GALOIS COHOMOLOGY IN DEGREE THREE AND HOMOGENEOUS VARIETIES*

by

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Abstract. — The central result of this paper is the following generalization of a result of the author on products of Severi-Brauer varieties. Let G be a semi-simple linear algebraic group over a field k. Let V be a generalized flag variety under G. Then there exist finite extensions k_i of k for $1 \leqslant i \leqslant m$, elements α_i in $\operatorname{Br} k_i$ and a natural exact sequence

$$\bigoplus_{i=1}^{m} k_i^* \xrightarrow{N_{k_i/k}(.\cup\alpha_i)} \operatorname{Ker}\left(H^3(k,\mathbf{Q}/\mathbf{Z}(2))\to H^3(k(V),\mathbf{Q}/\mathbf{Z}(2))\right)\to \mathrm{CH}^2(V)_{\mathrm{tors}}\to 0.$$

After giving a more explicit expression of the second morphism in a particular case, we apply this result to get classes in $H^3(Q,\mathbf{Q}/\mathbf{Z})$, which are k-negligible for any field k of characteristic different from 2 which contains a fourth root of unity, for a group Q which is a central extension of an $\mathbf{F}_2$ vector space by another.

Résumé. — Le résultat central de ce texte est la généralisation suivante d'un résultat de l'auteur sur les produits de variétés de Severi-Brauer. Soit G un groupe algébrique linéaire semi-simple sur un corps k. Soit V une variété de drapeaux généralisée sous G. Alors il existe des extensions finies k_i de k pour $1 \leqslant i \leqslant m$, des éléments α_i de $\operatorname{Br} k_i$ et une suite exacte naturelle

$$\bigoplus_{i=1}^{m} k_i^* \xrightarrow{N_{k_i/k}(.\cup\alpha_i)} \operatorname{Ker}\left(H^3(k,\mathbf{Q}/\mathbf{Z}(2))\to H^3(k(V),\mathbf{Q}/\mathbf{Z}(2))\right)\to \mathrm{CH}^2(V)_{\mathrm{tors}}\to 0.$$

Après avoir donné une description plus explicite du deuxième morphisme dans une cas particulier, nous utilisons ce résultat pour construire des classes dans $H^3(Q,\mathbf{Q}/\mathbf{Z})$ qui sont k-négligeables pour tout corps k de caractéristique différente de 2 et contenant une racine quatrième de l'unité, pour un groupe Q qui est extension centrale d'un $\mathbf{F}_2$ espace vectoriel par un autre.

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Contents

1. Introduction

The central result of this paper is a generalization of a previous result of the author about products of Severi-Brauer varieties [**Pe2**, theorem 4.1]. Let V be a generalized flag variety for a linear algebraic group G over a field k. Then the Picard group of V over a separable closure of k has a canonical basis B which is globally invariant under the action of the Galois group. Let E be the étale algebra corresponding to the Galois set B. There exists a class α in the Brauer group of E and an exact sequence

$$E^* \xrightarrow{N_{E/k}(.\cup\alpha)} \mathrm{Ker}(H^3(k,\mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(V),\mathbf{Q}/\mathbf{Z}(2))) \to \mathrm{CH}^2(V)_{\mathrm{tors}} \to 0.$$

In fact the part of the kernel given by the image of E^* comes from the kernel for Brauer groups in the following sense: this image is the union, for k' describing the separable finite extensions of k of the corestriction of the group $k'^* \cup \mathrm{Ker}(\mathrm{Br}\,k' \to \mathrm{Br}\,k'(V))$. This kernel has been studied independently by Merkur$'$ev in [**Me**].

After giving a few applications of this result and describing an explicit construction of the second map in a particular case, we turn to the second topic of this paper, the problem of totally k-negligible classes in the third cohomology group of some meta-abelian groups G, that is classes which vanish when lifted to the Galois cohomology of any extension of k. This notion of negligibility is weaker than the one introduced by Serre in [**Se**, §7]. In this article, the idea is to replace the field of invariants $k(W)^G$, on which the negligibility may be tested, by a function field $K(V)$ where K is purely transcendental over k and V a generalized flag variety over K for which we may apply the previous results of the paper. We then use this machinery to get a class in $H^3(G, \mathbf{Q}/\mathbf{Z})$ which is k-negligible for any field k of characteristic different from 2 and containing the fourth roots of one, where the group G is a central extension of an $\mathbf{F}_2$ vector space by another.

It is interesting to note that the first examples of geometrically negligible classes in the group $H^3(G, \mathbf{Q}/\mathbf{Z})$ were constructed by Saltman (see [**Sa2**, theorem 4.14]) for a 2-group G with a cyclic subgroup of index 2 using a kind of equivariant Chow group. Although the techniques used here are different from those of Saltman, the group $CH^2(V)$ which appears may also be interpreted as an equivariant Chow group; this seems to indicate the existence of a more general underlying structure.

Some of the results of this article have been stated with shortened proofs in [**Pe3**].

2. Notation and statement of the main result

Notation **.** — For any field L, $\overline{L}$ denotes an algebraic closure of L and L^s the separable closure of L in $\overline{L}$. For any discrete $\mathrm{Gal}(L^s/L)$-module M, set

$$H^i(L, M) = H^i(\mathrm{Gal}(L^s/L), M).$$

If the characteristic of L does not divide n, μ_n denotes the group of n-th roots of unity in L^s. If p is the exponential characteristic of L, which coincides with the usual one if it is different from 0 and is 1 otherwise, i a positive integer and j an integer, we put (see [**Kah**])

$$H^i(L, (\mathbf{Q}/\mathbf{Z})'(j)) = \varinjlim_{(p,n)=1} H^i(L, \mu_n^{\otimes j})$$

and, if $j = 0$, 1 or 2,

$$H^i(L, \mathbf{Q}_p/\mathbf{Z}_p(j)) = \varinjlim_{r} H^{i-j}(L, K_j(L^s)/p^r).$$

Then if $j = 0$, 1 or 2 we put

$$H^i(L, \mathbf{Q}/\mathbf{Z}(j)) = H^i(L, (\mathbf{Q}/\mathbf{Z})'(j)) \oplus H^i(L, \mathbf{Q}_p/\mathbf{Z}_p(j)).$$

If L' is a finite separable field extension of L then $N_{L'/L}$ denote the corestriction morphism from the Galois cohomology of L' to that of L.

If V is a variety over L then $L(V)$ is the function field of V. If moreover L' is a field extension of L then $V_{L'}$ denotes the product $V \times_{\operatorname{Spec} L} \operatorname{Spec} L'$ and V^s the variety V_{L^s}. For any nonnegative integer i, $V^{(i)}$ denotes the set of points of codimension i in V. The sheaf $\mathcal{K}_i$ is the sheaf on V for Zariski topology corresponding to the presheaf mapping U to $K_i(U)$, the i-th group of Quillen K-theory. If V is smooth, the codimension of support defines a decreasing filtration on $K^i(V)$ which is denoted by $K^i(V)^j$ (see [**Q**, §7.5]). The quotient $K^i(V)^j/K^i(V)^{j+1}$ is denoted by $K^i(V)^{(j/j+1)}$.

A generalized flag variety is a projective variety which is homogeneous under the action of a connected linear algebraic group G and such that V^s is isomorphic to the quotient of G^s by a standard parabolic subgroup. Without loss of generality, we may assume that the group G is semi-simple and simply connected.

From now on, G will denote a semi-simple simply connected linear algebraic group over a field k and V a generalized flag variety under G. We denote by $\mathcal{G}$ the Galois group of k^s over k.

The key result of this paper is the following theorem, the proof of which is given in section 5.

Theorem 2.1. — *With notation as above, the Picard group of V^s is a $\mathcal{G}$ permutation module. This means that there exist subgroups $\mathcal{H}_i$ of $\mathcal{G}$ of finite index such that $\operatorname{Pic} V^s$ is isomorphic to $\bigoplus_{i=1}^n \mathbf{Z}[\mathcal{G}/\mathcal{H}_i]$. Let k_i be the corresponding fields. Then for any i there is a class α_i of $\operatorname{Br} k_i$ such that:*

(i) In the natural exact sequence

$$\operatorname{Pic} V \to \operatorname{Pic}(V^s)^{\mathcal{G}} \xrightarrow{\rho} \operatorname{Br} k \to \operatorname{Br} k(V)$$

the natural generator of $\mathbf{Z}[\mathcal{G}/\mathcal{H}_i]^{\mathcal{G}}$ is sent to $N_{k_i/k}(\alpha_i)$ by ρ.

(ii) The homology of the complex

$$(\mathcal{C}) \qquad \bigoplus_{i=1}^m k_i^* \xrightarrow{N_{k_i/k}(.\cup\alpha_i)} H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(V), \mathbf{Q}/\mathbf{Z}(2))$$

is canonically isomorphic to the torsion subgroup of $\mathrm{CH}^2(V)$. In particular this homology is finite.

Remark 2.2. — Merkur′ev has proved in [**Me**] that the kernel of the first map in $(\mathcal{C})$ coincides with $H^1(V, \mathcal{K}_2)$.

3. $\mathcal{K}$-cohomology

We shall now consider the $\mathcal{G}$-module structure of the $\mathcal{K}$-cohomology groups of V^s.

We fix a parabolic subgroup P of G^s such that V^s is isomorphic to $P\backslash G^s$. Let B be a Borel subgroup of G^s contained in P, T be a maximal torus in B, Φ be the root system of T in G^s, and W be the corresponding Weyl group. The letter Δ denotes the basis of Φ corresponding to B. For any α in Δ, s_α denotes the corresponding generator of W and ϖ_α the corresponding fundamental weight. Let I be the subset of Δ corresponding to P (see [**Bor**, page 234]). For any subset J of Δ, the corresponding parabolic subgroup is denoted by P_J. Let V_J be the homogeneous variety $P_J\backslash G^s$ and π_J be the canonical projection $G^s \to V_J$. The subgroup generated by the s_α for $\alpha \in J$ is denoted by W_J and the set of the unique elements of minimal length in the classes $W_J w$ for w in W by W^J. Let w_J be the longest element in W_J.

Let $\mathfrak{g}$ be the Lie algebra of G^s and for any $\alpha \in \Phi$ let

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid \forall t \in T(k^s), \operatorname{Ad}(t)(X) = \alpha(t)X \}.$$

The unique subgroup of G^s normalized by T and having $\mathfrak{g}_\alpha$ as Lie algebra is denoted by U_α. If w belongs to W, then U'_w is the subgroup of G^s generated by the U_γ where γ varies over

$$\{ \delta \in \Phi \mid \delta > 0 \text{ and } w\delta < 0 \}.$$

By Bruhat's decomposition, G^s is the disjoint union of the double classes $C(w) = BwB$ for $w \in W$. Moreover by [**Bor**, theorem 14.12] there are isomorphisms

$$\begin{array}{rcl} B \times U'_w & \xrightarrow{\sim} & BwB \\ (b,u) & \mapsto & bwu \end{array} \tag{1}$$

and by [**Bor**, theorem 21.29], the sets $\pi_J(C(w))$ give a cellular decomposition of V_J as w varies over W^J. Moreover the isomorphism (1) yields an isomorphism from U'_w to $\pi_J(C(w))$. In particular by [**Bki**, chapitre VI, n° 1.6, page 158, corollaire 2],

$$\dim(\pi_J(C(w))) = \#\{ \gamma \in \Phi \mid \gamma > 0 \text{ and } w\gamma < 0 \} = l(w).$$

Let $X_{J,w} = \overline{\pi_J(C(w))}$. By [**Fu**, example 1.9.1], the Chow group $\mathrm{CH}_i(V_J)$, which by [**Q**, theorem 7.5.19] is isomorphic to $H^i(V_J, \mathcal{K}_i)$, is generated by the classes $[X_{J,w}]$ for

$$w \in W_i^J = \{ w \in W^J \mid l(w) = i \}$$

and by [**Dem**, corollaire page 69] these classes form a basis when $J = \varnothing$. We shall now generalize this result.

Proposition 3.1. — *With notation as above,* $\bigoplus_{i,j\geqslant 0} H^i(V_J, \mathcal{K}_{i+j})$ *is a free* $\bigoplus_{j\geqslant 0} K_j k^s$*-module with a canonical basis given by the classes* $[X_{J,w}]$ *in* $H^i(V_J, \mathcal{K}_i)$ *for* w *in* $W^J_{\dim V_J - i}$ *where* i *varies over* $\{0, \ldots, \dim V_J\}$*. Moreover the map* $w \mapsto w_J w w_\Delta$ *induces a bijection* $w \mapsto \bar{w}$ *from* W^J_i *to* $W^J_{\dim V_J - i}$ *and in the Chow ring* $\bigoplus_{i\geqslant 0} H^i(V_J, \mathcal{K}_i)$ *one has for any* $w \in W^J_i$ *and any* $w' \in W^J_{\dim V_J - i}$

$$[X_{J,w}].[X_{J,w'}] = \delta_{\bar{w},w'}[X_{J,e}]$$

where $[X_{J,e}]$ *is the class of a point.*

We first state two corollaries of this proposition.

Corollary 3.2. — *The* $\mathcal{G}$*-lattices* $H^i(V^s, \mathcal{K}_i)$ *are permutation modules.*

Proof. — Let $C^i_{\text{eff}} \subset H^i(V^s, \mathcal{K}_i) \otimes \mathbf{Q}$ be the cone of classes of effective divisors. Then $[X_{I,w}]$ belongs to this cone. On the other hand, let

$$\alpha = \sum_{w\in W^I_i} n_w [X_{I,w}]$$

be an element of $C^{\dim V - i}_{\text{eff}}$. Then, let E be an effective divisor representing a multiple of α. By [**Fu**, page 441] for any w belonging to $W^I_{\dim V - i}$, there exists g in $G(k^s)$ such that the intersection of $X_{I,w}.g$ with E is a union of points. Thus one has $[X_{I,w}].[E] \geqslant 0$ and hence $[X_{I,w}].\alpha \geqslant 0$. But, by the proposition, for any $w \in W^I_i$, $n_w = [X_{I,\bar{w}}].\alpha$. Therefore, we get

$$C^{\dim V - i}_{\text{eff}} = \sum_{w\in W^I_i} \mathbf{Q}_{\geqslant 0}[X_{I,w}].$$

The action of $\mathcal{G}$ on $H^i(V^s, \mathcal{K}_i) \otimes \mathbf{Q}$ leaves C^i_{eff} globally invariant. Its faces of dimension one remain also invariant and thus the basis $([X_{I,w}])_{w\in W^I_{\dim V - i}}$ is also globally invariant. $\square$

Corollary 3.3. — *For any positive integer* i*, one has*

$$H^1(\mathcal{G}, H^i(V^s, \mathcal{K}_{i+1})) = 0.$$

Proof. — By proposition 3.1, there are isomorphisms

$$H^i(V^s, \mathcal{K}_{i+1}) \xrightarrow{\sim} k^{s*} \otimes_{\mathbf{Z}} H^i(V^s, \mathcal{K}_i),$$

but the right term is a permutation module by corollary 3.2. Let $\mathcal{G}_j$ be the subgroups of $\mathcal{G}$ such that $H^i(V^s, \mathcal{K}_i) \xrightarrow{\sim} \bigoplus_{j=1}^m \mathbf{Z}[\mathcal{G}/\mathcal{G}_j]$. Then by Shapiro's lemma

$$H^i(V^s, \mathcal{K}_{i+1}) \xrightarrow{\sim} \bigoplus_{j=1}^m H^1(\mathcal{G}_j, k^{s*}).$$

But by Hilbert's theorem 90, the groups $H^1(\mathcal{G}_j, k^{s*})$ are trivial. □

Proof of proposition 3.1. — Let us first prove the assertions concerning the Chow ring. By [**Fu**, example 1.9.1], the classes $[X_{J,w}]$ generate the **Z**-module $\bigoplus_{i\in\mathbf{N}} H^i(V_J, \mathcal{K}_i)$. Let w belong to W_i^J and let us show that $w' = w_J w w_\Delta$ belongs to $W^J_{\dim V_J - i}$. By [**Bki**, chapitre IV, §1, exercice 3],

$$l(w_J w) = l(w_J) + l(w) = \dim V_\varnothing - \dim V_J + l(w)$$

and by [**Bki**, chapitre VI, n° 1.6, page 158, corollaire 3],

$$l(w_J w w_\Delta) = l(w_\Delta) - l(w_J w) = \dim V_J - l(w).$$

Let us write w' in the form $w_1 w_2$ with $w_1 \in W_J$ and $w_2 \in W^J$. Then

$$l(w_J w') = l(w_J w_1 w_2) = l(w_J) - l(w_1) + l(w_2).$$

Thus we have

$$l(w) = l(w_J w' w_\Delta) = l(w_\Delta) - l(w_J) + l(w_1) - l(w_2) = \dim V_J + l(w_1) - l(w_2)$$

but $l(w') = l(w_1) + l(w_2) = \dim V_J - l(w)$ and we get $2l(w_1) = 0$, which yields that w_1 is trivial.

Let us now consider elements w of W_i^J and w' of $W^J_{\dim V_J - i}$. We denote by $\pi_{\varnothing,J}$ the canonical projection from $V_\varnothing$ to V_J. By [**Bor**, proposition 21.29], one has the equality

$$\pi_{\varnothing,J}(\pi_\varnothing(C(w_J w))) = \pi_J(C(w))$$

since $l(w_J w) = l(w_J) + l(w) = \dim(V_\varnothing) - \dim(V_J) + l(w)$, we get that $\pi_{\varnothing,J}^{-1}(X_{J,w})$ is equal to $X_{\varnothing, w_J w}$. Then, using [**Dem**, §3.3, proposition 1] and [**Fu**, proposition

8.3], we obtain the following equalities

$$\begin{aligned}[X_{J,w}][X_{J,w'}] &= [X_{J,w}].\pi_{\varnothing,J*}([X_{\varnothing,w'}])\\ &= \pi_{\varnothing,J*}(\pi^*_{\varnothing,J}([X_{J,w}]).[X_{\varnothing,w'}])\\ &= \pi_{\varnothing,J*}([X_{\varnothing,w_J w}].[X_{\varnothing,w'}])\\ &= \pi_{\varnothing,J*}(\delta_{w_J w,w' w_\Delta}[X_{\varnothing,e}])\\ &= \delta_{\tilde{w},w'}[X_{J,e}].\end{aligned}$$

Thus the classes $[X_{J,w}]$ for $w \in W^J$ give a basis of the Chow ring and the intersection formula is proved.

