北京大学

26/6/2015 **VII Counter-examples**

In my talk at capital normal university, I described the counter-example of BATYREV & TSCHINKEL. Let me give you a variant of it

1) Variation on BATYREV–TSCHINKEL example

Let $V \subset \mathbb{P}^3_{\mathbb{K}} \times \mathbb{P}^3_{\mathbb{K}}$ be defined by the equation

$$\sum_{i=0}^{3} X_i Y_i^2 = 0$$

Again we consider the first projection
$\pi : V \longrightarrow \mathbb{P}^3_{\mathbb{K}}$ and, for $x \in \mathbb{P}^3(\mathbb{K})$,
$V_x = \pi^{-1}(\{x\})$

Let $U_0 : \prod_{i=0}^{3} X_i \neq 0 \quad \subset \mathbb{P}^3_{\mathbb{Q}} \qquad U = \pi^{-1}(U_0)$

If $x \in U_0(\mathbb{Q})$, V_x is a smooth diagonal quadric given by

$$V_x : \sum_{i=0}^{3} x_i Y_i^2 = 0.$$

But (F) is known for quadrics:
There are 3 cases:

1) $V_x(\mathbb{A}_{\mathbb{K}}) = \emptyset$ no rational points
2) $V_x(\mathbb{A}_{\mathbb{K}}) \neq \emptyset$ but $\mathrm{rk}(\mathrm{Pic}(V_x)) = 1$
   Then $\# V_x(\mathbb{K})_{H \leq B} \sim C_H(V_x) B$
3) V_x is split: $\mathrm{rk}(\mathrm{Pic}(V_x)) = 2$
   which means that $V_x \xrightarrow{\sim} \mathbb{P}^1_{\mathbb{K}} \times \mathbb{P}^1_{\mathbb{K}} \;/\mathbb{K}$
   Then $\# V_x(\mathbb{K})_{H \leq B} \sim C_H(V_x)\, B \log(B)$

When does case 3 occur? It occurs exactly when Q_x contains rational lines or equivalently
The quadratic form $\sum x_i Y_i^2$ may be written as $T_0^2 - T_1^2 + T_2^2 - T_3^2$ after a change of basis

But since we assume that the quadric has a rational point, that is the quadratic form $\sum x_i X_i^2$ is isotropic and thus isomorphic to
$$T_0^2 - T_1^2 + b_1 T_2^2 + b_2 T_3^2$$
So it is split if and only if its discriminant is a square, that is
$$\prod_{i=0}^{3} x_i \text{ is a square}$$
Let us consider
$$\begin{aligned} \psi \quad \mathbb{P}^3(\mathbb{K}) &\longrightarrow \mathbb{K}^*/\mathbb{K}^{*2} \cup \{0\} \\ [x_0 : \dots : x_3] &\longmapsto \Big[\prod_{i=0}^{3} x_i\Big] \end{aligned}$$
This map is well defined

and let
$$\mathbb{P}^3(\mathbb{K})^{\square} = \psi^{-1}(1).$$
This set is Zariski dense in $\mathbb{P}^3(\mathbb{Q})$.

<u>Prop</u>

Let H_1 be the height on $\mathbb{P}^3(\mathbb{Q})$ given by
$$\|(x_0, \dots, x_i)\|_\infty = \max_{0 \leq i \leq n} |x_i|$$
It is relative to $\mathcal{O}(1)$

Then
$$\mathbb{P}^3(\mathbb{Q})^{\square}_{H_1 \leq B} = O\left(B^{\frac{7}{2}}\right)$$

<u>NB</u>

For this height
$$\mathbb{P}^3(\mathbb{Q})_{H_1 \leq B} \sim C_{H_1}(\mathbb{P}^n_{\mathbb{Q}})\, B^4.$$

<u>Proof</u>

Pick up x_0, x_1, x_2, $|x_i| \leq B$

then let $\delta = \operatorname{sgn}(x_1 x_2 x_3) \times \prod_{\{p \mid v_p(x_1 x_2 x_3) \text{ odd}\}} p$

Then $\frac{x_3}{\delta}$ has to be a square

There are B^3 possibilities for x_0, x_1, x_2

at most $B^{1/2}$ ———— x_3 $\square$

Why is V a counter-example to the formula (F)?

By Lefschetz theorem, since $\dim(V) \geq 3$,

the restriction map

$$\rho : \mathrm{Pic}(\mathbb{P}^3_{\mathbb{K}} \times \mathbb{P}^3_{\mathbb{K}}) \to \mathrm{Pic}(V)$$

is an isomorphism. Therefore

$$rk(\mathrm{Pic}(V)) = 2$$

and (F) for U open in V would be

$$\# U(\mathbb{K})_{H \leq B} \underset{B \to +\infty}{\sim} C_H(V)\, B \log(B).$$

But, since the set $\mathbb{P}^n(\mathbb{K})^{\square}$ is Zariski dense,

$$\exists x \in \mathbb{P}^n(\mathbb{K})^{\square}, \quad Q_x \cap U \neq \emptyset$$

So $$\#(U \cap Q_x(\mathbb{K})_{H \leq B}) \underset{B \to +\infty}{\sim} C_H(Q_x)\, B \log(B)$$

This is known (remember: (F) is true for quadrics)

Reminder

(F) for all choice of norms on ω_V^{-1}

$\Rightarrow$ (E) $\Rightarrow$ for any closed $F \subsetneq V$,

$$\# F(\mathbb{K})_{H \leq B} = o(\# U(\mathbb{K})_{H \leq B})$$

Conclusion

There are many choices of the norm for which (F) is not true!

For this particular case, we have a very precise idea of what should be the correct formula

Conjecture (open)

$$\# U(\mathbb{K})_{H \leq B} \underset{B \to +\infty}{\sim} \Big(C_H(V) + \sum_{x \in \mathbb{P}^3(\mathbb{K})^{\square}} C_H(Q_x) \Big) B \log(B)$$

This formula seems to be out of reach, for now. The problem is that we have a bad control of the error term.

If we consider this example and the example given by BATYREV & TSCHINKEL, they have in common that there are too many points in a subset which is dense for Zariski topology but not dense for adelic topology. In fact in both cases they are <u>thin</u> subsets. Let me explain this notion:

2) Thin subsets

Definition (Reminder)

Let V be a nice variety / $\mathbb{K}$ number field

A subset $W \subset V(\mathbb{K})$ is said to be <u>thin</u> if there exists a morphism $\psi : X \to V$ such that

(i) ψ is generically finite;

($\eta : \mathbb{K}(V) \to V$ generic point, $\dim(X_\eta) = 0$)

(ii) ψ has no rational section;

(iii) $W \subset \psi(V(\mathbb{K}))$

Remarks

1) If ψ had a rational section

$$\begin{tikzcd} s : U \arrow[r] \arrow[d, phantom, "\cap"] & X \arrow[dl, "\psi"] \\ V & \end{tikzcd}$$

then $U(\mathbb{K}) \subset \psi(X(\mathbb{K}))$

and, in our setting, we do not want the rational points of an open subset to be thin.

