V The general setting, the program of BATYREV, MANIN, TSCHINKEL

To describe the general setting, I want to consider varieties over a number field. For that, let me describe the tools I need from number theory:

1) Survival kit in number theory

References

The book I prefer has a drawback it was written in french:

P. SAMUEL: Théorie algébrique des nombres
S. LANG: Algebraic Number theory
J. NEUKIRCH: Algebraic Number theory

Definitions & notations

- A number field is a finite field extension of $\mathbb{Q}$
- Let $\mathbb{K}$ be a number field

$\mathcal{O}_{\mathbb{K}}$ = ring of integers of $\mathbb{K}$
= integral closure of $\mathbb{Z}$ in $\mathbb{K}$
$= \{\alpha \in \mathbb{K} \mid \exists$ unitary $P \in \mathbb{Z}[X],\ P(\alpha)=0\}$

(unitary: the coefficient of the highest degree monomial is one: $P = X^d + \sum_{i=0}^{d-1} a_i X^i,\ a_i \in \mathbb{Z}$)

- A fractional ideal of $\mathbb{K}$ is a sub $\mathcal{O}_{\mathbb{K}}$-module of $\mathbb{K}$ which is finitely generated.
- Let $\mathcal{I}(\mathcal{O}_{\mathbb{K}})$ be the set of non zero fractional ideals in $\mathbb{K}$.

For $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}(\mathcal{O}_{\mathbb{K}})$ (FRAKTUR alphabet)
$\mathfrak{a}\mathfrak{b}$ is the sub-$\mathcal{O}_{\mathbb{K}}$-module of $\mathbb{K}$ generated by products xy with $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$

Proposition

$\mathcal{I}(\mathcal{O}_{\mathbb{K}})$ is a commutative group for the multiplication of fractional ideals. Its neutral element is $\mathcal{O}_{\mathbb{K}}$ and the inverse of $\mathfrak{a} \in \mathcal{I}(\mathcal{O}_{\mathbb{K}})$ is

$$\mathfrak{a}^{-1} = \{x \in \mathbb{K} \mid x\mathfrak{a}^{-1} \subset \mathcal{O}_{\mathbb{K}}\}$$

Notation (continued)

The map

$$\begin{aligned} \mathbb{K}^{\times} &\longrightarrow \mathcal{I}(\mathcal{O}_{\mathbb{K}}) \\ x &\longmapsto x\mathcal{O}_{\mathbb{K}} = (x) \end{aligned}$$

is a morphism of groups. Its image is denoted by $\mathcal{P}(\mathcal{O}_{\mathbb{K}})$ and is called the subgroup of principal ideals.

The quotient $\mathcal{I}(\mathcal{O}_{\mathbb{K}}) / \mathcal{P}(\mathcal{O}_{\mathbb{K}})$ is called the group of ideal classes of $\mathbb{K}$, it is denoted $Cl(\mathcal{O}_{\mathbb{K}})$.

Remark

$\mathrm{Spec}(\mathcal{O}_{\mathbb{K}})$ as a set is the set of prime ideals of $\mathcal{O}_{\mathbb{K}}$. It has dimension 1

In other words,

$\mathrm{Spec}(\mathcal{O}_{\mathbb{K}}) - \{(0)\}$ is the set of maximal ideals of $\mathcal{O}_{\mathbb{K}}$. I denote it as $\mathcal{M}_{\mathcal{O}_{\mathbb{K}}}$

Definition

A <u>Dedekind domain</u> is an integral domain wich is integrally closed, noetherian and any non zero prime ideal is maximal.

Examples

- fields
- $\mathcal{O}_{\mathbb{K}}$ is a Dedekind ring
- if R is a Dedekind ring
  and $S \subset R$ is a subset stable by multiplication
  so that $0 \notin S$
  The localization $R[S^{-1}]$ is a Dedekind ring

Remark

Let R be a Dedekind ring and $\mathbb{K} = \mathrm{Fr}(R)$
The set $\mathcal{I}(R)$ of fractional ideals of $\mathbb{K}$
with respect to R is also a group for the
multiplication

Theorem 1

Let R be a Dedekind ring and $\mathfrak{M}_R$ its set
of non zero prime ideals. The morphism
of groups

$$\mathrm{Div}(\mathrm{Spec}(R)) = \bigoplus_{\mathfrak{p} \in \mathfrak{M}_R} \mathbb{Z}\mathfrak{p} \longrightarrow \mathcal{I}(R)$$

$$(n_{\mathfrak{p}})_{\mathfrak{p} \in \mathfrak{M}_R} \longrightarrow \underbrace{\prod_{\mathfrak{p} \in \mathfrak{M}_R} \mathfrak{p}^{n_{\mathfrak{p}}}}_{\text{In fact it is a finite product.}}$$

is an isomorphism.

Corollary

We get a commutative diagram

$$\begin{tikzcd}
 & & \mathrm{Div}(\mathrm{Spec}(\mathcal{O}_{\mathbb{K}})) \arrow[r] \arrow[d, "\wr S"] & \mathrm{Pic}(\mathrm{Spec}(\mathcal{O}_{\mathbb{K}})) \arrow[r] \arrow[d, "\wr S"] & 0 \\
1 \arrow[r] & \mathcal{O}_{\mathbb{K}}^{*} \arrow[r] & \mathbb{K}^{*} \arrow[u, "\mathrm{div}"] \arrow[d] & & \\
 & & \mathcal{I}(\mathcal{O}_{\mathbb{K}}) \arrow[r] & \mathcal{C}\ell(\mathcal{O}_{\mathbb{K}}) \arrow[r] & 0
\end{tikzcd}$$

with exact lines.

Theorem 2

$Cl(G_{\mathbb{K}})$ is a finite group. Its order is denoted by h.

Remark

We get a map

$$\begin{aligned} \mathbb{K}^* &\longrightarrow \bigoplus_{\mathfrak{p} \in \mathfrak{M}_{G_{\mathbb{K}}}} \mathbb{Z}\mathfrak{p} \\ x &\longmapsto \sum_{\mathfrak{p}} v_{\mathfrak{p}}(x)\mathfrak{p} \end{aligned}$$

Put $v_{\mathfrak{p}}(0) = +\infty$

the map $v_{\mathfrak{p}} : \mathbb{K} \longrightarrow \mathbb{Z} \cup \{+\infty\}$ is a discrete valuation on $\mathbb{K}$ and defines a place of $\mathbb{K}$, which is denoted $\mathfrak{p}$ as well

Theorem 3

The map $\mathfrak{M}_{G_{\mathbb{K}}} \longrightarrow Pl(\mathbb{K})$ defined above is a bijection from $\mathfrak{M}_{G_{\mathbb{K}}}$ to the set of ultrametric places of $\mathbb{K}$, which I also denote by $Pl(\mathbb{K})_f$.

More notations

Let $\Sigma^{\mathbb{C}}_{\mathbb{K}/\mathbb{Q}}$ be the set of field morphisms from $\mathbb{K}$ to $\mathbb{C}$

Remark

- Since $\mathbb{C}$ is algebraically closed, Galois theory tells us that

$$\# \Sigma^{\mathbb{C}}_{\mathbb{K}/\mathbb{Q}} = [\mathbb{K} : \mathbb{Q}] \text{ the degree of } \mathbb{K}/\mathbb{Q}$$

- For $\sigma \in \Sigma^{\mathbb{C}}_{\mathbb{K}/\mathbb{Q}}$ $\quad \bar{\sigma} : \mathbb{K} \to \mathbb{C}$, $x \mapsto \overline{\sigma(x)}$ $\quad \in \Sigma^{\mathbb{C}}_{\mathbb{K}/\mathbb{Q}}$

this defines an action of $\mathbb{Z}/2\mathbb{Z} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\Sigma^{\mathbb{C}}_{\mathbb{K}/\mathbb{Q}}$

and

σ is a fixed point if and only if $\sigma(\mathbb{K}) \subset \mathbb{R}$

If $\sigma \in \Sigma^{\mathbb{C}}_{\mathbb{K}/\mathbb{Q}}$ $\quad |\cdot| \circ \sigma : \mathbb{K} \to \mathbb{R}_{\geq 0}$ is an absolute value on $\mathbb{K}$

Note that $|\cdot| \circ \bar{\sigma} = |\cdot| \circ \sigma$

Theorem 4

The map $\sigma \mapsto |\cdot| \circ \sigma$ defines a bijection from the orbits of the action of $\mathbb{Z}/2\mathbb{Z}$ on $\Sigma^{\mathbb{C}}_{\mathbb{K}/\mathbb{Q}}$ to the set of archimedean places of $\mathbb{K}$ which I denote by $\mathrm{Pl}(\mathbb{K})_\infty$

Definition

Let $\sigma \in \Sigma^{\mathbb{C}}_{\mathbb{K}/\mathbb{Q}}$ and v be the corresponding place

σ defines an isomorphism from $\mathbb{K}_v$ to

- $\mathbb{R}$ if $\sigma = \bar{\sigma}$, we say that v is real
- $\mathbb{C}$ if $\sigma \neq \bar{\sigma}$ ——— v is complex

r_1 (resp r_2) denotes the number of real (resp. complex) places

Fact

For any place w of $\mathbb{K}$, the induced topology on $\mathbb{Q}$ is the topology defined by the restriction of an absolute value defining w which is non trivial and define a place v of $\mathbb{K}$

we denote it $w|v$

Theorem 5

For any place v of $\mathbb{Q}$,

$$\mathbb{Q}_v \otimes_{\mathbb{Q}} \mathbb{K} \xrightarrow{\sim} \prod_{w|v} \mathbb{K}_w$$

as a $\mathbb{K}$ algebra.

In particular $[\mathbb{K}:\mathbb{Q}] = \sum_{w|v} [\mathbb{K}_w : \mathbb{Q}_v]$

$(r_1 + 2r_2 = [\mathbb{K}:\mathbb{Q}])$.