Let N be the cardinal of W^J. Let us now choose a bijection from $\{1,\dots,N\}$ to W^J such that $i \leqslant i'$ implies $l(w_i) \geqslant l(w_{i'})$, where w_i denotes the image of i. Then for any i between 1 and N we denote by O_i the open set $\bigcup_{j\leqslant i}\pi_J(C(w_j))$. We shall prove by induction on i that for any i such that $1 \leqslant i \leqslant N$ the $\bigoplus_{j\geqslant 0} K_j k^s$-module $\bigoplus_{j,l\geqslant 0} H^l(O_i, \mathscr{K}_{l+j})$ is free with a basis given by the classes

$$\left[\overline{\pi_J(C(w_j))}\right] \in H^{\dim V_J - l(w_j)}(O_i, \mathscr{K}_{\dim V_J - l(w_j)})$$

for j in $\{1,\dots,i\}$. For $i=1$, the open set O_1 is isomorphic to an affine space and the assertion is a consequence of the homotopy theorem for $\mathscr{K}$-cohomology (see [**Sh**, theorem 2.4]). Let us assume the result for $i-1$. Then $U_i = O_i - O_{i-1} = \pi_J(C(w_i))$ is isomorphic to an affine space of dimension $l(w_i)$. By the homotopy theorem one has

$$H^p(U_i, \mathscr{K}_q) = \begin{cases} K_q k^s & \text{if } p = 0,\\ 0 & \text{otherwise.}\end{cases}$$

Since the varieties U_i and O_i are smooth, there are long exact sequences

$$\cdots \to H^{p-d}(U_i, \mathscr{K}_{q-d}) \to H^p(O_i, \mathscr{K}_q) \to H^p(O_{i-1}, \mathscr{K}_q) \xrightarrow{\partial_i^{p,q}} H^{p+1-d}(U_i, \mathscr{K}_{q-d}) \to \cdots$$

where $d = \dim V_J - l(w_i)$ which are induced by the short exact sequences

$$0 \to \bigoplus_{x\in U_i^{(r-d)}} K_j k(x) \to \bigoplus_{x\in O_i^{(r)}} K_j k(x) \to \bigoplus_{x\in O_{i-1}^{(r)}} K_j k(x) \to 0.$$

For $q = p$ one has in particular

$$\dots \xrightarrow{\partial_i^{p-1,p}} H^{p-d}(U_i, \mathscr{K}_{p-d}) \to H^p(O_i, \mathscr{K}_p) \to H^p(O_{i-1}, \mathscr{K}_p) \to 0.$$

But, if $p > d$ the group $H^{p-d}(U_i, \mathcal{K}_{p-d})$ is trivial and $\partial_i^{p-1,p} = 0$. On the other hand, if $p = d$ then

$$\mathrm{rk}(H^p(V_J, \mathcal{K}_p)) = \# W_p^J = \sum_{\{i \mid l(w_i) = \dim V_J - p\}} \mathrm{rk}(\mathrm{Coker}\, \partial_i^{p-1,p}),$$

and the maps $\partial_i^{p-1,p}$ are trivial. But the morphisms ∂_i are $K_* k^s$ linear and by induction hypothesis $H^p(O_{i-1}, \mathcal{K}_{p+*})$ is a free $K_* k^s$-module. Therefore all maps $\partial_i^{p,q}$ are trivial. Hence we obtain the following commutative diagram the horizontal lines of which are exact:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^{p-d}(U_i, \mathcal{K}_{q-d}) & \longrightarrow & H^p(O_i, \mathcal{K}_q) & \longrightarrow & H^p(O_{i-1}, \mathcal{K}_q) & \longrightarrow 0 \\ & \uparrow \wr & & \uparrow & & \uparrow \wr & \\ 0 \rightarrow & K_{q-p} k^s \otimes H^{p-d}(U_i, \mathcal{K}_{p-d}) & \rightarrow & K_{q-p} k^s \otimes H^p(O_i, \mathcal{K}_p) & \rightarrow & K_{q-p} k^s \otimes H^p(O_{i-1}, \mathcal{K}_p) & \rightarrow 0. \end{array}$$

Therefore the vertical line in the middle is also an isomorphism. □

Concerning the Chow groups of V we simply note the following facts:

Proposition 3.4. — *Let k' be a separable finite extension of k which splits G.*

(i) The group $K_0(V)$ is without torsion. We may identify it with its image in $K_0(V_{k'})$.

(ii) One has the inclusions

$$[k' : k] K_0(V_{k'})^i \cap K_0(V) \subset K_0(V)^i \subset K_0(V_{k'})^i \cap K_0(V).$$

(iii) The kernel of the surjective map

$$\mathrm{CH}^i(V) \to K_0(V)^{(i/i+1)}$$

is killed by $(i-1)!$ and $[k' : k]$.

(iv) $K_0(V)^i = K_0(V_{k'})^i \cap K_0(V)$ if $i = 1$ or 2.

(v) $\mathrm{CH}^2(V)_{\mathrm{tors}} \xrightarrow{\sim} (K_0(V_{k'})^3 \cap K_0(V))/K_0(V)^3$.

Proof. — The assertion (i) is a consequence of Panin's result [**Pa**, theorem 4.2]. The assertions (ii), (iii) and (iv) are proved as the similar assertions of [**Pe2**, proposition 3.8]. For the last assertion, since $\mathrm{CH}^2(V_{k'})$ is without torsion and

$$\mathrm{Ker}(\mathrm{CH}^2(V) \to \mathrm{CH}^2(V_{k'}))$$

is killed by $[k':k]$, one has

$$\begin{aligned}\mathrm{CH}^2(V)_{\mathrm{tors}} &= \mathrm{Ker}(\mathrm{CH}^2(V) \to \mathrm{CH}^2(V_{k'}))\\ &= \mathrm{Ker}\left(K_0(V)^2/K_0(V)^3 \to K_0(V_{k'})^2/K_0(V_{k'})^3\right)\\ &= (K_0(V)^2 \cap K_0(V_{k'})^3)/K_0(V)^3\\ &= (K_0(V) \cap K_0(V_{k'})^3)/K_0(V)^3. \quad \square\end{aligned}$$

4. Explicit description of the Hochschild-Serre spectral sequence for hypercohomology

The construction of the morphism from the homology of the complex $(\mathscr{C})$ to the group $\mathrm{CH}^2(V)_{\mathrm{tors}}$ involves the Hochschild-Serre spectral sequence for relative hypercohomology groups of $k^s(V)/k^s$ with coefficients in the Lichtenbaum complex $\Gamma(2)$ (see [**Kah**, page 68]). Therefore we shall now give an explicit construction of this spectral sequence and then check that it is compatible with corestriction and cup-products. In fact, this construction is a simple generalization of the one of Hochschild and Serre and the proof of the compatibilities are quite straightforward but we include them for self-completeness.

For any group G and any G-module M, $C^i(G,M)$ denotes the group of normalized n-cochains; this means the group of functions $f : G^n \to M$ such that $f(g_1,\dots,g_n)$ is trivial whenever one of the g_i is e. Let G be a group and K be a normal subgroup of G. In the following a bounded complex of G-modules is a family $(M^i)_{i\in\mathbf{Z}}$ of G-modules, which are trivial except for a finite number of integers, equipped with a differential δ going from M^i to M^{i+1}. Let $(M^\bullet,\delta)$ be a bounded complex of G-modules. We consider the group

$$A(G) = \bigoplus_{\substack{i\geqslant 0\\ j\in\mathbf{Z}}} C^i(G,M^j)$$

which is a bicomplex for the differentials d' and d'' where

$$d' : C^i(G,M^j) \to C^{i+1}(G,M^j)$$

is the standard non homogeneous coboundary operator and

$$d'' : C^i(G,M^j) \to C^i(G,M^{j+1})$$

is $(-1)^i\delta_*$. We put $A^{(i,j)}(G) = C^i(G,M^j)$ if $i \geqslant 0$ and $j \in \mathbf{Z}$ and $A^{(i,j)}(G) = 0$ otherwise. As in [**HS**, page 119] one defines a decreasing filtration on A as follows: $A_l^{(i,j)}(G)$ is $A^{(i,j)}(G)$ if $l \leqslant 0$, $A_l^{(i,j)}(G)$ is the set of i-cochains $\gamma : G^i \to M^j$ such that $\gamma(g_1, \ldots, g_i)$ depends only on $g_1, \ldots, g_{i-l}, g_{i-l+1}K, \ldots, g_i K$ if $0 \leqslant l \leqslant i$ and is 0 otherwise. Then we put

$$A^n(G) = \bigoplus_{i+j=n} A^{(i,j)}(G), A_l^n(G) = \bigoplus_{i+j=n} A_l^{(i,j)}(G) \text{ and } A_l(G) = \bigoplus_{n\in\mathbf{Z}} A_l^n(G).$$

By [**HS**, page 119] this filtration is compatible with d'. The compatibility with d'' is clear. Therefore it is compatible with the total complex

$$\left(\bigoplus_{n\in\mathbf{Z}} A^n(G), d = d' + d''\right).$$

Moreover there is a natural morphism

$$\phi_l : A_l^{(i,j)}(G) \to C^l(G/K, C^{i-l}(K, M^j))$$

obtained by restricting the first $i-l$ coordinates of an element of $A_l^{(i,j)}(G)$ to K.

We recall that the corresponding spectral sequence is then defined by

$$Z_r^{p,q}(G) = \left\{ a \in A_p^{p+q}(G) \mid da \in A_{p+r}^{p+q+1}(G) \right\}$$

and

$$E_r^{p,q}(G/K) = Z_r^{p,q}(G)/(Z_{r-1}^{p+1,q-1}(G) + d(Z_{r-1}^{p-r+1,q+r-2}(G))).$$

The group G acts on the modules M^i and by conjugation on K. This yields an action of G on $C^i(K, M^j)$ compatible with the differentials and induces an action on the hypercohomology groups $\mathbf{H}^q(K, M^\bullet)$.

Lemma 4.1. — *The group K acts trivially on* $\mathbf{H}^q(K, M^\bullet)$.

Proof. — Consider the abelian category $\mathscr{C}_{\geqslant 0}$ of bounded complexes of G-modules $(N^\bullet, \delta)$ such that N^i is trivial if $n < 0$. Without loss of generality, we may assume that $M^\bullet$ is an object of this category. The hypercohomology functors $\mathbf{H}^i(K, -)$ when restricted to $\mathscr{C}_{\geqslant 0}$ are the right derived functors of the functor sending $(N^\bullet, \delta)$ on the group $\mathrm{Ker}(N^0 \to N^1)^K$ (see [**Mi**, appendix C]) and for any object $N^\bullet$ of $\mathscr{C}_{\geqslant 0}$, the group K acts trivially on the cohomology group $\mathbf{H}^0(K, N^\bullet)$. Thus, for any k in K, the induced automorphism of the cohomological functor $\mathbf{H}^*(K, -)$ is trivial in degree 0 and therefore in any degree [**CE**, Chapter III, proposition 5.2]. □

We get an action of G/K on $\mathbf{H}^q(K, M^\bullet)$ for any q in $\mathbf{Z}$.

Lemma 4.2. — *With notation as above, the map* ϕ_p *induces an isomorphism*

$$E_1^{p,q}(G/K) \xrightarrow{\sim} C^p(G/K, \mathbf{H}^q(K, M^\bullet))$$

and

$$E_2^{p,q}(G/K) \xrightarrow{\sim} H^p(G/K, \mathbf{H}^q(K, M^\bullet)).$$

Proof. — By construction, if we denote also by d' the map on $C^p(G/K, C^q(K, M^j))$ induced by the standard coboundary operator for K, then ϕ_p commutes with d'. Let d'' be defined as $(-1)^{p+q}\delta_*$ on $C^p(G/K, C^q(K, M^j))$. Then ϕ_p and d'' commute. Thus ϕ_p induces a canonical map

$$E_1^{p,q}(G/K) \to C^p(G/K, \mathbf{H}^q(K, M^\bullet)).$$

Let us check that it is an injection. Let us consider pairs $(\boldsymbol{f}, \boldsymbol{u})$ with

$$\boldsymbol{f} = (f_i)_{i\geqslant 0} \in \bigoplus_{i\geqslant 0} A_p^{(i,p+q-i)}(G) \text{ and } \boldsymbol{u} = (u_i)_{i\geqslant p} \in \bigoplus_{i\geqslant p} C^p(G/K, C^{i-p}(K, M^{p+q-1-i}))$$

such that $d\boldsymbol{f}$ belongs to $A_{p+1}^{p+q+1}(G)$ and $\phi_p(\boldsymbol{f}) = d(\boldsymbol{u})$ where d denotes the sum $d' + d''$. We put $u_i = 0$ if $i < p$. Since $M^\bullet$ is bounded, it is enough to prove by an increasing induction on m that for any such pair which verifies

$$\forall i \in \mathbf{Z}_{\geqslant 0},\ i \geqslant m \Rightarrow u_{i-1} = 0 \text{ and } f_i \in A_{p+1}^{(i,p+q-i)}(G),$$

there exists $\boldsymbol{h} \in A_p^{p+q-1}(G)$ such that

$$\boldsymbol{f} - d\boldsymbol{h} \in A_{p+1}^{p+q}(G).$$

It is true if m is zero. Let us assume it is true for m. Let $(\boldsymbol{f}, \boldsymbol{u})$ be a pair which verifies the condition for $m+1$. By definition of d one has that

$$d'(f_m) \in A_{p+1}^{(m+1,p+q-m)}(G) \text{ and } \phi_p(f_m) = d'(u_{m-1}).$$

By [**HS**, pages 121 and 122], there exists a cochain h_{m-1} in $A_p^{(m-1,p+q-m)}(G)$ such that

$$\phi_p(h_{m-1}) = u_{m-1} \text{ and } f_m - d'(h_{m-1}) \in A_{p+1}^{(m,p+q-m)}(G).$$

let $\boldsymbol{f}' = \boldsymbol{f} - d(h_{m-1})$. Then $\boldsymbol{f}' \in Z_1^{p,q}(G)$ and

$$\phi_p(\boldsymbol{f}') = \phi_p(\boldsymbol{f}) - \phi_p \circ d(h_{m-1}) = d(\boldsymbol{u}) - d \circ \phi_p(h_{m-1}).$$

It is enough to check the result for $\boldsymbol{f}'$. But

$$f'_m = f_m - d'(h_{m-1}) \in A_{(p+1)}^{(m,p+q-m)}(G)$$

and $\phi_p(\boldsymbol{f}') = d(\boldsymbol{u}')$ where

$$u'_i = \begin{cases} u_i & \text{if } i \leqslant m-2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we may apply the induction hypothesis.

Let us now prove the surjectivity. Let

$$(\boldsymbol{u}) = (u_i)_{i \geqslant 0} \in \bigoplus_{i \geqslant 0} C^p(G/K, C^{i-p}(K, M^{p+q-i}))$$

be such that $d(\boldsymbol{u}) = 0$. We shall prove by a decreasing induction on i the existence of

$$(f_j)_{j \geqslant i} \in \bigoplus_{j \geqslant i} C^j(G, M^{p+q-j})$$

such that if $j > i$

$$d''(f_j) + d'(f_{j-1}) \in A_{p+1}^{(j,p+q-j+1)}(G)$$

and, if $j \geqslant i$, we have $\phi_p(f_j) = u_j$. It is true for i big enough since $M^\bullet$ is bounded. Let us assume it is true for $i+1$. Then we have $\phi_p(f_{i+1}) = u_{i+1}$. But $d''(u_{i+1}) + d'(u_i) = 0$. Thus

$$\phi_p(d''(f_{i+1})) = d''(\phi_p(f_{i+1})) = d'(-u_i).$$

Moreover one has

$$d'(d''(f_{i+1})) = d''(-d''(f_{i+2}) - d'(f_{i+1})) \in A_{p+1}^{(i+2,p+q-i)}(G).$$

Therefore, by [**HS**, pages 121 and 122], there exists $f_i \in A_p^{(i,p+q-i)}(G)$ such that

$$\phi_p(f_i) = u_i \text{ and } d'(f_i) + d''(f_{i+1}) \in A_{p+1}^{(i+1,p+q-i)}(G)$$

and the result is proved for i. Thus we get an element

$$\boldsymbol{f} = (f_i)_{i \geqslant 0} \in \bigoplus_{i \geqslant 0} A_p^{(i,p+q-i)}(G)$$

such that

$$\phi_p(\boldsymbol{f}) = \boldsymbol{u} \text{ and } d(\boldsymbol{f}) \in A_{p+1}^{p+q}(G).$$

This proves the first assertion of the lemma.

It remains to show that we have a commutative diagram:

$$
\begin{array}{ccc}
C^p(G/K, \mathbf{H}^q(K, M^\bullet)) & \xrightarrow{(-1)^q d} & C^{p+1}(G/K, \mathbf{H}^q(K, M^\bullet)) \\
\uparrow \wr & & \uparrow \wr \\
E_1^{p,q}(G/K) & \xrightarrow{d} & E_1^{p+1,q}(G/K).
\end{array} \tag{2}
$$

As in [**HS**, page 123] we consider the partial coboundary operators defined, for any f in $C^{p+q-1}(G, M^n)$ by

$$
\begin{aligned}
\delta'_{p-1} f(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p) = {} & \alpha_1 . f(\alpha_2, \dots, \alpha_q, \beta_1, \dots, \beta_p) \\
& + \sum_{k=1}^{q-1} (-1)^k f(\alpha_1, \dots, \alpha_k \alpha_{k+1}, \dots, \alpha_q, \beta_1, \dots, \beta_p) \\
& + (-1)^q f(\alpha_1, \dots, \alpha_{q-1}, \beta_1, \dots, \beta_p)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_{p-1} f(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p) = {} & \beta_1 f(\beta_1^{-1} \alpha_1 \beta_1, \dots, \beta_1^{-1} \alpha_q \beta_1, \beta_2, \dots, \beta_p) \\
& + \sum_{k=1}^{p-1} (-1)^k f(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_k \beta_{k+1}, \dots, \beta_p) \\
& + (-1)^p f(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_{p-1}).
\end{aligned}
$$

For any subset $S = \{s_1, \dots, s_p\}$ of $\{1, \dots, p+q\}$ with $s_1 < \cdots < s_p$, let $S^* = \{s_1^*, \dots, s_q^*\}$ be its complement with $s_1^* < \cdots < s_q^*$. Write $i^* = s_i^* - i$ and $\nu(S) = \sum_{i=1}^q i^*$ and set $b_0 = 1$ and $b_k = \beta_1 \dots \beta_k$ for $1 \leqslant k \leqslant p$. As in [**HS**, page 123] one defines for any $g \in C^{p+q}(G, M^n)$

$$
g_S(\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q) = g(\gamma_1, \dots, \gamma_{p+q})
$$

where $\gamma_{s_i} = \beta_i$ for $1 \leqslant i \leqslant p$ and $\gamma_{s_i^*} = b_{i^*}^{-1} \alpha_i b_{i^*}$ for $1 \leqslant i \leqslant q$ and $g_{(p)} = \sum_S (-1)^{\nu(S)} g_S$ where S ranges over all the subsets of p elements from $(1, \dots, p+q)$. By [**HS**, page 123, proposition 2], for any f in $C^{p+q-1}(G, M^n)$ one has

$$
(d' f)_{(p)} = \delta'_{p-1}(f_{(p)}) + (-1)^q \partial_{p-1}(f_{(p-1)}).
$$

Let δ''_{p-1} be defined by

$$
(\delta''_{p-1} f)(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p) = (-1)^{p+q} \delta(f(\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p))
$$

and $\delta_p : A^{p+q}(G) \to A^{p+q+1}(G)$ by $\delta_p = \delta'_p + \delta''_p$. Then

$$\forall f \in A^{(p+i,q-i)}(G),\ (df)_{(p+1)} = \delta_p(f_{(p+1)}) + (-1)^i \partial_p(f_{(p)}). \tag{3}$$

Let $D_p : A(G) \to A(G)$ be defined as $(-1)^i \partial_p$ on $A^{(p+i,q-i)}(G)$. Then D_p induces on $C^p(G/K, C^i(K, M^{q-i}))$ a map D_p which commutes with the maps d' and d'' defined at the beginning of the proof. This is obvious for d'' and follows from straightforward computations for d'. The map D_p induces $(-1)^q d$ on $C^p(G/K, \mathbf{H}^q(K, M^\bullet))$. On the other hand, for any g belonging to $A_p^{(p+i,q-i)}(G)$, the natural image of $\phi_p(g)$ in $C^p(G, C^i(K, M^{q-i}))$ coincides with the restriction of $g_{(p)}$ to $K^i \times G^p$. Thus by (3) the map induced by $d = d' + d''$ on

$$\mathrm{Im}(C^p(G/K, \mathbf{H}^q(K, M^\bullet)) \to C^p(G, \mathbf{H}^q(K, M^\bullet)))$$

coincides with the one defined by D_p and this implies the commutativity of the diagram (2). □

Remark 4.3. — From this description of the spectral sequence it follows immediately that if H is a subgroup of G containing K then the spectral sequence is compatible with the restriction map from G to H and that it is functorial for maps of bounded complexes. Let us now state more precisely and prove the corresponding result for the corestriction.