2) 2 Let E be an elliptic curve (or an abelian variety) $/\mathbb{K}$. By Mordell-Weil's theorem, the quotient $A(\mathbb{K})/2A(\mathbb{K})$ is finite. Let $P_1, -, P_m \in A[\mathbb{K}]$ be such that

$$A[\mathbb{K}]/2A[\mathbb{K}] = \{[P_1], -, [P_m]\}$$

Then we may consider the map

$$\begin{aligned} \Psi \ \coprod_{i=1}^{m} A &\longrightarrow A \\ a &\longmapsto P_i + 2a \end{aligned}$$

$\Uparrow$
i-th component

Then $\Psi : \coprod_{i=1}^{m} A(\mathbb{K}) \longrightarrow A(\mathbb{K})$ is surjective but it has no rational section!

So $A(\mathbb{K})$ is thin according to the definition I gave.

Also if $V(\mathbb{K})$ is not Zariski dense in V, $V(\mathbb{K})$ is thin, because a closed immersion satisfies the condition of the definition.

<u>Examples</u>

1) $\{[x_0 : - : x_3] \in \mathbb{P}^3(\mathbb{Q}) \mid \forall i,j \ x_i/x_j \text{ is a cube}\}$ is thin. Indeed it is the image of the morphism

$$\begin{aligned} \mathbb{P}^3_{\mathbb{Q}} &\longrightarrow \mathbb{P}^3_{\mathbb{Q}} \\ [x_0 : - : x_3] &\longmapsto [x_0^3 : \_ : x_3^3] \end{aligned}$$

2) $\mathbb{P}^3(\mathbb{K})^{\square}$ is thin in $\mathbb{P}^3(\mathbb{K})$. The morphism is slightly more difficult to describe. Consider in $\mathbb{A}^5 - \{0\}$ the equation

$$W : X_0 X_1 X_2 X_3 = Y^2$$

There is an action $\mathbb{G}_m \circlearrowright W$ given by

$$\lambda(x_i, y) = (\lambda x_i, \lambda^2 y)$$

$X = W/\mathbb{G}_m$ is a singular toric variety

$\psi : X \longrightarrow \mathbb{P}^3_{\mathbb{K}}$ it is a map of degree 2

$[(x_i, y)] \longmapsto [x_0 : - : x_3]$

and $\mathbb{P}^3(\mathbb{K})^{\square} = \psi(X(\mathbb{K}))$.

Using

$$V \times_{\mathbb{P}^3} X \xrightarrow{\tilde{\psi}} V$$

we get that $\bigcup_{x \in \mathbb{P}^3(\mathbb{K})^{\square}} V_x(\mathbb{K})$ is also thin.

Theorem [COHEN – SERRE]

Let T be a thin set in $\mathbb{P}^n_{\mathbb{K}}$, then

$$\# T_{H \leq B} = O(B^{(n+1/2)} \log(B)^{\gamma})$$

with $\gamma < 1$

Method

It is an aplication of sieve method. using the following lemma:

Lemma

Assume we have a morphism $\psi : X \to V$ wich satisfies (i) – (iii) with X irreducible then there exists a finite Galois extension $\mathbb{L}/\mathbb{K}$ and a constant $c \in]0,1[$ so that for any prime ideal p of $G_{\mathbb{K}}$ which splits completely in $\mathbb{L}$ (ie $p G_{\mathbb{L}} = \prod_{i=1}^{d} \mathfrak{P}_i$ with $\#\{\mathfrak{P}_1, -, \mathfrak{P}_d\} = d$ and $d = [\mathbb{L} : \mathbb{K}]$)

$$\frac{\# \mathrm{red}_p(T)}{\# \mathbb{P}^n(\mathbb{F}_p)} < c$$

Remark

D. LOUGHRAN announced a more general result using similar techniques:
If V satisfies (E) for an open set U, then for any thin set T
$$\#(U(\mathbb{K})\cap T)_{H\leq B} = o(\#U(\mathbb{K})_{H\leq B})$$

General picture

For a variety V it is not enough to consider subvarieties. We have to consider non trivial morphisms

$$\begin{array}{l} X_1 \xrightarrow{\varphi_1} V \\ X_2 \xrightarrow{\varphi_2} V \\ X_3 \xrightarrow{\varphi_3} V \end{array}$$

with $X_1, \dots, X_m$ nice so that
$$(a(\varphi_i^*(\omega_V^{-1})), b(\varphi_i^*(\omega_V^{-1}))) \geq (1,t)$$
for lexicographic order

This gives accumulating thin sets. So now the best we can hope is that we only have to consider a finite number of such morphisms.

Refined heuristic

Does there exist a thin subset $T\subset V(\mathbb{K})$ such that

(F) $\quad \#(V(\mathbb{K})-T)_{H\leq B} \sim C_H(V)\, B \log(B)^{t-1}$

and

(E) $\quad \delta_{(V(\mathbb{K})-T)_{H\leq B}} \xrightarrow[B\to+\infty]{w} \mu$

Remark

On the positive side, no counter-example is known.
On the other hand, for a long time, no example

was known where there was a dense accumulating thin subset, and (F) was proven for the complement. The 1st example of such a result is due to C. LE RUDULIER who was a student of A CHAMBERT-LOIR and I would like to spend some time on her results.

3) Hilbert schemes

Let me start by explaining

a) Northcott theorem

Northcott theorem instead of considering rational points, consider algebraic points that is points on some finite extension

Definition

- Let $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$ $P = [x_0 : \dots : x_n]$, $x_0, \dots, x_n \in \overline{\mathbb{Q}}$, $x_i \neq 0$
  $\mathbb{K}(P) = \mathbb{Q}[x_0/x_i, \dots, x_n/x_i]$ number field
  it does not depend on the choice of i:
  $\mathbb{K}(P)$ is the intersection of the field $\mathbb{L} \subset \overline{\mathbb{Q}}$ such that $P \in im(\mathbb{P}^n(\mathbb{L}) \hookrightarrow \mathbb{P}^n(\overline{\mathbb{Q}}))$
  $d(P) = [\mathbb{K}(P) : \mathbb{Q}]$ is the degree of $P/\mathbb{Q}$
  (In fact, this corresponds to the residue field of a closed point)
- If $\mathbb{L}$ is a number field
  the normalized height $\overline{H}_{\mathbb{L}} : \mathbb{P}^n(\mathbb{L}) \to \mathbb{R}_{>0}$ is given by

$$\overline{H}_{\mathbb{L}}([x_0 : \dots : x_n]) = \left(\prod_{w \in Pl(\mathbb{L})} \max_{0 \leq i \leq n} |x_i|_w\right)^{\frac{1}{[\mathbb{L}:\mathbb{K}]}}$$

If $\mathbb{L}/\mathbb{K}$ is an extension of number fields and $\iota_{\mathbb{L}/\mathbb{K}} : \mathbb{P}^n(\mathbb{K}) \to \mathbb{P}^n(\mathbb{L})$ the induced map

$$\overline{H}_{\mathbb{L}} \circ i_{\mathbb{L}/\mathbb{K}} = \overline{H}_{\mathbb{K}}$$

So we can define

$$\overline{H} : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R}_{>0}$$

Z **Example**

$$\overline{H}([1 : \sqrt[n]{2}]) = \Big(\prod_{\zeta \ n^{th} \text{ root of } 1} |\zeta \sqrt[n]{2}| \Big)^{\frac{1}{n}} = \sqrt[n]{2} \xrightarrow[n \to +\infty]{} 1$$

So $\{ P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid \overline{H}(P) \leq B \}$ is not finite

Theorem (NORTHCOTT) Let $B > 0$, $d \geq 1$
$\{ p \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid \overline{H}(P) \leq B \text{ and } d(P) \leq d \}$
is a finite set.

