PRODUCTS OF SEVERI-BRAUER VARIETIES AND GALOIS COHOMOLOGY*

by

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Abstract. — Let $Y_1, \dots, Y_n$ be n Severi-Brauer varieties over a field k. Let $k(Y_1 \times \cdots \times Y_n)$ be the function field of their product. Using a recent result of Kahn we show that the quotient of the kernel of the restriction map

$$H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(Y_1 \times \cdots \times Y_n), \mathbf{Q}/\mathbf{Z}(2))$$

by the subgroup generated by the cup-products with the classes of $Y_1, \dots, Y_n$ in $\mathrm{Br}\, k$ is isomorphic to $\mathrm{CH}^2(Y_1 \times \cdots \times Y_n)_{\mathrm{tors}}$. We first apply this result to the product of two conics, using the fact that in this case $\mathrm{CH}^2(Y_1 \times Y_2)_{\mathrm{tors}}$ is trivial. Then we construct examples with three conics where this quotient is not trivial. We also show how, in the case of one conic, the restriction map fits into a longer exact sequence.

Résumé. — Soient $Y_1, \dots, Y_n$ n variétés de Severi-Brauer sur un corps k, Soit $k(Y_1 \times \cdots \times Y_n)$ le corps de fonctions de leur produit. En utilisant un résultat récent de Bruno Kahn, nous montrons que le quotient du noyau de l'application de restriction

$$H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(Y_1 \times \cdots \times Y_n), \mathbf{Q}/\mathbf{Z}(2))$$

par le sous-groupe engendré par les cup-produits des classes de $Y_1, \dots, Y_n$ dans $\mathrm{Br}\, k$ est isomorphe à $\mathrm{CH}^2(Y_1 \times \cdots \times Y_n)_{\mathrm{tors}}$. Nous appliquons ce résultat au produit de deux coniques en utilisant le fait que dans ce cas $\mathrm{CH}^2(Y_1 \times Y_2)_{\mathrm{tors}}$ est trivial. Nous construisons ensuite des exemples de produits de trois coniques pour lequel ce quotient n'est pas trivial.

*K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa-Barbara, 1992), Proc. Sympos. Pure Math., Vol. 58.2, AMS, Providence, 1995, pp. 369–401

Contents

1. Introduction

Let k be a field of characteristic different from 2 and M be a multiquadratic extension of k. Then the restriction and corestriction maps for the Galois cohomology of these fields were studied by Kahn, Merkur′ev, Shapiro, Tignol and Wadsworth ([**Kah1**], [**STW**], [**Ti**], [**MT**]) using various complexes which generalize the canonical long exact sequence associated to a quadratic extension. In particular in cohomological degree two one has the complex

$$\bigoplus_{u\in U} k(\sqrt{u})^* \xrightarrow{N} H^1(k,\mathbf{Z}/2\mathbf{Z})\otimes U \xrightarrow{\cup} H^2(k,\mathbf{Z}/2\mathbf{Z}) \xrightarrow{\mathrm{Res}}$$
$$\xrightarrow{\mathrm{Res}} H^2(M,\mathbf{Z}/2\mathbf{Z}) \xrightarrow{\mathrm{Cores}} \bigoplus_{g\in \mathrm{Gal}(M/k)-\{0\}} H^2(M^g,\mathbf{Z}/2\mathbf{Z})$$

where U is the kernel of the restriction map

$$H^1(k,\mathbf{Z}/2\mathbf{Z}) \to H^1(M,\mathbf{Z}/2\mathbf{Z})$$

and N is the sum of the maps

$$\begin{array}{rcl} k(\sqrt{u})^* & \to & H^1(k,\mathbf{Z}/2\mathbf{Z})\otimes U \\ a & \mapsto & N_{k(\sqrt{u})/k}(a)\otimes u. \end{array}$$

Tignol proved that this complex is exact when the degree of the extension is four but is not exact in general when it is greater than eight. The starting point of this problem is that M may be seen as the canonical splitting field for $U\subset H^1(k,\mathbf{Z}/2\mathbf{Z})$. Thanks to Amitsur's theorem (see [**Am**]), one gets that the function field of a product of Severi-Brauer varieties plays the same rôle for a finitely generated subgroup of the Brauer group of k. Indeed for any field

k, any elements $[Y_1], \dots, [Y_m]$ in $\mathrm{Br}\, k$ corresponding to Severi-Brauer varieties $Y_1, \dots, Y_m$ the function field $M = k(Y_1 \times \cdots \times Y_m)$ of $Y_1 \times \cdots \times Y_m$ verifies

$$\mathrm{Ker}\left(H^2(k, \mathbf{Q}/\mathbf{Z}(1)) \to H^2(M, \mathbf{Q}/\mathbf{Z}(1))\right) = < [Y_i], 1 \leqslant i \leqslant m >$$

and if k'/k is a field extension such that

$$< [Y_i], 1 \leqslant i \leqslant n > \subset \mathrm{Ker}(\mathrm{Br}\, k \to \mathrm{Br}\, k').$$

then there is a closed point P of $Y_1 \times \cdots \times Y_m$ and an embedding

$$k(P) \to k'$$

over k. By analogy with the case of multiquadratic extensions, we introduce the following complex:

$$(\mathscr{C}_\infty) \qquad \bigoplus_{u \in U} \bigoplus_{i \in \mathbf{N}} \left(A_u^{\otimes i}\right)^* \xrightarrow{\mathrm{Nrd}} k^* \otimes U \to H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(M, \mathbf{Q}/\mathbf{Z}(2))$$

where U is the subgroup of $\mathrm{Br}\, k$ generated by the $[Y_i]$ for $1 \leqslant i \leqslant m$, for any $u \in U$, A_u is a central simple algebra representing $u \in \mathrm{Br}\, k$ and Nrd denotes the sum of the morphisms

$$\begin{array}{ccc} \left(A_u^{\otimes i}\right)^* & \to & k^* \otimes U \\ a & \mapsto & \mathrm{Nrd}(a)^i \otimes u. \end{array}$$

If $U \subset \mathrm{Br}\, k_{(n)}$ and n is prime to the exponent characteristic of k, one may also consider the complex

$$(\mathscr{C}_n) \qquad \bigoplus_{u \in U} \bigoplus_{i \in \mathbf{N}} \left(A_u^{\otimes i}\right)^* \xrightarrow{\mathrm{Nrd}} H^1(k, \mu_n) \otimes U \xrightarrow{\cup} H^3(k, \mu_n^{\otimes 2}) \to H^3(M, \mu_n^{\otimes 2})$$

The exactness of this complex at the third term would mean that the kernel of the restriction map in degree three is simply given by sums of the form $\sum_{i=1}^m (a_i) \cup [Y_i]$. If $n = 2$ and U is generated by the class of a quaternion algebra $\left(\frac{a,b}{k}\right)$ then Arason [**Ar**] proved the exactness of $\mathscr{C}_2$ at the third term. Using a result of Merkur′ev and Suslin, Colliot-Thélène (1988, unpublished) proved the exactness of $\mathscr{C}_p$ at the third term when U is generated by the class of a cyclic central simple algebra of prime index. By Merkur′ev and Suslin [**MS1**, corollary 12.1], $\mathscr{C}_p$ is exact at the second term in this case. Therefore $\mathscr{C}_p$ is exact when U is generated by the class of a cyclic central simple algebra of prime index. Knus, Lam, Shapiro and Tignol proved in [**KLST**] that $\mathscr{C}_2$ is exact at the second term if U is generated by the tensor product of two quaternion algebras.

In section 2, we consider the case of one conic Y and, using computations of Suslin [**Su1**], construct morphisms

$$N : H^n_{\mathrm{nr}/k}(M, \mathbf{Z}/2\mathbf{Z}) \to H^n(k, \mathbf{Z}/2\mathbf{Z})$$

where $H^n_{\mathrm{nr}/k}(M, \mathbf{Z}/2\mathbf{Z})$ denotes the unramified cohomology groups of M over k. This enables us to fit $\mathscr{C}_2$ in a longer exact sequence of the form

$$A^* \xrightarrow{\mathrm{Nrd}} H^1(k, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\cup\alpha} H^3(k, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\mathrm{Res}} H^3_{\mathrm{nr}/k}(M, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{N}$$
$$\xrightarrow{N} \mathrm{Im}(H^1(k, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\cup\alpha} H^3(k, \mathbf{Z}/2\mathbf{Z})) \xrightarrow{\cup(-1)} H^4(k, \mathbf{Z}/2\mathbf{Z})$$

where A is the quaternion algebra corresponding to Y and α its class in $\mathrm{Br}\, k$.

In section 3, we compute in the general case the K-theory groups of $Y_1 \times \cdots \times Y_m$ and gather some information on the topological filtration of $K_0(Y_1 \times \cdots \times Y_m)$. These computations play a fundamental rôle in the following sections.

In section 4.1 we apply a theorem of Kahn to prove the main tool of this paper, namely the existence of a canonical isomorphism between the torsion part of $\mathrm{CH}^2(Y_1 \times \cdots \times Y_m)$ and the homology group of $\mathscr{C}_\infty$ at the third term.

In section 4.2 we use this theorem and results of Karpenko to show that if A is a central simple algebra over k such that the quotient of the index of A by its exponent is squarefree and if for any p dividing this quotient the p-primary component of the corresponding division algebra is decomposable then

$$\mathrm{Ker}\left(H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(Y), \mathbf{Q}/\mathbf{Z}(2))\right) = [A] \cup H^1(k, \mathbf{Q}/\mathbf{Z}(1))$$

where Y is the corresponding Severi-Brauer variety.

In part 5 we consider the case of two conics and, using the theorem of Knus, Lam, Shapiro and Tignol, prove the exactness of $\mathscr{C}_2$ in this case.

In section 6 we construct an example with three conics where $\mathscr{C}_\infty$ is not exact at the third term.

I would like to thank Colliot-Thélène and Kahn for several fruitful discussions.

1.1. Notations. — For any field L, we denote by L^s a separable closure of L and for any discrete $\mathrm{Gal}(L^s/L)$-module M,

$$H^i(L, M) = H^i(\mathrm{Gal}(L^s/L), M).$$

In particular the Brauer group of L is given by $\mathrm{Br}\, L = H^2(L, L^{s*})$. If the characteristic of L does not divide n then μ_n denotes the group of n-th roots of unity in L^s. If $j < 0$, we put $\mu_n^{\otimes j} = \mathrm{Hom}(\mu_n^{\otimes -j}, \mathbf{Z}/n\mathbf{Z})$. If L' is a finite field extension of L and $\phi : \operatorname{Spec} L' \to \operatorname{Spec} L$ the corresponding map, then the trace map (see [**SGA4**, exposé XVII, théorème 6.2.3])

$$\mathrm{Tr} : \phi_* \phi^* \mu_n^{\otimes j} \to \mu_n^{\otimes j}$$

induces a canonical map

$$\mathrm{Cores}_L^{L'} : H^i(L', \mu_n^{\otimes j}) \to H^i(L, \mu_n^{\otimes j})$$

which coincides with the usual corestriction map when the extension is separable. If L is a field of exponent characteristic p one defines (see [**Kah2**])

$$\begin{aligned} H^i(L, (\mathbf{Q}/\mathbf{Z})'(j)) &= \varinjlim_{(p,n)=1} H^i(L, \mu_n^{\otimes j}), \\ H^i(L, (\mathbf{Q}_p/\mathbf{Z}_p)(0)) &= \varinjlim_r H^i(L, \mathbf{Z}/p^r\mathbf{Z}), \\ H^i(L, (\mathbf{Q}_p/\mathbf{Z}_p)(1)) &= \varinjlim_r H^{i-1}(L, K_1(L^s)/p^r), \\ H^i(L, (\mathbf{Q}_p/\mathbf{Z}_p)(2)) &= \varinjlim_r H^{i-2}(L, K_2(L^s)/p^r) \end{aligned}$$

and, if $j = 0$, 1 or 2,

$$H^i(L, \mathbf{Q}/\mathbf{Z}(j)) = H^i(L, (\mathbf{Q}/\mathbf{Z})'(j)) \oplus H^i(L, (\mathbf{Q}_p/\mathbf{Z}_p)(j)).$$