Lemma 4.4. — *Let $K \subset H \subset G$ be three groups such that K is normal in G and H is of finite index in G and let $M^\bullet$ be a bounded complex of G-modules. Then the Hochschild-Serre spectral sequences*

$$E_2^{p,q}(H/K) = H^p(H/K, \mathbf{H}^q(K, M^\bullet)) \Rightarrow \mathbf{H}^{p+q}(H, M^\bullet)$$

and

$$E_2^{p,q}(G/K) = H^p(G/K, \mathbf{H}^q(K, M^\bullet)) \Rightarrow \mathbf{H}^{p+q}(G, M^\bullet)$$

are compatible with the corestriction map. More precisely, if $i \geqslant 2$, there are natural maps

$$\mathrm{Cores}_i : E_i^{p,q}(H/K) \to E_i^{p,q}(G/K)$$

such that one has commutative diagrams

$$\begin{array}{ccc} E_i^{p,q}(H/K) & \xrightarrow{\text{Cores}_i} & E_i^{p,q}(G/K) \\ {\scriptstyle d_i^{p,q}}\downarrow & & {\scriptstyle d_i^{p,q}}\downarrow \\ E_i^{p+i,q-i+1}(H/K) & \xrightarrow{\text{Cores}_i} & E^{p+i,q-i+1}(G/K) \end{array}$$

and

$$\begin{array}{ccc} E_2^{p,q}(H/K) & \xrightarrow{\text{Cores}_2} & E_2^{p,q}(G/K) \\ \downarrow\wr & & \downarrow\wr \\ H^p(H/K, \mathbf{H}^q(K, M^\bullet)) & \xrightarrow{\text{Cores}} & H^p(G/K, \mathbf{H}^q(K, M^\bullet)) \end{array}$$

and such that Cores_{i+1} *coincides with the map induced by* Cores_i. *This yields maps*

$$\text{Cores} : E_\infty^{p,q}(H/K) \to E_\infty^{p,q}(G/K).$$

Moreover the corestriction maps are compatible with the filtrations on $\mathbf{H}^p(H, M^\bullet)$ *and* $\mathbf{H}^p(G, M^\bullet)$ *and the diagram*

$$\begin{array}{ccc} E_\infty^{p,q}(H/K) & \xrightarrow{\text{Cores}} & E_\infty^{p,q}(G/K) \\ \downarrow\wr & & \downarrow\wr \\ \mathbf{H}^{p+q}(H, M^\bullet)^{(p/p+1)} & \xrightarrow{\text{Cores}} & \mathbf{H}^{p+q}(G, M^\bullet)^{(p/p+1)}. \end{array}$$

commutes.

Proof. — Let $\text{Ind}_H^G(M^\bullet)$ be the induced complex given by

$$\text{Ind}_H^G(M)^i = \text{Ind}_H^G(M^i)$$

where the induced module $\text{Ind}_H^G(M^i)$ is defined as the set of maps from G to M^i invariant under the $*$-action of H defined by the formula

$$\forall f \in \text{Map}(G, M^i),\ \forall h \in H,\ \forall g \in G,\ (h * f)(g) = h.(f(h^{-1}g))$$

equipped with the action of G defined by

$$\forall g \in G,\ \forall f \in \text{Ind}_H^G(M^i),\ \forall g' \in G,\ (g.f)(g') = f(g'g).$$

Then there are projections of H-modules

$$\begin{aligned} \text{Ind}_H^G(M^i) &\to M^i \\ f &\mapsto f(e) \end{aligned}$$

which gives an epimorphism of complexes $\mathrm{pr} : \mathrm{Ind}_H^G(M^\bullet) \to M^\bullet$. Composing it with the restriction map from G to H we get using remark 4.3 that the spectral sequences

$$H^p(H/K, \mathbf{H}^q(K, M^\bullet)) \Rightarrow \mathbf{H}^{p+q}(H, M^\bullet)$$

and

$$H^p(G/K, \mathbf{H}^q(K, \mathrm{Ind}_H^G M^\bullet)) \Rightarrow \mathbf{H}^{p+q}(G, \mathrm{Ind}_H^G M^\bullet)$$

are compatible with the isomorphisms of Shapiro's lemma.

But, since the M^i are G-modules the action $*$ extends to G and the group of invariants for this extended action is in bijection with M^i. This yields a map of G-modules $\mathrm{Tr} : \mathrm{Ind}_H^G(M^i) \to M^i$. But using the same argument as in the proof of lemma 4.1 we get that the corestriction from H to G is obtained as the composite of the map induced by Tr and the inverse of Shapiro's isomorphism. This completes the proof of the lemma. □

Remark 4.5. — Using an explicit description of the corestriction at the level of cocycles it is possible to show that the condition $i \geqslant 2$ is unnecessary.

It remains to prove the compatibility with cup-products. To this intent, we now give another filtration on the group $A(G)$ which produces the same spectral sequence and is compatible with cup-products (see [**HS**, page 118]). The group $B_l^{(i,j)}(G, M^\bullet)$ is $A^{(i,j)}(G)$ if $l \leqslant 0$, it is the set of all i-cochains $\gamma : G^i \to M^j$ such that $\gamma(g_1, \dots, g_i) = 0$ whenever $i - l + 1$ of the $g_1, \dots, g_i$ belongs to K if $0 \leqslant l \leqslant i$ and is 0 otherwise. Then

$$B_l^n(G, M^\bullet) = \bigoplus_{i+j=n} B_l^{(i,j)}(G, M^\bullet) \text{ and } B_l(G, M^\bullet) = \bigoplus_{n \in \mathbf{Z}} B_l^n(G, M^\bullet).$$

This filtration is compatible with d' and d'' and therefore with the total complex.

Let $M^\bullet$, $N^\bullet$, and $P^\bullet$ be three bounded complexes of G-modules. Then the tensor product $M^\bullet \otimes N^\bullet$ is given by

$$M^\bullet \otimes N^{\bullet n} = \bigoplus_{i+j=n} M^i \otimes N^j$$

equipped with the differential given by $\delta_M \otimes 1 + (-1)^i \otimes \delta_N$ on $M^i \otimes N^j$. Let ϕ be a morphism of G-complexes from $M^\bullet \otimes N^\bullet$ to $P^\bullet$. Then it induces a pairing

$$\cup_\phi : C^p(G, M^i) \times C^q(G, N^j) \to C^{p+q}(G, P^{i+j})$$

given by

$$f \cup_\phi g(\gamma_1, \dots, \gamma_{p+q}) = \phi(f(\gamma_1, \dots, \gamma_p) \otimes \gamma_1 \cdots \gamma_p . g(\gamma_{p+1}, \dots, \gamma_{p+q})).$$

One has the inclusion $B_l(G,M^\bullet)\cup_\phi B_s(G,N^\bullet)\subset B_{l+s}(G,P^\bullet)$ and this induces products for the corresponding spectral sequences

$$E_r^{i,j}(B_\bullet(G,M^\bullet))\otimes E_r^{k,l}(B_\bullet(G,N^\bullet))\to E_r^{i+k,j+l}(B_\bullet(G,P^\bullet)).$$

Lemma 4.6. — *The inclusion $A_l(G)\subset B_l(G,M^\bullet)$ gives a map of spectral sequences such that the induced morphisms*

$$E_r^{i,j}(G/K)\to E_r^{i,j}(B_\bullet(G,M^\bullet))$$

are isomorphisms for $r\geqslant 1$ and i,j in $\mathbf{Z}$.

Proof. — As in [**HS**, page 119], it is sufficient to prove that, for any l in $\mathbf{Z}_{\geqslant 0}$, the cohomology of the complex $(B_l(G,M^\bullet)/A_l(G),d)$ is trivial. Therefore, if $\boldsymbol{f}$ in $B_l^n(G,M^\bullet)$ is such that $d\boldsymbol{f}$ belongs to $A_l^{n+1}(G)$, we want to prove the existence of $\boldsymbol{g}$ in $B_l^{n-1}(G,M^\bullet)$ such that $\boldsymbol{f}-d\boldsymbol{g}$ belongs to $A_l^n(G)$. We shall prove by an increasing induction on m that for any such element

$$\boldsymbol{f}=(f_i)_{i\geqslant 0}\in\bigoplus_{i\geqslant 0}B_l^{(i,n-i)}(G,M^\bullet)$$

such that moreover

$$\forall i\in\mathbf{Z}_{\geqslant 0}, i\geqslant m\Rightarrow f_i\in A_l^{(i,n-i)}(G)$$

there exists $\boldsymbol{g}$ in $B_l^{n-1}(G,M^\bullet)$ such that $\boldsymbol{f}-d\boldsymbol{g}$ belongs to $A_l^n(G)$. It is verified for $m=0$. Let us assume it is true for m and let $\boldsymbol{f}$ verify the condition for $m+1$. Then

$$d'(f_m)+d''(f_{m+1})\in A_l^{(m+1,n-m)}(G).$$

Thus $d'(f_m)$ belongs to $A_l^{(m+1,n-m)}(G)$. By [**HS**, pages 119-120], there is g_{m-1} in the group $B_l^{(m-1,n-m)}(G,M^\bullet)$ such that $f_m-d'(g_{m-1})$ belongs to $A_l^{(m,n-m)}(G)$. Let $\boldsymbol{f}'$ be $\boldsymbol{f}-d(g_{m-1})$. Then one may apply the induction hypothesis to $\boldsymbol{f}'$. We get an element $\boldsymbol{g}'$ in $B_l^{n-1}(G,M^\bullet)$ such that $\boldsymbol{f}'-d(\boldsymbol{g}')$ belongs to $A_l^n(G)$. Then $\boldsymbol{f}-d(\boldsymbol{g}'+g_{m-1})$ belongs to $A_l^n(G)$. □

The next proposition follows directly from [**HS**, page 126].

Proposition 4.7. — *Let ρ be the induced isomorphisms from the group $E_1^{p,q}(B_\bullet(G,M^\bullet))$ (respectively $E_1^{p,q}(B_\bullet(G,N^\bullet))$, $E_1^{p,q}(B_\bullet(G,P^\bullet))$) to the group $C^p(G/K,\mathbf{H}^q(K,M^\bullet))$ (respectively $C^p(G/K,\mathbf{H}^q(K,N^\bullet))$, $C^p(G/K,\mathbf{H}^q(K,P^\bullet))$). Then*

$$\forall u\in E_1^{p,q}(B_\bullet(G,M^\bullet)),\ \forall v\in E_1^{r,s}(B_\bullet(G,N^\bullet)),\ \rho(u\cup_\phi v)=(-1)^{sp}\rho(u)\cup_\phi\rho(v).$$

5. Proof of the main statement

We shall decompose the proof of theorem 2.1 in a chain of lemmata. We first recall the following well known result:

Lemma 5.1. — *Let V be a nonsingular, proper and geometrically integral variety over k. Let $\mathscr{G}$ be the absolute Galois group of k. Then there exist a canonical exact sequence*

$$0 \to \operatorname{Pic} V \to \operatorname{Pic}(V_{k^s})^{\mathscr{G}} \to \operatorname{Br} k \to \operatorname{Br} k(V).$$

Proof. — Since V is nonsingular, proper and geometrically integral, we have an exact sequence

$$0 \to k^* \to k(V)^* \to \bigoplus_{P \in V^{(1)}} \mathbf{Z} \to \operatorname{Pic} V \to 0$$

as well as the corresponding one over k^s. Hilbert's theorem 90 then gives that

$$(k^s(V)^*/k^{s*})^{\mathscr{G}} = k(V)^*/k^*.$$

Therefore we get an exact sequence

$$0 \to k(V)^*/k^* \to \bigoplus_{P \in V^{s(1)}} \mathbf{Z} \to (\operatorname{Pic} V^s)^{\mathscr{G}} \to H^1(\mathscr{G}, k^s(V)^*/k^{s*}) \to 0$$

and thus

$$0 \to \operatorname{Pic} V \to (\operatorname{Pic} V^s)^{\mathscr{G}} \to H^1(\mathscr{G}, k^s(V)^*/k^{s*}) \to 0$$

is exact. But we have also an exact sequence

$$0 \to H^1(\mathscr{G}, k^s(V)^*/k^{s*}) \to H^2(\mathscr{G}, k^{s*}) \to H^2(\mathscr{G}, k^s(V)^*).$$

And Hilbert's theorem 90 implies that the map from $H^2(\mathscr{G}, k^s(V)^*)$ to $\operatorname{Br}(k(V))$ is injective. □

Lemma 5.2. — *If V is a generalized flag variety under a semi-simple linear algebraic group G, there exists a natural exact sequence*

$$(\operatorname{Pic} V^s \otimes k^{s*})^{\mathscr{G}} \to \operatorname{Ker}\left(H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(V), \mathbf{Q}/\mathbf{Z}(2))\right) \to \operatorname{CH}^2(V)_{\mathrm{tors}} \to 0.$$

Proof. — By [**CTR**, proposition 3.6], one has an exact sequence

$$H^1(V^s, \mathscr{K}_2)^{\mathscr{G}} \to H^1\left(\mathscr{G}, K_2(k^s(V))/H^0(V^s, \mathscr{K}_2)\right) \to$$
$$\to \operatorname{Ker}(\operatorname{CH}^2(V) \to \operatorname{CH}^2(V^s)) \to H^1(\mathscr{G}, H^1(V^s, \mathscr{K}_2)).$$

By corollary 3.3, $H^1(\mathscr{G}, H^1(V^s, \mathscr{K}_2))$ is trivial and proposition 3.1 gives isomorphisms

$$H^0(V^s, \mathscr{K}_2) \tilde{\to} K_2 k^s \text{ and } H^1(V^s, \mathscr{K}_2) \tilde{\to} \operatorname{Pic} V^s \otimes k^{s*}.$$

Moreover, since $\mathrm{CH}^2(V^s)$ is torsion-free, one has

$$\operatorname{Ker}(\mathrm{CH}^2(V) \to \mathrm{CH}^2(V^s)) = \mathrm{CH}^2(V)_{\mathrm{tors}}.$$

But by [**Kah**, corollaire 3.2] which is one of the key ingredient of the proof

$$H^1(\mathscr{G}, K_2(k^s(V))/K_2 k^s) \tilde{\to} \operatorname{Ker}\left(H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(V), \mathbf{Q}/\mathbf{Z}(2))\right)$$

which implies the exact sequence of the lemma. □

It remains to prove that the morphism from k_i^* to $H^3(k, \mathbf{Q}/\mathbf{Z}(2))$ is indeed the composition of a cup-product by the corestriction map. Let us first consider the case when $k_i = k$.

Lemma 5.3. *— Assume that $\mathscr{H}_i$ is equal to $\mathscr{G}$. Let α be the image of the natural generator of $\mathbf{Z}[\mathscr{G}/\mathscr{H}_i] \subset (\operatorname{Pic} V^s)^{\mathscr{G}}$ in the Brauer group of k. Then there is a commutative diagram*

$$\begin{array}{ccc} k^* & \longrightarrow & H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \\ & \searrow \quad \nearrow_{\cup} & \\ & k^* \otimes \alpha\mathbf{Z} & \end{array}$$

where the morphism at the top is the one defined in previous lemma.

Proof. — The proof of this lemma is exactly the same as in [**Pe2**, lemma 4.3] and uses compatibility with cup-products to get for any a in k^* a commutative diagram

$$\begin{array}{ccccccc} \mathbf{Z} & \longrightarrow & \operatorname{Pic}(V^s)^{\mathscr{G}} & \longrightarrow & H^1(\mathscr{G}, K_1(k^s(V))/H^0(V^s, \mathscr{K}_1)) & \to & H^2(k, \mathbf{Q}/\mathbf{Z}(1)) \\ \downarrow \cup a & & \downarrow \cup a & & \downarrow \cup a & & \downarrow \cup a \\ k^* & \to & (\operatorname{Pic} V^s \otimes k^{s*})^{\mathscr{G}} & \to & H^1(\mathscr{G}, K_2(k^s(V))/H^0(V^s, \mathscr{K}_2)) & \to & H^3(k, \mathbf{Q}/\mathbf{Z}(2)). \end{array}$$

But the morphism $\mathbf{Z} \to \operatorname{Br} k$ of lemma 5.1 coincides with the composition of the morphisms of the top row. □

It remains to prove the following lemma.

Lemma 5.4. — *For all i between 1 and m, one has a commutative diagram*

$$\begin{array}{ccc} (\mathbf{Z}[\mathscr{G}/\mathscr{H}_i]\otimes k^{s*})^{\mathscr{H}_i} & \rightarrow & H^3(k_i,\mathbf{Q}/\mathbf{Z}(2)) \\ {\scriptstyle N_{k_i/k}}\big\downarrow & & {\scriptstyle N_{k_i/k}}\big\downarrow \\ (\mathbf{Z}[\mathscr{G}/\mathscr{H}_i]\otimes k^{s*})^{\mathscr{G}} & \rightarrow & H^3(k,\mathbf{Q}/\mathbf{Z}(2)). \end{array}$$

Proof. — First let us recall a more precise description of the map

$$f:(\mathrm{Pic}(V^s)\otimes k^{s*})^{\mathscr{G}}\rightarrow H^3(k,\mathbf{Q}/\mathbf{Z}(2)).$$

We consider the Lichtenbaum complex $\Gamma(2,k^s)^i$ for k^s and the one for $k(V)^s$ (see [**Li1**], [**Li2**] and [**Li3**]). There is a canonical morphism

$$\Gamma(2,k^s)^i\rightarrow\Gamma(2,k(V)^s)^i$$

the cokernel of which will be denoted by $\Gamma(2,k(V)^s/k^s)^{i+1}$. Let $\mathbf{H}^i(k^s(V)/k^s,\Gamma(2))$ (respectively $\mathbf{H}^i(k(V)/k,\Gamma(2))$) be the hypercohomology groups corresponding to this complex of $\mathrm{Gal}(k(V)^s/k^s(V))$-modules (respectively $\mathrm{Gal}(k(V)^s/k(V))$-modules).