Proposition / Definition

Let $\mathfrak{p} \in Pl(\mathbb{K})_f$, $p \in \mathcal{P}$ the induced place on $\mathbb{Q}$

- We have a commutative diagram

$$\begin{tikzcd}
\mathbb{K}^* \arrow[r, two heads, "v_\mathfrak{p}"] & \mathbb{Z} \\
\mathbb{Q}^* \arrow[u] \arrow[r, two heads, "v_p"] & \mathbb{Z} \arrow[u, "\times e_\mathfrak{p}"']
\end{tikzcd}$$

$e_\mathfrak{p}$ is called the ramification index

- $\mathcal{O}_{\mathbb{K}}/\mathfrak{p}$ is a finite field extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

we denote it by $\mathbb{F}_\mathfrak{p}$, $f_\mathfrak{p} = [\mathbb{F}_\mathfrak{p} : \mathbb{F}_p]$

is called the residual degree

Proposition

$$[\mathbb{K}_\mathfrak{p} : \mathbb{Q}_p] = e_\mathfrak{p} f_\mathfrak{p}$$

and therefore

$$[\mathbb{K}:\mathbb{Q}] = \sum_{\mathfrak{p}|p} e_\mathfrak{p} f_\mathfrak{p}$$

Proposition

If $\mathbb{K}/\mathbb{Q}$ is galoisian then, for any $v \in Pl(\mathbb{Q})$, $Gal(\mathbb{K}/\mathbb{Q})$ acts transitively on $\{w|v\}$ and $[\mathbb{K}_w : \mathbb{Q}_v]$ (resp $f_\mathfrak{p}$ $e_\mathfrak{p}$) depends only on v (resp. p).

NB

Everything since last fact generalizes, mutatis mutandis, to an extension of number fields $\mathbb{L}/\mathbb{K}$.

Notation

For any $w \in Pl(\mathbb{K})$, let v be the induced place in $Pl(\mathbb{Q})$

For any $x \in \mathbb{K}_w$

$$|x|_w = |N_{\mathbb{K}_w/\mathbb{Q}_v}(x)|_v$$

If w is not complex $|\cdot|_w$ is an absolute value which defines w

But if w is complex

$$|z|_w = |z|^2 \text{ for } z \in \mathbb{C}$$

does not satisfy $|x+y|_w \leq |x|_w + |y|_w$!

and is not an absolute value.

However this notation is convenient for the following reasons.

Remark

- $\mathbb{K}_w$, as an additive group is a finite dimensional vector space on $\mathbb{Q}_v$

Thus it is locally compact and admits a Haar measure (that is a measure on the Borelian σ-algebra which is stable under translation, it is unique up to constant).

Let μ be such a measure

we have
$$\mu(aB) = |a|_v \, \mu(B)$$
for any borelian $B \subset \mathbb{Q}_w$.

- We have the formula

$$\forall x \in \mathbb{K}, \quad |N_{\mathbb{K}/\mathbb{Q}}(x)|_v = \prod_{w|v} |x|_w$$

This formula implies

<u>Proposition</u> (product formula for number fields)
$$\forall x \in \mathbb{K}^*, \quad \prod_{w \in \mathcal{P}\ell(\mathbb{K})} |x|_w = 1$$

⚠ It is not invariant under field extensions
For $x \in \mathbb{Q}_p$ $\quad |x|_w = |x|_v^{[\mathbb{K}_w : \mathbb{Q}_v]}$

Now I want to give a more complete description of the multiplicative group of $\mathbb{K}$.

<u>Notation</u>
Let
$$\underline{\log} : \mathbb{K}^* \longrightarrow \prod_{v \in \mathcal{P}\ell(\mathbb{K})_\infty} \mathbb{R} \qquad (\dim r_1 + r_2)$$
$$x \longmapsto (\log |x|_w)_{w \in \mathcal{P}\ell(\mathbb{K})_\infty}$$
and
$$\pi : \prod_{v \in \mathcal{P}\ell(\mathbb{K})_\infty} \mathbb{R} \longrightarrow \mathbb{R}$$
$$(x_v)_v \longmapsto \sum_{v \in \mathcal{P}\ell(\mathbb{K})_\infty} x_v$$

<u>Remark</u>
If $x \in \mathcal{O}_{\mathbb{K}}^*$, then $\forall p \in \mathcal{P}\ell(\mathbb{K})_f$, $v_p(x) = 0$, $|x|_p = 1$

So the product formula gives
$$\prod_{v \in \mathcal{P}\ell(\mathbb{K})_\infty} |x|_v = 1 \quad \text{and} \quad \pi \circ \underline{\log}(\mathcal{O}_{\mathbb{K}}^*) = \{0\}.$$

Theorem 6

$\mathrm{Ker}(\log_{|G^*_{\mathbb{K}}})$ is the group $\mu_\infty(\mathbb{K})$ of roots of unity in $\mathbb{K}$ which is finite and

$\mathrm{Im}(\log_{|G^*_{\mathbb{K}}})$ is a lattice Λ in $H = \mathrm{Ker}(\pi)$

(that is, it is generated by a basis of the $\mathbb{R}$ vector space H)

$\mathrm{Covol}(\Lambda) = \mathrm{Vol}(H/\Lambda)$ is called the regulator of $\mathbb{K}/\mathbb{Q}$.

To summarize

The structure of $\mathbb{K}^*$ is more or less given by two exact sequences:

$$1 \to G^*_{\mathbb{K}} \to \mathbb{K}^* \to \mathrm{Div}(\mathrm{Spec}(G_{\mathbb{K}})) \to \overbrace{\mathrm{Pic}(\mathrm{Spec}(G_{\mathbb{K}}))}^{\text{finite}} \to 0$$

and

$$1 \to \mu_\infty(\mathbb{K}) \to G^*_{\mathbb{K}} \xrightarrow{\log} \Lambda \to 0$$

$\cap$ lattice

H dim. $r_1 + r_2 - 1$

For the additive group, we have

Theorem 7 more easy

The injective morphism of $\mathbb{Q}$ algebras

$$\mathbb{K} \to \prod_{w|\infty} \mathbb{K}_w \simeq \mathbb{K} \otimes_{\mathbb{Q}} \mathbb{R}$$

maps $G_{\mathbb{K}}$ onto a lattice of $\mathbb{K} \otimes_{\mathbb{Q}} \mathbb{R}$.

The covolume of this lattice is $\sqrt{|d_{\mathbb{K}}|}$

where $d_{\mathbb{K}}$ is the discriminant of $\mathbb{K}$

(defined as $\det(\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(a_i a_j))$ where $(a_i)_{1 \leq i \leq [\mathbb{K}:\mathbb{Q}]}$ is a basis of $G_{\mathbb{K}}$ as a $\mathbb{Z}$-module)

18/5/2016 2) About models

a) Smooth models

Definition

Let R be an integral domain and $\mathbb{K} = \mathrm{Fr}(R)$

Let V be a variety $/\mathbb{K}$

A model of V over R is a scheme $\mathcal{V}$ over $\mathrm{Spec}(R)$ and an isomorphism

$$\varphi : \mathcal{V}_{\mathbb{K}} \xrightarrow{\sim} V.$$

Remark

If V is a projective variety, it is easy to produce such a model

Assume V to be projective $V \subset \mathbb{P}^N_{\mathbb{K}}$, defined by homogeneous polynomials

$$f_1, \dots, f_r \in \mathbb{K}[T_0, \dots, T_N].$$

But each f_i may be written as

$$f_i = \sum_j \alpha_{i,j} T_0^{j_0} \dots T_N^{j_N} \text{ and } \alpha_{i,j} = \frac{a_{i,j}}{b_{i,j}}$$

with $a_{i,j}, b_{i,j} \in R$, $b_{i,j} = 1$ if $\alpha_{i,j} = 0$.

By replacing f_i by $\left(\prod_j b_{i,j}\right) f_i$

we may assume

$$f_i \in R[T_0, \dots, T_n]$$

Then

$$\mathcal{V} = \mathrm{Proj}\left(R[T_0, \dots, T_n]/(f_1, \dots, f_r)\right)$$

is a model of V. If V is reduced, by increasing r, we may assume $(f_1, \dots, f_r) = \sqrt{(f_1, \dots, f_r)}$ and $\mathcal{V}$ reduced.

2 But in general, even if V is smooth $\mathcal{V}$ is not smooth. So the problem is to get a smooth projective model.

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Notation

Let $\mathbb{K}$ be a number field

Let S be a finite subset of $\mathcal{Pl}(\mathbb{K})_f$

Then

$$\mathcal{O}_S = \{ y \in \mathbb{K} \mid \forall \mathfrak{p} \in \mathcal{Pl}(\mathbb{K})_f - S,\ |x|_\mathfrak{p} \leq 1 \}$$

Remark

i) $\mathcal{O}_\emptyset = \mathcal{O}_\mathbb{K}$.

ii) Since $\mathrm{Pic}(\mathrm{Spec}(\mathcal{O}_\mathbb{K}))$ is finite and generated by $[\mathfrak{p}]$ for $\mathfrak{p} \in \mathcal{Pl}(\mathbb{K})_f$

There exists $S \subset \mathcal{Pl}(\mathbb{K})_f$ finite such that

$$\mathbb{K}^* \xrightarrow{\mathrm{div}} \bigoplus_{\mathfrak{p} \notin S} \mathbb{Z}\mathfrak{p}$$

is surjective. But $\mathcal{O}_S$ is a Dedekind ring as well so $\mathrm{Pic}(\mathrm{Spec}(\mathcal{O}_S)) = \{0\}$ and $\mathcal{O}_S$ is principal.

Proposition

Let V be a nice variety on the number field $\mathbb{K}$

Then there exists $S \subset \mathcal{Pl}(\mathbb{K})_f$ finite and a smooth and projective model of V over $\mathcal{O}_S$.

