/\mathbf{Z})_{\mathrm{n}},$$

and N^3G is a kind of equivariant Chow group.

The connection between Chow groups of codimension 2 and restriction maps in degree 3 appears also in [**Pe2**], [**Pe3**] and [**Pe4**], where we describe for any generalized flag variety V an exact sequence

$$\begin{aligned} &H^1_{\mathrm{Zar}}(V,\mathscr{K}_2) \xrightarrow{j} (\mathrm{Pic}\, V_{k^s}\otimes k^{s*})^{\mathscr{G}} \\ &\qquad \to \mathrm{Ker}\Big(H^3(\mathscr{G},\mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(V),\mathbf{Q}/\mathbf{Z}(2))\Big) \to \mathrm{CH}^2(V)_{\mathrm{tors}} \to 0\end{aligned}$$

where k^s is a separable closure of k, $\mathscr{G} = \mathrm{Gal}(k^s/k)$, and $\mathscr{K}_i$ is the sheaf associated to the presheaf of Quillen's K-groups $U \mapsto K_i(U)$. This exact sequence was obtained using the work of Colliot-Thélène and Raskind on the $\mathscr{K}$-cohomology (see [**CTR**]) and a result of Bruno Kahn based on Lichtenbaum's complexes (see [**Li1**], [**Li2**], [**Li3**] and [**Kah1**]). This sequence was also considered by Merkur′ev who proved in [**Me1**] that the map j is injective.

More recently, Kahn gave in [**Kah2**] a direct proof of this exact sequence and a description of the unramified cohomology group of degree three of these twisted generalized flag varieties using the Hochschild-Serre spectral sequence for the hypercohomology of Lichtenbaum's complexes.

One of the purposes of this text is to show that an easy generalization of the results of Kahn enables one to state the results for generalized flag varieties and for finite groups in a uniform way.

In fact we prove that if G is a finite group, W a faithful representation of G over an algebraically closed field k of exponential characteristic p such that the complement of the open set U on which G acts freely in W has a codimension bigger than 4, then there is an exact sequence

$$\begin{aligned} &0 \to \mathrm{CH}^2_G(k) \to H^3(G,\mathbf{Q}/\mathbf{Z}(2)) \\ &\qquad \to H^0_{\mathrm{Zar}}\Big(U/G,\mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))\Big) \to H^0_{\mathrm{Zar}}\Big(W,\mathscr{H}^3_{\text{ét}}(\mathbf{Q}_p/\mathbf{Z}_p(2))\Big)\end{aligned}$$

where U/G is the quotient of U by G, $\mathrm{CH}^2_G(k)$ is the equivariant Chow group of $\mathrm{Spec}\, k$ and $\mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))$ is the sheaf corresponding to the presheaf

$$V \mapsto H^3_{\text{ét}}(V,\mathbf{Q}/\mathbf{Z}(2)).$$

The connection with the results of Saltman becomes clear if one takes into account the inclusions

$$H^3_{\mathrm{nr}/k}(k(W)^G,\mathbf{Q}/\mathbf{Z}(2))\subset H^0_{\mathrm{Zar}}(U/G,\mathscr{H}^3_{\mathrm{\acute{e}t}}(\mathbf{Q}/\mathbf{Z}(2)))$$
$$\subset H^3(k(W)^G,\mathbf{Q}/\mathbf{Z}(2)).$$

The second section of this paper contains a partial description of the $\mathscr{K}$-cohomology groups of big open sets in cellular varieties followed by an easy generalization of the results of Kahn, the third applies the previous computations to the case of finite groups and makes explicit the connection with Saltman's work and the fourth extends the results to higher degrees using the work of Voevodsky.

2. Hochschild-Serre spectral sequence for Lichtenbaum's complex

2.1. Notations. — In the sequel we use the following notations:

Notation 2.1.1. — For any field L, let $\overline{L}$ be an algebraic closure of L and L^s be the separable closure of L in $\overline{L}$. For any variety V over L we denote by $L(V)$ the function field of V and for any extension L' of L by $V_{L'}$ the product $V\times_{\operatorname{Spec}L}\operatorname{Spec}L'$. We put $V^s=V_{L^s}$. One denotes by $V^{(i)}$ the set of points of codimension i in V, and, for any $x\in V$, by $\kappa(x)$ its residue field. The Chow groups of cycles of codimension i on V modulo rational equivalence are denoted by $\mathrm{CH}^i(X)$.

If L is a field, let p be the exponential characteristic of L, that is 1 if L is of characteristic 0 and the usual characteristic otherwise. If n is prime to p and V a variety over L, let μ_n be the étale sheaf of n-th roots of unity and for any r and i in $\mathbf{Z}_{\geqslant 0}$, let $W_r\Omega^i_{V,\log}$ be the logarithmic part of the corresponding De Rham-Witt sheaf $W_r\Omega^i_V$ (see [**Il**, §I.5.7]). By [**BK**, corollary 2.8], for $V=\operatorname{Spec}L$ one has

$$W_r\Omega^j_{L,\log}(L)\xleftarrow{\sim}K^M_j(L)/p^rK^M_j(L).$$

If $n=n'p^r$ with $(n',p)=1$, then one puts

$$\mathbf{Z}/n\mathbf{Z}(j)=\mu_n^{\otimes j}\oplus W_r\Omega^j_{V,\log}[-j].$$

One then defines

$$\mathbf{Q}/\mathbf{Z}(j)=\varinjlim_n\mathbf{Z}/n\mathbf{Z}(j),\quad(\mathbf{Q}/\mathbf{Z})'(j)=\varinjlim_{(n,p)=1}\mathbf{Z}/n\mathbf{Z}(j)$$

and if l is a prime number

$$\mathbf{Q}_l/\mathbf{Z}_l(j) = \varinjlim_r \mathbf{Z}/l^r\mathbf{Z}(j).$$

If F is one of the above complexes of étale sheaves, we put

$$H^i_{\text{ét}}(V, F) = \mathbf{H}^i_{\text{ét}}(V, F)$$

and if L is a field $H^i(L, F) = H^i_{\text{ét}}(\operatorname{Spec} L, F)$. The Zariski sheaf corresponding to the presheaf $U \mapsto H^i_{\text{ét}}(U, F)$ is denoted by $\mathscr{H}^i_{\text{ét}}(F)$.

If V is an algebraic variety over L and U a Galois covering of V^s with a finite Galois group G, then there exists a finite Galois extension L' of L and a Galois étale covering U' of $V_{L'}$ with Galois group G such that there exists an isomorphism from U'_s to U over V_s. We shall say that the pair (L', U') represents the étale covering $U \to V^s$. We shall denote by $\operatorname{Gal}(U/V)$ the profinite group

$$\varprojlim_{(L', U')} \operatorname{Gal}(U'/V)$$

where (L', U') is taken over the pairs representing U/V^s and such that U' is Galois over V.

2.2. $\mathscr{K}$-cohomology of big open sets. — The following well known result is a direct consequence of the Brown-Gersten-Quillen spectral sequence.

Proposition 2.2.1. — *If X is a smooth variety over a field k and Y a subvariety of codimension at least c in X then*

$$H^i_{\text{Zar}}(X, \mathscr{K}_j) \xrightarrow{\sim} H^i_{\text{Zar}}(X - Y, \mathscr{K}_j)$$

if $i \leqslant c - 2$.

Proof. — By Gersten's resolution the groups $H^i_{\text{Zar}}(X, \mathscr{K}_j)$ are isomorphic to the homology groups of the complex

$$\bigoplus_{x \in X^{(i-1)}} K_{j-i+1}\kappa(x) \xrightarrow{\partial_{i-1}} \bigoplus_{x \in X^{(i)}} K_{j-i}\kappa(x) \xrightarrow{\partial_i} \bigoplus_{x \in X^{(i+1)}} K_{j-i-1}\kappa(x).$$

Since the codimension of Y is at least c, we have for $j \leqslant c - 1$ the equality

$$(X - Y)^{(j)} = X^{(j)}$$

and the residue map ∂_i is the same for X and $X - Y$ if $i \leqslant c - 1$. Therefore the homology groups coincide. $\square$

Let us recall the definition of cellular varieties.

Definition 2.2.1. — A variety X over a field k is called k-cellular if and only if there exists a sequence of closed subsets of X

$$\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_n = X$$

such that for $1 \leqslant i \leqslant n-1$, $Z_{i+1} - Z_i$ is isomorphic to an affine space over k.

Corollary 2.2.2. — *If X is a smooth cellular variety over k and Y a subvariety of codimension at least c in X, then, if $i \leqslant c-2$,*

$$H^i_{\mathrm{Zar}}(X - Y, \mathscr{K}_j) \xrightarrow{\sim} \mathrm{CH}^i(X) \otimes K_{j-i}k$$

where $\mathrm{CH}^i(X)$ is a finitely generated free module over $\mathbf{Z}$.

Proof. — By [**Kah3**, lemma 3.3], the group $\mathrm{CH}^*(X)$ is a free $\mathbf{Z}$-module of finite type, Then, using the proof of [**Pe4**, proposition 3.1], we get that the module

$$\bigoplus_{i,j\geqslant 0} H^i_{\mathrm{Zar}}(X, \mathscr{K}_{i+j})$$

is a free $\bigoplus_{j\geqslant 0} K_j k$-module with a basis given by any basis of $\bigoplus_{i\geqslant 0}\mathrm{CH}^i(X)$ over $\mathbf{Z}$. By the last proposition, if $i \leqslant c-2$, we get

$$H^i_{\mathrm{Zar}}(X - Y, \mathscr{K}_j) \xrightarrow{\sim} H^i_{\mathrm{Zar}}(X, \mathscr{K}_j) \xrightarrow{\sim} \mathrm{CH}^i(X) \otimes K_{j-i}k. \quad \square$$

2.3. The main result in degree three. — Following the method described by Kahn in [**Kah2**] we shall now use the Hochschild-Serre spectral sequence for Lichtenbaum's complexes to get information about the kernel and cokernel of the map

$$H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3_{\mathrm{nr}/k}(k(V), \mathbf{Q}/\mathbf{Z}(2))$$

for varieties having an étale covering which is a big open set in a cellular variety.

Theorem 2.3.1. — *Let $U \to V$ be a finite étale Galois covering of smooth geometrically integral varieties over a perfect field k which is of the form*

$$U \to V_{k'} \to V$$

where k' is a finite separable extension of k. Let G be the Galois group of this covering. Assume that there is an embedding of U in a k'-cellular variety X such that

$$\mathrm{codim}_X(X - U) \geqslant 4,$$

and assume moreover that the action of G on U extends to an action of G on X over k. Let $\mathscr{G} = \mathrm{Gal}(U^s/V)$. Then the following assertions hold:

(i) There is an exact sequence

$$
\begin{aligned}
H^2(\mathcal{G},\mathbf{Q}/\mathbf{Z}(2)) &\to H^1_{\mathrm{Zar}}(V,\mathcal{K}_2) \to (\operatorname{Pic} X\otimes k')^G \\
&\to \operatorname{Ker}\Big(H^3(\mathcal{G},\mathbf{Q}/\mathbf{Z}(2)) \to H^3(k'(X)^G,\mathbf{Q}/\mathbf{Z}(2))\Big) \to \mathrm{CH}^2(V)_{\mathrm{tors}} \\
&\to H^1(G,\operatorname{Pic} X\otimes k'^*)
\end{aligned}
$$

(ii) There is a canonical morphism η from the group

$$
\frac{\operatorname{Ker}\Big(H^0_{\mathrm{Zar}}(V,\mathcal{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))) \to H^0_{\mathrm{Zar}}(X^s,\mathcal{H}^3_{\text{ét}}(\mathbf{Q}_p/\mathbf{Z}_p(2)))\Big)}{\operatorname{Im}\Big(H^3(\mathcal{G},\mathbf{Q}/\mathbf{Z}(2))\Big)}
$$

to the group

$$\mathrm{CH}^2(X)^G/\,\mathrm{CH}^2(V)$$

such that

$$\operatorname{Ker}\eta \subset \operatorname{Coker}\Big(\mathrm{CH}^2(V)_{\mathrm{tors}} \to H^1(G,\operatorname{Pic} X\otimes k'^*)\Big)$$

Remarks 2.3.2. — (i) The group $\mathrm{CH}^2(V)$ may in fact be interpreted as the equivariant Chow group of X. Indeed these two groups coincide when G is finite (see [**EG**, proposition 8]).

(ii) The group $H^0_{\mathrm{Zar}}\Big(X^s,\mathcal{H}^3_{\text{ét}}(\mathbf{Q}_p/\mathbf{Z}_p(2))\Big)$ is trivial if X^s is complete (see [**Kah2**, remark, page 397]).

(iii) The assumption that k is perfect is only needed for the p-part of the results.

Before proving this theorem, we give an example.