Then f is defined as the composition of natural morphisms

$$(\mathrm{Pic}(V^s)\otimes k^{s*})^{\mathscr{G}}\xrightarrow{f_1}H^1(V^s,\mathscr{K}_2)^{\mathscr{G}}\xrightarrow{f_2}H^1(\mathscr{G},K_2(k^s(V))/K_2(k^s))\xrightarrow{f_3}$$
$$\xrightarrow{f_3}H^1(\mathscr{G},\mathbf{H}^3(k^s(V)/k^s,\Gamma(2)))\xrightarrow{f_4}\mathbf{H}^4(k(V)/k,\Gamma(2))\xrightarrow{f_5}\mathbf{H}^4(k,\Gamma(2))\xrightarrow{f_6}H^3(k,\mathbf{Q}/\mathbf{Z}(2)).$$

The map f_2 is induced by the short exact sequence

$$0\rightarrow K_2k^s(V)/K_2k^s\rightarrow\mathscr{Z}\rightarrow H^1(V^s,\mathscr{K}_2)\rightarrow 0$$

where $\mathscr{Z}$ is the kernel of the map

$$\bigoplus_{x\in V^{s(1)}}k^s(x)^*\rightarrow\bigoplus_{x\in V^{s(2)}}\mathbf{Z}.$$

The morphism f_3 is induced by the natural morphism

$$\mathbf{H}^2(k^s(V),\Gamma(2))\rightarrow\mathbf{H}^3(k^s(V)/k^s,\Gamma(2))$$

and the isomorphism from $\mathbf{H}^2(k^s(V),\Gamma(2))$ to $K_2k^s(V)$, f_4 by the spectral sequence

$$H^p(\mathscr{G},\mathbf{H}^q(k^s(V)/k^s,\Gamma(2)))\Rightarrow\mathbf{H}^{p+q}(k(V)/k,\Gamma(2))$$

and f_6 by the canonical isomorphism

$$f_7:H^3(k,\mathbf{Q}/\mathbf{Z}(2))\xrightarrow{\sim}\mathbf{H}^4(k,\Gamma(2)).$$

The maps f_1, f_2 and f_3 are clearly compatible with the corestriction map. The map f_7 is induced by a morphism of complexes of Galois modules whereas the map f_5 is a cobordism for a short sequence of complexes. Therefore the maps f_5 and f_6 are compatible with the corestriction. Finally the compatibility of f_4 with the corestriction is a consequence of lemma 4.4. □

6. Connection with Panin's result

Let us first recall this result which yields the K-theory groups of the variety V.

***Theorem 6.1* (Panin, [Pa]).** — *If V is a generalized flag variety over a field k then there exists a natural separable algebra A over k and an isomorphism*

$$K_*V \xrightarrow{\sim} K_*A.$$

Proposition 6.2. — *Assume that the field k is perfect or that the center of G is a reduced k-group. With notation as in the preceding theorem and theorem 2.1, there exists a natural decomposition of A into the product of two separable algebras C and D such that one has $C \xrightarrow{\sim} \times_{i=1}^{m} C_i$ where C_i is a simple central algebra of centre k_i, the class of which is equal to α_i in $\mathrm{Br}\, k_i$.*

Remark 6.3. — The decomposition is given explicitly in the proof.

Proof. — We use the notation of section 3. We shall first recall Panin's construction of the algebra A. There exists a simply connected quasi-split k-form $\tilde{G}^q$ of G, a parabolic subgroup $\tilde{P}^q$ of $\tilde{G}^q$ and an element γ of $H^1(k, G^q)$ where $G^q = \tilde{G}^q/Z(\tilde{G}^q)$ such that V is the twisted k-form of $\tilde{P}^q\backslash\tilde{G}^q$ defined by γ and G the corresponding k-form of $\tilde{G}^q$. Let $\tilde{B}^q$ be a Borel subgroup of $\tilde{G}^q$ defined over k and contained in $\tilde{P}^q$ and $\tilde{T}^q$ be a maximal torus of $\tilde{B}^q$. We may assume that the isomorphism $G^s \xrightarrow{\sim} (\tilde{G}^q)^s$ sends $(\tilde{P}^q)^s$ (resp. $(\tilde{B}^q)^s$, $(\tilde{T}^q)^s$) on P (resp. B, T). We denote by $U'^{\,q}_w$ the images of U'_w in $(\tilde{G}^q)^s$. Let B^q be the image of $\tilde{B}^q$ in G^q and for any J in Δ, let P^q_J be the parabolic subgroup of G^q corresponding to J.

By [**St**, theorem 1.3], the ring of representations $R(P)$ of P over $\bar{k}$ has a canonical basis as a module over $R(G_{\bar{k}})$ which is defined as follows: let $W^{I'}$ be the set

$$\{\, w \in W \mid \forall \alpha \in I,\ w\alpha \in \Phi^+ \,\}$$

then the basis $(e'_w)_{w\in W^{I'}}$ is given by:

$$e'_w = \sum_{\lambda\in W_I.\lambda_w} \lambda \in \mathbf{Z}[X^*(T)]^{W_I} \xleftarrow{\sim} R(P)$$

where

$$\lambda_w = \sum_{\{\alpha\in\Delta | w^{-1}\alpha<0\}} w^{-1}\varpi_\alpha \in X^*(T).$$

The set $W^{I'}$ is globally invariant under the natural action of $\mathcal{G}$ on W. Let E be the commutative separable algebra corresponding to the $\mathcal{G}$-set $W^{I'}$. A representation of $\tilde{P}^q$ on E, when lifted to k^s, gives a family of representations indexed by $W^{I'}$. Thus it is characterized by a $\mathcal{G}$-invariant family of weights indexed by this set. The basis $(e'_w)_{w\in W^{I'}}$ defines a representation of $\tilde{P}^q$ over E. Let V be the induced representation from $\tilde{P}^q$ to $\tilde{G}^q$, A^q the ring $\mathrm{End}_E(V)$ and A^q_γ the twisted form of A^q by γ. The algebra $A = A^q_\gamma$ is the algebra constructed by Panin in [**Pa**, §12].

Lemma 6.4. — *The set $W^{I'}$ coincides with $(W^I)^{-1}$.*

Proof. — Let w belong to W^I. Assume that w^{-1} does not belong to $W^{I'}$. Let $\alpha\in I$ be such that $w^{-1}\alpha<0$. Then by [**Bki**, chapitre VI, n° 1.6, pages 157–158, corollaires 1 et 2]

$$l(w^{-1}s_\alpha) = \#\{\beta\in\Phi^+ \mid w^{-1}s_\alpha\beta<0\} = l(w^{-1})-1.$$

Then w is not of minimal length in its class $\bar{w}\in W_I\backslash W$ which is in contradiction with the hypothesis. thus $(W^I)^{-1}$ is a subset of $W^{I'}$. But $W^{I'}$ is a set of representing elements for W/W^I (see [**St**, lemma 2.5(a)]) and both sets have the same cardinal. □

In the sequel we put $e_w = e'_{w^{-1}}$ for all w in W^I.

Lemma 6.5. — *For any α in $\Delta-I$ one has that $w_{\Delta-\{\alpha\}}w_\Delta$ belongs to W^I and*

$$e_{w_{\Delta-\{\alpha\}}w_\Delta} = -\varpi_\alpha.$$

Proof. — By proposition 3.1, $w_{\Delta-\{\alpha\}}w_\Delta$ is of minimal length in its class modulo $W_{\Delta-\{\alpha\}}$, it is a fortiori of minimal length modulo W_I. Moreover

$$\{\beta\in\Delta \mid w_{\Delta-\{\alpha\}}w_\Delta\beta<0\} = \{\beta\in\Delta \mid w_{\Delta-\{\alpha\}}{}^\varepsilon\beta>0\} = \{{}^\varepsilon\alpha\}$$

where $\varepsilon:\Delta\to\Delta$ is the involutive bijection such that $w_\Delta.\alpha = -{}^\varepsilon\alpha$. Thus we get

$$\lambda_{w_{\Delta-\{\alpha\}}w_\Delta} = w_{\Delta-\{\alpha\}}w_\Delta\varpi_{^\varepsilon\alpha} = w_{\Delta-\{\alpha\}}.(-\varpi_\alpha) = -\varpi_\alpha.$$

But $-\varpi_\alpha$ is invariant under W_I and the lemma is proved. □

End of the proof of proposition 6.2. — Let C be the étale algebra constructed from the $\mathscr{G}$-set $(e_w)_{w\in\{w_{\Delta-\{\alpha\}}w_\Delta,\alpha\in\Delta-I\}}$. Then one may write A as $C\times D$ where D corresponds to the complementary set. The above $\mathscr{G}$-set is canonically isomorphic to the $\mathscr{G}$-set $(-\varpi_\alpha)_{\alpha\in\Delta-I}$. Thus we get a canonical isomorphism of étale algebras

$$Z(C)\tilde{\to}(\operatorname{Pic}V^s\otimes k^s)^{\mathscr{G}}.$$

Let F be this algebra. Let χ be the natural character $Z(\tilde{G}^q)\to\mathbf{G}_{mF}$ defined by the family $(-\varpi_\alpha)_{\alpha\in\Delta-I}$. Let η_1 be the image of γ by the composite map

$$H^1(k,G^q)\to H^1(F,G^q)\xrightarrow{\partial}H^2(F,Z(\tilde{G}^q))\xrightarrow{\chi^*}H^2(F,\mathbf{G}_m). \tag{4}$$

Then, thanks to the hypothesis on k, the components of η_1 in the decomposition

$$H^2(F,\mathbf{G}_m)\tilde{\to}\prod_{i=1}^m \operatorname{Br}k_i$$

are the classes of the algebras C_i (see [**Pa**, lemma 3.3] and [**Tit**, 4.2]). For any $\mathscr{G}$-module M, $\tilde{M}$ denotes the corresponding étale sheaf on $\operatorname{Spec}k$. By definition of the fields k_i there is an isomorphism from $H^0(F,\widetilde{\operatorname{Pic}V^s})$ onto $\bigoplus_{1\leqslant i,j\leqslant m}H^0(k_i,\mathbf{Z}[\mathscr{G}/\mathscr{H}_j])$. Let η_2 be the image of the sum of the elements $\mathscr{H}_i$ of $\mathbf{Z}[\mathscr{G}/\mathscr{H}_i]^{\mathscr{H}_i}$ by the composite morphism

$$\bigoplus_{1\leqslant i\leqslant m}H^0(k_i,\mathbf{Z}[\mathscr{G}/\mathscr{H}_i])\to H^0(F,\widetilde{\operatorname{Pic}V^s})\xrightarrow{\partial_1}H^1(F,\widetilde{k^s(V)^*/k^{s*}})\xrightarrow{\partial_2}H^2(F,\mathbf{G}_m)$$

where ∂_1 is the coboundary homomorphism for the short exact sequence

$$0\to k^s(V)^*/k^{s*}\to\operatorname{Div}V^s\to\operatorname{Pic}V^s\to 0$$

and ∂_2 the coboundary homomorphism for the short exact sequence

$$0\to k^{s*}\to k^s(V)^*\to k^s(V)^*/k^{s*}\to 0.$$

Then, by definition, the α_i are the components of η_2. Thus it remains to prove the following lemma:

Lemma 6.6. — *With notation as above, the classes η_1 and η_2 coincide.*

Proof. — Let us fix i in $\{1,\dots,m\}$. It is enough to prove that the image ξ_2 of the generator of $\mathbf{Z}[\mathscr{G}/\mathscr{H}_i]^{\mathscr{H}_i}$ by the composite map

$$\mathbf{Z}[\mathscr{G}/\mathscr{H}_i]^{\mathscr{H}_i}\to H^0(k_i,\operatorname{Pic}V^s)\to H^1(k_i,k^s(V)^*/k^{s*})\to H^2(k_i,k^{s*})$$

coincides with the component of η_1 in $\mathrm{Br}\, k_i$ which we denote by ξ_1. Let $\mathscr{C}$ be the orbit of $\mathscr{G}$ in $\Delta - I$ corresponding to k_i and $\chi_i : Z(\tilde{G}^q) \to \mathbf{G}_{mk_i}$ be the corresponding character. We denote also by

$$\gamma : \mathscr{G} \to G^q(k^s)$$

a cocycle which represents γ so that for any σ in $\mathscr{G}$ the composite map

$$\tilde{G}^q \times k^s \xrightarrow{\mathrm{Id}\times\sigma^{-1}} \tilde{G}^q \times k^s \tilde{\to} G \times k^s \xrightarrow{\mathrm{Id}\times\sigma} G \times k^s \tilde{\to} \tilde{G}^q \times k^s$$

coincides with the interior automorphism $\mathrm{Int}_{\gamma(\sigma)}$. Let $x \mapsto \tilde{x}$ be a set-theoritic section of the canonical map

$$\tilde{G}^q(k^s) \to G^q(k^s)$$

which is surjective, by the hypothesis on k. Then ξ_1 is the image by χ of the cocycle

$$(5) \qquad \begin{array}{rcl} \tilde{\xi}_1 : \mathscr{G}^2 & \to & Z\left(\tilde{G}^q\right)(k^s) \\ (\sigma_1, \sigma_2) & \mapsto & {}^{\sigma_1}\left(\widetilde{\gamma(\sigma_2)}\right)\widetilde{\gamma(\sigma_1\sigma_2)}^{-1}\widetilde{\gamma(\sigma_1)}. \end{array}$$

Let w'_Δ be a lifting of w_Δ in $\mathscr{N}_{\tilde{G}^q}(\tilde{T}^q)$. Let $s_1 : \mathrm{Pic}\, V^s \to \mathrm{Div}\, V^s$ be the section which sends the class of $\overline{\pi(BwB)}$ to $\overline{\pi(BwB)}$ for $w \in W^I_{\dim V-1}$. For all $\alpha \in \mathscr{C}$ let f_α be the function on $\tilde{G}^q$ defined by

$$\forall b \in \tilde{B}^q(k^s),\ \forall u \in U'^q_{w_\Delta}(k^s), f_\alpha(bw'_\Delta u) = \varpi_\alpha(b)^{-1},$$

where ϖ_α is the extension to $\tilde{B}^q$ of the corresponding character on $\tilde{T}^q$. By [**Pe4**, page 164] this function induces the section of the sheaf defined by ϖ_α which corresponds to the divisor $\overline{\pi(B^q\bar{s}_\alpha B^q)}$. The stabilizer of $\overline{\pi(B^q\bar{s}_\alpha B^q)} \subset \tilde{P}^q\backslash\tilde{G}^q$ contains B^q. It is thus a standard parabolic subgroup of G^q. But for any $\beta \in \Delta$ one has:

$$\begin{aligned} &\overline{\pi(B^q\bar{s}_\alpha B^q)}s_\beta \subset \overline{\pi(B^q\bar{s}_\alpha B^q)} \\ \Leftrightarrow\ & B^q s_\alpha w_\Delta B^q s_\beta \subset \overline{B^q s_\alpha w_\Delta B^q} \\ \Leftrightarrow\ & B^q w_\Delta s_{\varepsilon_\alpha} B^q s_\beta \subset \overline{B^q w_\Delta s_{\varepsilon_\alpha} B^q} \\ \Leftrightarrow\ & l(w_\Delta s_{\varepsilon_\alpha} s_\beta) = l(w_\Delta s_{\varepsilon_\alpha}) - 1 \\ \Leftrightarrow\ & \beta \in \Delta - {}^\varepsilon\alpha. \end{aligned}$$

where the third equivalence follows from the fact that, by [**Bki**, chapitre IV, §2, (3′) et théorème 2],

$$\dim(B^q w_\Delta s_{\varepsilon_\alpha} B^q s_\beta B^q) = \dim B^q + \sup(l(w_\Delta s_{\varepsilon_\alpha} s_\beta), l(w_\Delta s_{\varepsilon_\alpha})).$$

We then choose a section

$$s : P^q_{\Delta - \varepsilon_\alpha} \backslash G^q(k^s) \to G^q(k^s).$$

Then we may choose a section s_2

$$s_2 : k^s(V)^*/k^{s*} \to k^s(V)^*$$

such that, identifying $k^s(V)^*/k^{s*}$ with its image in $\operatorname{Div} V^s$, one has for any $g \in \tilde{G}^q(k^s)$

$$s_2(-\overline{\pi(B^q \bar{s}_\alpha B^q)} + \overline{\pi(B^q \bar{s}_\alpha B^q)}.g^{-1})(h) = f_\alpha(h)^{-1} f_\alpha\left(\widetilde{hs(\hat{g})}\right)$$

where $\hat{g}$ is the class of g in $P^q_{\Delta - \varepsilon_\alpha} \backslash G^q(k^s)$. We are now able to compute

$$\xi_2 = \partial_2 \circ \partial_1 \left(\left[\overline{\pi(B^q \bar{s}_\alpha B^q)} \right] \right).$$

First, using s_1, the class $\partial_1 \left(\left[\overline{\pi(B^q \bar{s}_\alpha B^q)} \right] \right)$ is represented by the cocycle

$$\sigma \mapsto -\overline{\pi(B^q \bar{s}_\alpha B^q)} + \overline{\pi(B^q \bar{s}_\alpha B^q)}.\gamma(\sigma)^{-1} \in \operatorname{Ker}(\operatorname{Div}(V^s) \to \operatorname{Pic}(V^s)).$$

Let $\tilde{\gamma}$ (resp. $\tilde{s}$) be the composite of γ (resp. s) with the section $x \mapsto \tilde{x}$. Then ξ_2 is represented by the cocycle

$$\begin{aligned}(\sigma_1, \sigma_2) \mapsto [g \mapsto & f_\alpha(g\tilde{\gamma}(\sigma_1))^{-1} f_\alpha\left(g\tilde{\gamma}(\sigma_1)\,{}^{\sigma_1}\tilde{s}\left(\widehat{\gamma(\sigma_2)}\right)\right) \\ & .f_\alpha(g) f_\alpha\left(g\tilde{s}\left(\widehat{\gamma(\sigma_1\sigma_2)}\right)\right)^{-1} \\ & .f_\alpha(g)^{-1} f_\alpha\left(g\tilde{s}\left(\widehat{\gamma(\sigma_1)}\right)\right)\Big].\end{aligned}$$

But for any g in $\tilde{G}^q(k^s)$, $g^{-1}\tilde{s}(\hat{g})$ belongs to $\tilde{P}^q_{\Delta - \varepsilon_\alpha}(k^s)$ and for any $p \in \tilde{P}^q_{\Delta - \varepsilon_\alpha}(k^s)$, the expression $\varpi_\alpha(w'_\Delta p w'_\Delta)$ is well defined and

$$\forall g \in \tilde{G}^q(k^s), f_\alpha(gp) = \varpi_\alpha(w'_\Delta p w'_\Delta)^{-1} f_\alpha(g).$$

We get that ξ_2 is represented by the cocycle

$$(\sigma_1,\sigma_2)\mapsto \Big[g\mapsto f_\alpha(g\tilde\gamma(\sigma_1))^{-1}f_\alpha(g\tilde\gamma(\sigma_1)^{\sigma_1}\tilde\gamma(\sigma_2))$$
$$.f_\alpha(g\tilde\gamma(\sigma_1\sigma_2))^{-1}f_\alpha(g\tilde\gamma(\sigma_1))$$
$$.\left({}^{\sigma_1}\varpi_\alpha\left(w'_\Delta\tilde\gamma(\sigma_2)^{-1}\tilde s\left(\widehat{\gamma(\sigma_2)}\right)w'_\Delta\right)^{-1}\right.$$
$$.\varpi_\alpha\left(w'_\Delta\tilde\gamma(\sigma_1\sigma_2)^{-1}\tilde s\left(\widehat{\gamma(\sigma_1\sigma_2)}\right)w'_\Delta\right)$$
$$\left.\varpi_\alpha\left(w'_\Delta\tilde\gamma(\sigma_1)^{-1}\tilde s\left(\widehat{\gamma(\sigma_1)}\right)w'_\Delta\right)^{-1}\right)\Big].$$

Then removing a coboundary and using (5) one gets that ξ_2 is represented by

$$(\sigma_1,\sigma_2)\mapsto -\varpi_\alpha(\tilde\xi_1(\sigma_1,\sigma_2)).\quad\square$$

7. A few examples

7.1. The low-dimensional cases

Corollary 7.1. — *With the notation of theorem 2.1, the complex $\mathscr{C}$ is exact if the dimension of V is* 1 *or* 2.