Old fashioned proof

As before may assume that V is defined by r homogeneous polynomials

$$f_1, \dots, f_r \in \mathcal{O}_\mathbb{K}[T_0, \dots, T_N] \text{ with } (f_0, \dots, f_r) = \sqrt{(f_0, \dots, f_r)}$$

and let $\mathcal{V}$ be the corresponding model of V over $\mathcal{O}_\mathbb{K}$

We are going to prove that there exists a finite set of ultrametric places S so that

$\mathcal{V}_{\mathcal{O}_S}$ is smooth.

In order to prove that I am going to use two things first the following characterisation of smoothness for V:

Reminder

For V as above, $n = \dim(V)$, V is smooth if and only if

$\forall (x_0, \dots, x_N) \in \mathbb{C}^{N+1} - \{0\}$,

$$\left(\forall i \in \{1, \dots, r\}, f_i(x_0, \dots, x_N) = 0\right) \Rightarrow rk\left(\left(\frac{\partial f_i}{\partial x_j}(x_0, \dots, x_N)\right)_{\substack{1 \le i \le r \\ 0 \le j \le N}}\right) = N-n$$

(Note that $N - n \le r$)

This can be expressed in terms of determinant of minors: This is equivalent to

For any $(x_0, \dots, x_N) \in \mathbb{C}^{n+1}$

if for $i \in \{1, \dots, r\}$ $f_i(x_0, \dots, x_N) = 0$

and for any $i_1, \dots, i_{N-n}$, $j_1, \dots, j_{N-n}$ such that

$1 \le i_1 < \dots < i_{N-m} \le r$, $0 \le j_1 < \dots < j_{N-n} \le N$

we have

$$\det\left(\frac{\partial f_{i_k}}{\partial X_{j_l}}(x_0, \dots, x_N)\right) = 0$$

then $(x_0, \dots, x_N) = 0$

The second tool I am going to use is

Hilbert Nullstellensatz

Let $g_1, \dots, g_m, f \in \mathbb{C}[X_0, \dots, X_N]$

such that

$\{x \in \mathbb{C}^{N+1} \mid \forall i \in \{1, \dots, m\}, g_i(x) = 0\} \subset \{x \in \mathbb{C}^{n+1} \mid f(x) = 0\}$

then

$$f \in \sqrt{(g_1, \dots, g_m)}$$

that is $\exists n$ such that

$$f^n \in (g_1, \dots, g_m)$$

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Remark

If $\mathbb{L}/\mathbb{K}$ is a field extension and E a $\mathbb{K}$-vector space, $F \subset E$ a subspace

$$\varphi : E \longrightarrow E \otimes_{\mathbb{K}} \mathbb{L}$$
$$x \longmapsto x \otimes 1$$

the extension of scalar map, then

$$\varphi^{-1}(\mathbb{L}\,\varphi(F)) = F$$

Thus for an ideal I of $\mathbb{K}[X_0, \dots, X_n]$

$$\sqrt{\mathbb{L} I} \cap \mathbb{K}[X_0, \dots, X_n] = \sqrt{I}$$

End of the proof of the proposition

so we get that there exists an integer m and polynomials $A_{u,i}$ $0 \leq u \leq N$, $1 \leq i \leq n$ and $B_{u,\underline{i},\underline{j}}$ $0 \leq u \leq N$ $\underline{i} = (i_1, \dots, i_{N-n})$ $\underline{j} = (j_1, \dots, j_{N-n})$

$1 \leq i_1 < \dots < i_{N-n} \leq n$ $0 \leq j_1 < \dots < j_{N-n} \leq N$

such that

$$(*) \quad X_u^m = \sum_i A_{u,i} f_i + \sum_{\underline{i},\underline{j}} B_{u,\underline{i},\underline{j}} \det\left(\frac{\partial f_{i_k}}{\partial X_{j_l}}\right)_{1 \leq k,l \leq N-n}$$

Now let D be the product of all denominators of all coefficients of the $A_{u,i}$ and $B_{u,\underline{i},\underline{j}}$ and let

$$S = \{\mathfrak{p} \mid v_{\mathfrak{p}}(D) \geq 1\}$$

it is a finite subset of $\mathrm{Pl}(\mathbb{K})_f$ and

$$A_{u,i}, B_{u,\underline{i},\underline{j}} \in \mathcal{O}_S[X_0, \dots, X_N]$$

and since $\mathcal{O}_S$ injects in $\mathbb{K}$

$(*)$ is true in $\mathcal{O}_S[X_0, \dots, X_N]$

But this implies that the morphism of vector bundles

$$T\mathbb{P}^N_{|\mathcal{V}_{G_S}} \xrightarrow{df} \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(d_i)_{|\mathcal{V}_{G_S}} \quad \text{where } d_i = \deg(f_i)$$

is of constant rank $N-n$.
Then we still have to check that by increasing S, we get that $\mathcal{V}_{G_S}$ is flat $/\operatorname{Spec}(G_S)$.
Using Prop. III.9.7 in HARTSHORNE's book since $\mathcal{V}$ is reduced, it suffices to prove that by increasing S, we may assume $\mathcal{V}$ irreducible. This will follow from the next lemma $\square$

Remark

In slightly more modern terms, the end of the proof could be rewritten as follows:
The equations

$$\det\left(\frac{\partial f_{ik}}{\partial X_{i\ell}}\right)_{1\leq k,\ell\leq N-n} = 0$$

defines a close subset $\mathcal{S}$ of $\mathcal{V}$ which do not meet the generic fibre V.
The structural map $\pi: \mathcal{V} \to \operatorname{Spec}(G_{\mathbb{K}})$ is projective and therefore proper. Thus $S = \pi(\mathcal{S})$ is a closed subset which does not contain the generic point (0) of $\operatorname{Spec}(G_{\mathbb{K}})$. Thus it is a finite subset of $P\ell(\mathbb{K})_f$ and $\mathcal{V}_{G_S}$ is smooth. $\square$
But in fact, this is Hilbert Nullstellensatz in in disguise;

But the argument using the Nullstellensatz generalizes easily in a non-projective setting

Lemma

Let $\mathbb{K}$ be a number field, S be a finite subset of $Pl(\mathbb{K})_f$ and $\mathcal{V}$ a noetherian scheme on $Spec(\mathcal{O}_S)$. If $\mathcal{V}_{\mathbb{K}} = \phi$ then the image of the structural morphism $\mathcal{V} \to Spec(\mathcal{O}_S)$ is finite.

Proof

Since $\mathcal{V}$ is noetherian it can be covered by a finite number of affine scheme and it is enough to prove the result when $\mathcal{V} = Spec(\mathcal{O}_S[T_1, \dots, T_N]/(f_1, \dots, f_r))$

But then

$\mathcal{V}_{\mathbb{K}} = Spec(\mathbb{K}[T_1, \dots, T_N]/(f_1, \dots, f_r)) = \phi$

means that (You may see that as a form of Hilbert Nullstellensatz)

$$1 \in (f_1, \dots, f_r)$$

Thus $\exists A_1, \dots, A_r \in \mathbb{K}[T_1, \dots, T_N]$, $1 = \sum_{i=1}^{N} A_i f_i$

Taking S' as the set of $p \ni D$ the product of the denominators of the A_i

$\mathcal{V}_{\mathcal{O}_{S'}} = \phi$ that the image of $\mathcal{V}$ is in $Spec(\mathcal{O}_S)$ is contained in $S' - S$. $\square$

Proposition 1

Let $\mathbb{K}$, S be as above and let $\mathcal{V}, \mathcal{W}$ be noetherian schemes over $Spec(\mathcal{O}_S)$

Let $\varphi_1, \varphi_2 : \mathcal{V} \to \mathcal{W}$ be morphism of schemes over $\mathcal{O}_S$, such that $\varphi_{1\,\mathbb{K}} = \varphi_{2\,\mathbb{K}} : \mathcal{V}_{\mathbb{K}} \to \mathcal{W}_{\mathbb{K}}$

then there exists a finite $S' \subset Pl(\mathbb{K})_f$ such that
$$\varphi_1{}_{\mathcal{O}_{S'}} = \varphi_2{}_{\mathcal{O}_{S'}}$$

Proof

Apply the lemma to the open subscheme of $\mathcal{V}$ defined by $\varphi_1(x) \neq \varphi_2(x)$ $\square$

Proposition 2

Let $\mathbb{K}$ be a number field and let S be a finite subset of $Pl(\mathbb{K})_f$, let $\mathcal{V}, \mathcal{W}$ be noetherian schemes over $\operatorname{Spec} \mathcal{O}_S$ and let $\varphi : \mathcal{V}_{\mathbb{K}} \to \mathcal{W}_{\mathbb{K}}$ be a morphism of $\mathbb{K}$ varieties then there exists a finite subset $S' \supset S$ in $Pl(\mathbb{K})_f$ and morphism $\tilde{\varphi} : \mathcal{V}_{\mathcal{O}_{S'}} \longrightarrow \mathcal{W}_{\mathcal{O}_{S'}}$ so that $\tilde{\varphi}_{\mathbb{K}} : \mathcal{V}_{\mathbb{K}} \longrightarrow \mathcal{W}_{\mathbb{K}}$ coincides with φ.