Example 2.3.1. — Let V be a generalized flag variety, that is a projective variety over k which is homogeneous under the action of a connected linear algebraic group G and such that the stabilizer of a point of $V(k^s)$ is a standard parabolic subgroup of G^s. Then Bruhat's decomposition yields a cellular decomposition of V over any Galois extension k' of k splitting the group G and over which V has a rational point. Moreover it yields a basis of $\operatorname{Pic} V^s$ which is globally invariant under the action of the Galois group of k. We get the following exact sequence

$$
\begin{aligned}
H^1_{\mathrm{Zar}}(V,\mathcal{K}_2) &\to (\operatorname{Pic} V^s\otimes k^{s*})^{\mathcal{G}} \\
&\to \operatorname{Ker}\Big(H^3(k,\mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(V),\mathbf{Q}/\mathbf{Z}(2))\Big) \to \mathrm{CH}^2(V)_{\mathrm{tors}} \to 0
\end{aligned}
$$

where $\mathcal{G} = \mathrm{Gal}(k^s/k)$. This sequence has been studied with more details in [**Pe4**] where it was obtained using results of Colliot-Thélène and Raskind [**CTR**] and Kahn [**Kah1**].

In section 3 we shall study the applications of theorem 2.3.1 to negligible classes and unramified cohomology. We now turn to its proof.

2.4. Proof of theorem 2.3.1. — The Hochschild-Serre spectral sequence for Lichtenbaum's complexes was described and used by Kahn in [**Kah1**] and [**Kah2**]. We use it in a slightly more general setting.

For any smooth connected variety X over k we consider Lichtenbaum's complex $\Gamma(2)=(\Gamma(2,X)^i)_{i\in\mathbf{Z}}$ (see [**Li1**], [**Li2**] and [**Li3**]). By [**Kah2**, theorem 1.1] the hypercohomology groups of these complexes are given by

$$\mathbf{H}^i_{\text{ét}}(X,\Gamma(2))=\begin{cases}0 & \text{for } i\leqslant 0,\\ K_3(k(X))_{\text{ind}} & \text{if } i=1,\\ H^0_{\text{Zar}}(X,\mathscr{K}_2) & \text{if } i=2,\\ H^1_{\text{Zar}}(X,\mathscr{K}_2) & \text{if } i=3,\\ \operatorname{Coker}\mathrm{cl}^2_X & \text{if } i=5,\\ H^{i-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(2)) & \text{if } i\geqslant 6,\end{cases}\tag{2.4.1}$$

where cl^2_X is the divisible cycle class map

$$\mathrm{CH}^2 X\otimes\mathbf{Q}/\mathbf{Z}\to H^4_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(2))$$

and there is an exact sequence

$$0\to\mathrm{CH}^2 X\to\mathbf{H}^4_{\text{ét}}(X,\Gamma(2))\to H^0_{\text{Zar}}\Big(X,\mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))\Big)\to 0.$$

As in [**Mi**, theorem III.2.20] and [**Kah2**] we get a Hochschild-Serre spectral sequence for these hypercohomology groups

$$E_2^{p,q}=H^p(\mathscr{G},\mathbf{H}^q_{\text{ét}}(U^s,\Gamma(2)))\Rightarrow\mathbf{H}^{p+q}_{\text{ét}}(V,\Gamma(2)).$$

By [**MS2**, §11], the canonical map

$$K_3(k^s)_{\text{ind}}\to K_3(k^s(U))_{\text{ind}}$$

is injective with a uniquely divisible cokernel. And by corollary 2.2.2 we have isomorphisms

$$\begin{aligned}H^0(U^s,\mathscr{K}_2)&\xrightarrow{\sim}\mathrm{CH}^0(X)\otimes K_2k^s\xrightarrow{\sim}K_2k^s,\\ H^1(U^s,\mathscr{K}_2)&\xrightarrow{\sim}\operatorname{Pic}(X)\otimes k^{s*}.\end{aligned}$$

If n is prime to p there is an exact sequence [**MS1**, theorem 11.5], [**Su2**], [**Le**]

$$0 \to \mu_n^{\otimes 2}(k) \to K_3(k)_{\mathrm{ind}} \xrightarrow{n} K_3(k)_{\mathrm{ind}} \to H^1(k, \mu_n^{\otimes 2}) \\ \to K^2(k) \xrightarrow{n} K^2(k) \to H^2(k, \mu_n^{\otimes 2}) \to 0.$$

We get that

$$(E_2^{q,1})' \xrightarrow{\sim} H^q(\mathcal{G}, (\mathbf{Q}/\mathbf{Z})'(2)) \quad \text{if } q \geqslant 2$$

and

$$(E_2^{q,2})' = 0 \quad \text{if } q \geqslant 1,$$

where the $'$ means that we consider only the prime to p part of the groups. Therefore the spectral sequence yields an exact sequence

$$H^2(\mathcal{G}, (\mathbf{Q}/\mathbf{Z})'(2)) \to H^1_{\mathrm{Zar}}(V, \mathcal{K}_2)' \to (\mathrm{Pic}(X^s) \otimes k^{s*})'^{\mathcal{G}} \to H^3(\mathcal{G}, (\mathbf{Q}/\mathbf{Z})'(2)) \\ \to \mathrm{Ker}\Big(\mathbf{H}^4_{\text{ét}}(V, \Gamma(2))' \to \mathbf{H}^4_{\text{ét}}(U^s, \Gamma(2))\Big) \to H^1(\mathcal{G}, \mathrm{Pic}(X^s) \otimes k^{s*})'.$$

On the other hand, by [**MS2**], the p-part of $K^3(k^s)_{\mathrm{ind}}$ is uniquely divisible and by [**Su2**] and Bloch-Kato's theorem there are exact sequences

$$0 \to K_2(k^s) \xrightarrow{p^r} K_2(k^s) \to W_r\Omega^2_{k^s,\log}(k^s) \to 0.$$

and

$$K_2(k) \xrightarrow{p^r} K_2(k) \to W_r\Omega^2_{k,\log}(k) \to 0$$

This implies that

$$E_2^{q,1} \otimes \mathbf{Z}_{(p)} = 0 \quad \text{if } q \geqslant 1,$$
$$E_2^{q,2} \otimes \mathbf{Z}_{(p)} \xrightarrow{\sim} H^{q+1}(\mathcal{G}, \mathbf{Q}_p/\mathbf{Z}_p(2)) \quad \text{if } q \geqslant 1.$$

Thus the p-part of the spectral sequence yields a similar exact sequence for the p-parts of the groups. We get an exact sequence

(2.4.2)
$$H^2(\mathcal{G}, \mathbf{Q}/\mathbf{Z}(2)) \to H^1_{\mathrm{Zar}}(V, \mathcal{K}_2) \to (\mathrm{Pic}(X^s) \otimes k^{s*})^{\mathcal{G}} \to H^3(\mathcal{G}, \mathbf{Q}/\mathbf{Z}(2)) \\ \to \mathrm{Ker}\Big(\mathbf{H}^4_{\text{ét}}(V, \Gamma(2)) \to \mathbf{H}^4_{\text{ét}}(U^s, \Gamma(2))\Big) \to H^1(\mathcal{G}, \mathrm{Pic}(X^s) \otimes k^{s*}).$$

Moreover, since k is perfect, by Bloch-Ogus spectral sequence [**BO**] and the corresponding one for $W_r\Omega^j_{X,\log}[-j]$ [**GS**, theorem 1.4] and by the fact that sheafification commutes with direct limits, we have that the group $H^0_{\mathrm{Zar}}(U^s, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2)))$ is isomorphic to $H^0_{\mathrm{Zar}}(X^s, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2)))$. But X^s contains an affine space $\mathbf{A}^N_{k^s}$ and therefore this group is contained in the group $H^0_{\mathrm{Zar}}(\mathbf{A}^N_{k^s}, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2)))$ the prime to p part of which is trivial by homotopy invariance. Therefore we get that there is an exact sequence

$$0 \to \mathrm{CH}^2(X^s) \to \mathbf{H}^4_{\text{ét}}(U^s, \Gamma(2)) \to H^0_{\mathrm{Zar}}(X^s, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}_{p'}/\mathbf{Z}_p(2))) \to 0. \tag{2.4.3}$$

We also have an exact sequence

$$0 \to \mathrm{CH}^2(V) \to \mathbf{H}^4_{\text{ét}}(V, \Gamma(2)) \to H^0_{\mathrm{Zar}}(V, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))) \to 0. \tag{2.4.4}$$

We consider the following groups

$$\begin{aligned}
A &= \mathrm{Coker}\Big((\mathrm{Pic}(X^s) \otimes k^{s*})^{\mathscr{G}} \to H^3(\mathscr{G}, \mathbf{Q}/\mathbf{Z}(2))\Big),\\
B &= \mathrm{Ker}\Big(H^0_{\mathrm{Zar}}(V, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))) \to H^0_{\mathrm{Zar}}(X^s, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}_{p'}/\mathbf{Z}_p(2)))\Big),\\
K &= \mathrm{Ker}(A \to B),\\
C &= \mathrm{Coker}(A \to B),\\
D &= \mathrm{Ker}(\mathrm{CH}^2(V) \to \mathrm{CH}^2(X^s)),\\
E &= \mathrm{Coker}(\mathrm{CH}^2(V) \to \mathrm{CH}^2(X^s)^{\mathscr{G}}),\\
M &= \mathrm{Ker}(\mathbf{H}^4_{\text{ét}}(V, \Gamma(2)) \to \mathbf{H}^4_{\text{ét}}(U^s, \Gamma(2))).
\end{aligned}$$

Then the sequences (2.4.3) and (2.4.4) and the snake lemma gives an exact sequence

$$0 \to D \to M \to B \to E.$$

But (2.4.2) implies the exactness of the complex

$$0 \to A \to M \to H^1(\mathscr{G}, \mathrm{Pic}(X^s) \otimes k^{s*}).$$

Moreover we get a canonical map

$$\psi : D \to H^1(\mathscr{G}, \mathrm{Pic}(X^s) \otimes k^{s*})$$

and we put $U = \operatorname{Ker}\psi$ and $V = \operatorname{Coker}\psi$. Thus we obtain a commutative diagram with exact lines

$$
\begin{tikzcd}
 & 0 \arrow[dr] & & & & & \\
0 \arrow[dr] & & K \arrow[dr] & 0 \arrow[d] & & & \\
 & U \arrow[dr] & & A \arrow[d] \arrow[dr] & & & \\
 & 0 \arrow[r] & D \arrow[r] \arrow[dr] & M \arrow[r] \arrow[d] & B \arrow[r] \arrow[dr] & E & \\
 & & & H^1(\mathcal{G}, \operatorname{Pic}(X^s) \otimes k^{s*}) \arrow[dr] & & C \arrow[u] \arrow[dr] & \\
 & & & & V \arrow[dr] & & 0 \\
 & & & & & 0 &
\end{tikzcd}
$$

and a little diagram chase yields an isomorphism from K to U and an injection

$$\operatorname{Ker}(C \to E) \to V.$$

Using the definitions, one gets an exact sequence

$$
\begin{aligned}
H^2(\mathcal{G}, \mathbf{Q}/\mathbf{Z}(2)) \to H^1_{\mathrm{Zar}}(V, \mathcal{K}_2) \to (\operatorname{Pic}(X^s) \otimes k^{s*})^{\mathcal{G}} \\
\to \operatorname{Ker}\Big(H^3(\mathcal{G}, \mathbf{Q}/\mathbf{Z}(2)) \to H^0_{\mathrm{Zar}}(V, \mathcal{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2)))\Big) \\
\to \operatorname{Ker}(\mathrm{CH}^2(V) \to \mathrm{CH}^2(X^s)^{\mathcal{G}}) \to H^1(\mathcal{G}, \operatorname{Pic}(X^s) \otimes k^{s*})
\end{aligned}
$$

and an injection from the homology of the complex

$$
\begin{aligned}
H^3(\mathcal{G}, \mathbf{Q}/\mathbf{Z}(2)) \to \operatorname{Ker}\Big(H^0_{\mathrm{Zar}}(V, \mathcal{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))) \to H^0_{\mathrm{Zar}}(X^s, \mathcal{H}^3_{\text{ét}}(\mathbf{Q}_p/\mathbf{Z}_p(2)))\Big) \\
\to \operatorname{Coker}(\mathrm{CH}^2(V) \to \mathrm{CH}^2(X^s)^{\mathcal{G}})
\end{aligned}
$$

to

$$\operatorname{Coker}\Big(\operatorname{Ker}\Big(\mathrm{CH}^2(V) \to \mathrm{CH}^2(X^s)\Big) \to H^1(\mathcal{G}, (\operatorname{Pic} X^s \otimes k^s)^*)\Big).$$

But $\mathrm{CH}^2(X^s)$ is a free abelian group, therefore

$$\mathrm{CH}^2(V)_{\mathrm{tors}} \subset \operatorname{Ker}(\mathrm{CH}^2(V) \to \mathrm{CH}^2(X^s)).$$

A transfer argument implies the inverse inclusion. We also have

$$\mathrm{CH}^2(X^s) = \mathrm{CH}^2(X) \quad \text{and} \quad \mathrm{CH}^2(X^s)^{\mathscr{G}} = \mathrm{CH}^2(X)^G.$$

Similarly $\mathrm{Pic}\, X^s = \mathrm{Pic}\, X$ and therefore

$$(\mathrm{Pic}\, X^s \otimes k^{s*})^{\mathscr{G}} = (\mathrm{Pic}\, X \otimes k')^G$$

and by using the inflation restriction exact sequence and Hilbert's theorem 90

$$H^1(\mathscr{G}, \mathrm{Pic}\, X^s \otimes k^{s*}) \xrightarrow{\sim} H^1(G, \mathrm{Pic}\, X \otimes k'^*).$$

Finally, since k is perfect, the Bloch-Ogus spectral sequence yields an embedding

$$H^0_{\mathrm{Zar}}(V, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))) \subset H^3(k(V), \mathbf{Q}/\mathbf{Z}(2)) = H^3(k'(X)^G, \mathbf{Q}/\mathbf{Z}(2)). \quad \square$$

3. Application to the case of finite groups

3.1. Negligible classes. — The notion of negligible classes has been introduced by Serre in his lecture at the Collège de France [**Se**]. We shall use a weaker condition than the one he used.