In particular there is a canonical isomorphism

$$H^2(L, \mathbf{Q}/\mathbf{Z}(1)) \xrightarrow{\sim} \operatorname{Br} L.$$

These definitions coincide with the usual ones if $p = 1$. We also define

$$\begin{aligned} H^i(L, \hat{\mathbf{Z}}'(j)) &= \varprojlim_{(n,p)=1} H^i(L, \mu_n^{\otimes j}), \\ H^i(L, \mathbf{Z}_p(1)) &= \varprojlim_r H^{i-1}(L, K_1(k^s)/p^r) \end{aligned}$$

and

$$H^i(L, \hat{\mathbf{Z}}(1)) = H^i(L, \hat{\mathbf{Z}}'(1)) \oplus H^i(L, \mathbf{Z}_p(1)).$$

There is a canonical morphism from L^* to $H^1(L,\hat{\mathbf{Z}}(1))$. Let $d,n,m\in\mathbf{N}$ be such that p is prime to nm and $d|n$ then one has a commutive diagram

$$\begin{array}{ccc}
\mu_{nm}^{\otimes i}\otimes\mu_d^{\otimes j} & \longrightarrow & \mu_d^{\otimes i+j}\\
\downarrow & & \downarrow\\
\mu_n^{\otimes i}\otimes\mu_d^{\otimes j} & \longrightarrow & \mu_d^{\otimes i+j}.
\end{array}$$

Hence we get a morphism

$$H^q(L,\hat{\mathbf{Z}}'(i))\otimes H^r(L,\mu_d^{\otimes j})\to H^{q+r}(L,\mu_d^{\otimes i+j}).$$

Moreover for any N such that $nm|N$ and N is prime to p one has a commutative diagram

$$\begin{array}{ccc}
\mu_N^{\otimes i}\otimes\mu_n^{\otimes j} & \longrightarrow & \mu_n^{\otimes i+j}\\
\downarrow & & \downarrow\\
\mu_N^{\otimes i}\otimes\mu_{nm}^{\otimes j} & \longrightarrow & \mu_{nm}^{\otimes i+j}.
\end{array}$$

Therefore we get a commutative diagram

$$\begin{array}{ccc}
H^q(L,\hat{\mathbf{Z}}'(i))\otimes H^r(L,\mu_n^{\otimes j}) & \longrightarrow & H^{q+r}(L,\mu_n^{\otimes i+j})\\
\downarrow & & \downarrow\\
H^q(L,\hat{\mathbf{Z}}'(i))\otimes H^r(L,\mu_{nm}^{\otimes j}) & \longrightarrow & H^{q+r}(L,\mu_{nm}^{\otimes i+j}).
\end{array}$$

and a morphism

$$H^q(L,\hat{\mathbf{Z}}'(i))\otimes H^r(L,\mathbf{Q}/\mathbf{Z}'(j))\to H^{q+r}(L,\mathbf{Q}/\mathbf{Z}'(i+j)).$$

If $j=0$ or 1 similar diagrams for the product

$$K_1(L^s)/p^s\otimes K_j(L^s)/p^s\to K_{j+1}(L^s)/p^s$$

induces

$$H^r(L,\mathbf{Z}_p(1))\otimes H^q(L,\mathbf{Q}_p/\mathbf{Z}_p(j))\to H^{q+r}(L,\mathbf{Q}_p/\mathbf{Z}_p(j+1))$$

and we get a canonical product

$$H^r(L,\hat{\mathbf{Z}}(1))\otimes H^q(L,\mathbf{Q}/\mathbf{Z}(j))\to H^{q+r}(L,\mathbf{Q}/\mathbf{Z}(j+1)).$$

In particular this induces a product

$$L^*\otimes\operatorname{Br}L\to H^3(L,\mathbf{Q}/\mathbf{Z}(2)).$$

The field L is a function field over K if and only if it is generated by a finite number of elements as a field over K. Let L be a function field over K. We denote by $\mathscr{P}(L/k)$ the set of discrete valuation rings A of rank one such that $K \subset A \subset L$ and the fraction field $\mathrm{Fr}(A)$ of A is L. If $A \in \mathscr{P}(L/K)$ then κ_A denotes the residue field. For any $i \in \mathbf{N} - 0$, $j \in \mathbf{Z}$ and any $n \in \mathbf{N}$ not divisible by the characteristic of L,

$$\partial_A : H^i(L, \mu_n^{\otimes j}) \to H^{i-1}(\kappa_A, \mu_n^{\otimes j-1})$$

denotes the residue map as defined in [**CTO**]. For any smooth variety X over a field k and any integer p one denotes by $X^{(p)}$ the set of points of codimension p in X. For any $i \in \mathbf{N}$, any integer n not divisible by the characteristic of k and any $j \in \mathbf{Z}$, the sheaf $\mathscr{H}^i(\mu_n^{\otimes j})$ is defined as the sheaf over X_{Zar} corresponding to the presheaf

$$U \mapsto H^i(U_{\text{ét}}, \mu_n^{\otimes j}).$$

For any i, j, n as above and any $l \in \mathbf{N}$, we put

$$H^l(X, \mathscr{H}^i(\mu_n^{\otimes j})) = H^l(X_{\mathrm{Zar}}, \mathscr{H}^i(\mu_n^{\otimes j})).$$

Similarly, $\mathscr{K}_j$ denotes the Zariski sheaf on X associated to the presheaf

$$U \mapsto K_j(H^0(U, \mathscr{O}_X))$$

and

$$H^l(X, \mathscr{K}_i) = H^l(X_{\mathrm{Zar}}, \mathscr{K}_i).$$

1.2. Basic facts on unramified cohomology

Definition 1.1. — If L is a function field over K, and n a positive integer prime to the exponent characteristic of K, the unramified cohomology groups are the groups

$$H^i_{\mathrm{nr}/K}(L, \mu_n^{\otimes j}) = \bigcap_{A \in \mathscr{P}(L/K)} \mathrm{Ker}(H^i(L, \mu_n^{\otimes j}) \xrightarrow{\partial_A} H^{i-1}(\kappa_A, \mu_n^{\otimes j-1}))$$

One defines similarly the unramified Brauer group.

We recall that two function fields L and M over K are stably isomorphic over K if and only if there exist indeterminates $U_1, \ldots, U_l$ and $T_1, \ldots, T_m$ and an isomorphism

$$L(U_1 \ldots, U_l) \xrightarrow{\sim} M(T_1, \ldots, T_m)$$

over K. A function field L over K is stably rational over K if it is stably isomorphic to K.

As was pointed out by Gabber, [**BO**] implies the following proposition:

Proposition 1.1 **(Bloch,Ogus).** — *If X is a smooth projective model of L over K then*

$$H^0(X, \mathscr{H}^i(\mu_n^{\otimes j})) \xrightarrow{\sim} H^i_{\mathrm{nr}/K}(L, \mu_n^{\otimes j}).$$

Proposition 1.2 **(Colliot-Thélène, Ojanguren [CTO])**

Let L and M be two function fields over K. If L and M are stably isomorphic over K then there exists an isomorphism

$$H^i_{\mathrm{nr}/K}(L, \mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{\mathrm{nr}/K}(M, \mu_n^{\otimes j}).$$

2. The case of one conic

Let us fix a field k of characteristic different from 2. In this section the coefficients of the cohomology groups are equal to $\mathbf{Z}/2\mathbf{Z}$. We shall omit them in the notation. The letter X denotes a conic over the field k given by the homogeneous equation

$$X_1^2 - aX_2^2 - bX_3^2 = 0.$$

We denote by (a, b) the corresponding symbol in $H^2(k)$ and by M the function field of X. We shall use results of Suslin [**Su1**] to fit the restriction maps

$$H^i(k) \to H^i(M)$$

into an infinite complex. Let $\overline{k}$ be an algebraic closure of k and $\overline{X} = X \times_k \overline{k}$. Since

$$H^q(\overline{X}_{\text{ét}}) = \begin{cases} \mathbf{Z}/2\mathbf{Z} & \text{if } q = 0 \text{ or } 2 \\ 0 & \text{otherwise,} \end{cases}$$

the Hochschild-Serre spectral sequence

$$H^p(k, H^q(\overline{X}_{\text{ét}})) \Rightarrow H^{p+q}(X_{\text{ét}})$$

gives a long exact sequence

$$\cdots \to H^n(k) \to H^n(X_{\text{ét}}) \to H^{n-2}(k) \xrightarrow{d_3} H^{n+1}(k) \to \cdots$$

and d_3 coincides with the cup-product $\cup(a, b, -1)$ (See [**Su1**, lemma 1]). Denote the generic point by $\eta : \operatorname{Spec} k(X) \to X$. By [**Su1**] the Leray spectral sequence

$$H^p(X, R^q\eta_*\mu_2) \Rightarrow H^{p+q}(k(X), \mu_2)$$

gives a long exact sequence

$$\cdots \to \bigoplus_{P\in X^{(1)}} H^{n-2}(k(P)) \to H^n(X_{\text{ét}}) \to H^n(k(X)) \xrightarrow{\oplus\partial_P} \bigoplus_{P\in X^{(1)}} H^{n-1}(k(P)) \to \cdots$$

Since the E_2 term of the Bloch-Ogus spectral sequence (see [**BO**, corollary 6.3])

$$E_1^{p,q} = \bigoplus_{x\in X^{(p)}} H^{q-p}(k(x)) \Rightarrow H^{p+q}(X_{\text{ét}})$$

is given by

$$E_2^{p,q} = H^p(X, \mathscr{H}^q)$$

one gets short exact sequences

$$0 \to H^1(X, \mathscr{H}^{n-1}) \to H^n(X_{\text{ét}}) \to H^0(X, \mathscr{H}^n) \to 0.$$

Lemma 2.1. — *The composite map*

$$H^1(X, \mathscr{H}^{n-1}) \to H^n(X_{\text{ét}}) \to H^{n-2}(k)$$

coincides with the map induced by

$$\bigoplus_{P\in X^{(1)}} H^{n-2}(k(P)) \xrightarrow{\oplus\,\mathrm{Cores}} H^{n-2}(k).$$

In the case $n = 3$ this is lemma 2 of [**Su1**].