Proof

Let $(\mathcal{V}_i)_{i\in I}$ (resp $(\mathcal{W}_j)_{j\in J}$) be a finite covering of $\mathcal{V}$ (resp $\mathcal{W}$) such that $\forall i \in I, \exists j \in J, \varphi(\mathcal{V}_{i\,\mathbb{K}}) \subset \mathcal{W}_{j\,\mathbb{K}}$

But if $\mathcal{V}_i = \operatorname{Spec}(\mathcal{O}_S[T_1, \dots, T_m]/(f_1, \dots, f_r))$
and $\mathcal{W}_j = \operatorname{Spec}(\mathcal{O}_S[T_1, \dots, T_n]/(g_1, \dots, g_s))$
$\varphi_{|\mathcal{V}_{i,\mathbb{K}}}$ corresponds to

$h_1, \dots, h_n \in \mathbb{K}[T_1, \dots, T_m]$
such that $\forall i \in \{1, \dots, s\}$ $g_i(h_1, \dots, h_n) \in (f_1, \dots, f_r)$
ie $\exists A_{i,j}$ such that
$$g_i(h_1, \dots, h_n) = \sum_j A_{ij} f_j$$

Taking S_i given by the product of all denominators of $h_1, \dots, h_n, A_{ij}$, we get

$$\varphi_i : \mathcal{V}_{i\,\mathcal{O}_{S'}} \longrightarrow \mathcal{W}_{j\,\mathcal{O}_{S'}}$$

extending $\varphi_{|V_{i\,\mathbb{K}}}$. Now put $S'' = \bigcup_{i \in I} S_i$
and we apply proposition 1 to

$$\varphi_i, \varphi_{i'} : \mathcal{V}_{i\,\mathcal{O}_{S''}} \cap \mathcal{V}_{i'\,\mathcal{O}_{S''}} \longrightarrow \mathcal{W}_{j\,\mathcal{O}_{S''}}$$

we get new finite sets $S_{i,i'}$ and we put $S' = \bigcup_{i,i'} S''_{i,i'}$
so that φ_i and $\varphi_{i'}$ coincide on $\mathcal{V}_{i\,\mathcal{O}_{S'}} \cap \mathcal{V}_{i'\,\mathcal{O}_{S'}}$

and they define $\varphi : \mathcal{V}_{\mathcal{O}_{S'}} \longrightarrow \mathcal{W}_{\mathcal{O}_{S'}}$ as wanted. $\square$

<u>Corollary</u>

Let V be a projective variety over the number field $\mathbb{K}$
Let S, S' be finite subsets of $Pl(\mathbb{K})_f$,
Let $\mathcal{V}$ be a projective model of V over $\mathcal{O}_S$
and $\mathcal{V}'$ ——— V ——— $\mathcal{O}_{S'}$
Let φ (resp φ') be the isomorphism

$$\mathcal{V}_{\mathbb{K}} \xrightarrow{\sim} V \quad (\text{resp } \mathcal{V}'_{\mathbb{K}} \xrightarrow{\sim} V)$$

then there exists $S'' \subset Pl(\mathbb{K})_f$ finite and containing $S \cup S'$ and an isomorphism

$$\rho : \mathcal{V}_{\mathcal{O}_{S''}} \xrightarrow{\sim} \mathcal{V}'_{\mathcal{O}_{S''}} \text{ which extends } \varphi'^{-1} \circ \varphi.$$

<u>Proof</u>

Apply proposition 2 to $\varphi'^{-1} \circ \varphi$ and $\varphi^{-1} \circ \varphi'$
to get ρ and ρ' and proposition 1 to $\rho \circ \rho'$ and $Id_{\mathcal{V}'}$
(resp. $\rho' \circ \rho$ and $Id_{\mathcal{V}}$). $\square$

So up to making S bigger the model is "unique".

b) Models of vector bundles

We can easily extend the notion of model to subcategories of the category of schemes

Definition

Let R be an integral domain, $\mathbb{K} = \mathrm{Fr}(R)$
Let V be a variety over $\mathbb{K}$,
$\mathcal{V}$ be a model of V over R
Let E be a vector bundle over V,
A model of E over $\mathcal{V}$ is a vector bundle $\mathcal{E}$ over R with an isomorphism of vector bundles from $\mathcal{E}_{\mathbb{K}}$ to E.

Of course, the question is: does it exists?

Proposition

Let V be a variety over a number field $\mathbb{K}$
Let E be vector bundle over V
There exist a finite set $S \subset \mathrm{Pl}(\mathbb{K})_f$ and a model $\mathcal{V}$ of V over $\mathcal{O}_S$ and a model $\mathcal{E}$ of E over $\mathcal{V}$.

Proof

Let $(U_i \xrightarrow{\alpha_i} V)_{i\in I}$ be a finite covering of V by open immersions, $U_i = \mathrm{Spec}(R_i)$ which trivializes E
and $(\varphi_i : E_{|U_i} \longrightarrow U_i \times \mathbb{A}^n_{\mathbb{K}})_{i\in I}$ a local trivialization of E.
Consider $U_{i,j} = \alpha_i^{-1}(\alpha_i(U_i) \cap \alpha_j(U_j))$ open in U_i
$$\varphi_{j,i} = \varphi_j \circ \varphi_i^{-1} : U_{i,j} \times \mathbb{A}^n_{\mathbb{K}} \longrightarrow U_{j,i} \times \mathbb{A}^n_{\mathbb{K}}$$
which is defined by
$$\psi_{j,i} : U_{i,j} \xrightarrow{\sim} U_{j,i} \text{ and } f_{j,i} : U_{i,j} \longrightarrow GL_{n,\mathbb{K}}$$
we fix i temporarily.
Write $R_i = \mathbb{K}[T_1, \dots, T_N]/(f_1, \dots, f_r)$.

and take S so that $f_1, \dots, f_r \in G_S[T_1, \dots, T_N]$
We get models of the $\mathcal{U}_i$.
$U_i - U_{ij}$ is closed in U_i, it is defined by the vanishing of some elements of $\mathbb{K}[T_1, \dots, T_N]$
by increasing S we may assume they are in $G_S[T_1, \dots, T_n]$ as well
we get $\mathcal{U}_{i,j} \subset \mathcal{U}_i$ open so that $\mathcal{U}_{i,j} \otimes \mathbb{K} = U_{i,j}$
Then we apply propositions 2 and 1 to extend $\varphi_{j,i}$ to $\mathcal{U}_{i,j} \times_{G_S} \mathbb{A}^n_{G_S} \longrightarrow \mathcal{U}_{j,i} \times_{G_S} \mathbb{A}^n_{G_S}$
as morphisms of vector bundles which satisfy the glueing condition

$$\tilde{\varphi}_{k,j} \circ \tilde{\varphi}_{j,i} = \tilde{\varphi}_{k,i} \text{ on } \mathcal{U}_{i,j} \cap \mathcal{U}_{i,k}$$

$\mathcal{E}$ (resp. $\mathcal{V}$) is obtained by glueing the $\mathcal{U}_i \times \mathbb{A}^n_{G_S}$ (resp. $\mathcal{U}_i$). $\square$

NB

Similarly one can get models of algebraic groups over $\mathbb{K}$, ...

Example

If $\mathcal{V}$ is a smooth projective model of a nice variety V over G_S then $T\mathcal{V}$ is a model of TV.

Remark

Again if we accept to add some primes to S, the models are unique.

3) Adelic norms and metrics

a) w-adic norms

Definition

Let $\mathbb{K}$ be a number field and let w be a place of $\mathbb{K}$.
Let E be a finite dimensional $\mathbb{K}_w$ vector space.
A norm on E is a map

$$\|\cdot\|_w : E \longrightarrow \mathbb{R}_{\geq 0}$$

such that

(i) $\|x\|_w = 0 \Leftrightarrow x = 0$

(ii) $\forall x \in E,\ \forall \lambda \in \mathbb{K}_w \quad \|\lambda x\|_w = |\lambda|_w \|x\|_w$

(iii) if w is ultrametric
$$\forall x, y \in E,\ \|x+y\|_w \leq \sup(\|x\|_w, \|y\|_w)$$

(iii') if w is real
$$\forall x, y \in E \quad \|x+y\|_w \leq \|x\|_w + \|y\|_w$$

(iii'') if w is complex
$$\forall x, y \in E \quad \|x+y\|_w^{1/2} \leq \|x\|_w^{1/2} + \|y\|_w^{1/2}.$$

NB

In particular $\|\cdot\|_w$ is continuous for the w-topology on E which implies the following proposition:

Proposition

Let $\|\cdot\|_w$ and $\|\cdot\|'_w$ be norms on E.
They are equivalent: $\exists\, C_1, C_2 \in \mathbb{R}_{>0}$ with $C_1 < C_2$ such that

$$\forall x \in E,\ C_1 \|x\|_w \leq \|x\|'_w \leq C_2 \|x\|_w.$$

Proof

We can define a continuous map

$$\mathbb{P}(E) = \{\text{subspaces of dim. } 1 \text{ in } E\} \longrightarrow \mathbb{R}_{>0}$$
$$\mathbb{K}_w x \longrightarrow \frac{\|x\|_w}{\|x\|'_w}$$

since $\mathbb{P}(E)$ is compact, this map reaches its minimum and its maximum. $\square$

Definition (continued)

The norm $\|\cdot\|_w$ will be said to be classical if:

(iv) if w is ultrametric

$$\mathrm{Im}(\|\cdot\|_w) \subset \mathrm{Im}(|\cdot|_w)$$

(iv') if w is real, $\|\cdot\|_w$ is euclidean: there exists a positive definite quadratic form q on E such that

$$\forall x \in E, \quad \|x\|_w = \sqrt{q(x)}$$

(iv'') if w is complex, there exists a positive definite hermitian form h on E such that

$$\forall x \in E, \quad \|x\|_w = h(x)$$

Remark

Let w be an ultrametric place and $\|\cdot\|_w$ be a classical norm on a $\mathbb{K}_w$ vector space E.

Then

$$\Lambda = \{x \in E \mid \|x_w\| \leq 1\}$$

is a sub-$\mathcal{O}_w$ module of E where $\mathcal{O}_w = \{x \in \mathbb{K}_w \mid |x|_w \leq 1\}$

let $e_1, \dots, e_n$ be a basis of E over $\mathbb{K}$

we define $\| \sum_{i=1}^{n} x_i e_i \|'_w = \sup_{1 \le i \le n} |x_i|_w$

Let $x \in E - \{0\}$ be such that

$\frac{\|x\|_w}{\|x\|'_w}$ is minimal

and let $\lambda \in \mathbb{K}_w$ be such that $\frac{\|x\|_w}{\|x\|'_w} = |\lambda|_w$

$\forall y \in \Lambda \quad \|\lambda y\|'_w = \frac{\|x\|_w}{\|x\|'_w} \|y\|'_w \le \|y\|_w \le 1$

So $\lambda\Lambda \subset \bigoplus_{i=1}^{n} \mathcal{O}_w e_i$

Since $\mathcal{O}_w$ is a local ring, it is principal and $\lambda\Lambda$ (and thus Λ) is a free $\mathcal{O}_w$ module of rank $\le n$.

but for $i \in \{1, \dots, n\}$ if $|\lambda_i|_w = \|e_i\|_w$ $\lambda_i^{-1} e_i \in \Lambda$

So Λ is a free $\mathcal{O}_w$ module of rank n

(we say that Λ is $\mathcal{O}_w$ lattice in E).