Definition 3.1.1. — Let H be a finite group, M be a H-module and E be a field. Then a class λ in $H^i(H, M)$ is said to be totally E-negligible if and only if for any extension F of E and any morphism

$$\rho : \mathrm{Gal}(F^s/F) \to H$$

the image of λ by ρ^* is trivial in $H^i(F, M)$.

In the following, we restrict ourselves to the case where E is separably closed and $M = \mathbf{Q}/\mathbf{Z}(i)$ for some integer i. The action of G on $(\mathbf{Q}/\mathbf{Z})'(i)$ is in fact trivial, but we keep the twist to get canonical morphisms.

If H is a finite group and W a faithful representation of G over a field E and $n \in \mathbf{Z}_{>0}$ then for any $g \in G$,

$$(W^n)^g = (W^g)^n,$$

where W^g is the subspace of invariant elements under g, and thus it has a codimension bigger or equal to n. Let U_n be the open set in W^n on which G acts freely. We get that $\mathrm{codim}_{W^n} W^n - U_n \geqslant n$. We recall the definition of equivariant Chow groups (see Edidin and Graham [**EG**, §2.2]).

Definition 3.1.2. — If Y is a smooth geometrically integral variety equipped with a G-action over k then

$$\mathrm{CH}^i_G(Y) = \mathrm{CH}^i((Y \times U_{i+1})/G).$$

We put $\mathrm{CH}^i_G(k) = \mathrm{CH}^i_G(\mathrm{Spec}\, k)$.

If k is separably closed, there is a natural notion of cycle class map going from $\mathrm{CH}^i_G(k)$ with value in $H^{2i-1}(G, (\mathbf{Q}/\mathbf{Z})'(i))$. Its construction is based on the following lemma:

Lemma 3.1.1. — *If k is a separably closed field of exponential characteristic p and n a positive integer with $(n,p) = 1$, then for any $j < i$,*

$$H^j(G, \mu_n^{\otimes m}) \xrightarrow{\sim} H^j_{\text{ét}}(U_i/G, \mu_n^{\otimes m}).$$

Proof. — By the Bloch-Ogus spectral sequence

$$\bigoplus_{x \in U_i^{(p)}} H^{q-p}(\kappa(x), \mu_n^{\otimes m-p}) \Rightarrow H^{p+q}_{\text{ét}}(U_i, \mu_n^{\otimes m})$$

and the similar one for W^i we get that if $j < i$

$$H^j_{\text{ét}}(U_i, \mu_n^{\otimes m}) \xrightarrow{\sim} H^j_{\text{ét}}(W^i, \mu_n^{\otimes m}) \xrightarrow{\sim} H^j_{\text{ét}}(k, \mu_n^{\otimes m}) = \begin{cases} 0 & \text{if } j \neq 0 \\ \mu_n^{\otimes m}(k) & \text{if } j = 0. \end{cases}$$

Using the Hochschild-Serre spectral sequence

$$H^p(G, H^q_{\text{ét}}(U_i, \mu_n^{\otimes m})) \Rightarrow H^{p+q}_{\text{ét}}(U_i/G, \mu_n^{\otimes m})$$

we get the isomorphism of the lemma. □

Definition 3.1.3. — Let k be a separably closed field of exponential characteristic p and G be a finite group. For any i in $\mathbf{Z}$ and any j in $\mathbf{Z}_{\geqslant 0}$ we put

$$H^j(G, \hat{\mathbf{Z}}'(i)) = \varprojlim_{(n,p)=1} H^j(G, \mu_n^{\otimes i}(k)).$$

If $j > 0$ there is a canonical isomorphism

$$H^j(G, (\mathbf{Q}/\mathbf{Z})'(i)) \xrightarrow{\sim} H^{j+1}(G, \hat{\mathbf{Z}}'(i)).$$

The cycle class map

$$\mathrm{cl}_i : \mathrm{CH}^i_G(k) \to H^{2i}(G, (\mathbf{Q}/\mathbf{Z})'(i))$$

is defined as the composite map

$$\mathrm{CH}^i_G(k) \xrightarrow{\sim} \mathrm{CH}^i(U_{2i+1}/G) \to \varprojlim_{(n,p)=1} \mathrm{CH}^i(U_{2i+1}/G)/n$$

$$\xrightarrow{\mathrm{cl}_i} \varprojlim_{(n,p)=1} H^{2i}_{\text{ét}}(U_{2i+1}/G, \mu_n^{\otimes i}) \xrightarrow{\sim} \varprojlim_{(n,p)=1} H^{2i}(G, \mu_n^{\otimes i}) \xrightarrow{\sim} H^{2i-1}(G, (\mathbf{Q}/\mathbf{Z})'(i)).$$

Remark 3.1.2. — if k is the field of complex numbers $\mathbf{C}$ then, modulo the isomorphism

$$\begin{array}{rcl} \mathbf{Q}/\mathbf{Z} & \to & \mathbf{Q}/\mathbf{Z}(1) \\ a & \mapsto & \exp(2i\pi a) \end{array}$$

the cycle class map coincides with the usual one from $\mathrm{CH}^i_G(\mathbf{C})$ to $H^{2i}(G, \mathbf{Z})$, defined using the classifying space (see [**To**]).

Example 3.1.1. — If k is separably closed, we consider the short exact sequence of G-modules

$$0 \to k^* \to k(W^2)^* \to \operatorname{Div} W^2 \to 0,$$

we get a long exact sequence

$$0 \to k^* \to k(U_2/G)^* \to \operatorname{Div}(U_2/G) \to H^1(G, k^*) \to H^1(G, k(W^2)^*).$$

But by Hilbert's theorem 90 $H^1(G, k(W^2)^*)$ is trivial and we get an isomorphism

$$\mathrm{cl}_1 : \operatorname{Pic}_G k \xrightarrow{\sim} H^1(G, \mathbf{Q}/\mathbf{Z}(1)),$$

which extends the cycle class map.

The theorem 2.3.1 has the following corollary.

Corollary 3.1.3. — *If k is an algebraically closed field and G a finite group, then the equivariant Chow group $\mathrm{CH}^2_G(k)$ is canonically isomorphic to the group of totally k-negligible classes in $H^3(G, \mathbf{Q}/\mathbf{Z}(2))$.*

Remarks 3.1.4. — (i) The map

$$\Phi_G : \mathrm{CH}^2_G(k) \to H^3(G, \mathbf{Q}/\mathbf{Z}(2))$$

extends the cycle class map cl_2.

(ii) The injectivity of this map follows from [**MS1**, corollary 18.3].

Proof. — First of all, as was pointed out by Serre, the group of totally k-negligible classes in $H^3(G,\mathbf{Q}/\mathbf{Z}(2))$ coincides with the kernel of the map

$$H^3(G,\mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(W)^G,\mathbf{Q}/\mathbf{Z}(2))$$

where W is an arbitrary faithful representation of G over k. Indeed one of the inclusion is obvious and if γ belongs to this kernel, if K is an extension of k, and

$$\rho : \mathrm{Gal}(K^s/K) \to G$$

any map then we may assume, without loss of generality, that ρ is surjective. Let K' correspond to the kernel of ρ. By the no-name lemma $K'(W)^G$ is rational over K. Thus the map

$$H^3(\mathrm{Gal}(K^s/K),\mathbf{Q}/\mathbf{Z}(2)) \to H^3(\mathrm{Gal}(K(W)^s/K'(W)^G),\mathbf{Q}/\mathbf{Z}(2))$$

is injective and there is a commutative diagram

$$\begin{tikzcd} H^3(G,\mathbf{Q}/\mathbf{Z}(2)) \arrow[r] \arrow[d] & H^3(\mathrm{Gal}(K^s/K),\mathbf{Q}/\mathbf{Z}(2)) \arrow[d] \\ H^3(\mathrm{Gal}(k(W)^s/k(W)^G),\mathbf{Q}/\mathbf{Z}(2)) \arrow[r] & H^3(\mathrm{Gal}(K(W)^s/K'(W)^G),\mathbf{Q}/\mathbf{Z}(2)). \end{tikzcd}$$

We may apply theorem 2.3.1 to $W' = W^4$ and we get an exact sequence

$$0 \to \mathrm{Ker}\Big(H^3(G,\mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(W')^G,\mathbf{Q}/\mathbf{Z}(2))\Big) \to CH^2_G(k)_{\mathrm{tors}} \to 0. \quad \square$$

Let us now compare corollary 3.1.3 with the corresponding result of Saltman. We first recall the definition of the groups considered by Saltman.

Definition 3.1.4. — Let l be a prime number, G an l-group, and W a faithful representation of G over $\mathbf{C}$. Then Z_lG is the free $\mathbf{F}_l$-module over the set of irreducible G-invariant subvarieties of codimension 2 in W. For any G-invariant closed irreducible subvariety of codimension 1 in W and any f in $\mathbf{C}(W)$ such that f^n is invariant under G for some $n \geqslant 1$, one defines $\mathrm{Div}(f)$ in $Z_l(G)$ as the class of

$$\sum_Y \nu_Y(f)Y$$

where Y goes over the set defining Z_lG. Let $R_l(G)$ be the subgroup of $Z_l(G)$ generated by these divisors. Then

$$N^3G = Z_lG/R_lG.$$

Let $H^3(G,\mathbf{Q}/\mathbf{Z}(i))_{\mathrm{n}}$ denote the set of totally $\mathbf{C}$-negligible classes. It contains two subgroups, namely the group of permutation negligible classes

$$H^3(G,\mathbf{Q}/\mathbf{Z}(i))_{\mathrm{p}} = \mathrm{Ker}(H^3(G,\mathbf{Q}/\mathbf{Z}(i)) \to H^3(G,\mathbf{C}(W)^*))$$

and

$$H^3(G,\mathbf{Q}/\mathbf{Z}(i))_{\mathrm{c}} = \sum_{H \subsetneq G} \mathrm{Cores}_H^G H^3(G,\mathbf{Q}/\mathbf{Z}(i))_{\mathrm{n}}.$$

Then in [**Sa2**, theorem 4.13], Saltman proved that there is a canonical isomorphism

$$H^3(G,\mathbf{Q}/\mathbf{Z})_{\mathrm{n}}/H^3(G,\mathbf{Q}/\mathbf{Z})_{\mathrm{p}} + H^3(G,\mathbf{Q}/\mathbf{Z})_{\mathrm{c}} \xrightarrow{\sim} N^3(G).$$

Proposition 3.1.5. — *If G is an l-group and W a representation of G of the form W'^4 where W' is a faithful representation of G over $\mathbf{C}$, then there is a commutative diagram*

$$\begin{array}{ccc}
\mathrm{CH}^2_G(\mathbf{C}) & \overset{\Phi_G}{\underset{\sim}{\longrightarrow}} & H^3(G,\mathbf{Q}/\mathbf{Z}(2))_n \\
\big\downarrow & & \big\downarrow \\
N^3(G) & \xrightarrow{\sim} & H^3(G,\mathbf{Q}/\mathbf{Z}(2))_n/\Big(H^3(G,\mathbf{Q}/\mathbf{Z}(2))_p + H^3(G,\mathbf{Q}/\mathbf{Z}(2))_c\Big).
\end{array}$$

In order to prove this proposition, we first need to prove that the canonical morphism Φ_G is compatible with corestriction and cup-product.

Notations 3.1.5. — If H is a subgroup of a finite group G, W a faithful representation of G, and U_i the open set in W^i on which G acts freely, then there is an étale covering

$$U_{i+1}/H \xrightarrow{\pi} U_{i+1}/G.$$

It induces a map

$$\pi_* : \mathrm{CH}^i(U_{i+1}/H) \to \mathrm{CH}^i(U_{i+1}/G)$$

and thus a map

$$\mathrm{CH}^i_H(k) \to \mathrm{CH}^i_G(k)$$

which will be denoted by Cores_H^G.