Proof. — In this case $K_0(V^s)^3=\{0\}$. Thus, by proposition 3.4 (v), $\mathrm{CH}^2(V)_{\mathrm{tors}}$ is trivial. The corollary is then a direct consequence of theorem 2.1. □

Corollary 7.2. — *If* $\dim V=3$ *then the homology of the complex $\mathscr{C}$ is either cyclic or trivial.*

Proof. — In this case $K_0(V^s)^3=\mathbf{Z}$. Thus proposition 3.4 (v) and theorem 2.1 imply the result. □

Remark 7.3. — The paragraph 6.2 of [**Pe2**] gives an example for which $\dim V=3$ and $(\mathscr{C})$ is not exact.

7.2. The case of orthogonal groups. — We shall first give an explicit description of the complex $(\mathscr{C})$ in the case of a flag variety under an orthogonal group.

Let k be a field of characteristic different from 2. Let q be a nondegenerate quadratic form of dimension n over k. We assume that n is strictly bigger than 2. Let G be the group $\mathrm{PSO}(q)$ and V be a flag variety under G. We denote by $C_0(q)$ the even Clifford algebra of q.

By [**MPW**, proposition 1.3], these varieties are characterized by the form q and their type over k^s. We recall their description (see [**MPW**, §5]). If n is odd, $n = 2m + 1$ then G is of type B_m. The root system is given in $\mathbf{R}^m$ by the basis $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ if $1 \leqslant i \leqslant m-1$ and $\alpha_m = \varepsilon_m$ where $(\varepsilon_i)_{1\leqslant i\leqslant m}$ is the standard basis of $\mathbf{R}^m$. The flag variety $X(q, n_1, \dots, n_l)$ corresponding to $\Delta - \{\alpha_{n_1}, \dots, \alpha_{n_l}\}$ with $1 \leqslant n_1 < \dots < n_l \leqslant m$ is such that

$$X(q, n_1, \dots, n_l)(k^s) = \{ (W_1, \dots, W_l) \mid W_1 \subset \dots \subset W_l \subset V \otimes k^s,\\ W_l \text{ is totally isotropic and } \dim_{k^s} W_i = n_i \}$$

as a set with Galois action.

If n is even, $n = 2m$, then G is of type D_m, the root system is given in $\mathbf{R}^m$ by the basis $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ if $1 \leqslant i \leqslant m-1$ and $\alpha_m = \varepsilon_{m-1} + \varepsilon_m$.

If the signed discriminant of q, $d_\pm q$, is a square then we are in the inner case. The algebra $C_0(q)$ may be written as $C_+(q)^2$. The variety of maximal totally isotropic spaces has two components. Over k^s, the form q may be written as $\sum_{i=1}^m x_i x_{2m-i}$ over a basis $(e_i)_{1\leqslant i\leqslant 2m}$. We choose the maximal torus to be the diagonal matrices $D(\alpha_1, \dots, \alpha_m, \alpha_m^{-1}, \dots, \alpha_1^{-1})$ and ε_i sends this diagonal element on α_i. Let us denote by $\mathscr{M}^+$ the component containing $< e_1, \dots, e_m >$ and by $\mathscr{M}^-$ the other one. If $1 \leqslant n_1 < \dots < n_l \leqslant m-1$ then $X(q, n_1, \dots, n_l)$ is the flag variety such that

$$X(q, n_1, \dots, n_l)(k^s) = \{ (W_1, \dots, W_l) \mid W_1 \subset \dots \subset W_l \subset V \otimes k^s,\\ W_l \text{ is totally isotropic and } \dim_{k^s} W_i = n_i \}.$$

If $n_l \leqslant m-2$ then the corresponding set of roots is $\Delta - \{\alpha_{n_1}, \dots, \alpha_{n_l}\}$; if $n_l = m-1$ then the corresponding set is $\Delta - \{\alpha_{n_1}, \dots, \alpha_{n_{l-1}}, \alpha_{m-1}, \alpha_m\}$ If $1 \leqslant n_1 < \dots < n_{l-1} \leqslant m-2$ and $n_l = m$, then

$$X^+(q, n_1, \dots, n_l)(k^s) = \{ (W_1, \dots, W_l) \mid W_1 \subset \dots \subset W_l \subset V \otimes k^s,\\ W_l \text{ is totally isotropic, } \dim_{k^s} W_i = n_i \text{ and } W_l \in \mathscr{M}^+(k^s) \}.$$

Similarly, one defines $X^-(q, n_1, \dots, n_l)$. The first variety corresponds to the set of roots $\Delta - \{\alpha_{n_1}, \dots, \alpha_{n_{l-1}}, \alpha_m\}$ and the second one to $\Delta - \{\alpha_{n_1}, \dots, \alpha_{n_{l-1}}, \alpha_{m-1}\}$.

If $d_\pm q$ is not a square we are in the outer case. Let L be the field $k(\sqrt{d_\pm q})$. The absolute Galois group of k acts through $\mathrm{Gal}(L/k)$ on the system of roots. If $1 \leqslant n_1 < \dots < n_l \leqslant m-1$ the variety $X(q, n_1, \dots, n_l)$ is defined as in the previous case. The set of roots is $\Delta - \{\alpha_{n_1}, \dots, \alpha_{n_l}\}$ if $n_l \neq m-1$ and $\Delta - \{\alpha_{n_1}, \dots, \alpha_{n_{l-1}}, \alpha_{m-1}, \alpha_m\}$ otherwise.

Lemma 7.4. — *Let V be one of the flag varieties described above. Let I be the corresponding set of roots. The non-trivial component of the algebra C given by the proposition 6.2 is similar to*

(i) $C_0(q)$ if $n = 2m+1$ and $\alpha_m \in \Delta - I$,

(ii) $C_0(q)$ if $n = 2m$ and $\{\alpha_m, \alpha_{m-1}\} \subset \Delta - I$,

(iii) $C_+(q)$ if $n = 2m$, $d_\pm q \in k^{*2}$ and either α_m or α_{m-1} does not belong to I.

In all other cases C is a product of trivial algebras.

Remark 7.5. — This result generalizes easily to the case of a central simple algebra of even degree with an involution of the first kind and of orthogonal type.

Proof. — The center of the algebra C is the étale algebra corresponding to the Galois set $\{\varpi_\alpha\}$ for $\alpha \in \Delta - I$. It is non-trivial only when $n = 2m$, $d_\pm q \notin k^{*2}$ and $\{\alpha_m, \alpha_{m-1}\} \subset \Delta - I$ in which case its non trivial component is $k(\sqrt{d_\pm q})$. By construction, (see (4)) the classes of the components of C depend only on the restriction of the characters ϖ_α to the center $\mathscr{Z}$ of $\mathrm{Spin}(q)$ for $\alpha \in \Delta - I$. By [**Bki**, planches II et IV], these restrictions are as follows:

— If $n = 2m+1$, then $\mathscr{Z} = \mathbf{Z}/2\mathbf{Z}$ and $\varpi_{i|\mathscr{Z}}$ generates $\mathrm{Hom}(\mathscr{Z}, \mathbf{G}_m)$ if and only if $i = m$.

— If $n = 2m$ with m even and $d_\pm q \in k^{*2}$ then $\mathrm{Hom}(\mathscr{Z}, \mathbf{G}_m) \tilde{\to} \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ so that the restriction of ϖ_i with $i < l-1$ is trivial if i is even and $(1,1)$ if i is odd, the restriction of ϖ_{l-1} corresponds to $(1,0)$ and the one of ϖ_l to $(0,1)$.

— if $n = 2m$ with m odd and $d_\pm q \in k^{*2}$ then $\mathrm{Hom}(\mathscr{Z}, \mathbf{G}_m) \tilde{\to} \mathbf{Z}/4\mathbf{Z}$ so that the restriction of ϖ_i with $i < l-1$ is trivial if i is even and equal to 2 if i is odd, the restriction of ϖ_{l-1} is 1 and the one of ϖ_l 3.

— If $n = 2m$ and $d_\pm q \notin k^{*2}$, then the above description is valid over $L = k(\sqrt{d_\pm q})$ and is compatible with the action of $\mathrm{Gal}(L/k)$ over $\Delta - I$ and $\mathbf{Z}/4\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

But, by [**MPW**, Proposition 2.2], the class of a component corresponding to a morphism ϕ from $\mathscr{Z}$ to $\mathbf{G}_{m,E}$ is given by the class of any central algebra A over E such that there exists a representation from $\mathrm{Spin}(q)_E$ to $\mathrm{GL}_1(A)$ which extends ϕ.

If n is odd the natural injection $\mathrm{Spin}(q) \to \mathrm{GL}_1(C_0(q))$ restricts itself to the generator of $\mathrm{Hom}(\mathscr{Z}, \mathbf{G}_m)$ and we get (i).

If n is even, $n = 2m$, and $d_{\pm}q \in k^{*2}$ then the injection

$$\mathrm{Spin}(q) \to C_0(q) \xrightarrow{\sim} C_+(q) \times C_+(q)$$

gives, by projection, the generators of $\mathrm{Hom}(\mathscr{Z}, \mathbf{G}_m)$. This implies (ii) and (iii). $\square$

In the case of quadrics, the results of Karpenko on the torsion subgroup in the second Chow group enables us to give a slight refinement of Arason's results. This refinement seems to be known but we give it as an illustration of our results.

For any $(a_1, \dots, a_n)$ in k^{*n} the n-Pfister form $\langle\langle a_1, \dots, a_n \rangle\rangle$ is the form

$$< 1, -a_1 > \otimes \cdots \otimes < 1, -a_n > .$$

A quadratic form q of dimension n is said to be a neighbour of a r-Pfister form q' if and only if $n > 2^{r-1}$ and q is similar to a subform of q'.

Proposition 7.6. — *Let q be an anisotropic quadratic form of dimension strictly bigger than two and let Q be the corresponding projective quadric then the following cases are possible:*

- *(i)* **(Arason** [**Ar**, Satz 5.4]**)** *If q is a neighbour of a 2-Pfister form $\langle\langle a, b \rangle\rangle$ then the sequence*

$$H^1(k, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\cup(a,b)} H^3(k, \mathbf{Z}/2\mathbf{Z}) \to H^3(k(Q), \mathbf{Z}/2\mathbf{Z})$$

  is exact.
- *(ii) If the dimension of q is four and q is not similar to a 2-Pfister form, let d be its discriminant and $c(q)$ its Witt invariant. Then the sequence*

$$H^1(k(\sqrt{d}), \mathbf{Z}/2\mathbf{Z}) \xrightarrow{N(.)\cup c(q)} H^3(k, \mathbf{Z}/2\mathbf{Z}) \to H^3(k(Q), \mathbf{Z}/2\mathbf{Z})$$

  is exact.
- *(iii)* **(Arason** [**Ar**, Satz 5.6]**)** *If the form is a neighbour of 3-pfister form $\langle\langle a, b, c \rangle\rangle$ then $\mathrm{Ker}(H^3(k, \mathbf{Z}/2\mathbf{Z}) \to H^3(k(Q), \mathbf{Z}/2\mathbf{Z}))$ is the subgroup generated by the symbol (a, b, c).*
- *(iv)* **(Arason** [**Ar**, Satz 5.6]**)** *In all other cases this kernel is trivial.*

Remark 7.7. — For assertion (iii), theorem 2.1 and the result of Karpenko implies only that the kernel is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. Using the bijectivity of Arason invariant proved independantly by Rost and by Merkur$'$ev and Suslin [**MS2**], the assertion (ii) implies a result of Fitzgerald on 3-fold Pfister forms [**Fi**, example, page 94].

Proof. — We only prove assertions (i), (ii) and (iv). By [**Kar**, theorem 6.1] the torsion subgroup of $\mathrm{CH}^2(Q)$ is trivial except when q is anisotropic and the neighbour of a 3-fold Pfister form in which case this group is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. Here we have

$$\Delta - I = \begin{cases} \{\alpha_1, \alpha_2\} & \text{if } n = 4, \\ \{\alpha_1\} & \text{otherwise.} \end{cases}$$

Thus, by lemma 7.4, the algebra C is nontrivial only if $\dim q = 3$ or 4 in which case the non trivial components are similar to

— $\left(\frac{a\,b}{k}\right)$ if q is a neighbour of $\langle\!\langle a, b\rangle\!\rangle$.

— $\left(\frac{a\,b}{k(\sqrt{d})}\right)$ if $\dim q = 4$ and $d = d_{\pm} q \notin k^{*2}$, where a, b are elements of k^* such that $(a, b) = c(q)_{k(\sqrt{d})}$.

We then apply theorem 2.1 to get (i), (ii), and (iv). □

We recall that q is an Albert form if the dimension of q is 6 and $d_{\pm} q \in k^{*2}$. In [**Lag**, Corollaire 6], Laghribi shows that if q is an Albert form and if L is the universal splitting field of q over k then one has an exact sequence

$$k^* \xrightarrow{.\cup c(q)} H^3(k, \mathbf{Z}/2\mathbf{Z}) \to H^3(L, \mathbf{Z}/2\mathbf{Z})$$

where $c(q)$ is the clifford invariant of q. By [**KR**] the field L coincides with the function field of the variety of Borel subgroups of $SO(q)$. We get the following result:

Proposition 7.8. — *Let q be an Albert form, let V be the variety of Borel subgroups in $SO(q)$ then*

$$\mathrm{CH}^2(V)_{\mathrm{tors}} = \{0\}.$$

8. An explicit expression in a particular case

Definitions 8.1. — For any field K a field extension L is a function field over K if and only if it is generated by a finite number of elements as a field over K. If L is a function field over K, then we denote by $\mathscr{P}(L/K)$ the set of discrete valuation rings of rank 1 such that

$$K \subset A \subset L \text{ and } \mathrm{Fr}(A) = L.$$

If A belongs to $\mathscr{P}(L/K)$ then κ_A denotes the residue field of A and if the characteristic of K does not divide n

$$\partial_A : H^i(L, \mu_n^{\otimes j}) \to H^{i-1}(\kappa_A, \mu_n^{\otimes j-1})$$

the residue map (see [**CTO**, §1]). The unramified cohomology groups of L over K are then defined by

$$H^i_{\mathrm{nr}/K}(L, \mu_n^{\otimes j}) = \bigcap_{A \in \mathscr{P}(L/K)} \mathrm{Ker}(\partial_A).$$

In the rest of this paragraph we assume that the field k is of characteristic different from 2, that the group G is of the form $\mathrm{SL}_1(A) \times G'$ where A is a quaternion algebra $\left(\frac{a\,b}{k}\right)$ and G' a semi-simple simply connected linear algebraic group and that V may be split into the product of a conic C by a homogeneous variety V' so that the action of G is the product of an action of $\mathrm{SL}_1(A)$ on C and an action of G' on V'. In this setting we shall now give a more explicit expression of the morphism from the homology of $(\mathscr{C})$ to $\mathrm{CH}^2(V)_{\mathrm{tors}}$.

By [**Ar**, Satz 5.4] and Merkur'ev and Suslin [**MS**, theorem 12.1], one has an exact sequence
(6)

$$\bigoplus_{P \in C^{(1)}_{k(V')}} k(V')(P)^* \xrightarrow{N_{k(V')(P)/k(V')}} k(V')^* \xrightarrow{\cup(a,b)} H^3(k(V'), \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(V), \mathbf{Q}/\mathbf{Z}(2)).$$

Let α be in the kernel of the canonical map from $H^3(k, \mathbf{Q}/\mathbf{Z}(2))$ to $H^3(k(V), \mathbf{Q}/\mathbf{Z}(2))$. Then $\alpha_{k(V')}$ may be written as (a, b, f) for some f in $k(V')$. Let D be the divisor of f on V'. Since α comes from $H^3(k, \mathbf{Q}/\mathbf{Z}(2))$, for any point P of codimension 1 in V one has

$$\partial_P(a, b, f) = 0.$$

By [**CTO**, proposition 1.3], one gets that

$$\forall P \in V'^{(1)}, \ (a, b)_{k(P)} \neq 0 \Rightarrow v_P(f) \text{ is even.}$$

Let $p_1 : V \to C$ and $p_2 : V \to V'$ be the natural projections. For any P in $V'^{(1)}$, the conic $C_{k(P)}$ has a rational point if and only if $(a, b)_{k(P)} = 0$. Thus we get that

$$D \in \mathrm{Im}(p_{2*} : \bigoplus_{P \in V^{(2)}} \mathbf{Z} \to \bigoplus_{P \in V'^{(1)}} \mathbf{Z}).$$

Let E belong to the inverse image of D and $[E]$ be its class in $\mathrm{CH}^2(V)/p_2^*(\mathrm{CH}^2(V'))$.

Lemma 8.1. — *With the above notations,* $[E]$ *depends only on* α.

Proof. — Let us first prove that for a fixed f the class $[E]$ is independant of the choice of E. Let E' in $\bigoplus_{P\in V^{(2)}}\mathbf{Z}$ be such that $p_{2*}(E')=p_{2*}(E)$. We may then write

$$E-E'=\left(\sum_{P\in V'^{(1)}}\sum_{\{Q\in V^{(2)}|p_{2*}(Q)\in\mathbf{Z}P\}} n_Q^P Q\right)+\sum_{P\in V'^{(2)}} n_P p_2^*(P)$$

where for any P in $V'^{(1)}$, $\sum_{\{Q\in V^{(2)}|p_{2*}(Q)\in\mathbf{Z}P\}} n_Q^P[k(Q):k(P)]=0$. But for any $P\in V'^{(1)}$, the Picard group of $C_{k(P)}$ is isomorphic to $\mathbf{Z}$ and thus there exists a function f_p on V_P such that

$$\mathrm{Div}(f_P)=\sum_{\{Q\in V^{(2)}|p_{2*}(Q)\in\mathbf{Z}P\}} n_Q^P Q.$$

Therefore the class of $E-E'$ in $\mathrm{CH}^2(V)/p_2^*(\mathrm{CH}^2(V'))$ is trivial.