(This is in fact true for any w-adic norm)

But since $\|\cdot\|_w$ is a classical norm,

$\|x\|_w = \min\{ |\lambda|_w, \lambda \in \mathbb{K}_w^*, \lambda^{-1} x \in \Lambda \}$

So we get a bijective map

$\mathcal{O}_w$ lattices in E $\Longleftrightarrow$ classical norms on E

23/5/2016 <u>Terminology</u>

A <u>w-adically normed space</u> [of finite dimension] is a $\mathbb{K}_w$ vector space equipped with a w-adic norm. All spaces I consider will be of finite dimension

Examples

a) $\mathbb{K}_w$ with $\|x\|_w = |x|_w$

b) E, F with classical norms $\|\cdot\|_w$, $\|\cdot\|'_w$

On $E \oplus F$

$$\|(x,y)\|''_w = \begin{cases} \sup(\|x\|_w, \|y\|'_w) & \text{if } w \text{ ultrametric},\\ \sqrt{\|x\|_w^2 + \|y\|'^2_w} & \text{if } w \text{ real},\\ \|x\|_w + \|y\|'_w & \text{if } w \text{ complex}. \end{cases}$$

$E \oplus F$ equipped with this norm is called the <u>direct sum</u> of the w-adic normed space E and F

If both norms are classic so is the norm on the direct sum. More precisely

If $\|\cdot\|_w$ is defined by a $\mathcal{O}_w$-module $\mathcal{E}$ and $\|\cdot\|'_w$ ——— by ——— $\mathcal{F}$ then the norm on $E \oplus F$ is defined by $\mathcal{E} \oplus \mathcal{F}$

c) Same notations as b)

Assume the norms are <u>classic</u>

There is a unique norm $\|\cdot\|''_w$ on $E \otimes F$ such that

$$\|x \otimes y\|''_w = \|x\|_w \otimes \|y\|'_w$$

it corresponds to

$$\begin{cases} q \otimes q' & \text{if } w \text{ is real} \\ h \otimes h' & \text{if } w \text{ is complex} \\ \mathcal{E} \otimes \mathcal{F} & \text{if } w \text{ is ultrametric} \end{cases}$$

d) If $F \subset E$ is a sub vector space the restriction of a w-adic norm is a w-adic norm, and the restriction of a classic norm is classic (given by $\mathcal{E} \cap F$ for w ultrametric)

e) quotient

$F \subset E$ subspace

$\|\cdot\|_w$ norm on E

Let $\pi : E \to E/F$ be the canonical projection

Let $z \in E/F$ and $y \in E$ such that $\pi(y) = z$

$$\{x \in E \mid \pi(x) = z \ \& \ \|x\|_w \leq \|y\|_w\}$$

is compact

Therefore $\{\|x\|_w, x \in \pi^{-1}(\{z\})\}$

has a minimal element

We define

$$\begin{aligned} \|\cdot\|'_w : E/F &\longrightarrow \mathbb{R}_{\geq 0} \\ z &\longmapsto \min\{\|x\|_w, x \in \pi^{-1}(\{z\})\} \end{aligned}$$

I claim this defines a w-adic norm on E/F.

(i) $\|z\|'_w = 0 \iff \exists x \in \pi^{-1}(\{z\}), \|x\|_w = 0$

$\iff 0 \in \pi^{-1}(\{z\})$

$\iff z = 0$

(ii) for $\lambda \in \mathbb{K}_w^*$

$$\pi^{-1}(\{\lambda z\}) = \lambda \pi^{-1}(\{z\})$$

so $\|\lambda z\|'_w = |\lambda|_w \|z\|'_w$

(iii) Let $z, z' \in E/F$ $y, y' \in E$

such that

$\pi(y) = z$, $\|y\|_w = \|z\|'_w$

$\pi(y') = z'$ $\|y'\|_w = \|z'\|'_w$

$$\|z + z'\|'_w \leq \|y + y'\|_w \leq \begin{cases} \sup(\|y\|_w, \|y'\|_w) & \text{ultrametric} \\ \|y\|_w + \|y'\|_w & \text{real} \\ (\|y\|_w^{1/2} + \|y'\|_w^{1/2})^2 & \text{complex} \end{cases}$$

Moreover $\|\cdot\|'$ is classic if $\|\cdot\|_w$ is

So that

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If w is ultrametric and $\|\cdot\|_w$ defined by an $\mathcal{O}_w$-module $\mathcal{E}$, $\|\cdot\|'_w$ is defined by $\mathcal{E}/\mathcal{E}\cap F$

If w is real (resp. complex) π induces an isomomorphism of euclidean (resp. hermitian) spaces from $F^\perp$ to E/F
(where $F^\perp$ is the orthogonal of F)

Terminology

A sequence of w-adic normed spaces

$$0 \to N \to E \to Q \to 0$$

is said to be exact if it is isomorphic (in the obvious sense) to a sequence of the form

$$0 \to F \to E \to E/F \to 0$$

Examples (continued)

f) E space $\|\cdot\|_w$ classic w-adic norm on E
we are going to define a w adic norm on the exterior product.

- If w is ultrametric, $\|\cdot\|'_w$ on $\Lambda^k E$ is defined by $\Lambda^k \mathcal{E}$ (if $(e_1, \dots, e_n)$ is a basis of the $\mathcal{O}_w$ module $\mathcal{E}$, $(e_{i_1} \wedge \dots \wedge e_{i_k})_{(1 \leq i_1 < \dots < i_k \leq n)}$ is a basis of $\Lambda^k \mathcal{E}$.
- If w is real (resp. complex), let $\langle \cdot, \cdot \rangle$ be the bilinear (resp. sesquilinear) form on E defining the norm.
Then there is a unique form on $\Lambda^k E$ such

that

$$\langle x_1\wedge\cdots\wedge x_k, y_1\wedge\cdots\wedge y_k\rangle = \det\left((\langle x_i, y_j\rangle)_{\substack{1\le i\le k\\ 1\le j\le k}}\right)$$

If $(e_1, \dots, e_n)$ is an orthonormal basis of E then

$(e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k})_{1\le i_1<\cdots<i_k\le n}$ is an orthonormal basis for $\Lambda^k E$

2

$$\Lambda^k E = E^{\otimes k}/I_k$$

$$I_k = \langle x\otimes x,\ x\in E\rangle \cap E^{\otimes k}$$

But the norm on $\Lambda^k E$ is <u>not</u> the quotient of the norm on E. in $E\otimes E$

For example, $\left\|\frac{1}{2}(e_1\otimes e_2 - e_2\otimes e_1)\right\| = \frac{1}{\sqrt{2}} < 1$ and this maps to $e_1\wedge e_2$ in $\Lambda^2 E$.

h) dual space

There exists a unique w-adic norm on $E^\vee$ such that

$$\forall\ y'\in E^\vee,\ y\in E \qquad \|y'\|_w\|y\|_w = |\langle y', y\rangle|_v$$
$$= |y'(y)|_v$$

<u>Remark</u>

We may define the grothendieck ring of <u>classic</u> w-adically normed spaces of finite dimension generated by isomorphism classes of w-adically normed spaces with relations

- $[E] = [N] + [Q]$
  if $0\to N\to E\to Q\to 0$ is exact.
- $[E]\times[F] = [E\otimes F]$

$\to \lambda_i([E]) = [\Lambda^i E]$, involution $[E]\mapsto[E^\vee]$

b) Adelic norms and metrics

We are now going to define norms on vector bundles

In this paragraph, $\mathbb{K}$ denotes a number field and V a nice variety over $\mathbb{K}$.

Definition

Let E be a vector bundle on V.

a w-adic norm on E is a continuous map

$$\|\cdot\|_w : E(\mathbb{K}_w) \longrightarrow \mathbb{R}_{\geq 0}$$

such that for any $x \in V(\mathbb{K}_w)$ the restriction of $\|\cdot\|_w$ to the $\mathbb{K}_w$-vector space $E(x)$ is a w-adic norm. It is said to be classic if $\|\cdot\|_{w|E(x)}$ is classic for any $x \in V(\mathbb{K}_w)$

Fundamental example

Assume that w is ultrametric and that $V_{\mathbb{K}_w}$ (resp $E_{\mathbb{K}_w}$) admits a model $\mathcal{V}$ (resp $\mathcal{E}$) on $\mathcal{O}_w$ (resp. $\mathcal{V}$), with $\mathcal{V}$ projective.

Let r be the rank of E.

Since $\mathcal{V}$ is projective, the natural map

$$\mathcal{V}(\mathcal{O}_w) \longrightarrow V(\mathbb{K}_w)$$

is bijective.

Let $x \in V(\mathbb{K}_w)$ and let $\tilde{x} \in \mathcal{V}(\mathcal{O}_w)$ be the corresponding point.