Lemma 3.1.6. *— If H is a subgroup of a finite group G, then the following diagram commutes:*

$$\begin{tikzcd} \mathrm{CH}^2_H(k) \arrow[r, "\Phi_H", "\sim"'] \arrow[d, "\mathrm{Cores}^G_H"'] & H^3(H, \mathbf{Q}/\mathbf{Z}(2))_n \arrow[d, "\mathrm{Cores}^G_H"'] \\ \mathrm{CH}^2_G(k) \arrow[r, "\Phi_G", "\sim"'] & H^3(G, \mathbf{Q}/\mathbf{Z}(2))_n. \end{tikzcd}$$

Proof. — The Hochschild-Serre spectral sequence

$$H^p(G, \mathbf{H}^q_{\text{ét}}(U_4, F)) \Rightarrow \mathbf{H}^p_{\text{ét}}(U_4/G, F)$$

where F is a complex of étale sheaves on U_4/G is compatible with restriction and morphisms of complexes.

If F is an étale sheaf on U_4/H, then we define $\mathrm{Ind}^G_H F$ as the direct image of F by the canonical projection

$$\pi : U_4/H \to U_4/G.$$

But for any étale map $U \to U_4/H$ there is a canonical map

$$U \to U \times_{U_4/G} U_4/H$$

which induces a map $\pi^*\pi_* F \to F$ and therefore maps

$$H^q_{\text{ét}}(U_4/G, \mathrm{Ind}^G_H F) \to H^q_{\text{ét}}(U_4/H, \pi^* \mathrm{Ind}^G_H F) \to H^q_{\text{ét}}(U_4/H, F).$$

Since π_* is exact in this case, the composite maps are isomorphisms, which is analoguous to Shapiro's lemma. The corresponding isomorphisms for hypercohomology

$$\mathbf{H}^q_{\text{ét}}(U_4/G, \mathrm{Ind}^G_H F) \xrightarrow{\sim} \mathbf{H}^q_{\text{ét}}(U_4/H, F)$$

are compatible with the Hochschild-Serre spectral sequence. If F is defined over U_4/G, then by [**SGA4**, exposé XVIII, théorème 2.9] there is a transfer map

$$\mathrm{Tr} : \pi_*\pi^* F \to F.$$

The corestriction may be defined as the composite of the map induced by Tr and the inverse of the Shapiro isomorphism. Thus the Hochschild-Serre spectral sequence is compatible with the corestriction. Therefore we get a commutative diagram for the prime to p part

$$\begin{tikzcd}
H^3(H,(\mathbf{Q}/\mathbf{Z})'(2)) \arrow[r, "\mathrm{Cores}_H^G"] \arrow[d] & H^3(G,(\mathbf{Q}/\mathbf{Z})'(2)) \arrow[d] \\
H^3\Big(H,\mathbf{H}^1_{\text{ét}}(U_4,\Gamma(2))\Big)' \arrow[r, "\mathrm{Cores}_H^G"] \arrow[d] & H^3\Big(G,\mathbf{H}^1_{\text{ét}}(U_4,\Gamma(2))\Big)' \arrow[d] \\
\mathbf{H}^4_{\text{ét}}(U_4,\Gamma(2))' \arrow[r, "\mathrm{Cores}_H^G"] & \mathbf{H}^4_{\text{ét}}(U_4,\Gamma(2))'
\end{tikzcd} \tag{3.1.1}$$

as well as a similar one for the p-part.

The long exact sequence of hypercohomology with support is also compatible with morphisms of complexes and contravariant for étale coverings. Thus they are compatible with corestrictions. Using [**CTHK**, §I.1], we get that the coniveau spectral sequence

$$E_1^{p,q} = \coprod_{x\in X^{(p)}} \mathbf{H}_x^{p+q}(U_4/G,\Gamma(2)) \Rightarrow \mathbf{H}^{p+q}_{\text{ét}}(U_4/G,\Gamma(2))$$

and the similar one for U_4/H are compatible with corestrictions. A similar statement holds for the isomorphisms (see [**Kah2**, theorem 6.1])

$$H^p(U_4/G,\mathscr{H}^q_{\text{ét}}(\Gamma(2))) \xrightarrow{\sim} E_2^{p,q}.$$

Therefore we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{CH}^2(U_4/H) \arrow[r, "\mathrm{Cores}_H^G"] \arrow[d] & \mathrm{CH}^2(U_4/G) \arrow[d] \\
\mathbf{H}^4_{\text{ét}}(U_4/H,\Gamma(2)) \arrow[r, "\mathrm{Cores}_H^G"] \arrow[d] & \mathbf{H}^4_{\text{ét}}(U_4/G,\Gamma(2)) \arrow[d] \\
H^0_{\mathrm{Zar}}\Big(U_4/H,\mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))\Big) \arrow[r, "\mathrm{Cores}_H^G"] \arrow[d] & H^0_{\mathrm{Zar}}\Big(U_4/G,\mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))\Big) \arrow[d] \\
0 & 0
\end{tikzcd} \tag{3.1.2}$$

where the vertical lines are exact by [**Kah2**, theorem 1.1].

The lemma follows from the commutativity of the diagrams (3.1.1) et (3.1.2). □

Notations 3.1.6. — The complexes

$$0 \to \mathbf{Z} \to \mathbf{Q} \to 0 \quad \text{and} \quad 0 \to \mathbf{Z} \xrightarrow{\Delta} \mathbf{Q} \oplus \mathbf{Q} \xrightarrow{\Sigma} \mathbf{Q} \to 0$$

are both quasi-isomorphic to $\mathbf{Q}/\mathbf{Z}[-1]$. Therefore in the category of bounded complexes of étale sheaves there is a canonical morphism

$$\mathbf{Q}/\mathbf{Z}[-1] \overset{\mathbf{L}}{\otimes} \mathbf{Q}/\mathbf{Z}[-1] \to \mathbf{Q}/\mathbf{Z}[-1].$$

Similarly one may define canonical products

$$\mathbf{Q}/\mathbf{Z}(i)[-1] \overset{\mathbf{L}}{\otimes} \mathbf{Q}/\mathbf{Z}(j)[-1] \to \mathbf{Q}/\mathbf{Z}(i+j)[-1].$$

Let $\Gamma(1)$ be the complex $\mathbf{G}_m[-1]$. There is also a product

$$\Gamma(1) \overset{\mathbf{L}}{\otimes} \Gamma(1) \to \Gamma(2)$$

and the natural morphisms

$$\mathbf{Q}/\mathbf{Z}(1)[-1] \to \Gamma(1) \quad \text{and} \quad \mathbf{Q}/\mathbf{Z}(2)[-1] \to \Gamma(2)$$

may be fitted into a commutative diagram

$$\begin{array}{ccc}
\mathbf{Q}/\mathbf{Z}(1)[-1] \overset{\mathbf{L}}{\otimes} \mathbf{Q}/\mathbf{Z}(1)[-1] & \longrightarrow & \mathbf{Q}/\mathbf{Z}(2)[-1] \\
\downarrow & & \downarrow \\
\Gamma(1) \overset{\mathbf{L}}{\otimes} \Gamma(1) & \longrightarrow & \Gamma(2).
\end{array}$$

The top horizontal line corresponds to a cup-product

$$\cup : H^p_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(1)) \otimes H^q_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(1)) \to H^{p+q+1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(2)).$$

Remark 3.1.7. — The product above may also be described as the composite map

$$\begin{aligned} H^p_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(1)) \otimes H^q_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(1)) &\to H^{p+1}_{\text{ét}}(X, \widehat{\mathbf{Z}}(1)) \otimes H^q_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(1)) \\ &\to H^{p+q+1}(X, \mathbf{Q}/\mathbf{Z}(2)) \end{aligned}$$

where

$$H^p_{\text{ét}}(X, \widehat{\mathbf{Z}}(j)) = \varprojlim_n H^p_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z}(j)).$$

Lemma 3.1.8. — *If G is a finite group and k a separably closed field, one has a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Pic}_G k \otimes_{\mathbf{Z}} \mathrm{Pic}_G k & \longrightarrow & \mathrm{CH}^2_G k \\
\downarrow \wr & & \downarrow \wr \\
H^1(G, \mathbf{Q}/\mathbf{Z}(1)) \otimes H^1(G, \mathbf{Q}/\mathbf{Z}(1)) & \xrightarrow{\cup} & H^3(G, \mathbf{Q}/\mathbf{Z}(2))
\end{array}$$

where the top map is given by the intersection product.

Proof. — Thanks to the compatibility of the coniveau spectral sequence with cup-products one gets for any a in $\mathrm{Pic}_G k$ a commutative diagram

$$\begin{array}{ccc}
\mathrm{Pic}_G k & \xrightarrow{.a} & \mathrm{CH}^2_G k \\
\| & & \| \\
\mathrm{Pic}\, U_4/G & \xrightarrow{.a} & \mathrm{CH}^2\, U_4/G \\
\downarrow \wr & & \downarrow \\
\mathbf{H}^2_{\text{ét}}(U_4/G, \Gamma(1)) & \xrightarrow{\cup a} & \mathbf{H}^4_{\text{ét}}(U_4/G, \Gamma(2)) \\
\uparrow \wr & & \uparrow \wr \\
H^2(G, \mathbf{Q}/\mathbf{Z}(1)[-1]) & \xrightarrow{\cup a} & H^4(G, \mathbf{Q}/\mathbf{Z}(2)[-1])
\end{array}$$

where a is successively seen as an element of

$$\mathrm{Pic}_G k, \quad \mathrm{Pic}\, U_4/G, \quad \mathbf{H}^2_{\text{ét}}(U_4/G, \Gamma(1)) \quad \text{and} \quad H^2(G, \mathbf{Q}/\mathbf{Z}(1)[-1]). \qquad \square$$

Proof of proposition 3.1.5. — The group $\mathrm{CH}^2_G(\mathbf{C})$ may be described as a quotient Z/R where Z is the free $\mathbf{Z}$-module over the set of G-orbits in $W^{(2)}$ and R is the subgroup generated by the divisors of functions f in $\mathbf{C}(Y)^{*\mathrm{Stab}_G Y}$ where Y goes over $W^{(1)}$. Then the obvious surjective map

$$Z \to Z_l G$$

sends R into $R_l G$. Indeed, if $Y \in W^{(1)}$ is not G-invariant and $f \in \mathbf{C}(Y)^{*\mathrm{Stab}_G Y}$, then for any C in $W^{(2)}$ invariant under G, $v_C(f)$ belongs to $l\mathbf{Z}$. Thus we get a surjective morphism

$$\mathrm{CH}^2_G(\mathbf{C}) \to N^3 G.$$

We put

$$\mathrm{CH}^2_G(\mathbf{C})_{\mathrm{c}} = \sum_{H \subsetneq G} \mathrm{Cores}^G_H \mathrm{CH}^2_H(\mathbf{C}).$$

By lemma 3.1.6 we have an isomorphism

$$\mathrm{CH}^2_G(\mathbf{C})_{\mathrm{c}} \xrightarrow{\sim} H^3(G, \mathbf{Q}/\mathbf{Z}(2))_{\mathrm{c}}.$$

By [**Sa2**, poposition 4.7], the permutation negligible classes may be described as

$$\mathrm{Ker}\Big(H^3(G, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(G, Q^*))\Big)$$

where Q^* is a G-module such that

$$\forall H \subset G, \quad H^1(H, Q^*) = 0$$

and there is an exact sequence

$$0 \to \mathbf{Q}/\mathbf{Z}(2) \to Q^* \to Q \to 0$$

where Q is a permutation module. It may be constructed as follows: let

$$Q = \bigoplus_{H \subset G} (\mathbf{Z}[G/H])^{H^1(H,\mathbf{Q}/\mathbf{Z}(2))}.$$

If H is a subgroup of G, any α in $H^1(H, \mathbf{Q}/\mathbf{Z}(2))$ defines canonically an element

$$\tilde{\alpha} \in \mathrm{Ext}^1_H(\mathbf{Z}, \mathbf{Q}/\mathbf{Z}(2)) \xrightarrow{\sim} \mathrm{Ext}^1_G(\mathbf{Z}[G/H], \mathbf{Q}/\mathbf{Z}(2))$$

where the isomorphism is given by Shapiro's lemma. We consider

$$\eta = \sum_{H \subset G} \sum_{\alpha \in H^1(H,\mathbf{Q}/\mathbf{Z}(2))} \tilde{\alpha} \in \mathrm{Ext}^1_G(Q, \mathbf{Q}/\mathbf{Z}(2))$$

the class η defines an extension

$$0 \to \mathbf{Q}/\mathbf{Z}(2) \to Q^* \to Q \to 0$$

unique up to isomorphism. But this yields

$$\begin{aligned} H^3(G, \mathbf{Q}/\mathbf{Z}(2))_{\mathrm{c}} &= \mathrm{Im}(H^2(G, Q) \xrightarrow{\partial} H^3(G, \mathbf{Q}/\mathbf{Z}(2))) \\ &= \sum_{\substack{H \subset G \\ \alpha \in H^1(H,\mathbf{Q}/\mathbf{Z}(2))}} \mathrm{Im}\Big(H^2(G, \mathbf{Z}[G/H]) \xrightarrow{\partial_\alpha} H^3(G, \mathbf{Q}/\mathbf{Z}(2))\Big) \end{aligned}$$

where ∂_α is the map defined by the short exact sequence

$$0 \to \mathbf{Q}/\mathbf{Z}(2) \to E_\alpha \to \mathbf{Z}[G/H] \to 0$$

associated to $\tilde{\alpha}$. On the other hand, we have a commutative diagram

$$\begin{tikzcd}
H^2(G, \mathbf{Z}[G/H]) \arrow[r, "\partial_\alpha"] & H^3(G, \mathbf{Q}/\mathbf{Z}(2)) \\
H^2(H, \mathbf{Z}) \arrow[u, "\wr"] \arrow[r, "\partial_\alpha"] & H^3(H, \mathbf{Q}/\mathbf{Z}(2)) \arrow[u, "\mathrm{Cores}_H^G"']
\end{tikzcd}$$

which follows from the commutative diagram of G-modules

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Q}/\mathbf{Z}(2)[G/H] \arrow[r] \arrow[d, "\mathrm{Tr}"] & E'_\alpha[G/H] \arrow[r] \arrow[d] & \mathbf{Z}[G/H] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{Q}/\mathbf{Z}(2) \arrow[r] & E_\alpha \arrow[r] & \mathbf{Z}[G/H] \arrow[r] & 0
\end{tikzcd}$$

where E'_α is the extension of H-modules defined by $\tilde{\alpha}$ and the fact that the corestriction is induced by the trace and the inverse of Shapiro's isomorphism. But for $G = H$ the map ∂_α is compatible with cup-products and therefore coincides with the cup-product by the class of α itself which is the image of 1 by the the map

$$H^0(H, \mathbf{Z}) \xrightarrow{\partial_\alpha} H^1(H, \mathbf{Q}/\mathbf{Z}(2)).$$