Reference
SERRE Lecture on Mordell-Weil theorem

Sketch of the proof

Def
$$S^d \mathbb{P}^n_{\mathbb{Q}} = (\mathbb{P}^n_{\mathbb{Q}})^d / \mathfrak{S}_d \subset \mathbb{P}^N_{\mathbb{Q}}$$

where $N+1$ is the dimension of the vector space of $P \in \mathbb{Q}[X_{i,j}, 1 \leq i \leq d, 0 \leq j \leq n]^{\mathfrak{S}_d}$ which are homogeneous of degree 1 in $(X_{i,0}, \dots, X_{i,n})$ for $i \in \{1, \dots, d\}$.

But the rational points of $S^d \mathbb{P}^n_{\mathbb{Q}}$ are

$$S^d \mathbb{P}^n(\mathbb{Q}) = S^d \mathbb{P}^n(\overline{\mathbb{Q}})^{Gal(\overline{\mathbb{Q}}/\mathbb{Q})} \subset \bigoplus_{P \in \mathbb{P}^n(\overline{\mathbb{Q}})} \mathbb{Z} P$$

$$= \{ f : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{N} \mid \sum_{P \in \mathbb{P}^n(\overline{\mathbb{Q}})} f(P) = d \ \& \ \forall \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}), f \circ \sigma = f \}$$

subsets of d elements, with multiplicities,

$$\supset \{ \mathbb{1}_{Gal(\overline{\mathbb{Q}}/\mathbb{Q}) . x} , \quad x \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid d(x) = d \}$$

So $\{x \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid d(x) = d\} \xrightarrow{d:1} S^d \mathbb{P}^n(\mathbb{Q})$

Now it remains to compare the normalized height of a point of degree d to the height of its image in $\mathbb{P}^N_{\mathbb{Q}}$

But the pull-back of $G(1)_{\mathbb{P}^N_{\mathbb{Q}}}$ to $(\mathbb{P}^n_{\mathbb{Q}})^d$ is $G(1, \dots, 1)$

So the theorem of Northcott follows from the finiteness of the number of points of bounded height in $\mathbb{P}^N(\mathbb{Q})$. $\square$

<u>Example</u>

$$S^d \mathbb{P}^1_{\mathbb{Q}} = \mathbb{P}^d(\mathbb{Q})$$

$$[(u_i : v_i)] \longmapsto \left[\prod_{i=1}^{d} (u_i V - v_i U)\right]$$

homogeneous polynomial of degree d vanishing at the given d points.

So it is quite natural to consider the asymptotic behaviour of points of fixed degree in the projective space

<u>Remark</u>

For $\mathbb{P}^1_{\mathbb{Q}}$ we get

$$\#\{P \in \mathbb{P}^1(\overline{\mathbb{Q}}) \mid d(P) = d \;\&\; \overline{H}(P) \leq B\}$$

$$\underset{B \to +\infty}{\sim} d\, C_H(\mathbb{P}^d_{\mathbb{Q}})\, B^{d(d+1)}.$$

and we have equidistribution on $\mathbb{P}^d_{\mathbb{Q}}$

The number of points in $\mathbb{P}^d_{\mathbb{Q}}$ coming from $(\mathbb{P}^1)^d(\mathbb{Q})$ is $\ll B^{2d}$ and is negligible if $d \geqslant 2$.

So everything is fine for $\mathbb{P}^1_{\mathbb{Q}}$, let us turn to the case of surfaces

b) Hilbert schemes and symmetric product for surfaces

In higher dimensions the symmetric product is singular but for surfaces we have a nice way to get a desingularization:

Definition

For a nice variety V over $\mathbb{K}$, and $d \geq 1$, the hilbert scheme $\mathrm{Hilb}^d(V)$ satisfies for any extension $\mathbb{L}$ of $\mathbb{K}$,

$$\mathrm{Hilb}^d(V)(\mathbb{L}) = \{\text{subschemes of dimension } 0 \text{ and degree } d \text{ in } V_{\mathbb{L}}\}$$

where, for $S \subset V_{\mathbb{L}}$ of dimension 0, $S = \mathrm{Spec}(A)$ we have $\deg(S) = \dim_{\mathbb{L}}(A)$.

Example

- $d=1$, $\mathrm{Hilb}^1 V \xrightarrow{\sim} V$
- $d=2$, $\mathrm{Hilb}^2 V(\mathbb{K})$ contains 3 types of points:
  a) the pairs $\{P_1, P_2\}$ with $P_1, P_2 \in V(\mathbb{K})$, $P_1 \neq P_2$
  b) the closed points $P \in V_{(0)}$ such that $\deg([\mathbb{K}(P) : \mathbb{K}]) = 2$
  $\mathbb{K}(P)$ residue field at P
  c) The pairs (P, D) with $P \in V(\mathbb{K})$
  $D \in \mathbb{P}(T_P V)$ corresponding to subschemes isomorphic to $\mathrm{Spec}\, \mathbb{K}[T]/(T^2)$

There is a natural map (Hilbert-Samuel map)

$$\sigma : \mathrm{Hilb}^d V \longrightarrow S^d V$$
$$S \longmapsto |S| \text{ support of } S$$

where $|S| = \sum_{P \in V_{(0)}} \dim_{\mathbb{K}(P)} (\mathcal{O}_S \otimes_{\mathcal{O}_V} \mathcal{O}_{V,P})\, P$

Remarks

1) For a curve C, $\mathrm{Hilb}^d C \cong S^d C$.
2) Let $U \subset S^d V$ be the open set image of $\{(P_1, -, P_d) \in V^d \mid P_i \neq P_j \text{ for } i \neq j\}$ then $\sigma_{|\sigma^{-1}(U)}$ is an isomorphism and σ is birational

Fact

If V is a surface then
$$\sigma : \mathrm{Hilb}^d V \longrightarrow S^d V$$
is a desingularization of $S^d V$.