Proof. — It is sufficient to show that for any $P \in X^{(1)}$ the composite map

$$H^{n-2}(k(P)) \to H^1(X, \mathscr{H}^{n-1}) \to H^n(X_{\text{ét}}) \to H^{n-2}(k)$$

coincides with the corestriction map. Let $k' = k(P)$, $X' = X \times_k k'$ and $\phi : X' \to X$ the canonical map. For any sheaf of 2-torsion on X the trace map

$$\mathrm{Tr} : \phi_*\phi^*F \to F$$

[**SGA4**, exposé XVII, théorème 6.2.3] yields a morphism

$$\mathrm{Tr} : H^n(X'_{\text{ét}}, \phi^*(F)) \to H^n(X_{\text{ét}}, F)$$

which is defined as the composite map

$$H^n(X'_{\text{ét}}, \phi^*(F)) \xrightarrow{\text{can}} H^n(X_{\text{ét}}, \phi_*\phi^*(F)) \xrightarrow{\mathrm{Tr}^*} H^n(X_{\text{ét}}, F)$$

Using the construction of the above spectral sequences (See [**HS**, chapter 8]) and the fact that ϕ_* preserves injectives and is exact since ϕ is finite, it is straightforward to show that the maps can and the corresponding morphisms for $\operatorname{Spec} k(P)$ and $\operatorname{Spec} k(X)$ are induced by morphisms of spectral sequences. Moreover the

Leray and Hochschild-Serre spectral sequences are functorial. Thus the morphisms induced by $\mathrm{Tr} : \phi_* \phi^* F \to F$ are also compatible with the spectral sequences. Therefore we get the commutative diagrams

$$\begin{array}{ccc}
\bigoplus_{\phi(P')=P} H^{n-2}(k'(P')) & \longrightarrow & H^n(X'_{\text{ét}}) \\
\big\downarrow \Sigma & & \big\downarrow \mathrm{Tr} \\
H^{n-2}(k(P)) & \longrightarrow & H^n(X_{\text{ét}})
\end{array}$$

and

$$\begin{array}{ccc}
H^n(X'_{\text{ét}}) & \longrightarrow & H^{n-2}(k(P)) \\
\big\downarrow \mathrm{Tr} & & \big\downarrow \mathrm{Cores} \\
H^n(X_{\text{ét}}) & \longrightarrow & H^{n-2}(k).
\end{array}$$

It is thus enough to prove the result for $k = k(P)$. For $n = 2$, one has

$$H^1(X, \mathscr{H}^{n-1}) \xrightarrow{\sim} \mathrm{Pic}\, X/2 \xrightarrow{\sim} \mathbf{Z}/2\mathbf{Z}$$

and the Bloch-Ogus spectral sequence gives an injection

$$\mathbf{Z}/2\mathbf{Z} \to H^2(X_{\text{ét}}).$$

Let ξ_X be the non-zero element in its image. In this case, X is isomorphic to $\mathbf{P}^1$ and by proposition 1.2

$$H^n(k) \to H^0(X, \mathscr{H}^n)$$

is an isomorphism. Moreover the Hochschild-Serre spectral sequence gives a short exact sequence

$$0 \to H^n(k) \to H^n(X_{\text{ét}}) \to H^{n-2}(k) \to 0$$

and the composite map

$$H^n(k) \to H^n(X_{\text{ét}}) \to H^0(X, \mathscr{H}^n) \xrightarrow{\sim} H^n(k)$$

is, by definition, the identity. Therefore the image of ξ_X by the map

$$H^2(X_{\text{ét}}) \to H^0(k) = \mathbf{Z}/2\mathbf{Z}$$

is non trivial and the result is proved for $n = 2$. Since both spectral sequences are compatible with cup-products, the map $H^{n-2}(k) \to H^n(X_{\text{ét}})$ coincides with $\cup \xi_X$ and the map $H^n(X_{\text{ét}}) \to H^{n-2}(k)$ sends $\alpha \cup \xi_X$ on α for any $\alpha \in H^{n-2}(k)$. $\square$

Definition 2.1. — We define the morphism N as the morphism from $H^0(X,\mathscr{H}^n)$ to $H^n(k)$ which fits into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & H^1(X,\mathscr{H}^{n-1}) & \to & H^n(X_{\text{ét}}) & \longrightarrow & H^0(X,\mathscr{H}^n) & \to & 0 \\
 & & & \searrow & \downarrow & & \downarrow N & & \\
 & & & & H^{n-2}(k) & \xrightarrow{\cup(a,b)} & H^n(k) & &
\end{array}$$

Indeed by lemma 2.1 the composite map $H^1(X,\mathscr{H}^{n-1})\to H^n(k)$ is zero and N is well defined.

Definition 2.2. — We denote by

$$\tau : \mathrm{Ker}(H^n(k)\to H^n(M))\to H^1(X,\mathscr{H}^{n-1})$$

the unique map fitting into the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ker}(H^n(k)\to H^n(M)) & \to & H^n(k) & & & & \\
 & & \downarrow\tau & & \downarrow & \searrow^{\mathrm{Res}} & & & \\
0 & \longrightarrow & H^1(X,\mathscr{H}^{n-1}) & \longrightarrow & H^n(X_{\text{ét}}) & \to & H^0(X,\mathscr{H}^n) & \to & 0.
\end{array}$$

Notation . — We consider the complex $\overline{\mathscr{C}_2}$:

$$\cdots\to H^{n-2}(k)\xrightarrow{\cup(a,b)} H^n(k)\xrightarrow{\mathrm{Res}} H^n_{\mathrm{nr}/k}(M)\xrightarrow{N}\mathrm{Im}(H^{n-2}(k)\xrightarrow{\cup(a,b)} H^n(k))\to$$
$$\xrightarrow{\cup(-1)}\mathrm{Ker}(H^{n+1}(k)\to H^{n+1}(M))\xrightarrow{\tau} H^1(X,\mathscr{H}^n)\xrightarrow{\oplus\,\mathrm{Cores}_k^{k(P)}} H^{n-1}(k)\to\cdots$$

and denotes by $\mathscr{H}_n(i)$ for $1\leqslant i\leqslant 6$ and $n\in\mathbf{N}$ the homology group of this complex at the $6n+i$-th term, with the convention that $H^l(k)=\{0\}$ if $l<0$ (for example $\mathscr{H}_n(1)=\mathrm{Ker}(\cup(a,b))/\mathrm{Im}(\oplus\,\mathrm{Cores}_k^{k(P)})$).

Proposition 2.2. — *These homology groups verify the following properties*
(1) $\mathscr{H}_n(i)=\{0\}$ *if* $n\leqslant 3$
(2) $\mathscr{H}_n(i)=\{0\}$ *if* $i=4,5$ *or* 6
(3) $\mathscr{H}_n(1)\tilde{\to}\mathscr{H}_n(3)$.

Remark 2.1. — The fact that $\mathscr{H}_2(2)=\{0\}$ is a particular case of Amitsur's theorem, $\mathscr{H}_3(2)=\{0\}$ is due to Arason (See [**Ar**, Satz 5.4]) and $\mathscr{H}_3(1)=\{0\}$ is due to Merkur′ev and Suslin (See [**MS1**, theorem 12.1]). The definition immediately implies the triviality of $\mathscr{H}_0(1)$, $\mathscr{H}_0(2)$, $\mathscr{H}_1(1)$, $\mathscr{H}_1(2)$ and $\mathscr{H}_2(1)$. Therefore it is sufficient to prove assertions 2 and 3 of the proposition.

Proof. — Let us first prove that $\mathscr{H}_n(4) = \{0\}$. This is a direct consequence of the commutativity of the diagram

$$\begin{tikzcd}
H^n(X_{\text{ét}}) \arrow[r] \arrow[d] & H^{n-2}(k) \arrow[r, "\cup(a,b,-1)"] \arrow[d] & H^{n+1}(k) \\
H^0(X, \mathscr{H}^n) \arrow[r, "N"] & H^n(k) \arrow[ur, "\cup(-1)"'] &
\end{tikzcd}$$

and of the exactness of its line.

Let $\alpha \in \operatorname{Ker} \tau$. Since the diagram

$$\begin{tikzcd}
 & \operatorname{Ker}(H^{n+1}(k) \to H^{n+1}(M)) \arrow[dl, "\tau"'] \arrow[d] \\
H^1(X, \mathscr{H}^n) \arrow[r] & H^{n+1}(X_{\text{ét}})
\end{tikzcd}$$

commutes, $\alpha \in \operatorname{Ker}(H^{n+1}(k) \to H^{n+1}(X_{\text{ét}}))$ and we get

$$\alpha \in \operatorname{Im}(H^{n-2}(k) \xrightarrow{(a,b,-1)} H^{n+1}(k)).$$

Thus $\mathscr{H}_n(5) = \{0\}$.

The triviality of $\mathscr{H}_n(6)$ follows from a diagram chase in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(H^{n+1}(k) \to H^{n+1}(M)) \arrow[r] \arrow[d, "\tau"] & H^{n+1}(k) \arrow[r, "\text{Res}"] \arrow[d] & H^{n+1}(M) & \\
0 \arrow[r] & H^1(X, \mathscr{H}^n) \arrow[r] \arrow[dr, "\oplus\,\text{Cores}"] & H^{n+1}(X_{\text{ét}}) \arrow[r] \arrow[d] & H^0(X, \mathscr{H}^{n+1}) \arrow[r] \arrow[u] & 0 \\
 & & H^{n-1}(k) & &
\end{tikzcd}$$

which has exact lines and column.

We now prove (3). The commutative diagram

$$\begin{tikzcd}
H^n(k) \arrow[r] \arrow[dr, "\text{Res}"] & H^n(X_{\text{ét}}) \arrow[r] \arrow[d] & H^{n-2}(k) \arrow[r] \arrow[d, "\cup(a,b)"] & H^{n+1}(k) \\
 & H^0(X, \mathscr{H}^n) \arrow[r, "N"] & H^n(k) \arrow[ur, "\cup(-1)"'] &
\end{tikzcd}$$

yields a surjective morphism

$$\psi : \operatorname{Ker}(. \cup (a, b)) \to \mathscr{H}_n(3).$$

Let $\alpha \in \operatorname{Ker}\psi$. Let $\beta \in H^n(X_{\text{ét}})$ represent α. Then the image of β in $H^n(M)$ is equal to the restriction of $\tilde{\beta} \in H^n(k)$. Thus we can assume that

$$\beta \in \operatorname{Ker}(H^n(X_{\text{ét}}) \to H^n(M)).$$

But in this case

$$\beta \in \operatorname{Im}(H^1(X, \mathscr{H}^{n-1}) \to H^n(X_{\text{ét}}))$$

and $\alpha \in \operatorname{Im}\left(\bigoplus_{P\in X^{(1)}} \operatorname{Cores}_k^{k(P)}\right)$. Thus

$$\operatorname{Ker}\psi \subset \operatorname{Im}\left(\bigoplus_{P\in X^{(1)}} \operatorname{Cores}_k^{k(P)}\right).$$

The other inclusion is straightforward. Therefore ψ induces an isomorphism

$$\mathscr{H}_n(1) \xrightarrow{\sim} \mathscr{H}_n(3). \quad \square$$

Remark 2.2. — Let us assume that $(a, b, -1) = 0$. In this case, by [**Su1**], Lemma 4 $(a, b) \in 2\operatorname{Br} k$. For example, if -1 is a square, $-1 = i^2$ then

$$(a, b) = 2[A_i(a, b)]$$

where $A_i(a, b)$ is the cyclic central algebra generated by two elements I and J with the relations

$$I^4 = -1, J^4 = -1 \text{ and } IJ = iJI.$$

Let $D \in \operatorname{Br}(k)$ be such that $2D = (a, b)$ then the image D_M of D in $\operatorname{Br} M$ belongs to $\operatorname{Br}(M)_{(2)} = H^2(M)$ and is unramified over k. We then get a surjection

$$H^{n-2}(k) \xrightarrow{.\cup D_M \circ \operatorname{Res}} H^n_{\mathrm{nr}/k}(M)/\operatorname{Ker} N.$$

We shall give in remark 6.3 an other description of the morphism N in degree 2 or 3 when -1 is a square.

3. K-theory of a product of Severi-Brauer varieties

3.1. K-theory groups. — We now go back to the case of m Severi-Brauer varieties $Y_1, \ldots, Y_m$ corresponding to elements $[Y_1], \ldots, [Y_m]$ in $\operatorname{Br} k$. As above, we denote by U the subgroup of $\operatorname{Br} k$ generated by $[Y_1], \ldots, [Y_m]$ and by M the function field of the product of $Y = Y_1 \times_k \cdots \times_k Y_m$. The purpose of this section is to describe the K-theory groups of Y and gather some information on their topological filtration. The K-theory of Y seems to be among the folklore and can be seen as a particular case of a much more general result of Panin [**Pa**].

The proof we give here for self-completeness follows the proof of Quillen (See [**Q**, §8.4]) step by step.