$\mathcal{E}(\tilde{x})$ is a $\mathcal{O}_w$-module projective of rank r. Since $\mathcal{O}_w$ is principal, $\mathcal{E}(\tilde{x})$ is free. Thus it is an $\mathcal{O}_w$ lattice

in $\mathcal{E}_{/\mathbb{K}}(x) \simeq E(x)$ and defines a norm
$\|\cdot\|_w : E(x) \to \mathbb{R}_{\geq 0}$ we get a w-adic norm
$$\|\cdot\|_w : E(\mathbb{K}_w) \longrightarrow \mathbb{R}_{\geq 0}$$
which is said to be <u>defined by the model $\mathcal{E}$</u>.
it is classic

<u>Particular case</u>

$\mathbb{P}^N$ and $G_{\mathbb{P}^N}(1)$ are defined over $\mathbb{Z}$, as well as $G(-1)$
Let p be a prime number and $x \in \mathbb{P}^n(\mathbb{Q}_p)^{\mathbb{P}^n}$
$x = [x_0 : - : x_p]$
$G_{\mathbb{P}^n_{\mathbb{Z}_p}}(-1)(x)$ corresponds to the $\mathbb{Z}_p$ module
$$\mathbb{Q}_p(x_0, -, x_n) \cap \mathbb{Z}_p^{n+1}$$
it is generated by $\left(\max\limits_{0 \leq i \leq n} |x_i|_p\right)(x_0, -, x_n)$

Remember that $|p^{-k}|_p = p^k$
In other words on $G_{\mathbb{P}^n}(-1)$ we get the norm
$$\|(y_0, -, y_n)\|_p = \max_{0 \leq i \leq n} |x_i|_p$$
as expected and, by duality on $G_{\mathbb{P}^n}(1)$
$$\|X_i(x)\|_p = \frac{|x_i|_p}{\max\limits_{0 \leq i \leq n} |x_i|_p}.$$
as I explained at the end of last chapter.

<u>Definition</u>

Let E be a vector bundle on V
An adelic norm on E is a family $(\|\cdot\|_w)_{w \in Pl(\mathbb{K})}$
where $\|\cdot\|_w$ is a w-adic norm on E,
such that there is a finite set $S \subset Pl(\mathbb{K})_f$,
and a model $\mathcal{E}$ of E over G_S such that

$\|\cdot\|_w$ is defined by $\mathcal{E}$ for $w \in Pl(\mathbb{K})_f - S$

An <u>adelic metric</u> on V is an adelic norm on TV

<u>Convention</u>

From now on, unless otherwise explicitely stated, all norms will be assumed to be <u>classical</u> !

<u>Remark</u>

(i) If $(\|\cdot\|_w)_{w \in Pl(\mathbb{K})}$ is an adelic norm on E then for any model $\mathcal{E}$ of E over $\mathcal{O}_S$, for some finite $S \subset Pl(\mathbb{K})_f$, there exists $S' \subset Pl(\mathbb{K})_f$ finite and containing S such that for any $w \in Pl(\mathbb{K})_f - S'$, $\|\cdot\|_w$ is defined by $\mathcal{E}$. As a consequence, if $(\|\cdot\|_w)_{w \in Pl(\mathbb{K})}$ and $(\|\cdot\|'_w)_{w \in Pl(\mathbb{K})}$ are adelic norms on E, $\|\cdot\|_w = \|\cdot\|'_w$ for almost all $w \in Pl(\mathbb{K})$.

(ii) If $x \in V(\mathbb{K})$,

$\Lambda = \{ y \in E(x) \mid \forall w \in Pl(\mathbb{K})_f, \ \|y\|_w \leq 1 \}$

defines an $\mathcal{O}_{\mathbb{K}}$ - submodule of $E(x)$, which is a $\mathbb{K}$ - vector space, which is locally free of rank r.

Choose S, finite, such that $\mathcal{O}_S$ is principal and let $(e_1, \ldots, e_r)$ be a basis of the free $\mathcal{O}_S$ - module $\Lambda \otimes_{\mathcal{O}_{\mathbb{K}}} \mathcal{O}_S \xrightarrow{\sim} \mathcal{O}_S \Lambda \subset E(x)$.

Let $y \in E(x) - \{0\}$ $y = \sum_{i=1}^{r} y_i e_i$ and for any $w \in Pl(\mathbb{K})_f - S$, we have $\|y\|_w = \max_{1 \leq i \leq r} |y_i|_w$

we get that $\|y\|_w = 1$ for almost all $w \in Pl(\mathbb{K})$

by almost all, I mean for any w outside a finite set of places.

(iii) Note that over $\mathbb{Q}$, a real norm $\|\cdot\|_\infty$ on TV is a continuous riemannian metric on $V(\mathbb{R})$. So adelic metric should be thought of as a generalisation of riemannian metrics

Terminology

I shall say "adelic bundle" for a vector bundle equipped with a [classical] adelic metric.

c) Examples

α) If $V = \mathrm{Spec}(\mathbb{K})$ (V is a point), then an adelic bundle on V is the same as a $\mathbb{K}$-vector space E of dim n with:

- An $\mathcal{O}_{\mathbb{K}}$-submodule $\mathcal{E} \subset E$ which is projective, of rank n
- for any real w, a euclidean norm on $E \otimes_{\mathbb{K}} \mathbb{K}_w$
- for any complex w, a positive definite hermitian form on $E \underset{\mathbb{K}}{\otimes} \mathbb{K}_w$

Note that the image of $\mathcal{E}$ in $E \otimes_{\mathbb{Q}} \mathbb{R} \cong \bigoplus_{w|\infty} E \otimes_{\mathbb{K}} \mathbb{K}_w$ is a lattice, in the usual sense, of the $\mathbb{R}$-vector space $E \underset{\mathbb{R}}{\otimes} \mathbb{R}$ which is of dimension $[\mathbb{K}:\mathbb{Q}]\, n$. and therefore it has a covolume

β) Let us consider the trivial line bundle $V \times \mathbb{A}^1_{\mathbb{K}}$. On it, we define the natural adelic metric by:

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For any $w \in Pl(\mathbb{K})$ and any $x \in V(\mathbb{K}_w)$ the fibre is canonically isomorphic to $\mathbb{K}_w$

$$\|y\|_w = |y|_w$$

γ) Using the constructions described for vector bundles and w-adic metrics, We can define direct sum $E \oplus F$, tensor products $E \otimes F$, exterior power $\Lambda^i E$, dual $E^\vee$, of adelic bundles.

Notations

We denote by $\widehat{K}_0(V)$ the Grothendieck ring of adelic bundles on V, equipped with the λ-operations, $\lambda^i([E]) = [\Lambda^i E]$. and by $\widehat{\mathrm{Pic}}(V)$ the group of adelic line bundles on V for the tensor product of adelic line bundles.

Remark

The neutral element in $\widehat{\mathrm{Pic}}(V)$ is the trivial line bundle with its natural adelic metric and the opposite of L is $L^\vee$.

References

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Summer School in GRENOBLE in june 2017.

S] Let $\varphi : X \to Y$ be a morphism of nice varieties and let E be an adelic bundle on Y.
then, for any $w \in Pl(\mathbb{K})$ and any $x \in X(\mathbb{K}_w)$ the fibre $\varphi^*(E)(x)$ is canonically isomorphic to $E(\varphi(x))$ and the norm $\|\cdot\|_w$ on $E(\varphi(x))$ defines a norm on $\varphi^*(E)(x)$
we get an adelic norm on $\varphi^*(E)$
and $\varphi^*(E)$ with this norm is called the <u>pull-back of E</u>

We get in that way morphisms
$\widehat{K}_0(Y) \to \widehat{K}_0(X)$ and $\widehat{Pic}(X) \to \widehat{Pic}(Y)$
so that $\widehat{K}_0$ and $\widehat{Pic}$ are contravariant functors.

d) first properties

Proposition 1

Let V be a nice variety $/\mathbb{K}$
Let E be a vector bundle on V
Then there exists an adelic metric on E.

Proof

We choose $S \subset Pl(\mathbb{K})_f$, finite and a model $\mathcal{E}$ of E over $\mathcal{O}_S$. By what I explained about models, I can do that.
This defines w-adic norms $\|\cdot\|_w$ on E for $w \in Pl(\mathbb{K}) - S$.

such that the map

$$\{y \in W \mid \forall i \in \{1, \ldots, r\}, f_i(y) = 0\} \longrightarrow W'$$
$$(y_1, \ldots, y_N) \longmapsto (y_{i_1}, \ldots, y_{i_{N-r}})$$

is a diffeomorphism.

To deal with the finite set of bad places I am going to use a few results from differential geometry, (in fact it is more than I need)

Theorem (Implicit function theorem)

Let $\mathbb{K}$ be a field which is complete for an absolute value $|\cdot|$ and $U \subset \mathbb{K}^N$ be an open subset. Let $f_1, \ldots, f_s : U \to \mathbb{K}$ be differentiable functions such that

$$rk \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{1 \leq i \leq s \\ 1 \leq j \leq N}} \text{ is constant on } U \text{ with value}$$

Then for any $x \in U$ such that $f_i(x) = 0$ for $i \in \{1, \ldots, s\}$ there exists $1 \leq i_1 < \cdots < i_{N-r} \leq N$, an open neighbourhood W of x in U an an open set $W' \subset \mathbb{K}^{N-r}$

This a generalization to complete valued fields of a classical result over $\mathbb{R}$.

This implies that the $\mathbb{K}$ points of any smooth varieties (in the algebraic sense) form a differential variety

<u>Corollary</u>

Let V be a nice variety over a number field $\mathbb{K}$. Let $w \in Pl(\mathbb{K})$ then there is a finite open covering $(U_i)_{i\in I}$ of $V(\mathbb{K}_w)$ such that each U_i is homeomorphic to open subset of $\mathbb{K}_w^n$. Moreover

— If w is archimedean, the transition maps are $\mathcal{C}^\omega$

— If w is ultrametric we may choose the covering so that $U_i \cap U_j \neq \phi$ if $i \neq j$. and that $U_i \simeq \mathcal{O}_w^n$.

<u>Remark</u>

In particular, it means that if there is a solution over $\mathbb{K}_w$, there are many of them. The adelic space is either empty or very big. Also Falting's theorem implies that curves of genus $\geqslant 2$ do not satisfy weak aproximation.