Notation

We now assume that V is a smooth Del Pezzo surface,
$$\pi : V^d \longrightarrow S^d V$$
$$\sigma : \mathrm{Hilb}^d V \longrightarrow S^d V$$
Let $E = \mathrm{Hilb}^d(V) - U$

For L is a line bundle $/ V$
$$\bigotimes_{i=1}^{d} pr_i^*(L) \longrightarrow V^d \quad \text{line bundle}$$
$$\circlearrowleft \; \mathfrak{S}^d \qquad\qquad \circlearrowleft \; \mathfrak{S}^d$$

and therefore there exist a line bundle $\tilde{L}$ on $S^d V$ such that $\pi^*(\tilde{L}) = \bigotimes_{i=1}^{d} pr_i^*(L)$

We get a morphism
$$\delta : \mathrm{Pic}(V) \longrightarrow \mathrm{Pic}(\mathrm{Hilb}^d V)$$
$$L \longmapsto \varepsilon^*(\tilde{L})$$

Proposition

(i) $\mathrm{Pic}(V^d) \cong \mathrm{Pic}(V)^d$

(ii) $\mathrm{Pic}(V) \oplus \mathbb{Z}E \cong \mathrm{Pic}(\mathrm{Hilb}^d V)$

$$L \longmapsto \rho(L)$$

(iii) $\omega^{-1}_{\mathrm{Hilb}^d(V)} = \rho(\omega_V^{-1})$

This means that if H is a height on V relative to ω_V^{-1}, on $\mathrm{Hilb}^d V$ the height which satisfies on U

$$H([(P_1, \dots, P_d)]) = \prod_{i=1}^{d} H(P_i)$$

is a height relative to $\omega^{-1}_{\mathrm{Hilb}^d(V)}$.

c) Counter example to (F) on open subsets

If the formula is true for S, since it is compatible with products, we would get that

$$\#[\varepsilon^{-1}(\pi(V^d(\mathbb{Q}))) \cap U(\mathbb{Q})]_{H \leq B} \sim \frac{c(V)^d}{d!} \times \frac{(t-1)!^d}{(dt-1)!} B \log(B)^{dt-1}$$

For any open $U \subset \mathrm{Hilb}^d V - E$, where $t = \mathrm{rk}(\mathrm{Pic}(V))$ and the expected formula for $\mathrm{Hilb}^n V$ is

$$\text{(F)} \quad U(\mathbb{Q})_{H \leq B} \sim c(\mathrm{Hilb}^d(V)) B \log(B)^{t+1-1}$$

Conclusion

Take $d \geq 2$

(i) If $t \geq 2$ or $d \geq 3$ contradiction with (F)

(ii) If $t = 1$ not negligible contradicts (E)

$T = \varepsilon^{-1}(\pi(V^d(\mathbb{Q})))$ is an accumulating thin subset.

d) (F) on complement of thin subsets

Theorem [C. LE RUDULIER]

If $V = \mathbb{P}^2_{\mathbb{Q}}$ or $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$, then there exists a non empty open subset $U \subset \mathrm{Hilb}^2 V$ such that

(F) $\#(U(\mathbb{Q}) - T)_{H \leq B} \sim c(\mathrm{Hilb}^2 V)\, B \log(B)^t$

where $t = \mathrm{rk}(\mathrm{Pic}(V)) = \mathrm{rk}(\mathrm{Pic}(\mathrm{Hilb}^2 V)) - 1$

Tools

It uses results of W. SCHMIDT on points of bounded height and degree.

29/6/2016 **Remarks**

1) For $V = \mathbb{P}^2_{\mathbb{Q}}$ only the constant is wrong, a, the power of B and b, the power of $\log(B)$ are correct

2) . For BATYREV & TSCHINKEL $\dim(V) = 5$

Here $\dim(\mathrm{Hilb}^2 V) = 4$

. For surfaces the formula is expected to work on an open subset

What of dimension 3

4 Fano volumes (varieties of dimension 3)

These varieties have been classified by MORI & MUKAI There are 105 deformation types

This classification has been investigated by LEHMANN, TANIMOTO, TSCHINKEL to find examples similar to the example of BATYREV & TSCHINKEL.

The point is that we are reduced to a geometric problem:

Definition

- Let V be nice over a field $\mathbb{K}$ with ω_V^{-1} extra big. Let $L \in \mathrm{Pic}(V) \cap \overline{C_{\mathrm{eff}}(V)}$
  - Let $Y \subset X$ be an irreducible proper subvariety and $\tilde{Y} \to Y$ a desingularization of Y (given by Hironaka's theorem)

  We get $\varepsilon_Y : \tilde{Y} \to X$; $a_Y^g(L) = a^g(\varepsilon_Y^*(L))$
  $b_Y^g(L) = b^g(\varepsilon_Y^*(L))$.
- L is balanced for Y [LEHMANN-TANIMOTO-TSCHINKEL] if
  $$(a_Y^g(L), b_Y^g(L)) < (a^g(L), b^g(L));$$
  for lexicographic order
- L is weakly balanced for Y if
  $$(a_Y^g(L), b_Y^g(L)) \leq (a^g(L), b^g(L)).$$
- Y is [geometrically] strongly accumulating for L if $a_Y^g(L) > a^g(L)$
- Y is [geometrically] accumulating for L if $(a_Y^g(L), b_Y^g(L)) > (a^g(L), b^g(L))$
- Y is [geometrically] weakly accumulating for L if $(a_Y^g(L), b_Y^g(V)) \geq (a^g(L), b^g(L))$

This covers more or less all cases.

- L is said to be balanced (respectively weakly balanced) if there exists a non empty open subset $U \subset V$ such that for any $Y \subset V$ such that $Y \cap U \neq \emptyset$, L is balanced (respectively weakly balanced for Y). I shall say that L is unbalanced (resp. weakly unbalanced) if L is not weakly balanced (resp. not balanced). I want that unbalanced implies weakly unbalanced

Remark

L is unbalanced (resp. weakly unbalanced) if and only if the accumulating (resp. weakly accumulating subvarieties) are Zariski dense.

Question

For which V is ω_V^{-1} unbalanced or weakly unbalanced?

Let me describe a few invariants of Fano 3-folds:

Notation

Let V be a smooth Fano 3-fold $/\mathbb{K}$.

- $\rho(V) = rk(Pic(V))$
- $r(V)$ = the largest $r \geq 1$ such that $[\omega_V^{-1}] = r[L]$ for some $[L] \in Pic(V)$
- $d(V) = (\omega_V^{-1})^3 = \omega_V^{-1} \cdot \omega_V^{-1} \cdot \omega_V^{-1}$

intersection product

It is the degree of V

V is called primitive if it is not the blowing up of another Fano 3-fold along a smooth irreducible curve. (It is automatic if $\rho(V)=1$)

Theorem [LEHMANN, TANIMOTO & TSCHINKEL]

Let V be a primitive Fano 3-fold with $\rho(V)=2$;

- If $d(V)=12, 14, 30, 48, 54, 56, 62$, ω_V^{-1} is balanced
- If $d(V)=24$ then ω_V^{-1} is weakly balanced and unbalanced
- If $d(V)=6$ then ω_V^{-1} is unbalanced.

If $\rho(V)=1$, we have the table

$r(V)$	$d(V)$	balanced	weak	unbalanced
4, 3	*	✓		
2	$\geqslant 15$		✓	
1	8			✓
1	$\geqslant 10$		✓	
1	8, 6			✓
	4			✓ if ω_V^{-1} very ample

So this shows that even in dimension 3 there are many examples with thin accumulating subsets. But the situation is even worse that that:

Even if ω_V^{-1} is balanced, there might exist dense accumulating thin subset:

For example, C. LE RUDULIER proved that

For any $Y \subset \mathrm{Hilb}^2(\mathbb{P}^2_{\mathbb{Q}})$

$$\#(Y \cap U(\mathbb{Q}))_{H \leq B} = o\left(U(\mathbb{Q})_{H \leq B}\right).$$

Problem

How can you characterize the accumulating thin subsets?