Let S be a scheme and $X \xrightarrow{\pi} S$ be a Severi-Brauer scheme over S of relative dimension $d-1$. Let $\mathscr{A}$ be the corresponding Azumaya algebra (See [**Gr**]) and $\mathscr{J}$ be the canonical vector bundle on X (see [**Q**, §8.4]). If $g : S' \to S$ is a faithfully flat map such that the product $X' = X \times_S S'$ is a projective bundle $\mathbf{P}(E)$ over S' then the inverse image of $\mathscr{J}$ is equal to $\mathscr{O}_{X'}(-1) \otimes_{S'} E$. Let $\mathscr{D}$ be an $\mathscr{O}_S$ algebra. We denote by $\mathscr{P}(S, \mathscr{D})$ the category of vector bundles over X which are left modules for $\mathscr{D}$.

Proposition 3.1. — *If S is quasi-compact then there is a canonical isomorphism*

$$\bigoplus_{0 \leqslant i \leqslant d-1} K_*(\mathscr{P}(S, \mathscr{A}^{\otimes i} \otimes_{\mathscr{O}_S} \mathscr{D})) \xrightarrow{\sim} K_*(\mathscr{P}(X, \mathscr{D})).$$

This isomorphism is given by

$$(x_i)_{1 \leqslant i \leqslant d-1} \mapsto \sum_{i=0}^{d-1} (\mathscr{J}^{\otimes i} \otimes_{\pi^*(\mathscr{A}^{\otimes i})} \pi^*(.))_*(x_i).$$

Corollary 3.2. — *If $Y_1, \dots, Y_m$ are Brauer-Severi schemes over a quasi-compact scheme S of relative dimension $d_1-1, \dots, d_m-1$, if $\mathscr{A}_1, \dots, \mathscr{A}_m$ are the corresponding Azumaya algebras and $\mathscr{J}_i$ the inverse image of the canonical vector bundle on Y_i by the projection $\pi_i : Y_1 \times \cdots \times Y_m \to Y_i$ then*

$$\bigoplus_{\substack{(k_i)_{1 \leqslant i \leqslant m} \\ 0 \leqslant k_i \leqslant d_i - 1}} K_*(\mathscr{A}_1^{\otimes k_1} \otimes \cdots \otimes \mathscr{A}_m^{\otimes k_m}) \xrightarrow{\sim} K_*(Y_1 \times_S \cdots \times_S Y_m)$$

where the isomorphism is given by

$$(x_{(k_i)})_{0 \leqslant k_i \leqslant d_i - 1} \mapsto \sum_{k_1=0}^{d_1-1} \cdots \sum_{k_m=0}^{d_m-1} \left(\bigotimes_{i=1}^{m} \mathscr{J}_i^{\otimes k_i} \otimes_{\pi^*(\mathscr{A}_1^{\otimes k_1} \otimes \cdots \otimes \mathscr{A}_m^{\otimes k_m})} \pi^*(.) \right) (x_{(k_i)}).$$

Proof of Proposition 3.1. — One has only to check that the constructions in [**Q**] are compatible with the structure of left $\mathscr{D}$-modules.

First the category $\mathscr{P}(X, \mathscr{D})$ is a full subcategory of the abelian category of left $\mathscr{D}$-modules and is closed by extensions. Moreover the forgetful functor $\mathbf{f}$ from $\mathscr{P}(X, \mathscr{D})$ to $\mathscr{P}(X)$ is an exact functor.

Let $S' \to S$ be a surjective étale morphism which splits X, $X' = X \times_S S'$ and $\tilde{g}$ be the induced morphism $X' \to X$. As in [**Q**] we define $\mathscr{R}_n(X, \mathscr{D})$ as the full subcategory of $\mathscr{P}(X, \mathscr{D})$ whose objects are the vector bundles $\mathscr{F}$ such that $\tilde{g}^*(\mathbf{f}(\mathscr{F}))(n)$ is regular and $\mathscr{P}_n(X, \mathscr{D})$ as the full subcategory of $\mathscr{P}(X, \mathscr{D})$ whose objects are the $\mathscr{F}$ such that, for any $q > 0$ and any $l \geqslant n$

$$R^q {\pi'}_* \left(\tilde{g}^*(\mathbf{f}(\mathscr{F}))(l)\right) = 0$$

where $\pi' : X' \to S'$ is the canonical morphism. By lemma 8.1.3 of [**Q**], one has

$$\mathscr{R}_n(X, \mathscr{D}) \subset \mathscr{P}_n(X, \mathscr{D}) \subset \mathscr{P}(X, \mathscr{D}).$$

As in [**Q**, §8.2] one gets

Lemma 3.3. — *For all n the canonical maps*

$$K_q(\mathscr{R}_n(X, \mathscr{D})) \to K_q(\mathscr{P}_n(X, \mathscr{D})) \to K_q(\mathscr{P}(X, \mathscr{D}))$$

are isomorphisms.

Proof. — For any object $\mathscr{F}$ of $\mathscr{P}(X, \mathscr{D})$ the sequence

$$0 \to \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{J}^\vee \to \cdots \to \mathscr{F} \otimes_{\mathscr{O}_X} \Lambda^r \mathscr{J}^\vee \to 0$$

is an exact sequence in $\mathscr{P}(X, \mathscr{D})$ where $\mathscr{J}^\vee$ denotes the dual bundle of $\mathscr{J}$. But the functor

$$u_p : \mathscr{P}_n(X, \mathscr{D}) \to \mathscr{P}_{n-1}(X, \mathscr{D})$$

which sends $\mathscr{F}$ on $\mathscr{F} \otimes \Lambda^p \mathscr{J}^\vee$ is exact and as in [**Q**, 8.2.2] we get a morphism

$$\sum_{p>0} (-1)^{p-1} u_p : K_q(\mathscr{P}_n(X, \mathscr{D})) \to K_q(\mathscr{P}_{n-1}(X, \mathscr{D}))$$

and the canonical morphism

$$K_q(\mathscr{P}_{n-1}(X, \mathscr{D})) \to K_q(\mathscr{P}_n(X, \mathscr{D}))$$

is an isomorphism. By [**Q**, 8.1.12.a], $\mathscr{P}(X, \mathscr{D})$ is the union of the $\mathscr{P}_n(X, \mathscr{D})$ and we get the second isomorphism. The proof of the isomorphisms for $\mathscr{R}_n(X, \mathscr{D})$ is similar. □

Lemma 3.4. — *There exits a sequence of functors*

$$T_i : \mathscr{R}_0(X, \mathscr{D}) \to \mathscr{P}(S, \mathscr{A}^{\otimes i} \otimes \mathscr{D}) \textit{ for } 0 \leqslant i \leqslant d-1$$

such that for any object $\mathscr{F}$ of $\mathscr{R}_0(X, \mathscr{D})$, one has a canonical exact sequence

$$0 \to \mathscr{J}^{\otimes d-1} \otimes_{\mathscr{A}^{\otimes d-1}} T_{d-1}(\mathscr{F}) \to \cdots \to \mathscr{O}_X \otimes_{\mathscr{O}_X} T_0(\mathscr{F}) \to \mathscr{F} \to 0$$

in $\mathscr{R}_0(X, \mathscr{D})$.

Proof. — As in [**Q**, 8.4.2] we define by induction

$$\begin{cases} T_i(\mathscr{F}) & = & f_*\left(\underline{\mathrm{Hom}}_X(\mathscr{J}^{\otimes i}, \mathscr{Z}_{i-1}(\mathscr{F}))\right) \\ \mathscr{Z}_i(\mathscr{F}) & = & \mathrm{Ker}(\mathscr{J}^{\otimes i} \otimes_{\mathscr{A}^{\otimes i}} T_i(\mathscr{F}) \to \mathscr{Z}_{i-1}(\mathscr{F})) \end{cases}$$

with $\mathscr{Z}_{-1}(\mathscr{F}) = \mathscr{F}$. These S-modules $T_i(\mathscr{F})$ have natural left $\mathscr{A}^{\otimes i}$-module and left $\mathscr{D}$-module structures and they are compatible. The end of the proof is then the same as in [**Q**, 8.4.2]. □

End of the proof of proposition 3.1. — As in [**Q**], one can show that these functors T_i yield the inverse morphism for the canonical map

$$\bigoplus_{0\leqslant i\leqslant d-1} K_*(\mathscr{P}(S, \mathscr{A}^{\otimes i} \otimes_{\mathscr{O}_S} \mathscr{D})) \to K_*(\mathscr{P}(X, \mathscr{D})). \quad \square$$

Notation . — From now on we take $S = \mathrm{Spec}\, k$ for a field k. The Azumaya algebras will be denoted by $A_1, \ldots, A_m$. We put $Y = Y_1 \times \cdots \times Y_m$ and denote by $g_{k_1,\ldots,k_m}$ the image in $K_0(Y)$ of the canonical generator of $K_0(A_1^{\otimes k_1} \otimes \cdots \otimes A_m^{\otimes k_m})$. In other words, if the tensor product $A_1^{\otimes k_1} \otimes \cdots \otimes A_m^{\otimes k_m}$ is isomorphic to $M_l(D)$ for a skew-field D, then

$$g_{k_1,\ldots,k_m} = \left[D^l \otimes_{A_1^{\otimes k_1} \otimes \cdots \otimes A_m^{\otimes k_m}} \mathscr{J}_1^{\otimes k_1} \otimes_Y \cdots \otimes_Y \mathscr{J}_m^{\otimes k_m} \right].$$

We get

Corollary 3.5. — *With notation as above,*

$$K_0(Y) = \bigoplus_{\substack{(k_i)_{1\leqslant i\leqslant m} \\ 0\leqslant k_i\leqslant d_i-1}} \mathbf{Z} g_{k_1,\ldots,k_m}$$

and the canonical map

$$K_0(Y) \to K_0(Y \times_k \overline{k})$$

is injective and sends $g_{k_1,\ldots,k_m}$ to

$$\mathrm{ind}(A_1^{\otimes k_1} \otimes \cdots \otimes A_m^{\otimes k_m}) \bigotimes_{i=1}^{m} \pi_i^*(\mathscr{O}_{\overline{Y}_i}(-1)^{\otimes k_i})$$

where $\pi_i : \overline{Y}_1 \times_{\overline{k}} \cdots \times_{\overline{k}} \overline{Y}_n \to \overline{Y}_i$ is the canonical projection.

Proof. — It remains to show the last assertion. Let $A = A_1^{\otimes k_1} \otimes \cdots \otimes A_m^{\otimes k_m}$, D the corresponding skew-field and $\mathscr{J} = \mathscr{J}_1^{\otimes k_1} \otimes \cdots \otimes \mathscr{J}_m^{\otimes k_m}$. We put $r = \operatorname{ind} A = \sqrt{\dim D}$ and $l = \prod_{i=1}^m d_i^{k_i}/r$. The pull-back of $\mathscr{J}$ over $\overline{Y} = Y \times_k \bar{k}$ is

$$\bigotimes_{i=1}^m \pi_i^*(\mathscr{O}_{\overline{Y}_i}(-1)^{\otimes k_i})^{d_i^{k_i}} = \mathscr{O}_{\overline{Y}}^{\prod_{i=1}^m d_i^{k_i}} \otimes \bigotimes_{i=1}^m \pi_i^*(\mathscr{O}_{Y_i}(-k_i)).$$

Thus the image of $g_{k_1,\ldots,k_m}$ in $K_0(\overline{Y})$ is the class of

$$\pi^*\left(M_r\left(\bar{k}\right)^l \otimes_{M_{lr}(\bar{k})} \bar{k}^{lr}\right) \otimes \bigotimes_{i=1}^m \pi_i^*(\mathscr{O}_{Y_i}(-k_i))$$

where $\pi : \overline{Y} \to \bar{k}$ is the canonical morphism. But there are isomorphisms

$$\begin{aligned} M_r\left(\bar{k}\right)^l \otimes_{M_{lr}(\bar{k})} \bar{k}^{lr} &\xrightarrow{\sim} \left(M_r\left(\bar{k}\right) \otimes_{\bar{k}} \bar{k}^l\right) \otimes_{M_r(\bar{k}) \otimes_{\bar{k}} M_l(\bar{k})} \left(\bar{k}^r \otimes_{\bar{k}} \bar{k}^l\right) \\ &\xrightarrow{\sim} \left(M_r\left(\bar{k}\right) \otimes_{M_r(\bar{k})} \bar{k}^r\right) \otimes_{\bar{k}} \left(\bar{k}^l \otimes_{M_l(\bar{k})} \bar{k}^l\right) \\ &\xrightarrow{\sim} \bar{k}^r. \quad \square \end{aligned}$$

3.2. The topological filtration on K_0. — Let us first give a complete description of the filtration by codimension of support in the split case. Let

$$Y = \mathbf{P}_k^{d_1-1} \times \cdots \times \mathbf{P}_k^{d_m-1}.$$

Let $\pi_i : Y \to \mathbf{P}_k^{d_i-1}$ be the canonical projection and $\mathscr{L}_i = \pi_i^*(\mathscr{O}_{\mathbf{P}_k^{d_i-1}}(-1))$. We denote by ξ_i the first Chern class of $\mathscr{L}_i$ in $H^1(Y, \mathscr{K}_1)$ and $z_i = [\mathscr{L}_i] \in K_0(Y)$.