Let us now prove that $[E]$ is independant of the choice of f. Let f' be an element of $k(V')$ such that

$$\alpha_{k(V')}=(a,b,f').$$

By the exact sequence (6), there exists a family $(f_P)_{P\in C^{(1)}_{k(V')}}$ in $\bigoplus_{P\in C^{(1)}_{k(V')}} k(V')(P)^*$ such that

$$f/f'=\prod_{P\in C^{(1)}_{k(V')}} N_{k(V')(P)/k(V')}(f_P).$$

Then $E'=E-\sum_{P\in C^{(1)}_{k(V')}}\mathrm{Div}(f_P)$ verifies $p_{2*}(E')=\mathrm{Div}(f')$, but its class in $\mathrm{CH}^2(V)$ is the same as the class of E. □

We denote by

$$\Phi:\mathrm{Ker}\left(H^3(k,\mathbf{Q}/\mathbf{Z}(2))\to H^3(k(V),\mathbf{Q}/\mathbf{Z}(2))\right)\to \mathrm{CH}^2(V)/\mathrm{CH}^2(V')$$

the induced map sending α on $[E]$.

Theorem 8.2. — *With notation as above, the natural morphism from the homology of the complex* $(\mathscr{C})$ *to* $\mathrm{CH}^2(V)/\mathrm{CH}^2(V')$ *defined by theorem 2.1 coincides with the map induced by* Φ.

Proof. — By [**Kah**, corollaire 3.2], there exists a canonical isomorphism

$$H^1(\mathscr{G}, K_2k^s(V)/K_2k^s(V')) \tilde{\to} \mathrm{Ker}(H^3(k(V'), \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(V), \mathbf{Q}/\mathbf{Z}(2))).$$

Moreover as in [**Pe2**, page 391] for any g in $k^s(V')$ there is a commutative diagram

$$\begin{array}{ccc} H^1(\mathscr{G}, K_1k^s(V)/K_1k^s(V')) & \tilde{\to} & \mathrm{Ker}(\mathrm{Br}(k(V')) \to \mathrm{Br}(k(V))) \\ \downarrow \cup g & & \downarrow \cup g \\ H^1(\mathscr{G}, K_2k^s(V)/K_2k^s(V')) & \tilde{\to} & \mathrm{Ker}(H^3(k(V'), \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(V), \mathbf{Q}/\mathbf{Z}(2))). \end{array}$$

But there is a surjection

$$(\mathrm{Pic}\, C^s)^{\mathscr{G}} \twoheadrightarrow H^1(\mathscr{G}, K_1k^s(V)/K_1k^s(V')).$$

Moreover $\mathrm{Pic}\, C^s$ is a free $\mathbf{Z}$-module of rank 1 with a trivial $\mathscr{G}$ action and the image of one of its generator in $\mathrm{Br}(k(V'))$ is (a, b).

We use again the symbols α, f and E introduced before lemma 8.1. Let β be the image of one generator of $\mathrm{Pic}\, C^s$ in $H^1(\mathscr{G}, K_1k^s(V)/K_1k^s(V'))$. Let γ be the image of α in $H^1(\mathscr{G}, K_2k^s(V)/K_2k^s)$. Then the image of γ in $H^1(\mathscr{G}, K_2k^s(V)/K_2k^s(V'))$ is $\beta \cup f$.

The conic C may be defined by the homogeneous equation

$$X^2 - aY^2 - bZ^2 = 0.$$

Let $\sqrt{a}$ be a square root of a in k^s. Then the cocycle

$$\begin{array}{rcl} \tilde{\beta} : \mathscr{G} & \to & k^s(V)^*/k^s(V')^* \\ \sigma & \mapsto & \begin{cases} [1] & \text{if } \sigma(\sqrt{a}) = \sqrt{a}, \\ \left[\frac{X - \sqrt{a}Y}{Z}\right] & \text{otherwise.} \end{cases} \end{array}$$

represents β. The element $\beta \cup f$ is thus given by the cocycle

$$\begin{array}{rcl} \widetilde{\beta \cup f} : \mathscr{G} & \to & K_2k^s(V)/K_2k^s(V') \\ \sigma & \mapsto & \begin{cases} [0] & \text{if } \sigma(\sqrt{a}) = \sqrt{a}, \\ \left[\left\{\frac{X - \sqrt{a}Y}{Z}, f\right\}\right] & \text{otherwise.} \end{cases} \end{array}$$

But the map

$$\Psi : H^1(\mathscr{G}, K_2(k^s(V))/K_2k^s) \to \mathrm{CH}^2(V)$$

is defined as follows (see [**CTR**, page 188]). One considers the kernel $\mathscr{Z}$ and the image $\mathscr{I}$ of the morphism

$$\bigoplus_{x\in V^{(1)}} k(x)^* \to \bigoplus_{x\in V^{(2)}} \mathbf{Z}$$

which appears in Gersten-Quillen spectral sequence, as well as the corresponding groups $\mathscr{Z}_{k^s}$ and $\mathscr{I}_{k^s}$ over k^s. We have a short exact sequence

$$0 \to K_2k^s(V)/K_2k^s \to \mathscr{Z}_{k^s} \to H^1(V^s, \mathscr{K}_2) \to 0.$$

Then the isomorphism

$$\Upsilon : H^1(\mathscr{G}, \mathscr{Z}_{k^s}) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{CH}^2(V) \to \mathrm{CH}^2(V^s))$$

is yielded by diagram chases in the following two commutative diagrams which have exact rows

(7)

$$\begin{array}{ccccccccc}
0\to & \mathscr{Z}_{k^s}^{\mathscr{G}} & \to & \left(\bigoplus_{x\in V_{k^s}^{(1)}} k^s(x)^*\right)^{\mathscr{G}} & \to & \mathscr{I}_{k^s}^{\mathscr{G}} & \xrightarrow{\partial} & H^1(\mathscr{G}, \mathscr{Z}_{k^s}) & \to 0 \\
 & \uparrow & & \uparrow\wr & & \uparrow & & & \\
0\to & \mathscr{Z} & \to & \bigoplus_{x\in V^{(1)}} k(x)^* & \to & \mathscr{I} & \to & 0 &
\end{array}$$

and

(8)

$$\begin{array}{ccccccc}
0\to & \mathscr{I}_{k^s}^{\mathscr{G}} & \to & \left(\bigoplus_{x\in V_{k^s}^{(2)}} \mathbf{Z}\right)^{\mathscr{G}} & \to & \mathrm{CH}^2(V_{k^s}) & \\
 & \uparrow & & \uparrow\wr & & \uparrow & \\
0\to & \mathscr{I} & \to & \bigoplus_{x\in V^{(2)}} \mathbf{Z} & \to & \mathrm{CH}^2(V) & \to 0.
\end{array}$$

We put $\mathscr{C}(1) = \bigoplus_{x\in V^{(1)}} k(x)^*$ and $\mathscr{C}(2) = \bigoplus_{x\in V^{(2)}} \mathbf{Z}$ and take a similar notation for the corresponding groups over k^s. Let $\mathscr{C}'(1)$, $\mathscr{C}'_{k^s}(1)$, $\mathscr{C}'(2)$, $\mathscr{C}'_{k^s}(2)$, $\mathscr{Z}'$, $\mathscr{Z}'_{k^s}$, $\mathscr{I}'$ and $\mathscr{I}'_{k^s}$ be the corresponding objects for V'. Since the map from $\mathscr{I}'$ to $\mathscr{I}$ is injective, we have exact sequences

$$0 \to \mathscr{Z}/\mathscr{Z}' \to \mathscr{C}(1)/\mathscr{C}'(1) \to \mathscr{I}/\mathscr{I}' \to 0$$

and

$$\mathscr{I}/\mathscr{I}' \to \mathscr{C}(2)/\mathscr{C}'(2) \to \mathrm{CH}^2(V)/\mathrm{CH}^2(V') \to 0$$

as well as the corresponding ones over k^s. Moreover Hilbert's theorem 90 implies that

$$\left(\mathscr{C}_{k^s}(1)/\mathscr{C}'_{k^s}(1)\right)^{\mathscr{G}} = \left(\bigoplus_{P\in V^{(1)}_{k^s}} k^s(P)^*/\bigoplus_{P\in V'^{(1)}_{k^s}} k^s(P)^*\right)^{\mathscr{G}}$$
$$= \bigoplus_{P\in V^{(1)}} k(P)^*/\bigoplus_{P\in V'^{(1)}} k(P)^* = \mathscr{C}(1)/\mathscr{C}'(1).$$

Thus we get the following two commutative diagrams of complexes the first of which has exact horizontal lines:

$$(9)\qquad \begin{array}{ccccccccc} 0\to & (\mathscr{Z}_{k^s}/\mathscr{Z}'_{k^s})^{\mathscr{G}} & \to & (\mathscr{C}_{k^s}(1)/\mathscr{C}'_{k^s}(1))^{\mathscr{G}} & \to & (\mathscr{I}_{k^s}/\mathscr{I}'_{k^s})^{\mathscr{G}} & \xrightarrow{\partial'} & H^1(\mathscr{G},\mathscr{Z}_{k^s}/\mathscr{Z}'_{k^s}) \\ & \uparrow & & \uparrow\wr & & \uparrow & & \\ 0\to & \mathscr{Z}/\mathscr{Z}' & \longrightarrow & \mathscr{C}(1)/\mathscr{C}'(1) & \longrightarrow & \mathscr{I}/\mathscr{I}' & \to 0 & \end{array}$$

and

$$(10)\qquad \begin{array}{ccccc} (\mathscr{I}_{k^s}/\mathscr{I}'_{k^s})^{\mathscr{G}} & \to & (\mathscr{C}_{k^s}(2)/\mathscr{C}'_{k^s}(2))^{\mathscr{G}} & \to & \mathrm{CH}^2(V_{k^s})/\mathrm{CH}^2(V'_{k^s}) \\ \uparrow & & \uparrow\wr & & \uparrow \\ \mathscr{I}/\mathscr{I}' & \to & \mathscr{C}(2)/\mathscr{C}'(2) & \to & \mathrm{CH}^2(V)/\mathrm{CH}^2(V') \end{array}$$

which defines a map from $\mathrm{Im}\,\partial'$ to $\mathrm{CH}^2(V)/\mathrm{CH}^2(V')$. Moreover there are obvious morphisms of diagrams from (7) to (9) and from (8) to (10). Thus we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Im}\,\partial' & \longrightarrow & \mathrm{CH}^2(V)/\mathrm{CH}^2(V') \\ \uparrow & & \uparrow \\ H^1(\mathscr{G},\mathscr{Z}_{k^s}) & \longrightarrow & \mathrm{CH}^2(V). \end{array}$$

It remains to show that the image of $\beta\cup f$ in $\mathrm{CH}^2(V)/\mathrm{CH}^2(V')$ is given by the class of E. The image of $\beta\cup f$ in $H^1(\mathscr{G},\mathscr{Z}_{k^s}/\mathscr{Z}'_{k^s})$ is given by the cocycle

$$\widetilde{\beta\cup f}':\sigma\mapsto\begin{cases}0 & \text{if } \sigma(\sqrt{a})=\sqrt{a},\\ \left[\sum_{P\in V^{(1)}_{k^s}}\lambda_P\right] & \text{otherwise}.\end{cases}$$

where

$$\lambda_P = \begin{cases} \left(\frac{Z}{X-\sqrt{a}Y}\right)^{\nu_Q(f)} \text{ if } P = p_2^{-1}(Q), \\ f \text{ if } P = p_1^{-1}((\sqrt{a}:1:0)), \\ f^{-1} \text{ if } P = p_1^{-1}((\sqrt{a}:-1:0)), \\ 1 \text{ otherwise.} \end{cases}$$

We may write E in the form $E_0 + \sum_{Q\in V'^{(1)}} E_Q$ where $p_{2*}(E_0) = 0$ and for any Q in $V'^{(1)}$, the support of E_Q is included in $p_2^{-1}(Q)$ and $p_{2*}(E_Q)$ is equal to $\nu_Q(f)Q$. Thus for any Q in $V'^{(1)}$ there exists a function g_Q in $k^s(Q)(C)$ such the divisor of g_Q over $k^s(Q)$ is

$$-\nu_Q(f)(\sqrt{a}:-1:0) + E_Q$$

and they may be choosen to be trivial except for a finite number. Then one considers the element γ of $\mathscr{C}_{k^s}(1)/\mathscr{C}'_{k^s}(1)$ defined by $\gamma = [\sum_{P\in V^{(1)}} \gamma_P]$ where

$$\gamma_P = \begin{cases} f \text{ if } P = p_1^{-1}((\sqrt{a}:-1:0)), \\ g_Q \text{ if } P = p_2^{-1}(Q), \\ 1 \text{ otherwise.} \end{cases}$$

Then for any σ in $\mathscr{G}$ such that $\sigma(\sqrt{a}) = -\sqrt{a}$ and any Q in $V'^{(1)}$ the divisor of ${}^\sigma g_Q g_Q^{-1}$ over $k^s(Q)$ coincides with the one of $\left(\frac{X-\sqrt{a}Y}{Z}\right)^{-\nu_Q(f)}$. Thus, for any $\sigma\in\mathscr{G}$,

$$\widetilde{\beta\cup f'}(\sigma) = {}^\sigma\gamma\gamma^{-1}.$$

Moreover the image of γ in $\mathscr{I}_{k^s}/\mathscr{I}'_{k^s}$ is invariant under $\mathscr{G}$ and its image in $\mathscr{C}_{k^s}(2)/\mathscr{C}'_{k^s}(2)$ comes from E by the natural map

$$\bigoplus_{P\in V^{(2)}} \mathbf{Z} \to (\mathscr{C}_{k^s}(2)/\mathscr{C}'_{k^s}(2))^{\mathscr{G}}$$

which implies the result. □

9. Application to negligible classes

We shall now apply the results of the preceding sections to the study of negligible classes in the cohomology of a central extension of an $\mathbf{F}_p$ vector space by another. Such groups have been used in [**Sa1**] to construct counter-examples to Noether's problem using classes in the unramified Brauer group. In fact one of the advantage of these extensions from the point of view of Galois cohomology is the fact that the unramified classes coming from the cohomology of the quotient may be characterized with linear algebra (see [**Bo**, lemma 5.1] and proposition 9.4 below).

9.1. Products of generic Severi-Brauer varieties. — Let p be a prime number and k be a field of characteristic different from p. If $p = 2$ we assume that the field contains a primitive fourth root of one and in general that it contains a primitive p-th root of one ξ. Let m be an integer, $X_1, \dots, X_n$ be indeterminates and K be the field $k(X_1, \dots, X_n)$. We fix an integer m and monomials A_i, B_i in the X_i for $1 \leqslant i \leqslant m$. We then consider the cyclic simple algebras $D_i = A_\xi(A_i, B_i)$ generated by two elements I and J with the relations

$$I^p = A_i, J^p = B_i \text{ and } IJ = \xi JI,$$

we denote by Y_i the corresponding Severi-Brauer variety and by Y the product of these varieties. Using Amitsur's theorem [**Am**, theorem 9.3] one gets (see [**Pe1**, lemma 8]) that

$$\mathrm{Ker}(\mathrm{Br}\, K \to \mathrm{Br}\, K(Y)) = < (A_i, B_i),\ 1 \leqslant i \leqslant m >$$

and the complex $\mathscr{C}$ may be written as

$$(11) \quad < (A_i, B_i),\ 1 \leqslant i \leqslant m > \otimes K^* \xrightarrow{\cup} H^3(K, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(K(Y), \mathbf{Q}/\mathbf{Z}(2)).$$

9.2. Connection with negligible classes. — Let U be an $\mathbf{F}_p$ vector space with a basis $u_1, \dots, u_n$. Let Φ^1_K be the morphism from the dual $U^\vee$ of U to K^*/K^{*p} which sends $u_i^\vee$ of the dual basis to (X_i). Let

$$\Phi^i_K : (\Lambda^i U)^\vee \to H^i(K, \mu_p)$$

be the induced map (see [**Pe1**, page 250]). This is an injection by [**Pe1**, lemma 7]. Moreover

$$\mathrm{Ker}\Big(\mathrm{Br}(K) \to \mathrm{Br}(K(Y))\Big) \subset \mathrm{Im}\, \Phi^2_K.$$

Let V be the dual of the inverse image of this kernel by Φ^2_K. We then have an injection

$$V^\vee \to (\Lambda^2 U)^\vee$$

and a surjective map

$$\Lambda^2 U \to V$$

which gives an element γ in $\Lambda^2(U^\vee)\otimes V$.

For any vector space W over $\mathbf{F}_p$ let $AC^*(W)$ be the quotient of the tensor algebra $T^*(W)$ by the ideal generated by $x\otimes y+y\otimes x$ for all x,y in W. Then there is a surjective morphism

$$AC^*(W)\to\Lambda^*(W)$$

which is an isomorphism if p is not 2 and a natural map of algebras

$$AC^*(W^\vee)\to H^*(W,\mathbf{F}_p)$$

which extends the isomorphism $W^\vee \xrightarrow{\sim} H^1(W,\mathbf{F}_p)$.

Let $\tilde\gamma$ be a lift of γ in $AC^*(U^\vee)\otimes V$. We also denote by $\tilde\gamma$ its image in $H^2(U,V)$. There is a natural surjection

$$H^2(U,V)\twoheadrightarrow\Lambda^2(U^\vee)\otimes V$$

(see [**Bro**, exercise IV.3.8]). The image of $\tilde\gamma$ by this map coincides with γ. Let

$$0\to V\xrightarrow{j}G\xrightarrow{\pi}U\to 0$$

be the central extension corresponding to $\tilde\gamma$.

Definition 9.1. — If H is a finite group, M a H-module and E a field, a class λ in $H^i(H,M)$ is said to be *totally E-negligible* if and only if for any extension F of E and any morphism

$$\rho:\mathrm{Gal}(F^s/F)\to H$$

the image of λ by ρ^* is zero in $H^i(F,M)$.

If E is a field over k and $\boldsymbol{a}=(a_i)_{1\leqslant i\leqslant n}$ a family of inversible elements of E, we denote by

$$\bigoplus_{i\geqslant 0}\Phi^i_{E,\boldsymbol{a}}:\bigoplus_{i\geqslant 0}\Lambda^i U^\vee\to\bigoplus_{i\geqslant 0}H^i(E,\mu_p^{\otimes i})$$

the morphism of graded algebras which sends $u_j^\vee$ onto a_j.

Proposition 9.1. — *Let $\tilde\lambda$ belong to $AC^iU^\vee$ and λ be its image in $\Lambda^iU^\vee$. The following three assertions are equivalent:*

(1) The image of $\tilde\lambda$ under inflation in $H^i(G,\mu_p^{\otimes i})$ is totally k-negligible.

(2) For any field E over k and any family $\boldsymbol{a}=(a_i)_{1\leqslant i\leqslant n}$ of inversible elements in E such that

$$\Phi^2_{E,\boldsymbol{a}}(V^\vee)=0$$

one has

$$\Phi^i_{E,\boldsymbol{a}}(\lambda) = 0.$$

(3) One has $\Phi^i_{K(Y),\boldsymbol{X}}(\lambda) = 0$.