Of course, as I explained, given a morphism from X to V, you have a test using geometric invariants to say whether it may give an accumulating subset. And, if you can prove equidistribution on the complement of a thin subset, You know that you have found all accumulating thin subsets.

But it is not satisfying in the sense that it does not give you a clear procedure to find the morphisms. If I give you a point in a variety how do you know if you have to remove it. This problem was the topic of my talk at capital Normal university and in the last chapter I want to explain why I think the concept of slopes may help.

VIII New perspectives, slopes

1) Quick reminder

Again, V is a nice variety $/\mathbb{K}$ number field which satisfies $\mathcal{H}$. $n = \dim(V)$

Definition

We equip V with an adelic metric, that is a classical adelic norm $(\|\cdot\|_w)_{w \in \mathcal{M}(\mathbb{K})}$ on TV

- This defines an adelic norm on $\omega_V^{-1} = \det(V)$ and therefore a height H on V.
- For any $x \in V(\mathbb{K})$ we get an $\mathcal{O}_{\mathbb{K}}$-lattice in $T_x V$

$$\Lambda_x = \{ y \in T_x V \mid \forall \mathfrak{p} \in \mathcal{M}(\mathbb{K})_f, \|y\|_{\mathfrak{p}} \leq 1 \}$$

and a euclidean structure on

$$T_x V \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \bigoplus_{w \mid \infty} T_x V \otimes_{\mathbb{K}} \mathbb{K}_w$$

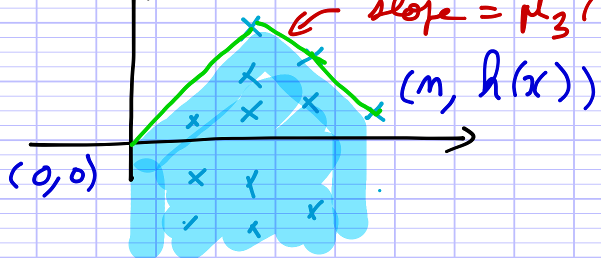
given by $\|(y_w)_{w\mid\infty}\| = \sqrt{\sum_{w\mid\infty} \|y\|_w^{[\mathbb{C}:\mathbb{K}_w]}}$

For F $\mathbb{K}$-subspace of $T_x V$

$$\widehat{\deg}(F) = -\log\left(\mathrm{Vol}(F/\Lambda \cap F)\right)$$

for the induced euclidean structure

$$\mathcal{P}(T_x V) = \text{Convex Hull of } \{(\mathrm{rk}(F), \widehat{\deg}(F)),\ F \subset T_x V\}$$



$h = \log \circ H$

For $t \in [0,n]$

$$m_{T_x V}(t) = \sup\{y \mid (t,y) \in \mathcal{P}(T_x V)\}$$

Slopes of x

$$\mu_i(x) = m_{T_x V}(i) - m_{T_x V}(i-1) \text{ for } i \in \{1,\dots,n\}$$

Remarks

$$\mu_1(x) \geq \dots \geq \mu_n(x) \qquad \sum_{i=1}^{n} \mu_i(x) = h(x)$$

and the slope of $T_x V$

$$\mu(T_x V) = \frac{\widehat{\deg}(T_x V)}{\dim(T_x V)} = \frac{1}{n} h(x)$$

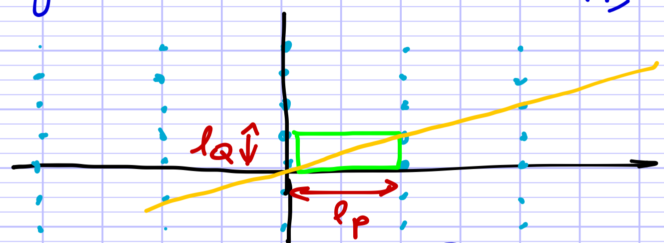
which also is the mean of the slopes

An easy example

For $V = \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$

$$T_{(P,Q)} \mathbb{P}^1 \times \mathbb{P}^1 = \underset{\underset{\Lambda_P}{\cup}}{T_P \mathbb{P}^1} \overset{\perp}{\oplus} \underset{\underset{\Lambda_Q}{\cup}}{T_Q \mathbb{P}^1}$$

Take F of dimension 1 in $T_{(P,Q)} \mathbb{P}^1 \times \mathbb{P}^1$



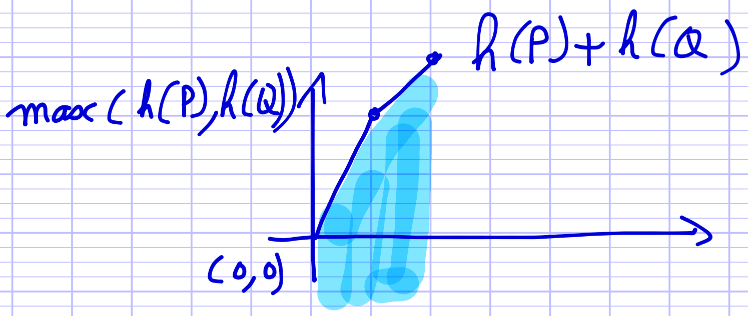
$$\mathrm{Covol}(\Lambda_{(P,Q)} \cap F) = \min\{\|x\|, x \in \Lambda_{(P,Q)} \cap F - \{0\}\}$$

But by Pythagoras theorem

$$\geq \min(l_P, l_Q)$$

So $\widehat{\deg}(F) \leq \max(-\underset{h(P)}{\underset{\parallel}{\log(l_P)}}, -\underset{h(Q)}{\underset{\parallel}{\log(l_Q)}})$

So, in that case Newton's polygon is



$$\mu_1(P,Q) = \max(h(P), h(Q))$$
$$\mu_2(P,Q) = \min(h(P), h(Q))$$
and their sum is $h(P)+h(Q) = h(P,Q)$.

Definition

The freeness of x is

$$\ell(x) = \begin{cases} 0 & \text{if } \mu_n(x) \leq 0 \\ \mu_n(x)/\mu(x) & \text{otherwise} \end{cases} \in [0,1]$$

Remark

$\ell(x) = 0 \iff \mu_n(x) \leq 0$

$\ell(x) = 1 \iff \mu_1(x) = \mu_2(x) = \dots = \mu_n(x)$
$= \mu(x)$
$\iff T_x V$ is semi-stable

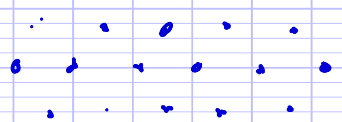
this defines the notion

Examples

- $\mathbb{Z}^n \subset \mathbb{R}^n$ is semi-stable

but it is not the only one

- the hexagonal lattice



$\Lambda = \mathbb{Z} \oplus \mathbb{Z} e^{i2\pi/3} \subset \mathbb{C}$
is semi stable

Remark

More generally if E is an adelic vector bundle on V one can consider the slopes
$$\mu_i(E_x) \text{ for } 1 \leq i \leq rk(E)$$

2) Slopes and successive minima

There is another notion which is natural for lattices in euclidean spaces, namely successive minima which generalizes to number fields:

Definition

If E is $\mathbb{K}$ vector space equipped with classical norms $(\|\cdot\|_w)_{w \in Pl(\mathbb{K})}$ such that there exist a basis $(e_1, \ldots, e_r)$ of E so that
$$\left\| \sum_{i=1}^{r} x_i e_i \right\|_w = \max_{1 \leq i \leq r} \|x_i\|_w \text{ for}$$
almost all w, then
$$\lambda_i(E) = \inf \left\{ \theta \in \mathbb{R}_{>0} \;\middle|\; \exists \text{ a linearly independent family } (u_1, \ldots, u_i) \in E^i \text{ such that } \forall j \in \{1, \ldots, i\} \prod_{w \in Pl(\mathbb{K})} \|u_j\| \leq \theta \right\}.$$

Proposition (E. GAUDRON using MINKOWSKI thm)
$$0 \leq \log(\lambda_i(E)) + \mu_i(E) \leq C_{\mathbb{K}}$$

Remark

There is also a duality formula
$$\mu_i(E) = -\mu_{r+1-i}(E^\vee)$$
combining this with the proposition, we get
$$\left| \mu_n(E) - \log\left(\min_{y \in E^\vee - \{0\}} \prod_w \|y\|_w \right) \right| < C_{\mathbb{K}}$$

In other words,

$$\left| \mu_r(E) - \min\left(h_1\left(\mathbb{P}(E^\vee)\right)\right)\right| < C_{\mathbb{K}}$$

So for our situation

$$\left| \mu_n(x) - \min\left(h_1(\mathbb{P}(T_x^\vee))\right)\right| < C_{\mathbb{K}}$$

But this is very hard to compute in general!

3) Slopes and change of metrics

Proposition

If we denote by μ_i and μ'_i the slopes corresponding to two adelic metrics on V there exist $C > 0$ such that

$$|\mu_i(x) - \mu'_i(x)| < C$$

for any $x \in V(\mathbb{K})$. Therefore

$$|\ell(x) - \ell'(x)| = O\left(\frac{1}{h(x)}\right)$$

Proof

This reduces to

$$0 < C_1 < \frac{\|\cdot\|'_w}{\|\cdot\|_w} < C_2$$

and $\|\cdot\|'_w = \|\cdot\|_w$ for almost all w. $\square$

4) Freeness and morphisms

Definition

Let $\varphi : \mathbb{P}^1_{\mathbb{K}} \to V$ be a non constant morphism. Write $\varphi^*(TV)$ as $\bigoplus_{i=1}^n \mathcal{O}(a_i)$ with $a_1 \geq \cdots \geq a_n$ and define

$$\ell(\varphi) = \begin{cases} 0 & \text{if } a_n \leq 0 \\ n\, a_n / \sum_{i=1}^n a_i & \text{otherwise} \end{cases}$$

Remark

$$\sum_{i=1}^{n} a_i = \deg_{\omega_V^{-1}}(\varphi)$$

Proposition 1

$$\ell(\varphi(x)) = \ell(\varphi) + O_\varphi\left(\frac{1}{h(x)}\right) \text{ for } x \in \mathbb{P}^1(\mathbb{K})$$

Proposition 2

Let C be a rigid rational curve on a surface S, then

$\{x \in C(\mathbb{K}) \mid \ell(x) > 0\}$ is finite

Proof

from the decomposition

$$\varphi^*(TV) \cong \bigoplus_{i=1}^{n} \mathcal{O}(a_i)$$

we get that $|\mu_i(\varphi(x)) - a_i h_1(x)| < C_\varphi$.

For proposition 2, we use the fact that $a_n < 0$! $\square$

Proposition 3

Let $\varphi : X \to Y$ be a morphism of nice varieties

Then there exist $c > 0$ so that for any $x \in X(\mathbb{K})$ in which $T_x\varphi : T_x X \to T_{\varphi(x)} Y$ is surjective, we have

$$\mu_{min}(x) \leq \mu_{min}(\varphi(x)) + c$$

In particular, if $y \in Y(\mathbb{K})$ is not a critical value of φ,

$$\ell(x) = O\left(\frac{1}{h(x)}\right) \text{ on the fibre } X_y(\mathbb{K}).$$

Proof

Under the hypothesis $T_y Y^\vee \hookrightarrow T_x X^\vee$

and thus $\min H_1(\mathbb{P}(TX^\vee)) \leq \min H_1(\mathbb{P}(TY^\vee)) + C$ $\square$

Remark

For the counter-examples like the one of BATYREV & TSCHINKEL which are given by a morphism the fibres of which are accumulating we get also that

$$\ell(x) \longrightarrow 0$$

in each of these accumulating subvariety.
This gives some support for the slogan

Slogan

"Bad points have a small freeness."
Like in politics slogan do not need to be true.
Moreover for the freeness to be useful we also need it to be big for good points

5) The case of the projective space

Proposition

Let $P \in \mathbb{P}^n(\mathbb{K})$

$$\ell(P) = \frac{n}{n+1} + \min_F \left(\frac{-n\,\widehat{\deg}(F)}{\operatorname{codim}(F)\, h(P)} \right)$$

where F goes over the subspaces of E such that $P \in \mathbb{P}(F)$ and $F \neq E$.

Proof

Remember that if P corresponds to a vector subspace $L \subset \mathbb{K}^{n+1} = E$ of dimension 1,

$$T_p \mathbb{P}^n \xrightarrow{\sim} \mathrm{Hom}(L, E/L) \xrightarrow{\sim} L^\vee \otimes E / L^\vee \otimes L$$

So each subspace $F' \subset T_p \mathbb{P}^n$ is isomorphic to $L^\vee \otimes F / L^\vee \otimes L$ for some subspace F such that

$$L \subset F \subsetneq E$$

and

$$\widehat{\deg}(F') = \widehat{\deg}(F) + \dim(F)\, \widehat{\deg}(L^\vee)$$

Indeed we have the general formula

$$\widehat{\deg}(E/F) = \widehat{\deg}(E) - \widehat{\deg}(F)$$

and $\widehat{\deg}(L \otimes E) = \widehat{\deg}(E) + \dim(E)\, \widehat{\deg}(L)$

On the other hand

$$h(P) = (n+1)\, \widehat{\deg}(L^\vee)$$

So we get

$$\mu(T_p \mathbb{P}^n / F') = \frac{\mathrm{codim}_E(F)\, \widehat{\deg}(L^\vee) - \widehat{\deg}(F)}{\mathrm{codim}_E(F)}$$

$$= -\widehat{\deg}(L) - \frac{\widehat{\deg}(F)}{\mathrm{codim}_E(F)}$$

the smallest slope is

$$\mu_{min}(T_p \mathbb{P}^n) = -\widehat{\deg}(L) + \min\left(\frac{-\widehat{\deg}(F)}{\mathrm{codim}_E F}\right)$$

and

$$\ell(P) = \frac{n\, \mu_{min}(T_p \mathbb{P}^n)}{-(n+1)\, \widehat{\deg}(L)}$$

<u>Corollary</u>

For each $P \in \mathbb{P}^n(\mathbb{K})$, one has

$$\ell(P) \geq \frac{n}{n+2}$$

<u>Proof</u>

$$\widehat{\deg}(F) = -\log\left(\mathrm{covol}(\mathcal{O}_K^{n+1} \cap F)\right)$$

and $\mathrm{covol}(\mathcal{O}_K^{n+1} \cap F) \geq 1$. $\square$

<u>Remark</u>

In fact, we have something somewhat more striking

$$\lim_{B\to+\infty} \frac{1}{\#\mathbb{P}^n(\mathbb{K})_{H\leq B}} \sum_{P\in\mathbb{P}^n(\mathbb{K})_{H\leq B}} \ell(P) \xrightarrow[B\to+\infty]{} 1$$

G of new formula?