Proposition 3.6. — *In the split case one has that*

(1) the $\mathscr{K}$-cohomology groups are given by

$$H^i(Y, \mathscr{K}_j) = \begin{cases} \{0\} & \text{if } i > j \text{ or } i \geqslant \sum_{l=1}^m d_l - m + 1 \\ \bigoplus_{\substack{\sum_{l=1}^m k_l = i \\ 0 \leqslant k_l \leqslant d_i - 1}} \left(\prod_{i=1}^m \xi_i^{k_i}\right) K_{j-i}(F) & \text{otherwise.} \end{cases}$$

(2) the i-th filtration group for the filtration by codimension of support is the ideal

$$K_*(Y)^i = (1 - z_1, \ldots, 1 - z_n)^i K_*(Y).$$

The first statement is a direct consequence of the following lemma.

Lemma 3.7. — *Let S be a scheme, $E_1,\dots,E_m$ be m vector bundles over S of relative dimension $d_1,\dots,d_m$. Let $Y_i=\mathbf{P}(E_i)$ and $Y=Y_1\times_S\cdots\times_S Y_m$. Let $\pi_i:Y\to Y_i$ be the canonical projection, L_i be the line bundle $\pi_i^*(\mathscr{O}_{Y_i}(-1))$ and ξ_i be the Chern class $c_1(L_i)\in H^1(Y,\mathscr{K}_1)$. Then the bigraded ring $H^*(Y,\mathscr{K}_*)$ is a free $H^*(S,\mathscr{K}_*)$-module with a basis given by*

$$\left(\prod_{1\leqslant i\leqslant n}\xi_i^{k_i}\right)_{\substack{(k_i)_{1\leqslant i\leqslant n}\\ 1\leqslant k_i\leqslant d_i-1}}$$

Proof. — This lemma is given by a straightforward induction from theorem 8.2 in [**Su2**] or theorem 3.1 in [**Sh**]. □

Proof of the second assertion of proposition 3.6. — As in [**Su2**, proposition 9.1], one can show that all differentials d_r in the Brown-Gersten-Quillen spectral sequence

$$H^p(Y,\mathscr{K}_{-q})\Rightarrow K_{p+q}(Y)$$

are zero if $r\geqslant 2$. Indeed, for any $(k_i)_{1\leqslant i\leqslant n}$, with $1\leqslant k_i\leqslant d_i-1$ for $1\leqslant i\leqslant m$ and $\sum_{i=1}^m k_i=p$,

$$\prod_{1\leqslant i\leqslant m}\xi_i^{k_i}\in H^p(Y,\mathscr{K}_p)$$

is in the kernel of d_r. Since d_r is $K_*(F)$-linear, $d_r=0$. On the other hand,

$$\prod_{i=1}^m(1-z_i)^{k_i}=(-1)^{\sum_{i=1}^m k_i}\left[\bigotimes_{i=1}^m\pi_i^*\mathscr{O}_{Z_i}(-k_i)\right]$$

where Z_i is a linear subspace of codimension k_i in $\mathbf{P}_k^{d_i-1}$. Thus

$$\prod_{i=1}^m(1-z_i)^{k_i}\in K_0(Y)^p$$

where $p=\sum_{i=1}^m k_i$. Moreover its class in $K_0(Y)^{(p/p+1)}=K_0(Y)^p/K_0(Y)^{p+1}$ is the image of

$$\prod_{i=1}^m\xi_i^{k_i}\in H^p(Y,\mathscr{K}_p)\tilde{\to}K_0(Y)^{(p/p+1)}.$$

Therefore an induction on i shows that

$$K_*(Y)^i\subset(1-z_1,\dots,1-z_m)^iK_*(Y).\quad\square$$

In the non-split case, we have the following partial results:

Proposition 3.8. — *Let $Y_1, \ldots, Y_m$ be m Severi-Brauer varieties corresponding to Azumaya algebras $A_1, \ldots, A_m$ over k. Let $Y = Y_1 \times \cdots \times Y_m$, $d_i = \dim Y_i$ and k' be a finite field extension of k which splits $Y_1, \ldots, Y_m$. As in section 3.1, $g_{k_1,\ldots,k_m}$ denote the canonical generators of $K_0(Y)$. Then*

(1) If we identify $K_0(Y)$ with its image in $K_0(Y_{k'})$ we get

$$[k' : k] K_0(Y_{k'})^i \bigcap K_0(Y) \subset K_0(Y)^i \subset K_0(Y_{k'})^i \bigcap K_0(Y).$$

(2) The kernel of the canonical surjection $\mathrm{CH}^i(Y) \to K_0(Y)^{(i/i+1)}$ is killed by $[k' : k]$ and $(i-1)!$.

(3) The first step of the filtration is given by

$$\begin{aligned} K_0(Y)^1 &= K_0(\overline{Y})^1 \bigcap K_0(Y) \\ &= \bigoplus_{\substack{(k_i)_{1 \leqslant i \leqslant m} \\ 0 \leqslant k_i \leqslant d_i - 1}} \left(g_{k_1,\ldots,k_m} - \mathrm{ind}\left(\bigotimes_{j=1}^{m} A_j^{\otimes k_j} \right) \right) \mathbf{Z}. \end{aligned}$$

(4) $K_0(Y)^2 = K_0(\overline{Y})^2 \bigcap K_0(Y)$.

(5) One has

$$CH^2(Y)_{\mathrm{tors}} = \mathrm{Ker}(CH^2(Y) \to \mathrm{CH}^2(Y_{k^s}))$$

and it is a finite group.

Proof. — (1). Since the morphism $\pi : Y_{k'} \to Y$ is flat, π^* preserves the topological filtration. Thus $K_0(Y)^i \subset K_0(Y_{k'})^i$. This morphism is also finite and hence π_* preserves the topological filtration. The map $\pi_* \circ \pi^*$ coincides with the multiplication by $[k' : k]$. Thus the composite map

$$K_0(Y_{k'}) \xrightarrow{\pi_*} K_0(Y) \xrightarrow{\pi^*} K_0(Y_{k'})$$

coincides with the multiplication by $[k' : k]$ on the image of $K_0(Y)$. But $K_0(Y_{k'})$ is a free $\mathbf{Z}$-module in which $\pi^*(K_0(Y))$ has finite index. Thus $\pi^* \circ \pi_*$ is also the multiplication by $[k' : k]$. Therefore

$$[k' : k] K_0(Y_{k'})^i = \pi^* \pi_* (K_0(Y_{k'})^i) \subset K_0(Y)^i.$$

(2). The first assertion of (2) is a consequence of the proof of proposition 3.6 (2). Indeed $\mathrm{CH}^i(Y)$ verifies

$$\mathrm{CH}^i(Y) \xrightarrow{\sim} H^i(Y, \mathcal{K}_i) \twoheadrightarrow K_0(Y)^{(i/i+1)}$$

and the morphisms d_r are killed by $[k' : k]$. The second one is a consequence of [**Su2**, proposition 9.3], which asserts that the composite map

$$CH^i(X) \to K_0(X)^{(i/i+1)} \xrightarrow{c_i} CH^i(Y)$$

coincides with the multiplication by $(-1)^i(i-1)!$.

(3). We have

$$\begin{aligned} K_0(Y)^1 &= \operatorname{Ker}(K_0(Y) \xrightarrow{\deg} \mathbf{Z}) \\ &= K_0(Y) \bigcap \operatorname{Ker}(K_0(\overline{Y}) \xrightarrow{\deg} \mathbf{Z}) \\ &= K_0(Y) \bigcap K_0(\overline{Y})^1. \end{aligned}$$

The second equality only uses the fact that, by corollary 3.5

$$\pi^* g_{k_1,\dots,k_m} = \operatorname{ind}\left(\bigotimes_{i=1}^m A_i^{\otimes k_i}\right) \prod_{i=1}^m z_i^{k_i}.$$

(4). Since Y is smooth and proper and $\overline{Y}$ is integral, one has an injection

$$\operatorname{Pic} Y \to \operatorname{Pic} \overline{Y}.$$

Hence the map $K_0(Y)^{(1/2)} \to K_0(\overline{Y})^{(1/2)}$ is an injection and

$$K_0(Y)^2 = K_0(Y) \bigcap K_0(\overline{Y})^2.$$

(5). If k'' is a finite field extension of k then

$$\operatorname{Ker}(\mathrm{CH}^2(Y) \to \mathrm{CH}^2(Y_{k'}))$$

is killed by $[k'' : k]$. Thus

$$\operatorname{Ker}(\mathrm{CH}^2(Y) \to \mathrm{CH}^2(Y_{k^s})) \subset \mathrm{CH}^2(Y)_{\mathrm{tors}}.$$

The other inclusion follows from proposition 3.6 which implies that $\mathrm{CH}^2(Y_{k'})_{\mathrm{tors}}$ is trivial. Since $K_0(Y)^2$ is finitely generated,

$$\mathrm{CH}^2(Y)_{\mathrm{tors}} \xrightarrow{\sim} K_0(Y)^{(2/3)}_{\mathrm{tors}}$$

is finite. □

As a corollary we give a result proved by Gabber in a letter to Colliot-Thélène using a slightly different method. Here $A_0(Y)$ denotes the kernel of the degree map $\mathrm{CH}_0(Y) \to \mathbf{Z}$.

Corollary 3.9. — *Let Y_1 and Y_2 be two conics. Then*

$$A_0(Y_1 \times Y_2) = \mathrm{CH}^2(Y_1 \times Y_2)_{\mathrm{tors}} = \{0\}.$$

Proof. — Here $\dim(Y) = 2$ and $\mathrm{CH}_0(Y) \xrightarrow{\sim} \mathrm{CH}^2(Y)$. By proposition 3.6, the group $\mathrm{CH}^2(\overline{Y}) = \mathbf{Z}$. Hence

$$A_0(Y) = \mathrm{CH}_0(Y)_{\mathrm{tors}} = \mathrm{CH}^2(Y)_{\mathrm{tors}}.$$

But in this case $K_0(\overline{Y})^3 = \{0\}$. Thus

$$K_0(Y)^3 = K_0(Y) \bigcap K_0(\overline{Y})^3 = \{0\}$$

and $\mathrm{Ker}(\mathrm{CH}^2(Y) \to \mathrm{CH}^2(\overline{Y})) = \{0\}$ which yields the second equality. □

4. The complex $\mathscr{C}_\infty$ and the second Chow group

4.1. The main tool. — We shall now prove the following theorem which is the main tool in the rest of this text.

Theorem 4.1. — *Let $Y_1, \ldots, Y_m$ be m-Severi-Brauer varieties. Let $[Y_1], \ldots, [Y_m]$ be their classes in* $\mathrm{Br}\,k$ *and $U \subset \mathrm{Br}\,k$ the subgroup generated by these classes. Let $Y = Y_1 \times_k \cdots \times_k Y_m$ and M be its function field. Then $\mathrm{CH}^2(Y)_{\mathrm{tors}}$ is canonically isomorphic to the homology group of the complex*

$$k^* \otimes U \to H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(M, \mathbf{Q}/\mathbf{Z}(2)).$$

In particular this homology is finite.