Therefore $H^3(G, \mathbf{Q}/\mathbf{Z}(2))_{\mathrm{p}}$ is given as

$$\sum_{H \subset G} \mathrm{Cores}_H^G \mathrm{Im}\Big(H^1(H, \mathbf{Q}/\mathbf{Z}(2)) \otimes H^2(H, \mathbf{Z}) \xrightarrow{\cup} H^3(H, \mathbf{Q}/\mathbf{Z}(2))\Big)$$

which is the same as

$$\sum_{H \subset G} \mathrm{Cores}_H^G \mathrm{Im}\Big(H^1(H, \mathbf{Q}/\mathbf{Z}(1)) \otimes H^1(H, \mathbf{Q}/\mathbf{Z}(1)) \xrightarrow{\cup} H^3(H, \mathbf{Q}/\mathbf{Z}(2))\Big).$$

We put

$$\mathrm{CH}^2_G(\mathbf{C})_{\mathrm{p}} = \sum_{H \subset G} \mathrm{Cores}_H^G \mathrm{Im}(\mathrm{Pic}_H \mathbf{C} \otimes \mathrm{Pic}_H \mathbf{C} \to \mathrm{CH}^2_H \mathbf{C}).$$

By lemma 3.1.8 we get an isomorphism

$$\mathrm{CH}^2_G(\mathbf{C})_{\mathrm{p}} \xrightarrow{\sim} H^3(G, \mathbf{Q}/\mathbf{Z}(2))_{\mathrm{p}}.$$

It remains to prove that

$$\mathrm{Ker}(\mathrm{CH}^2_G(\mathbf{C}) \to N^3(G)) = \mathrm{CH}^2_G(\mathbf{C})_{\mathrm{c}} + \mathrm{CH}^2_G(\mathbf{C})_{\mathrm{p}}. \tag{3.1.3}$$

Since G is a l-group, it has a subgroup of index l and since the composite map

$$\operatorname{Cores}_H^G \circ \operatorname{Res}_H^G$$

coincides with the multiplication by $[G:H]$, we get that $l\,\mathrm{CH}^2_G\,\mathbf{C}$ is contained in both sides of (3.1.3). Also if Y in $W^{(2)}$ is not G-invariant then the class of its orbit is the image of the class of Y in $\mathrm{CH}^2_{\mathrm{Stab}_G Y}\,\mathbf{C}$ by the corestriction. Therefore the group $\mathrm{CH}^2_G(\mathbf{C})_{\mathrm{c}}$ is generated by $l\,\mathrm{CH}^2_G(\mathbf{C})$ and the G-orbits in $W^{(2)}$ which are not reduced to one element. Thus the quotient

$$\mathrm{CH}^2_G(\mathbf{C})/\,\mathrm{CH}^2_G(\mathbf{C})_{\mathrm{c}}$$

may be described as the quotient of the $\mathbf{F}_l$-vector space $\tilde{F}_l$ with basis the elements of ${W^{(2)}}^G$ by the subspace $\tilde{R}_l$ generated by divisors of functions in $\mathbf{C}(Y)^G$ where Y goes over ${W^{(1)}}^G$.

The image of $\mathrm{CH}^2_G(\mathbf{C})_{\mathrm{p}}$ in this group coincides with the image of

$$\mathrm{Pic}_G\,\mathbf{C} \otimes \mathrm{Pic}_G\,\mathbf{C}$$

which is generated by the images of products $[y][z]$ with y and z in ${W^{(1)}}^G$. Moreover one has that

$$\mathrm{Pic}_G\,\mathbf{C} \xrightarrow{\sim} \mathrm{Pic}\,U_2/G.$$

Therefore one may assume that y is the inverse image of an element y' in ${W'^2}^{(1)}$ by the first projection $W \to W'^2$ whereas z comes from an element z' by the second one. Since the map

$$\mathbf{C}(W'^2)^* \to \mathrm{Div}(W'^2)$$

is surjective, z' is defined by a function f on W'^2 and $y.z$ is given as the divisor of the function $f \circ \mathrm{pr}_2$ restricted to y. Since z is G-invariant, one has

$$\forall g \in G, \quad {}^g\!f/f \in \mathbf{C}^*$$

and the map

$$\begin{array}{ccc} G & \to & \mathbf{C}^* \\ g & \mapsto & {}^g\!f/f \end{array}$$

is a morphism. Therefore $(f \circ \mathrm{pr}_2)^{|G|} \in \mathbf{C}(Y)^G$. We get that the image of the group $\mathrm{CH}^2_G(\mathrm{Spec}\,\mathbf{C})_{\mathrm{p}}$ in $\tilde{F}_l/\tilde{R}_l$ is contained in

$$\mathrm{Ker}(\tilde{F}_l/\tilde{R}_l \to N^3(G)).$$

Conversely, let Y belong to $(W^{(1)})^G$ and f be a function on Y such that

$$\exists n \in \mathbf{Z}_{>0}, \quad f^n \in \mathbf{C}(Y)^G$$

then

$$\forall g \in G, \quad {}^g f/f \in \mathbf{Q}/\mathbf{Z}(1)$$

and it defines an element of $H^1(G, \mathbf{Q}/\mathbf{Z}(1))$. Let Z' in $\mathrm{Div}(W'^2)$ represent the corresponding element of the group $\mathrm{Pic}_G \mathbf{C}$ and h' in $\mathbf{C}(W'^2)^*$ be such that

$$\mathrm{Div}\, h' = Z'.$$

We may choose λ and μ in $\mathbf{C}$ so that the divisor of

$$h = \lambda h' \circ \mathrm{pr}_1 + \mu h' \circ \mathrm{pr}_2$$

intersects properly with Y. By construction we have that

$$\forall g \in G, \quad ({}^g h/h)_{|Y} = {}^g f/f$$

and therefore

$$h_{|Y}/f \in \mathbf{C}(Y)^G.$$

Then the image of $\mathrm{Div} f$ in $\tilde{F}_l/\tilde{R}_l$ coincides with the one of $h_{|Y}$ which is the image of the product $\mathrm{Div}\, h.y$ and we get that

$$\mathrm{Ker}(\tilde{F}_l/\tilde{R}_l \to N^3(G)) = \mathrm{Im}(\mathrm{CH}^2_G(\mathbf{C})_\mathrm{p} \to \tilde{F}_l/\tilde{R}_l)$$

as wanted. □

Example 3.1.2. — If G is an $\mathbf{F}_l$-vector space with $l \neq 2$ by [**Bro**, page 60], one has an isomorphism

$$H^n(G, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \mathrm{Hom}(H_n(G, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}).$$

By [**Car**, théorème 1], we get that

$$S^2 G^\vee \oplus \Lambda^3 G^\vee \xrightarrow{\sim} H^3(G, \mathbf{Q}/\mathbf{Z})$$

where the isomorphism is given by the map

$$\Lambda^3 G^\vee \xrightarrow{\sim} \Lambda^3 H^1(G, \mathbf{Z}/p\mathbf{Z}) \to H^3(G, \mathbf{Z}/p\mathbf{Z}) \to H^3(G, \mathbf{Q}/\mathbf{Z})$$

and the map

$$S^2 G^\vee \xrightarrow{\sim} S^2 H^2(G, \mathbf{Z}) \xrightarrow{\cup} H^4(G, \mathbf{Z}) \xrightarrow{\sim} H^3(G, \mathbf{Q}/\mathbf{Z}).$$

By [**Pe1**, lemma 7], the map

$$\Lambda^3 G^\vee \to H^3(k(W)^G, \mathbf{Q}/\mathbf{Z})$$

is injective. But by lemma 3.1.8, the elements in the image of $S^2G^\vee$ are permutation negligible and we get that

$$S^2G^\vee \xrightarrow{\sim} S^2 \operatorname{Pic}_G k \xrightarrow{\sim} \mathrm{CH}^2_G k.$$

More generally, Totaro has given a description of the map

$$\mathrm{cl}_i : \mathrm{CH}^i_G(\mathbf{C}) \to H^{2i-1}(G, \mathbf{Q}/\mathbf{Z}(i))$$

in this case.

Example 3.1.3. — If G has a 2-dimensional representation P then we may assume that there is a surjective map $W \to P$. Its kernel defines an element of $\mathrm{CH}^2_G(k)$. If G is a 2-group having a cyclic subgroup of index 2 this is the example of Saltman [**Sa2**, theorem 4.14] who proved that the element obtained in N^3G is non-trivial.

Finally we want to give another description of the negligible classes when k is algebraically closed of characteristic 0.

Notation 3.1.7. — Let $\mathscr{R}(G)$ be the ring of representations of G over $\mathbf{C}$. The ring $\mathscr{R}(G)$ has a canonical structure of augmented λ-ring (see [**At**, §12]). The first steps of the filtration are obtained as follows:

$$\begin{aligned}\mathscr{R}(G)^0 &= \mathscr{R}(G),\\ \mathscr{R}(G)^1 &= \operatorname{Ker}(\dim),\\ \mathscr{R}(G)^2 &= \operatorname{Ker}(\det).\end{aligned}$$

where $\dim : \mathscr{R}(G) \to \mathbf{Z}$ is the dimension homorphism and

$$\det : \mathscr{R}(G) \to \operatorname{Hom}(G, \mathbf{C}^*)$$

is defined by

$$\det R = \Lambda^{\dim R} R$$

for any representation R of G.

Corollary 3.1.9. — *If k is an algebraically closed field of characteristic* 0*, then there is a canonical surjective map*

$$\mathscr{R}(G)^2 \twoheadrightarrow H^3(G, \mathbf{Q}/\mathbf{Z}(2))_n.$$

Remark 3.1.10. — In fact this map is induced by the second Chern class c_2.

Proof. — By corollary 3.1.3, one has

$$\mathrm{CH}^2_G(k) \xrightarrow{\sim} H^3(G, \mathbf{Q}/\mathbf{Z}(2))_{\mathrm{n}}$$

and, for $i \leqslant 2$, we have

$$\mathrm{CH}^i_G(k) \xrightarrow{\sim} \mathrm{CH}^i(U_3/G).$$

For any smooth variety X over a field, let $K_0(X)^i$ be the i-th filtration group for the filtration by codimension of support and denote by $K_0(X)^{(i/i+1)}$ the quotient group $K_0(X)^i/K_0(X)^{i+1}$. Then by [**Su1**, proposition 9.3], the composite map

$$\mathrm{CH}^i(U_3/G) \twoheadrightarrow K_0(U_3/G)^{(i/i+1)} \xrightarrow{c_i} \mathrm{CH}^i(U_3/G)$$

is the multiplication by $(-1)^i(i-1)!$. Therefore the group $K_0(U_3/G)^{(i/i+1)}$ is isomorphic to $\mathrm{CH}^i_G(k)$ if $i \leqslant 2$.