Proof. — • Let us first prove the equivalence of the first two assertions. Let E be an extension of k. By Kummer theory, there is a natural correspondance between the morphisms ρ_1 from $\mathrm{Gal}(F^s/F)$ to U and the families $(a_i)_{1\leqslant i\leqslant n}$ of elements of E^*/E^{*p}. Let us first prove the following lemma:

Lemma 9.2. — *With the notation as above, the morphism*

$$\rho_1 : \mathrm{Gal}(F^s/F) \to U$$

may be lifted in a morphism

$$\rho : \mathrm{Gal}(F^s/F) \to G$$

if and only if

$$\Phi^2_{E,\boldsymbol{a}}(V^\vee) = 0.$$

Proof. — The morphism ρ_1 may be lifted to G if and only if the image of $\tilde{\gamma}$ in $H^2(F, V)$ is trivial. Let us choose a basis $(v_1, \dots, v_m)$ of V and let $(v_1^\vee, \dots, v_m^\vee)$ be the dual basis. The condition is equivalent to the triviality of $v^\vee_{i\,*}(\rho_1^*(\tilde{\gamma}))$ for $1 \leqslant i \leqslant m$ that is the triviality of $\rho_1^*(v^\vee_{i\,*}(\tilde{\gamma}))$ for $1 \leqslant i \leqslant m$ which is equivalent to $\Phi^2_{E,\boldsymbol{a}}(v_i^\vee) = 0$. □

End of the proof of proposition 9.1. — Let us assume that the first assertion is true and let E be a field over k and $\boldsymbol{a}$ a family of inversible elements such that the hypothesis of the second assertion are verified. By the preceding lemma, the morphism $\mathrm{Gal}(E^s/E) \to U$ defined by $\boldsymbol{a}$ may be lifted to G. But by the first assertion the image of $\tilde{\lambda}$ in $H^i(G, \mathbf{Z}/p\mathbf{Z})$ is sent to 0 in $H^i(E, \mu_p^{\otimes i})$.

Let us assume that the second assertion is true, let E be a field extension of k and $\rho : \mathrm{Gal}(E^s/E) \to G$ be a morphism. The map ρ induces a morphism

$$\mathrm{Gal}(E^s/E) \to U$$

corresponding to a family $\boldsymbol{a}$. By the lemma $\Phi^2_{E,\boldsymbol{a}}(V^\vee) = 0$. Thus $\Phi^i_{E,\boldsymbol{a}}(\lambda) = 0$ and the image of $\tilde{\lambda}$ in $H^i(E, \mu_p^{\otimes i})$ is trivial.

• It is clear that the second assertion implies the third. Let us prove the opposite implication. First the fact that $\Phi^i_{K(Y),\boldsymbol{X}}(\lambda) = 0$ implies that $\Phi^i_{E\otimes_k K(Y),\boldsymbol{X}}(\lambda) = 0$. Thus we are reduced to the case where $k = E$.

Let $A_1 \in \mathscr{P}(K/k)$ be the discrete valuation ring of rank one defined by the divisor $X_1 = a_1$. Consider the algebras $\mathscr{A}_\xi(A_j, B_j)$ defined over A_1 by the generators I and J and the relations

$$I^p = A_j, J^p = B_j \text{ and } IJ = \xi JI.$$

It defines a Severi-Brauer scheme $\mathscr{Y}_j^1$ over $\operatorname{Spec} A_1$. Let $\mathscr{Y}^1$ be the product of these schemes. Then the special fiber of $\mathscr{Y}^1$ defines a local ring B_1 in $\mathscr{P}(K(Y)/k)$ over A_1 such that B_1 is unramified over A_1 and the residue field κ_{B_1} is the function field over $k(X_2, \ldots, X_n)$ of the product Y^1 of the Severi-Brauer varieties defined by the algebras

$$A_\xi(A_j(a_1, X_2, \ldots, X_n), B_j(a_1, X_2, \ldots, X_n)).$$

In a similar way we construct discrete valuation rings of rank one A_j in $\mathscr{P}(\kappa_{A_{j-1}}/k)$ and $B_j \in \mathscr{P}(\kappa_{B_{j-1}}/k)$ so that $\kappa_{A_j} = k(X_{j+1}, \ldots, X_n)$ and $\kappa_{B_j} = \kappa_{A_j}(Y^j)$ where Y^j is defined as Y^1. We put $\kappa_{A_0} = K$ and $\kappa_{B_0} = K(Y)$. Let us assume that

$$\Phi^i_{\kappa_{B_j},(a_1,\ldots,a_j,X_{j+1},\ldots,X_n)}(\lambda) = 0.$$

Then taking the completion of κ_{B_j} for B_{j+1}

$$\Phi^i_{\kappa_{\hat{B}_j},(a_1,\ldots,a_j,X_{j+1},\ldots,X_n)}(\lambda) = 0.$$

But the field $\kappa_{\hat{B}_j}$ is isomorphic to $\kappa_{B_{j+1}}((X_{j+1} - a_{j+1}))$. Thus the natural surjection

$$\operatorname{Gal}(\hat{\kappa}^s_{B_j}/\kappa_{\hat{B}_j}) \to \operatorname{Gal}(\kappa^s_{B_{j+1}}/\kappa_{B_{j+1}})$$

has a section and the map

$$H^i(\kappa_{B_{j+1}}, \mu_p^{\otimes i}) \to H^i(\kappa_{\hat{B}_j}, \mu_p^{\otimes i})$$

is an injection. Thus

$$\Phi^i_{\kappa_{B_{j+1}},(a_1,\ldots,a_j,a_{j+1},X_{j+2},\ldots,X_n)}(\lambda) = 0.$$

Therefore by induction, we get that $\Phi^i_{\kappa_{B_n},\boldsymbol{a}}(\lambda) = 0$. But, since $\Phi^2_{k,\boldsymbol{a}}(V^\vee) = 0$, κ_{B_n} is rational on k and $\Phi^i_{k,\boldsymbol{a}}(\lambda) = 0$. □

Lemma 9.3. — *The images in* $\Lambda^3(U^\vee)$ *of the kernels of the natural maps*

$$H^3(U,\mathbf{Z}/p\mathbf{Z}) \to H^3(G,\mathbf{Z}/p\mathbf{Z})$$

and

$$H^3(U,\mathbf{Z}/p\mathbf{Z}) \to H^3(G,\mathbf{Q}/\mathbf{Z})$$

are equal to $U^\vee \wedge V^\vee$.

Proof. — The element $\tilde{\gamma}$ in $AC^2(U^\vee)\otimes V$ defines a morphism

$$\tilde{\gamma} : V^\vee \to AC^2(U^\vee)$$

which is injective. We then consider the Hochschild-Serre spectral sequences

$$E_2^{p,q}(\mathbf{Z}/p\mathbf{Z}) = H^p(U,H^q(V,\mathbf{Z}/p\mathbf{Z})) \Rightarrow H^{p+q}(G,\mathbf{Z}/p\mathbf{Z})$$

and

$$E_2^{p,q}(\mathbf{Q}/\mathbf{Z}) = H^p(U,H^q(V,\mathbf{Q}/\mathbf{Z})) \Rightarrow H^{p+q}(G,\mathbf{Q}/\mathbf{Z}).$$

Moreover the natural map $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Q}/\mathbf{Z}$ induces a morphism of spectral sequences from the first one to the second one. If $p = 2$ there is a natural isomorphism

$$S^p(U^\vee)\otimes S^q(V^\vee) \xrightarrow{\sim} H^p(U,H^q(V,\mathbf{Z}/2\mathbf{Z}))$$

(see [**Car1**, théorème 2]) and if $p \neq 2$ isomorphisms

$$U^\vee \oplus \Lambda^2(U^\vee) \xrightarrow{\delta\oplus\cup} H^2(U,\mathbf{Z}/p\mathbf{Z})$$

where δ is the Bockstein operator and by [**Car2**, théorème 2],

$$U^\vee\otimes U^\vee \oplus \Lambda^3 U^\vee \xrightarrow{\delta.\cup.\oplus.\cup.\cup.} H^3(U,\mathbf{Z}/p\mathbf{Z}).$$

- By [**Bro**, page 60], one has an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(H_{n-1}(U,\mathbf{Z}),\mathbf{Q}/\mathbf{Z}) \to H^n(U,\mathbf{Q}/\mathbf{Z}) \to \mathrm{Hom}(H_n(U,\mathbf{Z}),\mathbf{Q}/\mathbf{Z}) \to 0.$$

But $\mathrm{Ext}^1_{\mathbf{Z}}(H_{n-1}(U,\mathbf{Z}),\mathbf{Q}/\mathbf{Z})$ is trivial and by [**Bro**, pages 122 and 123], $\Lambda^i U$ is isomorphic to $H_i(U,\mathbf{Z})$ if $i = 1$ or 2 and $\Lambda^i U \to H_i(U,\mathbf{Z})$ for $i \geqslant 3$. We get isomorphisms

$$H^0(U,\mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z},\ H^1(U,\mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} U^\vee \text{ and } H^2(U,\mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \Lambda^2 U^\vee$$

as well as surjections

$$H^i(U,\mathbf{Q}/\mathbf{Z}) \twoheadrightarrow \Lambda^i U^\vee.$$

Similarly, one gets

$$H^1(U,H^1(V,\mathbf{Q}/\mathbf{Z})) \xrightarrow{\sim} U^\vee\otimes V^\vee \text{ and } H^2(V,\mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \Lambda^2 V^\vee. \tag{12}$$

• We shall now prove that the map

$$V^\vee \to AC^2(U^\vee)$$

induced by the first spectral sequence is the one defined by $-\tilde{\gamma}$. We use the same filtration on $\bigoplus_{n\in\mathbf{N}} C^n(G,\mathbf{Z}/p\mathbf{Z})$ as in section 4. Let us write $\tilde{\gamma} = \sum_{i=1}^t f_i.g_i \otimes v_i$ with f_i, g_i in $U^\vee$ and v_i in V. Let $s : U \to G$ be a set-theoritic section of π such that $s(0) = e$ and

$$\forall u,u' \in U,\ s(u)s(u') = j\left(\sum_{i=1}^t f_i(u)g_i(u')v_i\right) s(uu').$$

Let h belong to $V^\vee \xrightarrow{\sim} H^0(U, H^1(V,\mathbf{Z}/p\mathbf{Z})) \xrightarrow{\sim} E_2^{0,1}(G/V)$. For any g in G let $\tau(g)$ in V denote $gs(\pi(g))^{-1}$. Then h is represented in $C^1(G,\mathbf{Z}/p\mathbf{Z})$ by the cocycle $\tilde{h}$ defined by

$$\begin{array}{ccc} G & \to & \mathbf{Z}/p\mathbf{Z} \\ g & \mapsto & h(\tau(g)). \end{array}$$

But $\tau(gg') = \tau(g) + \tau(g') + \sum_{i=1}^t f_i(\pi(g))g_i(\pi(g'))v_i$. Thus

$$\begin{aligned} d\tilde{h}(g,g') &= h(\tau(g')) - h(\tau(gg')) + h(\tau(g)) \\ &= -h\left(\sum_{i=1}^t f_i(\pi(g))g_i(\pi(g'))v_i\right) \end{aligned}$$

which is sent in $H^2(U,\mathbf{Z}/p\mathbf{Z})$ on the image of the element of $AC^2(U^\vee)$ given by

$$-\sum_{i=1}^t h(v_i)f_i.g_i.$$

But it is the opposite of the image of h by $\tilde{\gamma}$.

• Since the Hochschild-Serre spectral sequence is compatible with the cup-product, the map

$$H^1(U, H^1(V,\mathbf{Z}/p\mathbf{Z})) \to H^3(U,\mathbf{Z}/p\mathbf{Z})$$

is induced by the map

$$\begin{array}{ccc} U^\vee \otimes V^\vee & \to & AC^3U^\vee \\ u\otimes v & \mapsto & -u.\tilde{\gamma}(v). \end{array} \tag{13}$$

• Also thanks to this compatibility, the composite morphism

$$AC^2(V^\vee) \to H^0(U, H^2(V,\mathbf{Z}/p\mathbf{Z})) \to H^2(U, H^1(V,\mathbf{Z}/p\mathbf{Z}))$$

is given by the opposite of the map

$$\begin{array}{rcl} AC^2(V^\vee) & \to & AC^2(U^\vee)\otimes V^\vee \\ xy & \mapsto & \tilde\gamma(x)\otimes y-\tilde\gamma(y)\otimes x. \end{array}$$

- By the expression of the map

$$H^1(U,H^1(V,\mathbf{Z}/p\mathbf{Z}))\to H^3(U,\mathbf{Z}/p\mathbf{Z})$$

we get that $U^\vee\wedge V^\vee$ is contained in the image in $\Lambda^3U^\vee$ of the kernel of the inflation map

$$H^3(U,\mathbf{Z}/p\mathbf{Z})\to H^3(G,\mathbf{Z}/p\mathbf{Z}).$$

Thus it suffices to prove that the image in $\Lambda^3U^\vee$ of

$$\operatorname{Ker}(H^3(U,\mathbf{Z}/p\mathbf{Z})\to H^3(G,\mathbf{Q}/\mathbf{Z}))$$

is contained in $U^\vee\wedge V^\vee$. But, by (12) and (13), we already know that the image of the composite map

$$H^1(U,H^1(V,\mathbf{Q}/\mathbf{Z}))\to H^3(U,\mathbf{Q}/\mathbf{Z})\to\Lambda^3U^\vee$$

is contained in $U^\vee\wedge V^\vee$. Also the computation of the map

$$AC^2(V^\vee)\to H^2(U,H^1(V,\mathbf{Z}/p\mathbf{Z}))$$

shows that the map

$$H^2(V,\mathbf{Q}/\mathbf{Z})\to H^2(U,H^1(V,\mathbf{Q}/\mathbf{Z}))$$

is induced by the map

$$\begin{array}{rcl} \Lambda^2V^\vee & \to & AC^2(U^\vee)\otimes V^\vee \\ x\wedge y & \mapsto & \tilde\gamma(y)\otimes x-\tilde\gamma(x)\otimes y \end{array}$$

which is injective, since $\tilde\gamma$ is injective. Therefore $E_3^{0,2}(\mathbf{Q}/\mathbf{Z})$ is trivial and the result is proved. □

Notation . — Let $(V^\vee\wedge U^\vee)^\perp_{\mathrm{dec}}\subset\Lambda^3U$ be the subgroup of $(V^\vee\wedge U^\vee)^\perp$ generated by the elements of the form $u\wedge v$, for $u\in\Lambda^2U$, $v\in U$. Let $K^3_{\max}$ be its orthogonal in $\Lambda^3U^\vee$.

Proposition 9.4. — *The inverse image in $\Lambda^3U^\vee$ of $H^3_{\mathrm{nr}/k}(K(Y),\mu_p^{\otimes 3})$ is equal to $K^3_{\max}$. In particular one has*

$$V^\vee\wedge U^\vee\subset\operatorname{Ker}\Phi^3_{K(Y),X}\subset K^3_{\max}.$$

Moreover the quotient $\operatorname{Ker}\Phi^3_{K(Y),X}/V^\vee \wedge U^\vee$ *is imbedded in* $\mathrm{CH}^2(Y)_{\mathrm{tors}}$.

Proof. — The inclusion $V^\vee \wedge U^\vee \subset \operatorname{Ker}\Phi^3_{K(Y),X}$ is clear. Therefore

$$(\operatorname{Ker}\Phi^3_{K(Y),X})^\perp \subset (V^\vee \wedge U^\vee)^\perp$$

and

$$(\operatorname{Ker}\Phi^3_{K(Y),X})^\perp_{\mathrm{dec}} \subset (V^\vee \wedge U^\vee)^\perp_{\mathrm{dec}}.$$

Thus for any f in $K^3_{\max}$, the restriction of f to $(\operatorname{Ker}\Phi^3_{K(Y),X})^\perp_{\mathrm{dec}}$ is trivial and by [**Pe1**, theorem 2 and remark p. 251] the image of f in $H^3(K(Y),\mu_p^{\otimes 3})$ is unramified over k.

Let $\lambda \notin K^3_{\max}$. We want to show that its image is ramified. There exist $p \in \Lambda^2 U$ and $u'_1 \in U$ such that $<\lambda, p \wedge u'_1> \neq 0$ and $p \wedge u'_1$ belongs to $(V^\vee \wedge U^\vee)^\perp$. We complete (u'_1) in a basis $(u'_1,\dots,u'_n)$ of U. We may choose these elements so that they can be lifted in a basis of $\mathbf{Z}^n$. Let $M_1,\dots,M_n$ be the monomials in $X_1,\dots,X_n,X_1^{-1},\dots,X_n^{-1}$ given by this basis of $\mathbf{Z}^n$. Then $\Phi^1_K(u_i'^\vee) = (M_i)$ and $k(X_1,\dots,X_n) = k(M_1,\dots,M_n)$. Let A be the discrete valuation ring corresponding to M_1 in $k(M_1,\dots,M_n)$. One may write the symbols (A_i,B_i) as (A'_i,B'_i) where the A'_i are monomials in $M_2,\dots,M_n$ and B'_i monomials in $M_1,\dots,M_n$. Moreover, by taking suitable powers of B'_i and A'_i, we may assume that $v_A(B'_i) \in \{0,1\}$. Then let I'_i and J'_i be generators of D_i such that

$$I_i'^p = A'_i,\ J_i'^p = B'_i \text{ and } I'_iJ'_i = \xi J'_iI'_i \text{ for } 1 \leqslant i \leqslant m$$

and let $\mathscr{D}_i$ be the order of D_i over A generated by I'_i and J'_i. It is a maximal order of D_i. Indeed, if $v_A(B'_i) = 0$, this follows from the fact that $\mathscr{D}_i/(M_1)$ is a skew field. Otherwise, define a function v on $D_i - \{0\}$ by

$$v\Big(\sum_{\substack{0\leqslant j<p\\0\leqslant l<p}} a_{j,l}I_i^jJ_i^l\Big) = \inf_{\substack{0\leqslant j<p\\0\leqslant l<p}}\left(v_A(a_{j,l}) + \frac{l}{p}\right)$$

then v verifies $v(a+b) \geqslant \inf(v(a),v(b))$, and for any a, b in $D_i - \{0\}$, write $a = J_i^r\alpha$ and $b = J_i^s\beta$ with $v(\alpha) = v(\beta) = 0$. Then

$$\alpha = \sum_{\substack{0\leqslant j<p\\0\leqslant l<p}} \alpha_{j,l}I_i^jJ_i^l \text{ and } \beta = \sum_{\substack{0\leqslant j<p\\0\leqslant l<p}} \beta_{j,l}I_i^jJ_i^l$$

with $\inf_{0\leqslant j<p}\nu_A(\alpha_{j,0}) = \inf_{0\leqslant j<p}\nu_A(\beta_{j,0}) = 0$. Since $A[{A'_i}^{1/p}]/(\mathfrak{m}_A)$ is a field, we get that $\nu(\sum_{0\leqslant j,l<p}(\alpha_{j,0}\beta_{l,0})I_i^{j+l}) = 0$ and then that $\nu(ab) = \nu(a)+\nu(b)$. Thus ν is a valuation and $\mathscr{D}_i$ the maximal order of D_i (see [**Re**, §12]). Let $\mathscr{Y}_i$ be the connected component of the corresponding Severi-Brauer scheme on $\operatorname{Spec} A$ which contains the generic fiber (see [**Art**], [**Brz**] and [**Fr**, page 37]). This is Artin's model of Y_i. Let $\mathscr{Y}$ be the products of the $\mathscr{Y}_i$. If M_1 does not divide B'_i then the special fibre $\mathscr{Y}_i^0$ of $\mathscr{Y}_i$ is the Severi-Brauer variety corresponding to the algebra $\mathscr{D}_i\otimes\kappa_A$ and otherwise, by [**Art**, theorem 1.4], $\mathscr{Y}_i^{0^s}$ has p irreducible components birationally equivalent to $\mathbf{P}^{p-1}_{\kappa_A^s}$. They may be described as follows: $\mathscr{D}_i\otimes\kappa_A$ is the algebra generated by I'_i and J'_i with the relations

$$I_i'^p = A'_i,\ J_i'^p = 0 \text{ and } I'_iJ'_i = \xi J'_iI'_i.$$

Let us consider the algebra $\mathscr{D}'_i = \mathscr{D}_i\otimes\kappa_A({A'_i}^{1/p})$. A maximal set of orthogonal idempotents e_j of $\mathscr{D}'_i$ is given by

$$e_j = \prod_{\substack{l\in\mathbf{Z}/p\mathbf{Z}\\ l\neq j}}\frac{I'_i-\xi^l{A'_i}^{1/p}}{(\xi^j-\xi^l){A'_i}^{1/p}}.$$

This follows from the existence of an isomorphism from $D_i\times K({A'_i}^{1/p})$ to $M_p(K({A'_i}^{1/p}))$ which maps I_i to the diagonal matrix $D({A'_i}^{1/p},\xi{A'_i}^{1/p},\dots,\xi^{p-1}{A'_i}^{1/p})$. Therefore the points of $\mathscr{Y}_i^{0^s}$ which correspond to right ideals L for which $\dim Le_l = 1$ for any l (see [**Art**, lemma 3.3]) are given by $de_j\mathscr{D}'_i$ for $1\leqslant j\leqslant p$ and d in $\mathscr{D}'_i$ such that $de_j\mathscr{D}'_ie_l\neq 0$ for $1\leqslant l\leqslant p$. Therefore the components of $\mathscr{Y}_i^{0^s}$ are permuted cyclically by $\operatorname{Gal}(\kappa_A({A'_i}^{1/p})/\kappa_A)$. Thus the special fiber $\mathscr{Y}^0$ of $\mathscr{Y}$ is integral over κ_A and defines a discrete valuation ring B over A which is unramified over A.