Remark 4.1. — If $U \subset \mathrm{Br}(k)_{(n)}$ and the characteristic of k does not divide n then there is a commutative diagram

$$\begin{tikzcd}
k^* \otimes U \arrow[r] \arrow[d] & H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \arrow[r] & H^3(M, \mathbf{Q}/\mathbf{Z}(2)) \\
H^1(k, \mu_n) \otimes U \arrow[r] & H^3(k, \mu_n^{\otimes 2}) \arrow[r] \arrow[u] & H^3(M, \mu_n^{\otimes 2}) \arrow[u]
\end{tikzcd}$$

By the main theorem of Merkur′ev and Suslin [**MS1**], if m and n are prime to the exponent characteristic of k, the map $H^2(k, \mu_{nm}^{\otimes 2}) \to H^2(k, \mu_m^{\otimes 2})$ is surjective and hence $H^3(k, \mu_n^{\otimes 2}) \to H^3(k, \mathbf{Q}/\mathbf{Z}(2))$ is injective. A diagram chase which uses this injectivity and the surjectivity $k^* \otimes U \to H^1(k, \mu_n) \otimes U$ gives a canonical injection from the homology of the second line to that of the first. We shall give in section 6.2 an example where this injection is not surjective.

Theorem 4.1 is a consequence of the following theorem

Theorem 4.2 **(Kahn,** [**Kah2**, corollaire 3.2]**).** — *Let X be a geometrically integral variety over a field E. We denote by $\mathcal{G}$ the absolute Galois group of E. Then there is a canonical isomorphism*

$$H^1\left(\mathcal{G}, K_2(E^s(X))/K_2(E^s)\right) \xrightarrow{\sim} \operatorname{Ker}\left(H^3(E, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(E(X), \mathbf{Q}/\mathbf{Z}(2))\right).$$

Proof of theorem 4.1. — Let $\mathcal{G}$ be the absolute Galois group of k. According to [**CTR**, proposition 3.6], one has an exact sequence

$$\begin{aligned} H^1(Y_{k^s}, \mathcal{K}_2)^{\mathcal{G}} &\to H^1\left(\mathcal{G}, K_2(k^s(Y))/H^0(Y_{k^s}, \mathcal{K}_2)\right) \to \\ &\to \operatorname{Ker}(\mathrm{CH}^2(Y) \to \mathrm{CH}^2(Y_{k^s})) \to H^1(\mathcal{G}, H^1(Y_{k^s}, \mathcal{K}_2)). \end{aligned}$$

By proposition 3.6, $H^0(Y_{k^s}, \mathcal{K}_2) \xrightarrow{\sim} K_2(k^s)$ and $H^1(Y_{k^s}, \mathcal{K}_2) \xrightarrow{\sim} k^{s*m}$. Therefore $H^1(Y_{k^s}, \mathcal{K}_2)^{\mathcal{G}} \xrightarrow{\sim} k^{*m}$ and by Hilbert's theorem 90 $H^1(\mathcal{G}, H^1(Y_{k^s}, \mathcal{K}_2)) = 0$. By proposition 3.8 (5),

$$\mathrm{CH}^2(Y)_{\mathrm{tors}} = \operatorname{Ker}(\mathrm{CH}^2(Y) \to \mathrm{CH}^2(Y_{k_s}))$$

Thus we get an exact sequence

$$k^{*m} \to H^1(\mathcal{G}, K_2(k^s(Y))/K_2k^s) \to \mathrm{CH}^2(Y)_{\mathrm{tors}} \to 0.$$

It remains to prove the following lemma

Lemma 4.3. — *The composite map*

$$k^{*m} \to H^1\left(\mathcal{G}, K_2(k^s(Y))/K_2k^s\right) \to H^3(k, \mathbf{Q}/\mathbf{Z}(2))$$

fits into a diagram of the form

$$\begin{tikzcd} k^{*m} \arrow[d] \arrow[dr] & \\ k^* \otimes U \arrow[r, "\cup"] & H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \end{tikzcd}$$

where the map $k^{*m} \to k^* \otimes U$ is surjective.

Proof. — Let us first recall the construction of the composite map

$$k^{*m} \xrightarrow{f} H^1(Y_{k^s}, \mathcal{K}_2)^{\mathcal{G}} \xrightarrow{g} H^1\left(\mathcal{G}, K_2(k^s(Y))/K_2(k^s)\right) \xrightarrow{h} H^3(k, \mathbf{Q}/\mathbf{Z}(2)).$$

The map f is induced by the composite map

$$k^{*m} \to k^{s*m} \xrightarrow{\sim} \left(\bigoplus_{i=1}^{m} \xi_i \mathbf{Z}\right) \otimes K_1(k^s) \xrightarrow{\sim} H^1(Y_{k^s}, \mathcal{K}_1) \otimes H^0(Y_{k^s}, \mathcal{K}_1) \xrightarrow{\sim} H^1(Y_{k^s}, \mathcal{K}_2).$$

The map g is the coboundary map for the short exact sequence of $\mathscr{G}$-modules

$$0 \to K_2 k^s(Y)/H^0(Y_{k^s}, \mathscr{K}_2) \to \mathscr{Z} \to H^1(Y_{k^s}, \mathscr{K}_2) \to 0$$

where $\mathscr{Z}$ is the kernel of the canonical map

$$\bigoplus_{x \in Y^{(1)}_{k^s}} k(x)^* \to \bigoplus_{x \in Y^{(2)}_{k^s}} \mathbf{Z}.$$

Similarly there is a canonical map $\sigma : \mathbf{Z}^m \to H^1(\mathscr{G}, K_1(k^s(Y))/K_1 k^s)$ defined as the composite map

$$\mathbf{Z}^m \tilde{\to} H^1(Y_{k^s}, \mathscr{K}_1)^{\mathscr{G}} \to H^1(\mathscr{G}, K_1(k^s(Y))/K_1 k^s)$$

where the second map is the coboundary map for the short exact sequence of $\mathscr{G}$-modules

$$0 \to k^s(Y)^*/k^{s*} \to \operatorname{Div} Y_{k^s} \to \operatorname{Pic} Y_{k^s} \to 0.$$

Since $H^1(\mathscr{G}, \operatorname{Div} Y_{k^s}) = \{0\}$ the map σ is surjective.

For the map h we need to recall some facts about Lichtenbaum complexes $\Gamma(i) = \Gamma(i, L^s)$ for $i \leqslant 2$ and a fixed field L (See [**Li1**], [**Li2**], [**Li3**] and [**Kah2**]). The complex $\Gamma(0)$ is $\mathbf{Z}$ in degree 0 and $\Gamma(1)$ is L^{s*} in degree 1. The complex $\Gamma(2)$ is acyclic outside $[1, 2]$. As in [**Kah2**, §3], if $i \leqslant 2$, we consider $C^\bullet(L, \Gamma(i))$ the total complex for the bicomplex

$$\bigoplus_{j,l \in \mathbf{N}} C^j(\operatorname{Gal}(L^s/L), \Gamma(i, L^s)^l).$$

and for any extension of fields F/E such that E is algebraically closed in F, the cokernel $C(F/E, \Gamma(i))$ of the morphism

$$C(E, \Gamma(i))[1] \to C(F, \Gamma(i))[1].$$

The homology of this complex is denoted by $\mathbf{H}^j(F/E, \Gamma(i))$ and is called the relative hypercohomology of F/E with value in $\Gamma(i)$. One has a canonical long exact sequence

$$\cdots \to \mathbf{H}^j(E, \Gamma(i)) \to \mathbf{H}^j(F, \Gamma(i)) \to \mathbf{H}^{j+1}(F/E, \Gamma(i)) \to \mathbf{H}^{j+1}(E, \Gamma(i)) \cdots$$

which yields isomorphisms

$$\mathbf{H}^j(F/E, \Gamma(i)) = 0 \text{ if } j \leqslant 1 \text{ and } i = 1 \text{ or } 2, \tag{4.1}$$

$$\mathbf{H}^2(F/E, \Gamma(1)) \tilde{\to} F^*/E^*, \tag{4.2}$$

$$\mathbf{H}^3(F/E, \Gamma(1)) \tilde{\to} \mathrm{Ker}(\mathrm{Br}(E) \to \mathrm{Br}(F)), \tag{4.3}$$

$$\mathbf{H}^2(F/E, \Gamma(2)) \tilde{\to} K_3(F)_{\mathrm{ind}}/K_3(E)_{\mathrm{ind}}, \tag{4.4}$$

$$\mathbf{H}^3(F/E, \Gamma(2)) \tilde{\to} K_2(F)/K_2(E) \tag{4.5}$$

and

$$\mathbf{H}^4(F/E, \Gamma(2)) \tilde{\to} \mathrm{Ker}\left(H^3(E, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(F, \mathbf{Q}/\mathbf{Z}(2))\right). \tag{4.6}$$

If E' is a Galois extension of E and $F' = E'F$ then there is a Hochschild-Serre spectral sequence

$$H^p\left(G, \mathbf{H}^q(F'/E', \Gamma(i))\right) \Rightarrow \mathbf{H}^{p+q}(F/E, \Gamma(i))$$

where $G = \mathrm{Gal}(E'/E)$. By (4.1) this spectral sequence for $i = 1$ yields a morphism

$$H^1\left(\mathscr{G}, \mathbf{H}^2(k^s(Y)/k^s, \Gamma(1))\right) \to \mathbf{H}^3(k(Y)/k, \Gamma(1))$$

which is an isomorphism, as $\mathrm{Br}(k^s) = 0$. By (4.4), since $K_3(k^s(Y))_{\mathrm{ind}}/K_3(k^s)_{\mathrm{ind}}$ is uniquely divisible (See [**MS2**]), this spectral sequence also yields a morphism

$$H^1\left(\mathscr{G}, \mathbf{H}^3(k^s(Y)/k^s, \Gamma(2))\right) \to \mathbf{H}^4(k(Y)/k, \Gamma(2))$$

which by (4.5) and (4.6) gives the map

$$H^1(\mathscr{G}, K_2(k^s(Y))/K_2(k^s)) \to \mathrm{Ker}\left(H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(Y), \mathbf{Q}/\mathbf{Z}(2))\right).$$

For any $a \in k^*$, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{Z} \arrow[r, "a^{\cdot}"] \arrow[d, "\xi_i"] & k^* \arrow[d] \\
H^1(Y_{k^s}, \mathcal{K}_1)^{\mathcal{G}} \arrow[r, "\cup a"] \arrow[d, two heads] & H^1(Y_{k^s}, \mathcal{K}_2)^{\mathcal{G}} \arrow[d] \\
H^1(\mathcal{G}, K_1(k^s(Y))/H^0(Y_{k^s}, \mathcal{K}_1)) \arrow[r, "\cup a"] \arrow[d, "\wr"] & H^1(\mathcal{G}, K_2(k^s(Y))/H^0(Y_{k^s}, \mathcal{K}_2)) \arrow[d, "\wr"] \\
H^1\left(\mathcal{G}, \mathbf{H}^2(k^s(Y)/k^s, \Gamma(1))\right) \arrow[r, "\cup a"] \arrow[d, "\wr"] & H^1\left(\mathcal{G}, \mathbf{H}^3(k^s(Y)/k^s, \Gamma(2))\right) \arrow[d] \\
\mathbf{H}^3(k(Y)/k, \Gamma(1)) \arrow[r, "\cup a"] \arrow[d, "\wr"] & \mathbf{H}^4(k(Y)/k, \Gamma(2)) \arrow[d, "\wr"] \\
\operatorname{Ker}\left(H^2(k, \mathbf{Q}/\mathbf{Z}(1)) \to H^2(k(Y), \mathbf{Q}/\mathbf{Z}(1))\right) \arrow[r, "\cup a"] & \operatorname{Ker}\left(H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(Y), \mathbf{Q}/\mathbf{Z}(2))\right)
\end{tikzcd}$$

where a is respectively seen as an element of k^*, $H^0(Y_{k^s}, \mathcal{K}_1)^{\mathcal{G}}$, $H^0(\mathcal{G}, \mathbf{H}^1(k^s, \Gamma(1)))$, $\mathbf{H}^1(k, \Gamma(1))$ and $H^1(k, \hat{\mathbf{Z}}(1))$. But the column on the right side yields a surjective morphism

$$\mathbf{Z}^m \to \operatorname{Ker}\left(H^2(k, \mathbf{Q}/\mathbf{Z}(1)) \to H^2(k(Y), \mathbf{Q}/\mathbf{Z}(1))\right) = U. \quad \square$$

4.2. First applications. — The following result was proved by Arason for a quaternion algebra and by Colliot-Thélène using a result of Merkur′ev and Suslin for a cyclic central simple algebra of prime index.