if $\mathscr{X}$ is a scheme equipped with an action of a group scheme $\mathscr{G}$, then Thomason has developed in [**Th**] equivariant K-theory groups $K_i(\mathscr{G}, \mathscr{X})$ and $K'_i(\mathscr{G}, \mathscr{X})$ (see also [**Me2**, §2]). By [**Me2**, corollary 2.12], there is a canonical isomorphism

$$\mathscr{R}(G) \xrightarrow{\sim} K'_O(G, \operatorname{Spec} k),$$

where we identify $\mathscr{R}(G)$ with the ring of G-representations over k. By [**Me2**, corollary 2.8] one has an isomorphism

$$K'_0(G, \operatorname{Spec} k) \xrightarrow{\sim} K'_0(G, W^3),$$

by [**Th**, theorem 2.7] there is a surjection

$$K'_0(G, W^3) \twoheadrightarrow K'_0(G, U^3),$$

and, by [**Me2**, proposition 2.4], isomorphisms

$$K'_0(G, U_3) \xleftarrow{\sim} K'_0(U_3/G) \xleftarrow{\sim} K_0(U_3/G).$$

Therefore we get a surjective map

$$\mathscr{R}(G) \twoheadrightarrow K_0(U_3/G)$$

which sends a G-representation R on the vector bundle

$$(R \times U_3)/G \to U_3/G.$$

But we have

$$\mathscr{R}(G)^{(i/i+1)} \xrightarrow{\sim} K_0(U_3/G)^{(i/i+1)}$$

if $i \leqslant 1$. Indeed this follows from the commutativity of the diagrams

$$\begin{array}{ccc} K_0(U_3/G)^{(0/1)} & \xrightarrow[\sim]{\deg} & \mathbf{Z} \\ \uparrow\!\!\uparrow & & \| \\ \mathscr{R}(G)^{(0/1)} & \xrightarrow[\sim]{\dim} & \mathbf{Z} \end{array}$$

and

$$\begin{array}{ccc} K_0(U_3/G)^{(1/2)} & \xrightarrow{\sim} & \mathrm{Pic}_G\, k \\ \uparrow\!\!\uparrow & & \uparrow\wr \\ \mathscr{R}(G)^{(1/2)} & \xrightarrow[\sim]{\det} & \mathrm{Hom}(G, \mathbf{C}^*) \end{array}$$

Therefore we have a surjective map

$$\mathscr{R}(G)^2 \twoheadrightarrow K_0(U_3/G)^2. \quad \square$$

Example 3.1.4. — By [**At**, page 23 and Appendix], if $H^q(G, \mathbf{Z}) = 0$ for all odd q, the chern map

$$c_2 : \mathscr{R}(G)^2 \to H^3(G, \mathbf{Q}/\mathbf{Z})$$

is surjective. In this case, one gets

$$H^3(G, \mathbf{Q}/\mathbf{Z})_{\mathrm{n}} = H^3(G, \mathbf{Q}/\mathbf{Z}).$$

This applies in particular to the groups with periodic cohomology, in which case this equality may also be deduced from [**Sa2**, theorem 4.14].

3.2. Unramified cohomology groups. — Let us first recall the definition of higher unramified cohomology groups (see [**CTO**]).

Definition 3.2.1. — If K is a function field over k, that is generated by a finite number of elements as a field over k then one considers the set $\mathscr{P}(K/k)$ of discrete valuation rings A of rank one such that

$$k \subset A \subset K \quad \text{and} \quad \mathrm{Fr}(A) = K.$$

For any A in $\mathscr{P}(K/k)$ and any n prime to the exponential characteristic p of k, one considers the residue map

$$\partial_A : H^i(K, \mu_n^{\otimes j}) \to H^{i-1}(\kappa_A, \mu_n^{\otimes j-1})$$

where κ_A denotes the residue field of A. The unramified cohomology groups of K over k are defined as

$$H^i_{\mathrm{nr}/k}(K,\mu_n^{\otimes j}) = \bigcap_{A\in\mathscr{P}(K/k)} \operatorname{Ker}\partial_A.$$

We shall also consider

$$H^i_{\mathrm{nr}/k}(K,(\mathbf{Q}/\mathbf{Z})'(j)) = \varinjlim_{(n,p)=1} H^i_{\mathrm{nr}/k}(K,\mu_n^{\otimes j}).$$

Remarks 3.2.1. — By [**CTO**], the unramified cohomology groups are invariant for stable rationality. In particular, it follows from the no-name lemma that, for any finite group G the unramified cohomology group

$$H^i_{\mathrm{nr}/k}(k(W)^G,(\mathbf{Q}/\mathbf{Z})'(j))$$

where W is a faithful representation of G over k depends only on k and G.

If $i=2$, Bogomolov [**Bo**, theorem 3.1] proved that

$$H^2_{\mathrm{nr}/\mathbf{C}}(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z}(1)) = \operatorname{Ker}\Big(H^2(G,\mathbf{Q}/\mathbf{Z}(1)) \to \prod_{B\in\mathscr{B}} H^2(B,\mathbf{Q}/\mathbf{Z}(1))\Big)$$

where $\mathscr{B}$ denotes the set of bicyclic groups in G, that is abelian subgroups generated by two elements.

Theorem 2.3.1 implies the following generalization of a result of Saltman:

Corollary 3.2.2. — *(See Saltman* [**Sa2**, theorem 5.3]*) If G a finite group and W a faithful representation of G over a separably closed field k, then*

$$H^3_{\mathrm{nr}/k}(k(W)^G,(\mathbf{Q}/\mathbf{Z})'(2)) \subset \operatorname{Im}(H^3(G,(\mathbf{Q}/\mathbf{Z})'(2)) \to H^3(k(W)^G,(\mathbf{Q}/\mathbf{Z})'(2)).$$

Proof. — Let U be the open set in W on which G acts freely. We may assume that $\operatorname{codim}_W(W-U)\geqslant 4$ and we may apply theorem 2.3.1 to U and get that

$$H^0_{\mathrm{Zar}}\Big(U/G,\mathscr{H}^3_{\text{ét}}((\mathbf{Q}/\mathbf{Z})'(2))\Big)$$

is contained in

$$\operatorname{Im}\Big(H^3(G,(\mathbf{Q}/\mathbf{Z})'(2)) \to H^3(k(W)^G,(\mathbf{Q}/\mathbf{Z})'(2))\Big).$$

But it follows from Bloch-Ogus spectral sequence that

$$H^3_{\mathrm{nr}/k}(k(W)^G,(\mathbf{Q}/\mathbf{Z})'(2)) \subset H^0_{\mathrm{Zar}}\Big(U/G,\mathscr{H}^3_{\text{ét}}((\mathbf{Q}/\mathbf{Z})'(2))\Big)$$

and this implies the corollary. □

4. Application of Voevodsky's motivic complexes

In this section, we want to generalize the results of the previous sections to the fourth and fifth cohomology groups using the Hochschild-Serre and coniveau spectral sequences for Voevodky's étale complexes $\mathbf{Z}(3)$ and $\mathbf{Z}(4)$ [**Vo3**, §2.1].

Let us first recall a few facts about the coniveau spectral sequence.

4.1. Reminder on the coniveau spectral sequence. —

Notations 4.1.1. — From now on, we assume that the characteristic of k is 0. Let $\mathbf{Z}(n)_{\text{ét}}$ be Voevodsky's étale motivic complex of weight n [**Vo3**, §2.1]. Then for any smooth variety X, one puts

$$\mathbf{H}^q_{\text{ét}}(X, \mathbf{Z}(n)) = \mathbf{H}^q_{\text{ét}}(X, \mathbf{Z}(n)_{\text{ét}}).$$

Let α be the canonical morphism from the big étale site to the big Zariski one. Then there is a Leray spectral sequence (see [**Mi**, theorem III.1.18])

$$E_2^{p,q}(n) = H^p_{\text{Zar}}(X, \mathscr{H}^q_{\text{ét}}(\mathbf{Z}(n))) \Rightarrow \mathbf{H}^{p+q}_{\text{ét}}(X, \mathbf{Z}(n))$$

where $\mathscr{H}^q_{\text{ét}}(\mathbf{Z}(n))$ is the Zariski sheaf corresponding to the presheaf $\mathbf{H}^q_{\text{ét}}(\mathbf{Z}(n))$ given by

$$U \mapsto \mathbf{H}^q_{\text{ét}}(U, \mathbf{Z}(n)).$$

Indeed, by [**Mi**, proposition III.1.13], this sheaf coincides with $\mathbf{R}^q\alpha_*\mathbf{Z}(n)_{\text{ét}}$. Using [**Vo2**, §3.3] and the proof of [**Vo1**, theorem 5.3], we get that $\mathbf{H}^q_{\text{ét}}(\mathbf{Z}(n))$ has a canonical structure of homotopy invariant pretheory and by [**Vo1**, proposition 4.26] this is also the case of $\mathscr{H}^q_{\text{ét}}(\mathbf{Z}(n))$ (see also [**Vo3**, page 10]). By [**Vo1**, theorem 4.37], there is a Gersten resolution of $\mathscr{H}^p_{\text{ét}}(\mathbf{Z}(n))$ and we get a coniveau spectral sequence

$$E_1^{p,q}(n) = \bigoplus_{x \in X^{(p)}} (i_x)_*(\mathscr{H}^q_{\text{ét}}(\mathbf{Z}(n))_{-p}) \Rightarrow \mathbf{H}^{p+q}_{\text{ét}}(X, \mathbf{Z}(n))$$

where for any pretheory F, the pretheory F_{-1} is given by

$$F_{-1}(U) = \operatorname{Coker}\Big(F(U \times \mathbf{A}^1) \xrightarrow{\text{Res}} F(U \times (\mathbf{A}^1 - \{0\}))\Big).$$

Theorem 4.1.1. — *(See Kahn* [**Kah3**]*) With notations as above, if $n \geqslant 1$, one has*

(i) the group $E_1^{p,q}(n)$ is uniquely divisible if $p > q$ and $0 \leqslant p \leqslant n-2$,

(ii) the group $E_1^{p,q}(n)$ is uniquely 2-divisible if $p = q$ and $0 \leqslant p \leqslant n-2$,

(iii) one has

$$E_1^{p,q}(n)\otimes \mathbf{Z}_{(2)} = \bigoplus_{x\in X^{(p)}} K^M_{n-p}(\kappa(x))\otimes \mathbf{Z}_{(2)}$$

if $q=n$ and $0\leqslant p\leqslant n-3$,

(iv) there are canonical isomorphisms

$$E_1^{p,q}(n) = \bigoplus_{x\in X^{(p)}} K^M_{n-p}(\kappa(x))$$

if $q=n$ and $n-2\leqslant p\leqslant n$,

(v) the group $E_1^{p,q}(n)\otimes \mathbf{Z}_{(2)}$ is trivial if $q=n+1$ and $0\leqslant p\leqslant n-3$,

(vi) one has

$$E_1^{p,q}(n) = \bigoplus_{x\in X^{(p)}} H^{q-1}(\kappa(x),\mathbf{Q}/\mathbf{Z}(n))$$

if $0\leqslant p\leqslant q-1$ and $q>n+1$,

(vii) all other $E_1^{p,q}(n)$ with $q=n+1$ or $p\geqslant q$ are trivial.

Remark 4.1.2. — The Milnor-Bloch-Kato conjecture is used in all the assertions where the prime 2 plays a special rôle. If we assumed that this conjecture held for any prime, we would be able to simplify the assertions accordingly.

Proof. — By [**Vo3**, lemma 2.9] and comparison theorems between Nisnevich and Zariski topology [**Vo1**, theorem 5.7], one has that if $i\leqslant n$,

$$\mathscr{H}^q_{\text{ét}}(\mathbf{Z}(n))_{-i} = \mathscr{H}^{q-i}_{\text{ét}}(\mathbf{Z}(n-i)).$$

Therefore, if $p\leqslant n$, one has

$$E_1^{p,q}(n) = \bigoplus_{x\in X^{(p)}} \mathbf{H}^{q-p}_{\text{ét}}(\kappa(x),\mathbf{Z}(n-p)).$$

But for any positive m there is a distinguished triangle

$$\mathbf{Z}(n)\xrightarrow{\times m}\mathbf{Z}(n)\to \mathbf{Z}/m\mathbf{Z}(n)\to \mathbf{Z}(n)[1]$$

yielding for any field K a long exact sequence

$$\text{(4.1.1)}\quad \mathbf{H}^q_{\text{ét}}(K,\mathbf{Z}(n))\xrightarrow{\times m}\mathbf{H}^q_{\text{ét}}(K,\mathbf{Z}(n))\to \mathbf{H}^q_{\text{ét}}(K,\mathbf{Z}/m\mathbf{Z}(n))\to \mathbf{H}^{q+1}_{\text{ét}}(K,\mathbf{Z}(n))$$

and by [**Vo3**, theorem 2.6],

$$\text{(4.1.2)}\qquad \mathbf{H}^q_{\text{ét}}(K,\mathbf{Z}/m\mathbf{Z}(n))\xrightarrow{\sim} H^q(K,\mu_m^{\otimes n})$$

which is trivial if $q<0$. This implies assertion (i).

Assertion (ii) is proved in [**Kah4**, theorem 3.1 (a)].

By Beilinson-Lichtenbaum conjecture [**Vo3**, theorem 2.11], if $0 \leqslant q \leqslant n$,

$$\mathbf{H}^q_{\mathrm{Nis}}(K, \mathbf{Z}(n)) \otimes \mathbf{Z}_{(2)} \xrightarrow{\sim} \mathbf{H}^q_{\text{ét}}(K, \mathbf{Z}(n)) \otimes \mathbf{Z}_{(2)}$$

and by [**SV**, proposition 3.2] one has

$$K^M_n(K) \xrightarrow{\sim} \mathbf{H}^n_{\mathrm{Nis}}(K, \mathbf{Z}(n))$$

this implies (iii). The same argument implies the case $p = n - 2$ of (iv).