<u>Theorem</u> (Partition of unity)

Let K be a compact topological space, in fact normal space is enough, let $(U_i)_{i\in I}$ be a finite open covering of K, then there exists a family of functions $(f_i)_{i\in I}$ from K to $[0,1]$ such that

(i) $\forall i \in I, \forall x \in K \quad f_i(x) > 0 \Rightarrow x \in U_i$;

(ii) $\forall x \in K \quad \sum_{i\in I} f_i(x) = 1$.

Now I can go back to my proof:

End of the proof of the proposition

Take an open covering of V, for Zariski topology which trivializes E.

This means that for each $i \in I$, I may choose r sections

$$s_{i,1}, \ldots, s_{i,r} \in \Gamma(U_i, E)$$

such that for any point x of U_i over a field $\mathbb{L}$ $(s_{i,1}(x), \ldots, s_{i,r}(x))$ is a basis of $E(x)$

For $w \in Pl(\mathbb{K})$, $x \in U_i(\mathbb{K}_w)$ $y \in E(x)$ write $y = \sum_{j=1}^{r} \lambda_j s_{i,j}(x)$ $(\lambda_1, \ldots, \lambda_r) \in \mathbb{K}_w^r$

and put

$$\|y\|_w^i = \begin{cases} \max_{1 \leq j \leq r} |y_j|_w & \text{if } w \text{ ultrametric} \\ \sqrt{\sum_{j=1}^{r} y_j^2} & \text{if } w \text{ is real} \\ \sum_{j=1}^{r} y_j \bar{y}_j & \text{if } w \text{ is complex} \end{cases}$$

If w is archimedean let $(\rho_i)_{i \in I}$ be a partition of 1 for the covering $(U_i(\mathbb{K}_w))_{i \in I}$ we define

$$\|y\|_w = \begin{cases} \sum_{\{i \in I \mid x \in U_i\}} \sqrt{\rho_i(x) \left(\|y\|_w^i\right)^2} & \text{if } w \text{ is real} \\ \sum_{\{i \in I \mid x \in U_i\}} \rho_i(x) \|y\|_w^i & \text{if } w \text{ is complex} \end{cases}$$

If w is ultrametric, using the corollary, we choose a refinement $(W_k)_{k \in K}$ of $(U_i(\mathbb{K}_w))_{i \in I}$ which is a covering by disjoint open sets and for any $k \in K$, choose $i \in I$ such that $W_k \subset U_i(\mathbb{K}_w)$ and write, for $x \in W_k$, $y \in E(x)$,

$$\|y\|_w = \|y\|_w^i. \qquad \square$$

25/5/2016 **Proposition**

Let V be a nice variety on the number field $\mathbb{K}$, and let E, F be adelic bundles on V.
Let $\varphi : E \to F$ be a morphism of vector bundles.
Then there is a family $(C_w)_{w \in Pl(\mathbb{K})} \in \mathbb{R}_{>0}^{(Pl(\mathbb{K}))}$ with $C_w = 1$ for almost all $w \in Pl(\mathbb{K})$ such that

$$\forall w \in Pl(\mathbb{K}), \forall x \in V(\mathbb{K}_w), \forall y \in E(x), \|\varphi(x)\|_w \leq C_w \|x\|_w$$

Proof

- We may consider the projective bundle associated to E:

$$\begin{array}{c} \mathbb{P}(E) = \mathrm{Proj}(\underbrace{\mathrm{Sym}^*(E^\vee)}_{\text{Symmetric } \mathcal{O}_V \text{ graded algebra generated by the sections of } E^\vee}) \\ \big\downarrow \pi \\ V \end{array}$$

For any $\mathbb{K}$-algebra A and any point $x \in V(A)$, $\mathbb{P}(E)(x) = \pi^{-1}(x)$ may be identified with the set of direct factors of $E(x)$ (seen as a projective A-module) of rank 1.

- Then, for $w \in Pl(\mathbb{K})$, we have a map

$$\begin{array}{ccc} \mathbb{P}(E)(\mathbb{K}_w) & \longrightarrow & \mathbb{R} \\ \mathbb{K}_w y & \longmapsto & \dfrac{\|\varphi(y)\|_w}{\|y\|_w} \text{ for } y \in E(x) - \{0\} \\ \cap & & \\ E(x) & & \end{array}$$

But $\mathbb{P}(E)(\mathbb{K}_w)$ is compact so this function admits a maximal value C_w

- It remains to prove that $C_w \leq 1$ for almost all w.

There is a finite set of places $S \subset Pl(\mathbb{K})_f$ so that V (resp E, F) has a model $\mathcal{V}$ (resp $\mathcal{E}$, $\mathcal{F}$) on $\mathcal{O}_S$ and there is a morphism

$$\tilde{\varphi} : \mathcal{E} \longrightarrow \mathcal{F}$$

so that $\tilde{\varphi}_{\mathbb{K}} = \varphi$

Moreover we may assume that the norms are defined by $\mathcal{E}$ and $\mathcal{F}$ outside S

Let $w \in Pl(\mathbb{K})_f - S$.

For any $x \in V(\mathbb{K}_w)$ corresponding to $\tilde{x} \in \mathcal{V}(\mathcal{O}_w)$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}(\tilde{x}) & \xrightarrow{\tilde{\varphi}} & \mathcal{F}(\tilde{x}) \\ \downarrow & & \downarrow \\ E(x) & \xrightarrow{\varphi} & F(x) \end{array}$$

So for $y \in E(x)$ $\|y\|_w \leq 1 \Rightarrow \|\varphi(y)\|_w \leq 1$ which implies that

$$\forall y \in E(x) \quad \|\varphi(y)\|_w \leq \|y\|_w$$

In other words $C_w \leq 1$. $\square$

If $C_w \leq 1$, I may take $C_w = 1$ instead.

<u>Corollary</u>

Let E be a vector bundle on V and let $(\|\cdot\|_w)_{w \in Pl(\mathbb{K})}$ and $(\|\cdot\|'_w)_{w \in Pl(\mathbb{K})}$ be adelic metrics on E then there exist constants $(C_w)_{w \in Pl(\mathbb{K})}$ and $(C'_w)_{w \in Pl(\mathbb{K})}$ such that

(i) $C_w = C'_w = 1$ for almost all w

(ii) $\forall w \in Pl(\mathbb{K})$, $\forall y \in E(\mathbb{K}_w)$, $C_w \|y\|_w \leq \|y\|'_w \leq C'_w \|y\|_w$

<u>Proof</u>

Apply prop. to Id_E twice. $\square$

4) Heights, height zeta function

a) Height pairing

Again $\mathbb{K}$ denotes a number field and V a nice variety $/\mathbb{K}$

Definition

Let E be an adelic bundle on V

Let $x \in V(\mathbb{K})$. Let $y \in E(x) - \{0\}$

$\{w \in Pl(\mathbb{K}) \mid \|y\|_w \neq 1\}$ is finite

and $\forall \lambda \in \mathbb{K}^*$ $\prod_{w \in Pl(\mathbb{K})} \|\lambda y\|_w = \left(\prod_{w \in Pl(\mathbb{K})} |\lambda|_w\right) \prod_{w \in Pl(\mathbb{K})} \|y\|_w$

$$= \prod_{w \in Pl(\mathbb{K})} \|y\|$$

Therefore this product depends only on x.

We define the exponential height of x relative to E as the product

$$H_E(x) = \prod_{w \in Pl(\mathbb{K})} \|y\|_w^{-1}$$

The corresponding logarithmic height is $h_E = \log \circ H_E$

We get a map

$$\widehat{Pic}(V) \times V(\mathbb{K}) \longrightarrow \mathbb{R}_{>0}$$
$$(E, x) \longmapsto H_E(x)$$

which is called the height pairing

For any given x the map which sends E onto $h_E(x)$ is a morphism of group, so we may see this pairing as a map

$$V(\mathbb{K}) \longrightarrow Hom_{gr}(\widehat{Pic}(V), \mathbb{R}_{>0})$$

Remarks

(i) Let us say that E and E' are equivalent if there exists an isomorphism of vector bundles $\varphi : E \xrightarrow{\sim} E'$ and $(\lambda_w)_{w \in Pl(\mathbb{K})} \in \mathbb{R}_{>0}^{(Pl(\mathbb{K}))}$

with $\lambda_w = 1$ for almost all w
such that
(i) $\prod_{w \in Pl(\mathbb{K})} \lambda_w = 1$
(ii) $\forall w \in Pl(\mathbb{K})\ \forall y \in E(\mathbb{K}_w) \quad \|\varphi(y)\|'_w = \lambda_w \|y\|_w$
Equivalent line bundles define the same height
If we define
$$\mathcal{H}(V) = \widehat{\mathrm{Pic}}(V)/\underbrace{\sim}_{\text{this equivalence}}$$
We get a map
$$V(\mathbb{K}) \longrightarrow \mathrm{Mor}(\mathcal{H}(V), \mathbb{R}_{>0})$$
(iii) There is a natural action of $\mathbb{R}_{>0}$ on $\mathcal{H}(V)$
Let $\lambda \in \mathbb{R}^*$. choose any $w_0 \in Pl(\mathbb{K})_\infty$
and maps the class of E equipped with $(\|\cdot\|_w)_{w \in Pl(\mathbb{K})}$
to E with $(\|\cdot\|'_w)_{w \in Pl(\mathbb{K})}$
with
$$\|\cdot\|'_w = \begin{cases} \|\cdot\|_w & \text{if } w \neq w_0 \\ \lambda \|\cdot\|_{w_0} & \text{for } w = w_0 \end{cases}$$
We get
$$V(\mathbb{K}) \longrightarrow \mathrm{Hom}_{\underbrace{\mathbb{R}_{>0}\text{-}gr}_{\text{group with } \mathbb{R}^* \text{ action}}}(\mathcal{H}(V), \mathbb{R}_{>0}) = \mathcal{H}(V)^\vee$$

Reminder

The functor which maps an adelic bundle to the corresponding vector bundle define morphisms
$$o : \widehat{K}_0(V) \longrightarrow K_0(V) \quad \text{and} \quad o : \widehat{\mathrm{Pic}}(V) \to \mathrm{Pic}(V)$$

Proposition

Let E, E' be adelic line bundles on V
such that $o(E) = o(E')$ then there exists
$C_1, C_2 \in \mathbb{R}_{>0}$ such that
$$\forall x \in V(\mathbb{K}) \qquad C_1 < \frac{H_E(x)}{H_{E'}(x)} < C_2$$

So, if we change the adelic norm, the change to the height is bounded and the line bundle determines the height up to a bounded function!