Let $\varepsilon :]1,+\infty[\longrightarrow \mathbb{R}_{>0}$ be a decreasing function such that

(i) $\varepsilon(t) \xrightarrow[t\to+\infty]{} 0$

(ii) $\log(t)^\alpha \varepsilon(t) \xrightarrow[t\to+\infty]{} +\infty$ if $\alpha>0$

So a slowly decreasing function

Let $V(\mathbb{K})^{\varepsilon-\ell}_{H\leq B} = \{P\in V(\mathbb{K}) \mid H(P)\leq B \ \& \ \ell(P)\geq \varepsilon(B)\}$

(F_ε) $V(\mathbb{K})^{\varepsilon-\ell}_{H\leq B} \sim C_H(V) B \log(B)^{r-1}$

(E_ε) $\delta_{V(\mathbb{K})^{\varepsilon-\ell}_{H\leq B}} \longrightarrow \mathbb{N}$

Proposition

(F_ε) and (E_ε) are compatible with the product of varieties.

Z Because $\ell(P)$ is hard to compute I can not yet claim that this works better than removing a thin subset.

There is still much to be done
It works for quadrics but

Open question

For a smooth hypersurface of degree $d \geqslant 3$
$N+1 > 2^d(d-1)$ is (F_ε) correct?

22/4/2015
北京

Curve statistics

As you probably know, to determine if a variety is rational (that is birational to $\mathbb{P}^n$) is extremely hard. On the other hand, we know that Fano varieties are rationally connected. So it is quite natural to see how we can strengthen the notion of rational connectivity.

Prop

Let $V/\mathbb{C}$ be a smooth, projective, geometrically integral variety which is Rationally Connected then for any subscheme $S \subset \mathbb{P}^1$ of dim 0 and any $\psi: S \to V$, there exists $\varphi: \mathbb{P}^1 \to V$ such that $\varphi_{|S} = \psi$.

Of course, such a φ may be of very high degree. A slightly less well known fact, but which has been noticed several times is the occurrence of stabilisation phenomena as the degree goes to ∞. It was probably first noted by G. Segal. Let me explain it on

1) Sand box example. $V = \mathbb{P}^n$

I need to introduce the ring $\mathcal{M}_{\mathbb{C}}$:

As a group

- generated by $[V]$ V variety $/\mathbb{C}$
- relations
  - $[V] = [V']$ if $V \simeq V'$
  - $[V] = [U] - [F]$

if $F \subset V$ closed subvariety $U = V - F$

- ring structure given by $[V][V'] = [V \times V']$

Then we consider

$M_d = [\underline{\mathrm{Mor}}_d(\mathbb{P}^1, \mathbb{P}^n)] \in \mathcal{M}_k$. $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$

On the other hand, there is a very elementary description of this space

$$\mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^n) \simeq \underbrace{\{(P_0, -, P_n) \in \mathbb{C}[T]^n \mid \begin{cases} \gcd(P_i) = 1 \\ \max(\deg(P_i)) = d \end{cases}\}}_{W_d} / \mathbb{G}_m$$

But if we define

we get a map ($d \geq 0$)

$$\coprod_{0 \leq k \leq d} W_{d-k} \times \underbrace{\mathbb{A}^k_{\mathbb{C}}}_{\substack{\text{unitary polynomials} \\ \text{of degree } k}} \longrightarrow \underbrace{\mathbb{A}^{(n+1)(d+1)} - \mathbb{A}^{(n+1)d}}_{\substack{(n+1)\text{-uples of polyn.} \\ \text{of max. deg } d}}$$

In $\mathcal{M}_{\mathbb{C}}[[T]]$, we get

$$\left(\sum_k \mathbb{L}^k T^k\right) \times \left(\sum_k [W_k] T^k\right) = \left(\mathbb{L}^{n+1} - 1\right) \sum_d \mathbb{L}^{(n+1)d} T^d$$

from which we get:

$$[W_d] = (\mathbb{L}^{n+1} - 1)\left(\mathbb{L}^{(n+1)d} - \mathbb{L}\,\mathbb{L}^{(n+1)(d-1)}\right) \text{ if } d \geq 1$$

and $[W_d]\,\mathbb{L}^{-(n+1)d} = (\mathbb{L}^{n+1} - 1)(1 - \mathbb{L}^{-n})$ if $d \geq 1$

Thus

$$M_d\, \mathbb{L}^{-(n+1)d} = \frac{\mathbb{L}^{n+1} - 1}{\mathbb{L} - 1}(1 - \mathbb{L}^{-n}) \text{ for } d \geq 1$$

Note that intersection cohomology factors through the ring $\mathcal{M}_{\mathbb{C}}$ so such a formula descends to cohomology. The generality of this phenomena is still unknown but let me explain the framework for this generalisation: First of all in general we do not such a simple stable formula but only a limit in a topological ring constructed using motives.

2) General Framework

a) Topological K_0 of varieties

- We can define a filtration by dimensions on $\mathcal{M}_k[\mathbb{L}^{-1}]$

$$F^k(\mathcal{M}_{\mathbb{C}}[\mathbb{L}^{-1}]) = \underset{\text{Subgp}}{\langle} \mathbb{L}^{-i}[V],\ i - \dim(V) \geq k \rangle$$

$$\hat{\mathcal{M}}_{\mathbb{C}} = \varprojlim \mathcal{M}_{\mathbb{C}}[\mathbb{L}^{-1}] / F^k \mathcal{M}_{\mathbb{C}}[\mathbb{L}^{-1}]$$

which is a topological ring where something goes to 0 if its dimension goes to $-\infty$

<u>Note</u> $\mathcal{M}_{\mathbb{C}} \longrightarrow \hat{\mathcal{M}}_{\mathbb{C}}$ is not injective which means a loss of information

$$\mathcal{M}_{\mathbb{C}} \xrightarrow{\text{not injective}} \mathcal{M}_{\mathbb{C}}[\mathbb{L}^{-1}] \nearrow \hat{\mathcal{M}}_{\mathbb{C}}$$

It turns out that this ring is still not flexible enough. Let me introduce a variant of it

$K_0(M_{\mathbb{C}}^{ch})$ Grothendieck group for the category of Chow motives $/\mathbb{C}$

We get a map

$$\Psi : \mathcal{M}_{\mathbb{C}}[\mathbb{L}^{-1}] \longrightarrow K_0(M_{\mathbb{C}}^{ch})[\mathbb{L}^{-1}]$$

$$\tilde{\mathcal{M}}_{\mathbb{C}} = \varprojlim \Psi(\mathcal{M}_{\mathbb{C}}[\mathbb{L}^{-1}]) / \Psi(F_k \mathcal{M}_{\mathbb{C}}[\mathbb{L}^{-1}])$$