Assertions (iv) and (vii) for $p = n$ or $n - 1$ follow from the isomorphisms

$$\mathbf{Z}(1) \xrightarrow{\sim} \mathbf{G}_m[-1] \quad \text{and} \quad \mathbf{Z}(0) \xrightarrow{\sim} \mathbf{Z}.$$

Assertion (v) follows from Hilbert's theorem 90 [**Vo3**, theorem 4.1] which also implies (vii) for $p \geqslant n - 2$ and $q = n + 1$.

From the distinguished triangle

$$\mathbf{Z}(n) \to \mathbf{Q}(n) \to \mathbf{Q}/\mathbf{Z}(n) \to \mathbf{Z}(n)[1]$$

and the comparison theorem for $\mathbf{Q}(n)$ [**Vo3**, theorem 2.5], one gets that the motivic cohomology group $\mathbf{H}^q_{\text{ét}}(K, \mathbf{Z}(n))$ is torsion for $q > n + 1$. Then (4.1.1) and (4.1.2) gives an isomorphism

$$\mathbf{H}^q_{\text{ét}}(K, \mathbf{Z}(n)) \xrightarrow{\sim} H^{q-1}_{\text{ét}}(K, \mathbf{Q}/\mathbf{Z}(n))$$

if $q \leqslant n$. This yields (vi) for $p \leqslant n$.

The exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$$

implies that

$$\mathscr{H}^q_{\text{ét}}(\mathbf{Z}) \xrightarrow{\sim} \begin{cases} \mathscr{H}^{q-1}_{\text{ét}}(\mathbf{Q}/\mathbf{Z}) & \text{if } q \geqslant 2, \\ \mathbf{Z} & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

But by [**Vo1**, §3.4], one has

$$\mathscr{H}^q_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(i))_{-j} = \mathscr{H}^{q-j}_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(i-j))$$

and we get the assertions (vi) and (vii) for $p > n$. □

***Notation 4.1.2.* —** We define $H^i_{\mathrm{Zar}}(X, \mathscr{K}^M_q)$ as the i-th homology group of the complex (see [**Ka**, page 242])

$$0 \to \bigoplus_{x \in X^{(0)}} K^M_q \kappa(x) \to \bigoplus_{x \in X^{(1)}} K^M_{q-1} \kappa(x) \to \cdots \to \bigoplus_{x \in X^{(q)}} \mathbf{Z} \to 0.$$

Corollary 4.1.3. — *One has that*

(i) $E_2^{p,q}(n)$ *is uniquely divisible if* $p > q+1$ *and* $0 \leqslant p \leqslant n-2$.

(ii) $E_2^{p,q}$ *is 2-divisible if* $q \leqslant p \leqslant q+1$ *and* $0 \leqslant p \leqslant n-2$.

(iii) $E_2^{p,q}(n) \otimes \mathbf{Z}_{(2)} = H^p_{\mathrm{Zar}}(X, \mathscr{K}^M_q) \otimes \mathbf{Z}_{(2)}$ *if* $q = n$ *and* $0 \leqslant p \leqslant n-2$,

(iv) $E_2^{p,q}(n) = H^p_{\mathrm{Zar}}(X, \mathscr{K}_q)$ *if* $q = n$ *and* $n-1 \leqslant p \leqslant n$,

(v) $E_2^{p,q}(n) \otimes \mathbf{Z}_{(2)}$ *is trivial if* $q = n+1$ *and* $0 \leqslant p \leqslant n-3$,

(vi) $E_2^{p,q}(n) = H^p_{\mathrm{Zar}}(X, \mathscr{H}^q_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(n)))$ *if* $0 \leqslant p \leqslant q-1$ *and* $q > n+1$.

(vii) all other $E_2^{p,q}(n)$ *with* $q = n+1$ *or* $p \geqslant q$ *are trivial.*

Corollary 4.1.4. — *If* X *is a smooth variety over a field* k *of characteristic* 0 *and* Y *a subvariety of codimension at least* c *in* X, *then for any positive integers* n *and* q *such that* $c > \sup(n, q)$ *the natural restriction map*

$$\mathbf{H}^q_{\text{ét}}(X, \mathbf{Z}(n)) \xrightarrow{\sim} \mathbf{H}^q_{\text{ét}}(X - Y, \mathbf{Z}(n))$$

is an isomorphism.

Proof. — This follows from assertion (vii) of theorem 4.1.1 as in proposition 2.2.1. □

***Corollary 4.1.5* (Kahn, [Kah3, §5, $n = 3$]).** — *With notations as above, there is an exact sequence*

$$\begin{aligned} 0 &\to H^2_{\mathrm{Zar}}(X, \mathscr{K}_3) \otimes \mathbf{Z}_{(2)} \to \mathbf{H}^5_{\text{ét}}(X, \mathbf{Z}_{(2)}(3)) \to H^0_{\mathrm{Zar}}(X, \mathscr{H}^4_{\text{ét}}(\mathbf{Q}_2/\mathbf{Z}_2(3))) \\ &\to \mathrm{CH}^3(X) \otimes \mathbf{Z}_{(2)} \to \mathrm{Ker}\Big(\mathbf{H}^6_{\text{ét}}(X, \mathbf{Z}_{(2)}(3)) \to H^0_{\mathrm{Zar}}(X, \mathscr{H}^5_{\text{ét}}(\mathbf{Q}_2/\mathbf{Z}_2(3)))\Big) \\ &\to H^1_{\mathrm{Zar}}(X, \mathscr{H}^4_{\text{ét}}(\mathbf{Q}_2/\mathbf{Z}_2(3))) \to 0. \end{aligned}$$

Corollary 4.1.6. — *With notations as above, there is a canonical isomorphism*

$$H^1_{\mathrm{Zar}}(X, \mathscr{K}^M_3) \otimes \mathbf{Z}_{(2)} \xrightarrow{\sim} \mathbf{H}^4_{\text{ét}}(X, \mathbf{Z}_{(2)}(3)).$$

Corollary 4.1.7. — *(See Kahn,* [**Kah3**, §5, $n = 4$]*) With notation as above, there is a canonical exact sequence*

$$\begin{aligned} 0 &\to H^2_{\mathrm{Zar}}(X, \mathscr{K}^M_4) \otimes \mathbf{Z}_{(2)} \to \mathbf{H}^6_{\text{ét}}(X, \mathbf{Z}_{(2)}(4)) \to H^0_{\mathrm{Zar}}(X, \mathscr{H}^5_{\text{ét}}(\mathbf{Q}_2/\mathbf{Z}_2(4))) \\ &\to H^3_{\mathrm{Zar}}(X, \mathscr{K}_4) \otimes \mathbf{Z}_{(2)} \to \mathrm{Ker}\Big(\mathbf{H}^7_{\text{ét}}(X, \mathbf{Z}_{(2)}(4)) \to H^0_{\mathrm{Zar}}(X, \mathscr{H}^6_{\text{ét}}(\mathbf{Q}_2/\mathbf{Z}_2(4))\Big) \\ &\to H^1_{\mathrm{Zar}}(X, \mathscr{H}^5_{\text{ét}}(\mathbf{Q}_2/\mathbf{Z}_2(3))) \to \mathrm{Ker}\Big(\mathrm{CH}^4(X) \otimes \mathbf{Z}_{(2)} \to \mathbf{H}^8_{\text{ét}}(X, \mathbf{Z}_{(2)}(4))\Big). \end{aligned}$$

Remark 4.1.8. — The maps $\mathrm{CH}^i(X) \to \mathbf{H}^{2i}_{\text{ét}}(X, \mathbf{Z}(i))$ which appear in these corollaries may be interpreted as cycle class maps.

4.2. Finite groups and motivic cohomology. — We now want to relate the cohomology of finite groups with coefficients in $\mathbf{Q}/\mathbf{Z}(n)$ to integral motivic cohomology.

Proposition 4.2.1. — *If W is a faithful representation of a finite group G over an algebraically closed field k of characteristic 0, such that the open set U on which G acts freely verifies $\operatorname{codim}_W W - U > i$ and $i > n$ then*

$$H^{i-1}(G, \mathbf{Q}_2/\mathbf{Z}_2(n)) \xrightarrow{\sim} \mathbf{H}^i_{\text{ét}}(U/G, \mathbf{Z}_{(2)}(n))$$

Proof. — We consider the Hochschild-Serre spectral sequence

$$H^p(G, \mathbf{H}^q_{\text{ét}}(U, \mathbf{Z}(n))) \Rightarrow \mathbf{H}^{p+q}_{\text{ét}}(U/G, \mathbf{Z}(n)).$$

By corollary 4.1.4, we have that if $j \leqslant i$

$$\mathbf{H}^j_{\text{ét}}(U, \mathbf{Z}(n)) \xrightarrow{\sim} \mathbf{H}^j_{\text{ét}}(W, \mathbf{Z}(n)) \xrightarrow{\sim} \mathbf{H}^j_{\text{ét}}(k, \mathbf{Z}(n))$$

where the second isomorphism is given by homotopy invariance. Using the distinguished triangle

$$\mathbf{Z}(n) \xrightarrow{\times m} \mathbf{Z}(n) \to \mathbf{Z}/m\mathbf{Z}(n) \to \mathbf{Z}(n)[1]$$

and the isomorphisms

$$H^j_{\text{ét}}(k, \mu_m^{\otimes n}) \xrightarrow{\sim} \mathbf{H}^j_{\text{ét}}(k, \mathbf{Z}/m\mathbf{Z}(n))$$

we get that the groups $\mathbf{H}^j_{\text{ét}}(k, \mathbf{Z}(n))$ are uniquely divisible for $j \neq 0$ and $j \neq 1$. By [**Kah4**, theorem 3.1 (a)] the group $\mathbf{H}^0_{\text{ét}}(k, \mathbf{Z}_{(2)}(n))$ is also uniquely divisible. We obtain a short exact sequence

$$0 \to \mathbf{Q}_2/\mathbf{Z}_2(n)(k) \to \mathbf{H}^1_{\text{ét}}(k, \mathbf{Z}_{(2)}(n)) \to \mathbf{H}^1_{\text{ét}}(k, \mathbf{Q}(n)) \to 0.$$

By [**Vo3**, theorem 2.5], we have

$$\mathbf{H}^i_{\text{Zar}}(k, \mathbf{Q}(n)) \xrightarrow{\sim} \mathbf{H}^i_{\text{ét}}(k, \mathbf{Q}(n))$$

which, by construction, is 0 if $i \geqslant n+1$. Therefore we get that

- $H^p(G, \mathbf{H}^q_{\text{ét}}(k, \mathbf{Z}_{(2)}(n)))$ is uniquely divisible if $p = 0$, $q \leqslant n$, $q \neq 1$,
- $\mathbf{H}^1_{\text{ét}}(k, \mathbf{Z}_{(2)}(n))$ is divisible,
- $H^p(G, \mathbf{H}^1_{\text{ét}}(k, \mathbf{Z}_{(2)}(n))) = H^p(G, \mathbf{Q}_2/\mathbf{Z}_2(n))$ if $p > 0$,
- $H^p(G, \mathbf{H}^q_{\text{ét}}(k, \mathbf{Z}_{(2)}(n))) = 0$ otherwise.

The spectral sequence yields for $i > n+1$ isomorphisms

$$H^{i-1}(G, \mathbf{Q}_2/\mathbf{Z}_2(n)) \xrightarrow{\sim} \mathbf{H}^i_{\text{ét}}(U/G, \mathbf{Z}_{(2)}(n))$$

and an exact sequence

$$\mathbf{H}^n_{\text{ét}}(k, \mathbf{Z}_{(2)}(n)) \xrightarrow{\psi} H^n(G, \mathbf{Q}_2/\mathbf{Z}_2(n)) \to \mathbf{H}^{n+1}_{\text{ét}}(U/G, \mathbf{Z}_{(2)}(n)) \to 0.$$

But the first group is divisible and the second killed by $\#G$, therefore ψ is trivial. □

4.3. Equivariant $\mathscr{K}$-cohomology. — The following proposition is classical (see [**To**, §1]).

Proposition 4.3.1. — *Let W and W' be two faithful representations of a finite group G over a field k such that there are open sets U and U' on which G acts freely with*

$$\operatorname{codim}_W W - U \geqslant q+2 \quad \textit{and} \quad \operatorname{codim}_{W'} W' - U' \geqslant q+2,$$

then for any smooth G-variety there is a canonical isomorphism

$$H^q((Y \times U)/G, \mathscr{K}_n^M) \xrightarrow{\sim} H^q((Y \times U')/G, \mathscr{K}_n^M).$$

Before proving this proposition let us recall a result of Rost:

***Proposition 4.3.2* (Rost).** — *If $X \to Y$ is a vector bundle, then for any $q \geqslant 0$, $p \geqslant 0$, one has*

$$H^p(X, \mathscr{K}_q^M) \xrightarrow{\sim} H^p(Y, \mathscr{K}_q^M).$$

Proof. — This follows from theorem 1.4, remark 2.4, and proposition 8.6 in [**Ro**]. □

Proof of proposition 4.3.1. — As in [**To**, §1] or [**EG**] we use Bogomolov's double fibration argument. By the proof of proposition 2.2.1, if $W = W'$, we have isomorphisms

$$\begin{aligned} H^q((Y \times U)/G, \mathscr{K}_n^M) &\xrightarrow{\sim} H^q((Y \times (U \cap U'))/G, \mathscr{K}_n^M) \\ &\xrightarrow{\sim} H^q((Y \times U')/G, \mathscr{K}_n^M) \end{aligned}$$

thus this group does not depend on the choice of U. But the canonical map

$$(Y \times U \times W')/G \to (Y \times U)/G$$

is a vector bundle. Using proposition 4.3.2, we get isomorphisms

$$\begin{aligned} H^q((Y \times U)/G, \mathscr{K}_n^M) &\xrightarrow{\sim} H^q((Y \times U \times W')/G, \mathscr{K}_n^M) \\ &\xrightarrow{\sim} H^q((Y \times W \times U')/G, \mathscr{K}_n^M) \\ &\xrightarrow{\sim} H^q((Y \times U')/G, \mathscr{K}_n^M). \quad \square \end{aligned}$$

Definition 4.3.1. — With notation as in the proposition, we define the equivariant Milnor $\mathscr{K}$-cohomology group as

$$H^p_G(Y, \mathscr{K}_q^M) = H^p_{\mathrm{Zar}}((Y \times U)/G, \mathscr{K}_q^M)$$

and put $H^p_G(k, \mathscr{K}_q^M) = H^p_G(\operatorname{Spec} k, \mathscr{K}_q^M)$.