We may reduce to the case where $M_1|B'_i$ if and only if $1\leqslant i\leqslant l$ for some l between 1 and m. Then κ_B is rational over

$$\kappa_A((A'_i)^{1/p}, 1\leqslant i\leqslant l)\Big(\prod_{l+1\leqslant i\leqslant m}\mathscr{Y}_i^0\Big).$$

By [**Pe1**, lemma 6]

$$\partial_B(\Phi^3_{k(Y),X}(\lambda)) = \Phi^2_{\kappa_B}(\hat{u}'_1(\lambda))$$

where $\Phi^*_{\kappa_B} : \Lambda^* {u'_1}^{\perp} \to H^*(\kappa_B, \mu_p^{\otimes *})$ is the morphism which sends u'_i over M_i for $2 \leqslant i \leqslant n$ and for any w of $\Lambda^i U^\vee$, $\hat{u}'_1(w)$ is the unique element of $\Lambda^{i-1}({u'_1}^{\perp})$ such that

$$w - {u'_1}^\vee \wedge \hat{u}'_1(w) \in \Lambda^i({u'_1}^{\perp}).$$

We want to show that the image of λ is ramified at B. Therefore, we shall now describe the kernel of $\Phi^2_{\kappa_B}$.

Lemma 9.5. — *With notation as above,*

$$\mathrm{Ker}(\Phi^2_{\kappa_B}) = \hat{u}'_1(V^\vee) \wedge ({u'_1}^{\perp}) + V^\vee \cap \Lambda^2({u'_1}^{\perp}).$$

Proof. — Let a_i (resp. b_i) be the inverse image of (A'_i) (resp. (B'_i)) in $U^\vee$. Then the right hand side coincides with

$$\sum_{i=1}^{l} a_i \wedge ({u'_1}^{\perp}) + \sum_{i=l+1}^{m} \mathbf{F}_p a_i \wedge b_i$$

which is contained in $\mathrm{Ker}\,\Phi^2_{\kappa_B}$. Moreover Amitsur's theorem implies (see [**Pe1**, lemma 8]) that

$$\mathrm{Ker}\left(H^2(\kappa_A((A'_i)^{1/p}, 1 \leqslant i \leqslant l), \mu_p^{\otimes 2}) \to H^2(\kappa_B, \mu_p^{\otimes 2})\right)$$

is generated by the symbols (A'_i, B'_i) for $i \geqslant l+1$. Therefore we only have to consider the case $m = l$.

In this case, let $w_1^\vee, \dots w_r^\vee$ be a basis of $\hat{u}'_1(V^\vee)$. We complete it in a basis $(w_i^\vee)_{1 \leqslant i \leqslant m-1}$ of ${u'_1}^{\perp}$. Let $w_1, \dots, w_{m-1}$ be the dual basis and let η belong to

$$\Lambda^2({u'_1}^{\perp}) - \hat{u}'_1(V^\vee) \wedge ({u'_1}^{\perp}).$$

Then there exist i and j strictly bigger than r such that

$$< \eta | w_i \wedge w_j > \neq 0.$$

We may construct a discrete valuation ring A' (resp. B') in $k(M_2, \dots, M_n)$ (resp. κ_B) such that

$$v_{A'}(\Phi^1_{\kappa_A}(w_k)) = \delta_{i,k} \text{ and } \kappa_{A'} = k(N_1, \dots, \hat{N}_i, \dots, N_{n-1})$$

where $N_1, \dots, N_{n-1}$ are monomials corresponding to $w_1, \dots, w_{n-1}$, such that B' is unramified on A', which is possible since $i > r$ and $\kappa_{B'}$ is rational over

$$k(N_1, \dots, \hat{N}_i, \dots, N_n)(N_1^{1/p}, \dots, N_r^{1/p}).$$

Thus

$$\partial_{B'}(\Phi^2_{\kappa_B}(\eta)) = \Phi^1_{\kappa_{B'}}(\sum_{l\neq i} <\eta|w_i\wedge w_l> w_l) = (\prod_{l\neq i} N_l^{<\eta|w_i\wedge w_l>}) \neq 0$$

and

$$\Phi^2_{\kappa_B}(\eta)\neq 0. \quad \square$$

End of the proof of proposition 9.4. — We want to show that the class $\Phi^2_{\kappa_B}(\hat{u}'_1(\lambda))$ is non zero, which, by lemma 9.5, is equivalent to

$$\hat{u}'_1(\lambda) \notin \hat{u}'_1(V^\vee)\wedge {u'_1}^\perp + V^\vee \cap \Lambda^2({u'_1}^\perp).$$

Taking the inverse image by $\hat{u}'_1$ is is sufficient to show that

$$\lambda \notin V^\vee \wedge {u'_1}^\perp + V^\vee \wedge {u'_1}^\vee + \Lambda^3({u'_1}^\perp) = V^\vee\wedge U^\vee + \Lambda^3({u'_1}^\perp).$$

But, by hypothesis, $p\wedge u'_1$ belongs to $(V^\vee\wedge U^\vee)^\perp$. Thus $p\wedge u'_1$ is zero on

$$V^\vee\wedge U^\vee + \Lambda^3({u'_1}^\perp).$$

Since $<\lambda, p\wedge u'_1>\neq 0$, the first assertion is proved. It implies the following inclusions.

It remains to prove the last assertion. The kernel of $\Phi^3_{K(Y),X}$ is the inverse image in $\Lambda^3 U^\vee$ of

$$\mathrm{Ker}(H^3(K,\mathbf{Q}/\mathbf{Z}(2)) \to H^3(K(Y),\mathbf{Q}/\mathbf{Z}(2))).$$

Let λ in $\Lambda^3 U^\vee$ be such that

$$\Phi^3_K(\lambda) \in <(A_i,B_i), 1\leqslant i\leqslant m>\otimes K^*,$$

then we may write

$$\Phi^3_K(\lambda) = \Phi^3_K(w) + \sum_{i=1}^m (A_i,B_i,P_i)$$

where w belongs to $V^\vee\wedge U^\vee$ and P_i is a polynomial which is not divisible by any of the X_j. Then taking successive residues at places defined by indeterminates X_i, X_j, and X_l with i, j, and l distincts in $\{1,\dots,n\}$, we get that

$$(\lambda - w)(u_i\wedge u_j\wedge u_k) = 0$$

and therefore $\lambda = w$. Thus the last assertion follows from theorem 2.1 and (11). $\square$

9.3. An explicit example. — We now assume that U is an $\mathbf{F}_2$ vector space of dimension 6 and V is the dual of the subspace of $\Lambda^2 U$ generated by the elements

$$u_2^\vee \wedge u_5^\vee,\ u_4^\vee \wedge u_1^\vee,\ u_6^\vee \wedge u_3^\vee,\ (u_2^\vee + u_4^\vee + u_6^\vee) \wedge (u_1^\vee + u_3^\vee + u_5^\vee).$$

Lemma 9.6. — *With the notations of section 9.2*

$$K^3_{\max}/V^\vee \wedge U^\vee \xrightarrow{\sim} \mathbf{Z}/2\mathbf{Z}.$$

Proof. — We have

$$\begin{aligned} V^\vee \wedge U^\vee = <& u_1^\vee \wedge u_2^\vee \wedge u_4^\vee,\ u_1^\vee \wedge u_2^\vee \wedge u_5^\vee,\ u_1^\vee \wedge u_3^\vee \wedge u_4^\vee,\\ & u_1^\vee \wedge u_3^\vee \wedge u_6^\vee,\ u_1^\vee \wedge u_4^\vee \wedge u_5^\vee,\ u_1^\vee \wedge u_4^\vee \wedge u_6^\vee,\\ & u_2^\vee \wedge u_3^\vee \wedge u_5^\vee,\ u_2^\vee \wedge u_3^\vee \wedge u_6^\vee,\ u_2^\vee \wedge u_4^\vee \wedge u_5^\vee,\\ & u_2^\vee \wedge u_5^\vee \wedge u_6^\vee,\ u_3^\vee \wedge u_4^\vee \wedge u_6^\vee,\ u_3^\vee \wedge u_5^\vee \wedge u_6^\vee,\\ & u_1^\vee \wedge u_2^\vee \wedge u_3^\vee + u_1^\vee \wedge u_5^\vee \wedge u_6^\vee,\ u_2^\vee \wedge u_3^\vee \wedge u_4^\vee + u_1^\vee \wedge u_2^\vee \wedge u_6^\vee,\\ & u_3^\vee \wedge u_4^\vee \wedge u_5^\vee + u_1^\vee \wedge u_2^\vee \wedge u_3^\vee,\ u_4^\vee \wedge u_5^\vee \wedge u_6^\vee + u_2^\vee \wedge u_3^\vee \wedge u_4^\vee > . \end{aligned}$$

Thus

$$\begin{aligned} (V^\vee \wedge U^\vee)^\perp = <& u_1 \wedge u_2 \wedge u_3 + u_3 \wedge u_4 \wedge u_5 + u_5 \wedge u_6 \wedge u_1,\\ & u_2 \wedge u_3 \wedge u_4 + u_4 \wedge u_5 \wedge u_6 + u_6 \wedge u_1 \wedge u_2,\\ & u_2 \wedge u_4 \wedge u_6, u_1 \wedge u_3 \wedge u_5 > . \end{aligned}$$

Let

$$\begin{aligned} g_1 &= u_1 \wedge u_2 \wedge u_3 + u_3 \wedge u_4 \wedge u_5 + u_5 \wedge u_6 \wedge u_1,\\ g_2 &= u_2 \wedge u_3 \wedge u_4 + u_4 \wedge u_5 \wedge u_6 + u_6 \wedge u_1 \wedge u_2,\\ h_1 &= u_2 \wedge u_4 \wedge u_6 \end{aligned}$$

and

$$h_2 = u_1 \wedge u_3 \wedge u_5.$$

Since

$$g_1 + g_2 + h_1 + h_2 = (u_1 + u_4) \wedge (u_2 + u_5) \wedge (u_3 + u_6),$$

we get that

$$< h_1, h_2, g_1 + g_2 + h_1 + h_2 > \subset (V^\vee \wedge U^\vee)^\perp_{\mathrm{dec}}.$$

But by [**Pe1**, pages 265 and 266]

$$(\sum_{i=1}^{6} a_i u_i) \wedge (\alpha g_1 + \beta g_2 + \gamma h_1 + \delta h_2) = 0$$

is equivalent to

$$\begin{array}{rclrcl} a_1\beta + a_4\alpha & = & 0 & a_1\alpha + a_4\gamma & = & 0 \\ a_2\gamma + a_5\alpha & = & 0 & a_3\alpha + a_6\gamma & = & 0 \\ a_3\beta + a_6\alpha & = & 0 & a_3\delta + a_6\beta & = & 0 \\ a_1\delta + a_4\beta & = & 0 & a_2\beta + a_5\delta & = & 0 \\ a_2\alpha + a_5\beta & = & 0. & & & \end{array}$$

Let us assume that $\alpha = 1$ and $\beta = 0$. Then $a_4 = a_2 = a_6 = 0$ and this implies that $a_3 = a_5 = a_1 = 0$. Thus the elements of the form $g_1 + \gamma h_1 + \delta h_2$ are not decomposable. This is also the case for the elements of the form $g_2 + \gamma h_1 + \delta h_2$. Thus

$$[g_1] \in (V^\vee \wedge U^\vee)^\perp / < h_1, h_2, g_1 + g_2 + h_1 + h_2 >$$

does not lift to a decomposable element. This implies the equality

$$< h_1, h_2, g_1 + g_2 + h_1 + h_2 > = (V^\vee \wedge U^\vee)^\perp_{\text{dec}}. \quad \square$$

Proposition 9.7. — *With notation as above*

$$\operatorname{Ker}(\Phi^3_{k(Y),X}) = K^3_{\max} \neq V^\vee \wedge U^\vee.$$

Proof. — The group $K^3_{\max}$ is the orthogonal of $(V^\vee \wedge U^\vee)^\perp_{\text{dec}}$. Therefore we have

$$K^3_{\max} = V^\vee \wedge U^\vee + < u_2^\vee \wedge u_3^\vee \wedge u_4^\vee + u_5^\vee \wedge u_6^\vee \wedge u_1^\vee > .$$

Thus it is enough to show that

$$(X_2, X_3, X_4) + (X_5, X_6, X_1)$$

is trivial in $H^3(K(Y), \mathbf{Z}/2\mathbf{Z})$. Let Y' be the product of the conics C_1, C_2, C_3 given by the homogeneous equations

$$\begin{aligned} (C_1) &: T_{1,1}^2 - X_2 T_{1,2}^2 - X_5 T_{1,3}^2 = 0, \\ (C_2) &: T_{2,1}^2 - X_4 T_{2,2}^2 - X_1 T_{2,3}^2 = 0, \\ (C_3) &: T_{3,1}^2 - X_6 T_{3,2}^2 - X_3 T_{3,3}^2 = 0. \end{aligned}$$

Then a direct computation using the assertion (ii) of proposition 3.4 for quadratic extensions yields an explicit element of $\mathrm{CH}^2(Y)$ which is either 0 or the unique nontrivial element of $\mathrm{CH}^2(Y)_{\text{tors}}$. Then the method described in section 8 used backwards yields the function f of $k(Y')$ defined by

$$f = \left(\frac{T_{1,1}}{T_{1,3}}\right)^2 X_1 - \left(\frac{T_{2,1}}{T_{2,3}}\right)^2 X_5 = \left(\frac{T_{2,1}T_{1,2}}{T_{2,3}T_{1,3}}\right)^2 X_2 - \left(\frac{T_{1,1}T_{2,2}}{T_{1,3}T_{2,3}}\right)^2 X_4$$

in $k(C_1 \times \cdots \times C_3)$. But by [**Lam**, chapter 10, proposition 1.3],

$$\begin{aligned}(f, X_1X_3X_5, X_2X_4X_6) &= \left(X_2\left(\frac{T_{2,1}T_{1,2}}{T_{2,3}T_{1,3}}\right)^2 - X_4\left(\frac{T_{1,1}T_{2,2}}{T_{1,3}T_{2,3}}\right)^2, X_2X_4, X_1X_3X_5\right)\\ &\quad + \left(X_1\left(\frac{T_{1,1}}{T_{1,3}}\right)^2 - X_5\left(\frac{T_{2,1}}{T_{2,3}}\right)^2, X_6, X_1X_5\right)\\ &= (X_2, X_4, X_1X_3X_5) + (X_1, X_5, X_6)\\ &= (X_2, X_3, X_4) + (X_1, X_5, X_6).\end{aligned}$$

This element is therefore trivial in $H^3(k(Y), \mathbf{Z}/2\mathbf{Z})$. □

Corollary 9.8. — *Consider $U = \bigoplus_{i=1}^{6} \mathbf{F}_2 u_i$, $V = \bigoplus_{i=1}^{4} \mathbf{F}_2 v_i$ and G the central extension of U by V corresponding to the following element of $H^2(U, V)$:*

$$u_2^\vee \cup u_5^\vee \otimes v_1 + u_4^\vee \cup u_1^\vee \otimes v_2 + u_6^\vee \cup u_3^\vee \otimes v_3 + (u_1^\vee + u_3^\vee + u_5^\vee) \cup (u_2^\vee + u_4^\vee + u_6^\vee) \otimes v_4.$$

Then for any field k of characteristic different from 2 and containing a primitive fourth root of one, $u_1^\vee \cup u_2^\vee \cup u_3^\vee + u_4^\vee \cup u_5^\vee \cup u_6^\vee$ gives a non-trivial totally k-negligible element in $H^3(G, \mathbf{Q}/\mathbf{Z})$.

Remark 9.9. — If we consider the group $H^3(G, \mathbf{Q}/\mathbf{Z})_{\mathrm{perm}}$ of permutation negligible classes introduced by Saltman in [**Sa2**] and which corresponds to classes vanishing in the cohomology group $H^3(G, \mathbf{C}(W)^*)$ for any faithful representation W of G over $\mathbf{C}$, then it is possible to show using computations in the cohomology of G that the class obtained is not permutation negligible. The first examples of geometrically negligible elements in degree three which are not permutation negligible are given by Saltman in [**Sa2**, theorem 4.14] for 2-groups having a cyclic subgroup of index 2.

Proof. — The corollary follows from proposition 9.7, lemma 9.3 and proposition 9.1. □

Corollary 9.10. — *For any field k of characteristic different from 2 and containing the fourth roots of one, for any family $(a_1, \ldots, a_6)$ of elements in k^* such that*

$$(a_1, a_4) = (a_2, a_5) = (a_3, a_6) = (a_2a_4a_6, a_1a_3a_5) = 0$$

one has

$$(a_1, a_2, a_3) + (a_4, a_5, a_6) = 0.$$

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1998

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