Proposition 4.4. — *Let A be a central simple algebra over k such that the quotient of the index of A by its exponent is squarefree and for any prime p dividing this quotient the p-primary component of the corresponding division algebra is decomposable. Denote by $[A]$ its class in* $\operatorname{Br} k$ *and by Y the corresponding Severi-Brauer variety then*

$$\operatorname{Ker}\left(H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(k(Y), \mathbf{Q}/\mathbf{Z}(2))\right) = [A] \cup H^1(k, \mathbf{Q}/\mathbf{Z}(1)).$$

Proof. — According to [**Kar**], $K_0(Y)^{(2/3)} \xrightarrow{\sim} \mathbf{Z}$ and therefore $\mathrm{CH}^2(Y)$ has no torsion. This result is then a direct consequence of theorem 4.1. $\square$

Corollary 4.5. — *If A is the product of two quaternion algebras over a field of characteristic different from two, then the complex*

$$(\mathscr{C}_2) \qquad \bigoplus_{u\in U} A_u^* \xrightarrow{\mathrm{Nrd}} H^1(k)\otimes U \xrightarrow{\cup} H^3(k) \to H^3(M)$$

is exact.

Proof. — In this case, $\mathrm{ind}(A)/\exp(A) \leqslant 2$ and A is decomposable. Therefore proposition 4.4 gives the exactness at the second term of the complex. The exactness at the first is due to Knus, Lam, Shapiro and Tignol [**KLST**]. □

5. The case of two conics

5.1. The result. — From now on k is a field of characteristic different from 2. As before, we shall omit the coefficients in the cohomology groups when they are equal to $\mathbf{Z}/2\mathbf{Z}$. The purpose of this section is to show the following result

Theorem 5.1. — *Let k be a field of characteristic different from 2 and a_1, b_1, a_2, b_2 belong to k^*. We denote by Y_i the conic defined by the homogeneous equations*

$$X_1^2 - a_i X_2^2 - b_i X_3^2 = 0$$

for $i = 1$ or 2, $Y = Y_1 \times Y_2$, M the function field of Y and U the subgroup of $H^2(k)$ generated by (a_1, b_1) and (a_2, b_2). For any $u \in U$, let A_u be a simple central algebra which represents u. Then the complex

$$(\mathscr{C}_2) \qquad \bigoplus_{u\in U} A_u^* \xrightarrow{\mathrm{Nrd}} H^1(k)\otimes U \xrightarrow{\cup} H^3(k) \to H^3(M)$$

is exact.

5.2. Exactness of the first part of the complex. — If $a_1, \dots, a_n$ belong to k then the corresponding Pfister form is defined by

$$\langle\!\langle a_1, \dots, a_n \rangle\!\rangle = < 1, -a_1 > \otimes \cdots \otimes < 1, -a_n > .$$

Lemma 5.2. — *Let k be a field of characteristic different from 2 and let $a_1, \dots, a_{n-1}$, $b_1, \dots, b_{n-1}$ belong to k^*. Assume that for any $c_1, \dots, c_n, d_1, \dots, d_n$ in k^* the equality*

$$(c_1, \dots, c_n) = (d_1, \dots, d_n)$$

in $H^n(k)$ implies that the Pfister forms $\langle\!\langle c_1, \dots, c_n \rangle\!\rangle$ and $\langle\!\langle d_1, \dots, d_n \rangle\!\rangle$ are isomorphic. Then for any $x, y \in k^*$ such that

$$(a_1, \dots, a_{n-1}, x) = (b_1, \dots, b_{n-1}, y)$$

in $H^n(k)$ there exists $u \in k^*$ such that

$$(a_1,\dots,a_{n-1},x) = (a_1,\dots,a_{n-1},u) = (b_1,\dots,b_{n-1},u) = (b_1,\dots,b_{n-1},y).$$

Remark 5.1. — By [**AEJ**, theorem 1] or [**JR**, page 554], the assumption is verified for any field if $n \leqslant 4$. This was proved by Merkur′ev for $n = 3$ and by Arason, Elman and Jacob for $n = 4$.

Proof. — By assumption we have that

$$\langle\langle a_1,\dots,a_{n-1},x\rangle\rangle \cong \langle\langle b_1,\dots,b_{n-1},y\rangle\rangle$$

as quadratic forms. Let q_1 be the pure subform of $\langle\langle a_1,\dots,a_{n-1}\rangle\rangle$. It is characterized by the isomorphism

$$<1> \oplus q_1 \cong \langle\langle a_1,\dots,a_{n-1}\rangle\rangle.$$

Let q_2 be the pure subform of the Pfister form $\langle\langle b_1,\dots,b_{n-1}\rangle\rangle$ and q the pure subform of the form $\langle\langle a_1,\dots,a_{n-1},x\rangle\rangle$. Then one has the isomorphisms

$$\begin{aligned} -q &\cong -q_1 \oplus x\langle\langle a_1,\dots,a_{n-1}\rangle\rangle \\ &\cong -q_2 \oplus y\langle\langle b_1,\dots,b_{n-1}\rangle\rangle. \end{aligned}$$

Let X_i be coordinates corresponding to the first decomposition and Y_i coordinates corresponding to the second one. Let V be the subspace given by the equations

$$X_1 = \cdots = X_{2^{n-1}-1} = Y_1 = \cdots = Y_{2^{n-1}-1} = 0$$

then $\dim V \geqslant 2^n - 1 - 2(2^{n-1} - 1) = 1$. If $q_{|V}$ is isotropic then $\langle\langle a_1,\dots,a_{n-1}\rangle\rangle$ is isotropic and $\langle\langle a_1,\dots,a_{n-1}\rangle\rangle = 0$ in $W(k)$. By [**Ar**, Satz 1.6], $(a_1,\dots,a_{n-1}) = 0$ and $u = y$ verifies the conclusion of the lemma. Otherwise let $v \in V - \{0\}$ and $u = -q(v)$. Then $\langle\langle a_1,\dots,a_{n-1}\rangle\rangle$ represents ux^{-1} and $\langle\langle b_1,\dots,b_{n-1}\rangle\rangle$ represents uy^{-1}. By [**Lam**, chapter 10, corollary 1.6], we get that

$$\begin{aligned} \langle\langle a_1,\dots,a_{n-1},ux^{-1}\rangle\rangle &\cong \langle\langle b_1,\dots,b_{n-1},uy^{-1}\rangle\rangle \\ &\cong 0 \end{aligned}$$

in $W(k)$ and therefore

$$(a_1,\dots,a_{n-1},x) = (a_1,\dots,a_{n-1},u) = (b_1,\dots,b_{n-1},u) = (b_1,\dots,b_{n-1},y). \quad \square$$

Proof of the exactness of the first part of the complex. — Let

$$\alpha \in \operatorname{Ker}\left(H^1(k) \otimes U \xrightarrow{\cup} H^3(k)\right).$$

Then α may be written as

$$\alpha = x \otimes (a_1, b_1) + y \otimes (a_2, b_2)$$

with $x, y \in H^1(k)$ such that

$$(x, a_1, b_1) = (y, a_2, b_2).$$

By lemma 5.2, there exists $u \in k^*$ such that

$$(x, a_1, b_1) = (u, a_1, b_1) = (u, a_2, b_2) = (y, a_2, b_2).$$

By the theorem of Knus, Lam, Shapiro and Tignol [**KLST**], one has an exact sequence

$$\left(\left(\frac{a_1, b_1}{k}\right) \otimes \left(\frac{a_2, b_2}{k}\right)\right)^* \xrightarrow{\mathrm{Nrd}} H^1(k) \xrightarrow{\cup((a_1,b_1)+(a_2,b_2))} H^3(k).$$

Therefore

$$u \in \mathrm{Nrd}\left(\left(\left(\frac{a_1, b_1}{k}\right) \otimes \left(\frac{a_2, b_2}{k}\right)\right)^*\right).$$

By Merkur′ev and Suslin's result [**MS1**, corollary 12.1]

$$x/u \in \mathrm{Nrd}\left(\frac{a_1, b_1}{k}\right)^* \text{ and } y/u \in \mathrm{Nrd}\left(\frac{a_2, b_2}{k}\right)^*$$

and we get

$$\alpha \in \mathrm{Im}\left(\bigoplus_{u \in U} A_u^* \xrightarrow{\mathrm{Nrd}} H^1(k) \otimes U\right)$$

as wanted. □

5.3. Exactness of the second part of the complex. — This is a direct corollary of corollary 3.9 and theorem 4.1. However using the Lichtenbaum complex in this case is like shooting sparrows with cannons. Therefore we shall now give a direct proof of 4.1 in this particular case.

Lemma 5.3. — *With the notation of theorem 5.1 the sequence*

$$H^1(k) \otimes U \to H^3(k) \to H^3(M)$$

is exact.

Proof. — We denote by M_1 the field $k(Y_1)$. Let $\alpha \in \mathrm{Ker}(H^3(k) \to H^3(M))$. By Arason's theorem [**Ar**, Satz 5.4], the image α_{M_1} of α may be written as $\alpha = (a_2, b_2, y)$ with $y \in M_1^*$. But $\alpha_{M_1} \in H^3_{\mathrm{nr}/k}(M_1)$. And by [**CTO**, proposition 13], for any $P \in Y_1^{(1)}$,

$$\partial_P(\alpha) = v_P(y)(a_2, b_2)_{k(P)}.$$

Therefore $v_P(y)$ is even for all $P \in Y_1^{(1)}$ such that $(a_2, b_2)_{k(P)} \neq 0$. We have

$$\mathrm{CH}_0(Y_1) = \mathrm{Pic}(Y_1) = \mathbf{Z}$$

thus $\mathrm{Ker}(\pi_{1*} : \mathrm{CH}_0(Y_1 \times Y_2) \to \mathrm{CH}_0(Y_1))$ coincides with $A_0(Y_1 \times Y_2)$ and by corollary 3.9 is trivial. We may then apply propositions 2.1 and 4.2 of [**CTS**] and we get that

$$y \in k^* \,\mathrm{Nrd}\left(\frac{a_2, b_2}{M_1}\right)^*.$$

Thus there exists $z \in k^*$ such that $\alpha_{M_1} = (a_2, b_2, z)_{M_1}$. Therefore

$$\alpha - (a_2, b_2, z) \in \mathrm{Ker}(H^3(k) \to H^3(M_1))$$

and applying Arason's theorem once more, we get the result. □

6. The case of three conics

6.1. Generalities

Proposition 6.1. — *Let k be a field of characteristic different from 2 and let a_1, b_1, a_2, b_2, a_3 and b_3 belong to k^*. As above Y_i denotes the conic defined by the form $< 1, -a_i, -b_i >$, $Y = Y_1 \times Y_2 \times Y_3$, M the function field of Y and U the subgroup of $H^2(k)$ generated by the symbols (a_i, b_i), $1 \leqslant i \leqslant 3$. Let $D_i = \left(\frac{a_i, b_i}{k}\right)$ be the corresponding algebra for $1 \leqslant i \leqslant 3$. Then the homology of the complex*

$$k^* \otimes U \to H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(M, \mathbf{Q}/\mathbf{Z}(2))$$

has order 1 or 2. It is equal to $\{0\}$ if and only if the three algebras $\left(\frac{a_i, b_i}{k}\right)$ for $1 \leqslant i \leqslant 3$ are split by a field extension of degree $d_0 m$ where

$$d_0 = \operatorname*{lcm}_{i \in \{0,1\}^3} \left(\mathrm{ind}\left(\bigotimes_{j=1}^{3} D_j^{\otimes i_j}\right)\right)$$

and m is an odd number.