Example 4.3.1. — If $p = q$, we obtain the usual equivariant Chow group.

Example 4.3.2. — If $p = q - 1$, then we have

$$H^p_G(Y, \mathscr{K}_{p+1}^M) = H^p(Y \times U/G, \mathscr{K}_{q+1})$$

which coincides with the equivariant higher Chow group $\mathrm{CH}^p_G(Y, 1)$ (see [**EG**]).

Proposition 4.3.3. — *For any finite group G and any algebraically closed field k, one has*

$$H^1_G(k, \mathscr{K}_2^M) \xrightarrow{\sim} H^2(G, \mathbf{Q}/\mathbf{Z}(2)).$$

Proof. — We use the Hochscild-Serre spectral sequence

$$H^p(G, \mathbf{H}^q_{\text{ét}}(U, \Gamma(2))) \Rightarrow \mathbf{H}^{p+q}_{\text{ét}}(U/G, \Gamma(2))$$

and the results of Kahn (2.4.1) to get the isomorphism. $\square$

Definition 4.3.2. — Similarly we define the equivariant $\mathscr{H}_{\text{ét}}$-cohomology groups

$$H^p_G(Y, \mathscr{H}^q_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(n))) = H^p_{\mathrm{Zar}}((Y \times U)/G, \mathscr{H}^q_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(n)))$$

and put $H^p_G(k, \mathscr{H}^q_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(n))) = H^p(\operatorname{Spec} k, \mathscr{H}^q_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(n)))$.

Remark 4.3.4. — One has the inclusions

$$H^p_{\mathrm{nr}/k}(k(W)^G, \mathbf{Q}/\mathbf{Z}(n)) \subset H^0_G(k, \mathscr{H}^p_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(n))) \subset H^p(k(W)^G, \mathbf{Q}/\mathbf{Z}(n)).$$

Example 4.3.3. — if $p=2$ we have
$$H^2(k(W)^G,\mathbf{Q}/\mathbf{Z}(2))=\mathrm{Br}(k(W)^G)$$
which is in general infinite,
$$H^2(G,\mathbf{Q}/\mathbf{Z}(1))\xrightarrow{\sim}H^0_G(k,\mathscr{H}^2_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(1)))$$
and, by Bogomolov's result,
$$H^2_{\mathrm{nr}/k}(k(W)^G,\mathbf{Q}/\mathbf{Z}(1))=\mathrm{Ker}\Big(H^2(G,\mathbf{Q}/\mathbf{Z}(1))\to\prod_{B\in\mathscr{B}}H^2(B,\mathbf{Q}/\mathbf{Z}(1))\Big)$$
where $\mathscr{B}$ is the set of bicyclic groups in G.

4.4. Application to negligible classes. — In degree 4, we get the following results:

Theorem 4.4.1. — *If G is a finite group and k an algebraically closed field of characteristic 0, then there is a canonical exact sequence*
$$\begin{aligned}0\to H^2_G(k,\mathscr{K}_3)\otimes\mathbf{Z}_{(2)}\to H^4(G,\mathbf{Q}_2/\mathbf{Z}_2(3))\to H^0_G(k,\mathscr{H}^4_{\text{ét}}(\mathbf{Q}_2/\mathbf{Z}_2(3)))\\ \to\mathrm{CH}^3_G(k)\otimes\mathbf{Z}_{(2)}\to H^5(G,\mathbf{Q}_2/\mathbf{Z}_2(3))\end{aligned}$$
and a canonical isomorphism
$$H^1_G(k,\mathscr{K}_3^M)\otimes\mathbf{Z}_{(2)}\xrightarrow{\sim}H^3(G,\mathbf{Q}_2/\mathbf{Z}_2(3)).$$

Proof. — The first assertion follows from proposition 4.2.1 and corollary 4.1.5 and the second from the same proposition and corollary 4.1.6. □

Notations 4.4.1. — Let $H^i(G,\mathbf{Q}_2/\mathbf{Z}_2(i-1))_{\mathrm{n}}$ be the kernel of the map
$$H^i(G,\mathbf{Q}_2/\mathbf{Z}_2(i-1))\to H^i(k(W)^G,\mathbf{Q}_2/\mathbf{Z}_2(i-1))$$
and $H^i(G,\mathbf{Q}_2/\mathbf{Z}_2(i-1))_{\mathrm{c}}$ be
$$\sum_{H\subsetneq G}\mathrm{Cores}_H^G H^i(H,\mathbf{Q}_2/\mathbf{Z}_2(i-1))_{\mathrm{n}}.$$
We also consider the group
$$\begin{aligned}&H^i(G,\mathbf{Q}_2/\mathbf{Z}_2(i-1))_{\mathrm{p}}\\ &\qquad=\sum_{H\subset G}\mathrm{Cores}_H^G(H^1(H,\mathbf{Q}_2/\mathbf{Z}_2(1))\cup H^{i-2}(H,\mathbf{Q}_2/\mathbf{Z}_2(i-2)))\end{aligned}$$

where the product is defined as in notations 3.1.6. As in paragraph 3.1 we may define for any subgroup H of G a map

$$\mathrm{Cores}_H^G : H_H^p(k, \mathscr{K}_q^M) \to H_G^p(k, \mathscr{K}_q^M)$$

and there is a natural product (see [**Ro**, remark 2.4 and §14])

$$H_G^i(k, \mathscr{K}_p^M) \otimes H_G^j(k, \mathscr{K}_q^M) \to H_G^{i+j}(k, \mathscr{K}_{p+q}^M).$$

We define

$$H_G^p(k, \mathscr{K}_q)_{\mathrm{c}} = \sum_{H \subsetneq G} \mathrm{Im}\,\mathrm{Cores}_H^G$$

and

$$H_G^2(k, \mathscr{K}_3)_{\mathrm{p}} = \sum_{H \subset G} \mathrm{Cores}_H^G(\mathrm{Pic}_G \operatorname{Spec} k \cup H_G^1(k, \mathscr{K}_2)).$$

Remark 4.4.2. — The group $H^i(G, \mathbf{Q}_2/\mathbf{Z}_2(i-1))_{\mathrm{p}}$ coincides also with the kernel of a map

$$H^i(G, \mathbf{Q}_2/\mathbf{Z}_2(i-1)) \to H^i(G, k(W)^*).$$

Proposition 4.4.3. — *With the notations of theorem 4.4.1 the canonical isomorphism*

$$H_G^2(k, \mathscr{K}_3^M) \otimes \mathbf{Z}_{(2)} \xrightarrow{\sim} H^4(G, \mathbf{Q}_2/\mathbf{Z}_2(3))_n$$

induces an isomorphism from

$$H_G^2(k, \mathscr{K}_3^M)/\Big(H_G^2(k, \mathscr{K}_3)_p + H_G^2(k, \mathscr{K}_3)_c\Big) \otimes \mathbf{Z}_{(2)}$$

to

$$H^4(G, \mathbf{Q}_2/\mathbf{Z}_2(3))_n/\Big(H^4(G, \mathbf{Q}_2/\mathbf{Z}_2(3))_p + H^4(G, \mathbf{Q}_2/\mathbf{Z}_2(3))_c\Big).$$

Proof. — This follows easily from the next two lemmata:

Lemma 4.4.4. — *With notations as above, there is a commutative diagram*

$$\begin{tikzcd} H_H^2(k, \mathscr{K}_3^M) \otimes \mathbf{Z}_{(2)} \arrow[r] \arrow[d, "\mathrm{Cores}_H^G"] & H^4(H, \mathbf{Q}_2/\mathbf{Z}_2(3)) \arrow[d, "\mathrm{Cores}_H^G"] \\ H_G^2(k, \mathscr{K}_3^M) \otimes \mathbf{Z}_{(2)} \arrow[r] & H^4(G, \mathbf{Q}_2/\mathbf{Z}_2(3)). \end{tikzcd}$$

Proof. — As the proof of lemma 3.1.6 this follows from the compatibility of the spectral sequences with corestriction. □

Lemma 4.4.5. — *With notations as above, there is a commutative diagram*

$$\begin{array}{ccc}
H^1_G(k,\mathscr{K}_2^M)\otimes \mathrm{Pic}_G k\otimes \mathbf{Z}_{(2)} & \longrightarrow & H^2_G(k,\mathscr{K}_3^M)\otimes \mathbf{Z}_{(2)}\\
\downarrow\wr & & \downarrow\wr\\
H^2(G,\mathbf{Q}/\mathbf{Z}(2))\otimes H^1(G,\mathbf{Q}_2/\mathbf{Z}_2(1)) & \longrightarrow & H^4(G,\mathbf{Q}_2/\mathbf{Z}_2(3))_n.
\end{array}$$

Proof. — As in the proof of lemma 3.1.8, the compatibility of the coniveau spectral sequence with cup-product (see also [**We**]) yields a commutative diagram

$$\begin{array}{ccc}
H^3(G,\mathbf{Q}_2/\mathbf{Z}_2(2)[-1])\otimes H^2(G,\mathbf{Q}_2/\mathbf{Z}_2(1)[-1]) & \longrightarrow & H^5(G,\mathbf{Q}_2/\mathbf{Z}_2(3)[-1])\\
\downarrow & & \downarrow\\
\mathbf{H}^3_{\text{ét}}(U/G,\mathbf{Z}(2))\otimes \mathbf{H}^2_{\text{ét}}(U/G,\mathbf{Z}(1)) & \longrightarrow & \mathbf{H}^5_{\text{ét}}(U/G,\mathbf{Z}(3))\\
\uparrow & & \uparrow\\
H^1(U/G,\mathscr{K}_2)\otimes \mathrm{Pic}(U/G) & \longrightarrow & H^2(U/G,\mathscr{K}_3).
\end{array}$$ □

In degree 5 we get the following results:

Theorem 4.4.6. — *If G is a finite group and k an algebraically closed field of characteristic* 0, *then there is a canonical exact sequence*

$$\begin{aligned}0\to H^2_G(k,\mathscr{K}_4^M)\otimes\mathbf{Z}_{(2)}\to H^5(G,\mathbf{Q}_2/\mathbf{Z}_2(4))\to H^0_G(k,\mathscr{H}^5_{\text{ét}}(\mathbf{Q}_2/\mathbf{Z}_2(4)))\\
\to H^3_G(k,\mathscr{K}_4^M)\otimes\mathbf{Z}_{(2)}\to H^6(G,\mathbf{Q}_2/\mathbf{Z}_2(4)).\end{aligned}$$

Proof. — This follows from corollary 4.1.7 and proposition 4.2.1. □

Notation 4.4.2. — We put

$$H^2_G(k,\mathscr{K}_4^M)_{\mathrm{p}}=\sum_{H\subset G}\mathrm{Cores}_H^G(\mathrm{Pic}_G k\cup H^1_G(k,\mathscr{K}_3^M)).$$

Proposition 4.4.7. — *The canonical isomorphism*

$$H^2_G(k,\mathscr{K}_4^M)\otimes\mathbf{Z}_{(2)}\xrightarrow{\sim} H^5(G,\mathbf{Q}_2/\mathbf{Z}_2(4))_n$$

induces an isomorphism from

$$H^2_G(k,\mathscr{K}_4^M)/\Big(H^2_G(k,\mathscr{K}_4)_p+H^2_G(k,\mathscr{K}_4)_c\Big)\otimes\mathbf{Z}_{(2)}$$

to

$$H^5(G,\mathbf{Q}_2/\mathbf{Z}_2(4))_n/\Big(H^5(G,\mathbf{Q}_2/\mathbf{Z}_2(4))_p+H^5(G,\mathbf{Q}_2/\mathbf{Z}_2(4))_c\Big).$$

Proof. — The proof is similar to the one of proposition 4.4.3. □

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