So we loose even more information

<u>b) Multidegree</u>

$$\underline{\deg} : \operatorname{Mor}(\mathbb{P}^1, V) \longrightarrow \operatorname{Pic}(V)^\vee_{\mathbb{R}} = \operatorname{Hom}(\operatorname{Pic}(V), \mathbb{R})$$
$$\varphi \longmapsto (L \mapsto \deg(\varphi^*(L)))$$

$$C_{eff}(V)^\vee = \{ v \in \operatorname{Pic}(V)^\vee_{\mathbb{R}} \mid \forall D \text{ effective } \langle v, D \rangle \geq 0 \}$$

$\partial C_{eff}(V)^\vee$ its boundary

<u>Space</u> :

$\underline{d} \in C_{eff}(V)^\vee$

$\mathcal{M}or^{vf}_{\underline{d}}(\mathbb{P}^1, V)$ = space of very free morphism from $\mathbb{P}^1$ to V of multidegree $\underline{d}$

(very free : $\varphi^*(T_V) = \bigoplus_{i=1}^{n} \mathcal{O}(a_i)$ $\quad a_1 \geq \cdots \geq a_n > 0$)

Following ideas of V. BATYREV, J. ELLENBERG, D. BOURQUI

Question

When is $[\underline{\mathrm{Mor}}_{\underline{d}}(\mathbb{P}^1, V)]\,\mathbb{L}^{-\langle \underline{d}, \omega_V^{-1}\rangle} \xrightarrow[\substack{\underline{d} \in C_{eff}(V) \\ \mathrm{dist}(\underline{d}, \partial C_{eff}(V)) \to +\infty}]{} C(V)$ (explicit too)

in $\widetilde{\mathcal{M}}_{\mathbb{C}}$?

I will give later turn back to the meaning of $C(V)$

3) Evidence

- True for $\mathbb{P}^n$
- compatible with results of Kapranov on motivic Eisenstein series for split $V = G/P$ G reductive linear alg group, P maximal parabolic subgroup.
- True for split toric variety (D. BOURQUI) using descent met ~~ho~~ similar to the case of $\mathbb{P}^n$
- some evidence for equivariant compactification of affine spaces (CHAMBERT-LOIR, LOESER) using a motivic POISSON formula

Open but might be doable

hypersurface in $\mathbb{P}^n$ $n \gg \deg$ via a motivic circle method (using Poisson formula).

4) About the constant $C(V)$

a) Equidistribution principle

This is related to the proposition I stated at the very beginning of the talk

Equidistribution question

Take $S \subset \mathbb{P}^1$ subsdome of dim 0.
and $F \subset \underline{\mathrm{Mor}}(S, V)$ closed
$\mathrm{Mor}_{\underline{d}}^{vf}(\mathbb{P}^1, V, F)$ extra condition: $\varphi_{|S} \in W$

$$\text{“}\frac{[\mathrm{Mor}_{\underline{d}}^{vf}(\mathbb{P}^1, V, F)]}{[\mathrm{Mor}_{\underline{d}}^{vf}(\mathbb{P}^1, V)]} \overset{?}{\longrightarrow} \frac{[F]}{[\underline{\mathrm{Mor}}(S, V)]}\text{”}$$

ie

$$\left([\mathrm{Mor}_{\underline{d}}^{vf}(\mathbb{P}^1, V, F)][\underline{\mathrm{Mor}}(S, V)] - [\mathrm{Mor}_{\underline{d}}^{vf}(\mathbb{P}^1, V)][F]\right)\mathbb{L}^{-\langle \underline{d}, \omega_V^{-1}\rangle} \underset{?}{\longrightarrow} 0$$

It more or less says that the germs condition at different points are totally independent...

b) Back to $C(V)$

What we would like to do

$$C(V) = \text{something simple} \times \prod_{P \in \mathbb{P}^1_{(0)}} [V_{\mathbb{C}(P)}]\mathbb{L}^{-\dim V_{\mathbb{C}(P)}}$$

but it can't work like that

Problems

1) Product over uncountable set
Use formal formula

$$\prod_{P \in \mathbb{P}^1} f(P) = \exp\Big(\underbrace{\int_{\mathbb{P}^1}}\log(f(P))\Big)$$

meaning through motivic integration

2) This can not converge with $f(P) = [V_{\mathbb{C}(P)}]\mathbb{L}^{-\dim(V_{\mathbb{C}(P)})}$

even for $\mathbb{P}^n$! $f(P) = 1 + \mathbb{L}^{-\deg(P)} + \cdots + \mathbb{L}^{-n\deg(P)}$

Easy computation

$$Z(\lambda) = \Big[\prod_{P \in \mathbb{P}^1} \left(1 - \mathbb{L}^{\lambda \deg(P)}\right)\Big]^{-1} = \sum_d \sum_{P \in (\mathbb{P}^1)^{(d)}} \mathbb{L}^{\lambda d}$$

(like $\Big[\prod_p (1-p^{-s})\Big]^{-1} = \sum_n \frac{1}{n^s}$) $\quad \|$

$$= \sum_d [\mathbb{P}^d]\, \mathbb{L}^{\lambda d}$$

$$= \sum_d \frac{\mathbb{L}^{d+1} - 1}{\mathbb{L} - 1} \mathbb{L}^{\lambda d}$$

which diverges if $\lambda = -1$

But if $\lambda < -1$

$$= \frac{1}{\mathbb{L}-1}\left(\frac{\mathbb{L}}{1 - \mathbb{L}^{\lambda+1}} - \frac{1}{1-\mathbb{L}^{\lambda}}\right)$$

$$= \frac{1}{(1-\mathbb{L}^{\lambda})(1-\mathbb{L}^{\lambda+1})}$$

we get $C(\mathbb{P}^n) = \dfrac{1}{(\mathbb{L}-1)\,\mathbb{L}^n}\, Z(n+1)^{-1}$

$$= \frac{1}{(\mathbb{L}-1)\,\mathbb{L}^n} \times \prod_{P \in \mathbb{P}^1_{(0)}} \left(1 - \mathbb{L}^{\deg(P)}\right) f(P).$$

In general, for a variety $V/_{\mathbb{C}}$ with

$H^1(V, \mathcal{O}_V) = H^2(V, \mathcal{O}_V) = 0$, $\mathrm{Pic}(V)$ torsion free

(automatic if V Fano).

There exists a versal torsor $\mathcal{T} \to V$ under

$T_{NS} = \mathrm{Spec}\,(\mathbb{C}[\mathrm{Pic}(V)]) \simeq \mathbb{G}_m^t$

"Def"

$$C(V) = \frac{1}{(\mathbb{L}-1)^t\, \mathbb{L}^{\dim(V)}} \prod_{P \in \mathbb{P}^1_{(0)}} \mathrm{Vol}\left(\mathcal{T}(\mathcal{O}_P)\right)$$

Remark

(a) Extensions to $\overset{\text{char } 0}{\widehat{K}} \neq \mathbb{C}$, $C \neq \mathbb{P}^1$

(b) Question : loose less information? $\mathcal{V} \to C$, $\mathbb{F}_\rho$ ($\widetilde{M}_K$ instead of M_K)

3 νf enough? $a_n \to +\infty$?

4 Fano too strict

a_v^{-1} = ample + $d n c$?

(more than big).