Proof. — By theorem 4.1, the homology of the complex is isomorphic to the group $\mathrm{CH}^2(Y)_{\mathrm{tors}}$, which by proposition 3.8 (5) is equal to $\mathrm{Ker}(\mathrm{CH}^2(Y)\to \mathrm{CH}^2(\overline{Y}))$ and hence by proposition 3.8 (2) and (4) to

$$\mathrm{Ker}(K_0(Y)^{(2/3)}\to K_0(\overline{Y})^{(2/3)})\tilde{\to} K_0(\overline{Y})^3\bigcap K_0(Y)/K_0(Y)^3.$$

But here, with the notation of section 3.2, $K_0(\overline{Y})^3$ is generated by

$$(1-z_1)(1-z_2)(1-z_3)$$

and by corollary 3.5 the image of g_{i_1,i_2,i_3} in $K_0(\overline{Y})$ is $\mathrm{ind}(\bigotimes_{j=1}^3 D_j^{\otimes i_j})\prod_{j=1}^3 z_j^{i_j}$. Therefore, d_0 being defined by the formula of the proposition, one has

$$\begin{aligned}K_0(Y)\bigcap K_0(\overline{Y})^3 &= d_0(1-z_1)(1-z_2)(1-z_3)\mathbf{Z}\\ &= \sum_{i\in\{0,1\}^3}\frac{(-1)^{\sum_{j=1}^3 i_j}d_0}{\mathrm{ind}\left(\bigotimes_{j=1}^3 D_j^{\otimes i_j}\right)}g_i\mathbf{Z}.\end{aligned}$$

Let $D=D_1\otimes D_2\otimes D_3$. Let $d=\mathrm{ind}(D)$ and k' be an extension of k of degree d which splits D. Then over k' we have

$$(a_1,b_1)+(a_2,b_2)=(a_3,b_3).$$

By [**Lam**, chapter 11, lemma 4.11], there exist $t,x,y\in k'^*$ such that $(a_1,b_1)=(t,x)$ and $(a_2,b_2)=(t,y)$. Then $k'(\sqrt{t})$ splits D_1, D_2 and D_3. By proposition 3.8 (1),

$$2d(1-z_1)(1-z_2)(1-z_3)\in K_0(Y)^3$$

thus the order of the homology group divides $2d/d_0\leqslant 2$.

It remains to show that the last assertion of the proposition is equivalent to

$$(*)\qquad \sum_{i\in\{0,1\}^3}\frac{(-1)^{\sum_{j=1}^3 i_j}d_0}{\mathrm{ind}\left(\bigotimes_{j=1}^3 D_j^{\otimes i_j}\right)}g_i\in K_0(Y)^3.$$

Let us first show that the last assertion of the proposition implies $(*)$. Let k' be a finite field extension which splits D_1, D_2 and D_3. Then by proposition 3.8 (1)

$$[k':k](1-z_1)(1-z_2)(1-z_3)\in K_0(Y)^3.$$

Since $k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$ splits D_1, D_2 and D_3, we have

$$8(1-z_1)(1-z_2)(1-z_3) \in K_0(Y)^3.$$

and the last assertion of the proposition implies

$$d_0 m(1-z_1)(1-z_2)(1-z_3) \in K_0(Y)^3$$

Since m is odd, this implies $(*)$.

We now show the converse. Assume that $(*)$ is true. Since the composite map

$$\mathrm{CH}^3(Y) \twoheadrightarrow K_0(Y)^{(3/4)} \xrightarrow{\sim} K_0(Y)^3$$

is surjective, we may take α in the inverse image of $\{d_0(1-z_1)(1-z_2)(1-z_3)\}$. The degree of α is then d_0. Thus there exists a closed point P in Y such that $[k(P):k] = d_0 m$ for an odd number m. Let P_i be the projection of P on Y_i for $i = 1, 2$ or 3. The field $k(P_i)$ is a subfield of $k(P)$ and splits D_i. Thus $k(P)$ splits the algebras D_1, D_2 and D_3. □

Corollary 6.2. — *If* $\mathrm{ind}(D_1 \otimes D_2 \otimes D_3) = 8$ *or if* $a_1 = a_2 = a_3$ *then the complex*

$$k^* \otimes U \to H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(M, \mathbf{Q}/\mathbf{Z}(2))$$

is exact.

The cases $\mathrm{ind}(D_1 \otimes D_2 \otimes D_3) = 8$ and $a_1 = a_2 = a_3$ may be seen as the extremal ones.

Proof. — In the first case $d_0 = 8$ and $k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$ splits D_1, D_2 and D_3. In the second case, namely $a_1 = a_2 = a_3$, either the three conics are split and the result is trivial or $d_0 = 2$ and $k(\sqrt{a_1})$ splits D_1, D_2 and D_3. □

6.2. A counterexample for $\mathscr{C}_\infty$. —

Proposition 6.3. — *Let k be a field of characteristic different from* 2 *and containing a fourth root of the unity. Let a, b, c belong to k^* and assume that the symbol (a, b, c) in $H^3(k)$ is not trivial. We use the notation of section 6.1 with*

$$a_1 = b_3 = a,\ a_2 = b_1 = b \text{ and } a_3 = b_2 = c.$$

then the homology group of the complex

$$k^* \otimes U \to H^3(k, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(M, \mathbf{Q}/\mathbf{Z}(2))$$

is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ *and generated by the class of an element of order 4 in* $H^3(k, \mathbf{Q}/\mathbf{Z}(2))$.

Remark 6.1. — This also gives an example where $\mathrm{CH}^2(Y)_{\mathrm{tors}} \neq \{0\}$.

Proof. — Let $M_1 = k(Y_1)$. The conic Y_1 is defined by the homogeneous equation

$$X_1^2 - aX_2^2 - bX_3^2 = 0.$$

Let $X = \frac{X_2}{X_1}$, $Y = \frac{X_3}{X_1}$ and $\alpha = (a, Y) + (b, X)$. The only points where α may have non trivial residues are

$$P_1 : X_1 = 0,\ P_2 : X_2 = 0 \text{ and } P_3 : X_3 = 0.$$

But

$$\partial_{P_1}(\alpha) = v_{P_1}(Y)(a) + v_{P_1}(X)(b) = (ab)$$

and $k(P_1) = k\left(\sqrt{-\frac{a}{b}}\right) = k(\sqrt{ab})$ since there is a fourth root of unity i in k.

$$\partial_{P_2}(\alpha) = v_{P_2}(X)(b) = (b)$$

and $k(P_2) = k(\sqrt{b})$. Similarly $\partial_{P_3}(\alpha) = (a) = 0$. Therefore $\alpha \in H^2_{\mathrm{nr}/k}(k(Y_1))$. Let $K = k\left(\frac{X_2}{X_1}\right)$ then

$$\begin{aligned} M &= K\left(\sqrt{\frac{1}{b} - \frac{a}{b}\left(\frac{X_2}{X_1}\right)^2}\right) \\ &= K\left(\frac{X_3}{X_1}\right). \end{aligned}$$

We have the following formula

$$\begin{aligned} \mathrm{Cores}_K^M(\alpha) &= \mathrm{Cores}_K^M((a, Y)) \\ &= (a, -Y^2) \\ &= \left(a, -\frac{1}{b} + \frac{a}{b}\left(\frac{X_2}{X_1}\right)^2\right) \end{aligned}$$

Since $i \in k$, we get

$$\mathrm{Cores}_K^M(\alpha) = \left(-\frac{1}{b}\frac{a}{b}, -\frac{1}{b} + \frac{a}{b}\left(\frac{X_2}{X_1}\right)^2\right).$$

By [**Lam**, chapter 10, proposition 1.3]

$$\mathrm{Cores}_K^M(\alpha) = \left(-\frac{1}{b}, \frac{a}{b}\right) = (a, b) \neq 0.$$

Thus α does not belong to the image of $H^2(k)$. By remark 2.2, if we denote by $(a)_4$ the image of a in $H^1(k,\mu_4)$ and by $(a,b)_4$ the cup-product $(a)_4 \cup (b)_4$ in $H^2(k,\mu_4^{\otimes 2})$, which is isomorphic to $H^2(k,\mu_4)$ by the choice of i, we get that

$$\alpha - \mathrm{Res}((a,b)_4) \in \mathrm{Im}(H^2(k) \to H^2(M_1))$$

Therefore α is the image of an element $\tilde{\alpha}$ of order four in $H^2(k,\mathbf{Q}/\mathbf{Z}(1))$ and such that $\tilde{\alpha} - (a,b)_4$ belongs to $H^2(k)$. Let $\beta = \tilde{\alpha} \cup (c)_4$ in $H^3(k,\mu_4^{\otimes 2})$.

$$\beta_{M_1} = \alpha \cup (c)_2 = (a,Y,c) + (b,X,c).$$

Therefore $\beta_{M_1} \in \mathrm{Ker}(H^3(M_1) \to H^3(M))$ and

$$\beta \in \mathrm{Ker}\left(H^3(k,\mathbf{Q}/\mathbf{Z}(2)) \to H^3(M,\mathbf{Q}/\mathbf{Z}(2))\right).$$

However $\beta - (a,b,c)_4$ is of order at most two. Since $(a) \cup (b) \cup (c)$ is not trivial, β is of order four and does not belong to

$$\mathrm{Im}\left(U \otimes k^* \xrightarrow{\cup} H^3(k,\mathbf{Q}/\mathbf{Z}(2))\right).$$

Hence the order of the homology group is bigger than two and by corollary 6.1 is equal to two. □

Remark 6.2. — This proof gives also that the homology is generated by an element α in $H^3(k,\mu_4^{\otimes 2})$ such that

$$\alpha - (a,b,c)_4 \in H^3(k).$$

Remark 6.3. — The image by the corestriction map of an element in $H^n_{\mathrm{nr}/k}(M_1)$ is in $H^n_{\mathrm{nr}/k}(K)$ and by proposition 1.2 comes from a unique element in $H^n(k)$. Let N' be the induced morphism from $H^n_{\mathrm{nr}/k}(M_1)$ to $H^n(k)$. We put $\gamma = \mathrm{Res}(a,b)_4$. By the preceding proof we see that $N'(\gamma) = (a,b)$ and N' is trivial on the image of $H^2(k)$. therefore N' coincides with N in degree two if -1 is a square. In degree 3, N' is trivial on $\mathrm{Im}\,\mathrm{Res} = \mathrm{Ker}\,N$ and coincides with N on $\gamma \cup \mathrm{Res}\,H^1(k)$ since both maps are compatible with cup-products by elements of $H^1(k)$. By remark 2.2

$$\gamma \cup \mathrm{Res}\,H^1(k) \to H^3_{\mathrm{nr}/k}(M_1)/\,\mathrm{Ker}\,N$$

is surjective. Hence these maps coincide also in degree three if k contains a fourth root of unity.

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