Proof

Apply last corollary. □

Corollary

More generally, if there exists $n \geq 1$ such that $o(E^{\otimes n}) \simeq o(E'^{\otimes n})$ then $H_{E'}/H_E$ is bounded.

I am going to admit the following

Theorem

$\mathrm{Pic}(V)$ is a finitely generated group.

Idea

There is an exact sequence

$$0 \longrightarrow \underbrace{\underbrace{\mathrm{Pic}^0(V)(\mathbb{K})}_{\text{abelian variety}} \longrightarrow \mathrm{Pic}(V)}_{\text{finitely generated}} \longrightarrow \underbrace{NS(V)}_{\text{finitely generated}} \longrightarrow 0$$

Definition

A <u>system of heights</u> is a section $L \mapsto \tilde{L}$ of

$$\alpha : \mathrm{Pic}(V) \longrightarrow \mathrm{Pic}(V) \longrightarrow \mathrm{Pic}(V)/\mathrm{Pic}(V)_{(\mathrm{tors})}$$

It defines a map

$$\begin{tikzcd}
V(\mathbb{K}) \arrow[r] \arrow[d, "\mathbb{H}"] & \mathrm{Mor}_{gr}(\mathrm{Pic}(V), \mathbb{R}^*) \arrow[r, "\sim"] & \mathrm{Mor}_{gr}(\mathrm{Pic}(V), \mathbb{R}) \arrow[d, "\wr"] \\
\mathrm{Mor}_{gr}(\mathrm{Pic}(V)\otimes\mathbb{C}, \mathbb{C}^*) & \mathrm{Mor}_{\mathbb{C}\text{-ev}}(\mathrm{Pic}(V)\otimes_{\mathbb{Z}}\mathbb{C}, \mathbb{C}) \arrow[l] & \mathrm{Mor}_{gr}(\mathrm{Pic}(V), \mathbb{C}) \arrow[l, "\sim"']
\end{tikzcd}$$

$\varphi \longmapsto \log \circ \varphi$

$\exp \circ \psi \longleftarrow\!\shortmid\ \psi$

In other words,

$$\mathbb{H}(x)(L^{\otimes s}) = H_L(x)^s = \exp(s\, h_L(x))$$

To give you an example which uses the flexibility of the notion of heights we are using, let me give one example

b) Particular heights

We shall consider the following particular case:

We assume that there exists a morphism

$$\varphi : V \longrightarrow V$$

$d \geqslant 2$ and a line bundle L on V with an isomorphism

$$L^{\otimes d} \xrightarrow[\sim]{\psi} \varphi^*(L)$$

Example

take $c \in \mathbb{Q}(i)$

$$\varphi : \mathbb{P}^1_{\mathbb{Q}(i)} \longrightarrow \mathbb{P}^1_{\mathbb{Q}(i)}$$

$$[x : y] \longmapsto [x^2 + y^2 c : y^2]$$

$$\varphi^*(\mathcal{O}(1)) = \mathcal{O}(2).$$

Let $(\|\cdot\|_w)_{w \in \mathrm{Pl}(\mathbb{K})}$ be an adelic metric on L then there are constants $(C^1_w)_{w \in \mathrm{Pl}(\mathbb{K})}$ and $(C^2_w)_{w \in \mathrm{Pl}(\mathbb{K})}$ almost all equal to 1 such that for any $w \in \mathrm{Pl}(\mathbb{K})$

$$\forall \underset{\underset{L(x)}{\cap}}{y} \in E(\mathbb{K}_w) \quad C^1_w \|y\|^d_w \leqslant \|\underbrace{\psi(y^{\otimes d})}_{\in L(\varphi(x))}\| \leqslant C^2_w \|y\|^d_w$$

Taking logarithms we have

$$\left| \frac{1}{d} \log(\|\psi(y^{\otimes d})\|_w) - \log \|y\|_w \right| \leqslant \frac{1}{d} C$$

where $C = \max(|\log(C^1_w)|, |\log(C^2_w)|)$.

let us consider the sequence $\left(\frac{1}{d^k} \log(\|\psi^k(y^{\otimes d^k})\|_w)\right)_{k \geqslant 0}$

we have for $p \leq q$

$$\left| \frac{1}{d^p} \log \| \psi^p (y^{\otimes d^p}) \|_w - \frac{1}{d^q} \log \| \psi^q (y^{\otimes d^q}) \|_w \right|$$

$$\leq \frac{1}{d^p} \left(\sum_{k=1}^{q-p} \frac{1}{d^k} \right) C \leq \frac{C}{d^p (d-1)}$$

which proves that the sequence converges uniformly and we may define

$$\| y \|'_w = \lim_{k \to +\infty} \| \psi^k (y^{\otimes d^k}) \|_w^{1/d^k}$$

$$\cap \qquad L(\psi^k(x))$$

we get an adelic norm $(\| \cdot \|'_w)_{w \in \mathcal{M}(\mathbb{K})}$ on L

2 not necessarily classical

such that

$$\forall w \in \mathcal{M}(\mathbb{K})\ \forall y \in E(\mathbb{K}_w) \quad \| \psi (y^{\otimes d}) \|'_w = \left(\| y \|'_w \right)^d$$

This implies that the corresponding height satisfies

$$H'(\psi(x)) = H'(x)^d .$$

<u>Particular case</u>

Take an abelian variety $A / \mathbb{K}$ (say a projective algebraic group over $\mathbb{K}$) and L an ample symmetric line bundle on A (that is $[-1]^* L \simeq L$ where $[n] : A \longrightarrow A$, $P \longmapsto nP$)

Then one can show that $[2]^* L \simeq L^{\otimes 4}$

we get a logarithmic height

$$h : A(\mathbb{K}) \longrightarrow \mathbb{R}$$

so that $\forall x \in A(\mathbb{K})$, $h(2x) = 4 h(x)$.

Theorem (NÉRON – LANG)

h defines a positive definite quadratic form on $A(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{R}$.

This defines the Neron – Tate pairing on $A(\mathbb{K})$

Corollary

h defines a euclidean structure on $A(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{R}$ and $A(\mathbb{K})/A(\mathbb{K})_{tors}$ embeds as a lattice in $A(\mathbb{K})_{\mathbb{Z}} \mathbb{R}$

Then

$$\# A(\mathbb{K})_{H' \leq B} \underset{B \to +\infty}{\sim} \frac{\#(A(\mathbb{K})_{tors})}{\operatorname{covol}(A(\mathbb{K})/A(\mathbb{K})_{tors})} \log(B)^{\frac{r}{2}}$$

where $r = \operatorname{rk}(A(\mathbb{K}))$

Remark

It is one of the very few cases where we have a behaviour with $\log(B)$ at the power half an integer

Proof of the corollary

Use MASSER & VAALER. $\square$

c) The height zeta function

Definition

If you remember to get equidistribution on the variety one has to consider open subsets

More generally, let V be a nice variety with a system of heights

For $W \subset V(\mathbb{K})$ and $s \in \mathrm{Pic}(V) \otimes_{\mathbb{Z}} \mathbb{C}$ we define

$$\zeta_W(s) = \sum_{P \in W} \frac{1}{\mathbb{H}(P)(s)} \quad \text{if this}$$

series converges. If $X \subset V$ is a subscheme, (closed or open) we write ζ_X for $\zeta_{X(\mathbb{K})}$

Remark

We are going to relate the properties of this function to the asymptotic behaviour of

$$\# W_{H_L \leq B} = \#\{P \in W \mid \mathbb{H}(P)(L) \leq B\}$$

Notation

any set, any map

For $H : W \longrightarrow \mathbb{R}_{>0}$

Assume $\forall B \in \mathbb{R}_{>0}$ $W_{H \leq B} = \{P \in W, H(P) \leq B\}$ is finite

Define $a_W(H) = \overline{\lim_{B \to +\infty}} \log(\# W_{H \leq B}) / \log(B) \leq +\infty$

Remark

If $\# W_{H \leq B} \underset{B \to +\infty}{\sim} C B^a \log(B)^{b-1}$

then $a_H(W) = a$ So it is the power of B in the asymptotic behaviour

Proposition With the preceding notations

Assume $a_H(W) < +\infty$. The series

$$\sum_{P \in W} \frac{1}{H(P)^s}$$

(i) converges absolutely if $\mathrm{Re}(s) > a_H(W)$

(ii) diverges if $s \in \mathbb{R}$, $s < a_H(W)$

Proof

Remember the 2^{nd} lecture:

Sam using STIELTJES integrals

Let $g(t) = \# W_{H \leq t}$ and $f(t) = \frac{1}{t^s}$

Then, by definition

$$\sum_{P \in W_{H \leq B}} \frac{1}{H(P)^s} = \int_0^B f(t)\, dg(t)$$

$$= \left[f(t) g(t) \right]_0^B - \int_0^B f'(t) g(t)\, dt$$

this is in fact a form of Abel summation formula

$$= \frac{g(B)}{B^s} + s \int_0^B \frac{g(t)}{t^{s+1}}\, dt \qquad (*)$$

(i) Let $\eta > 0$ $g(B) \ll_\eta B^{a_H(W)+\eta/2}$

so if $Re(s) > a_H(W) + \eta$ $(*)$ converges

(ii) if $s \in \mathbb{R}$, $s < a_W(H)$ $\eta = (a_W(H) - s)/2$

$\forall A \in \mathbb{R}_{>0}$ $\exists B \in \mathbb{R}$, $B > A$ and $g(B) \gg B^{s+\eta}$

Thus

$$\overline{\lim_{B \to +\infty}} \frac{g(B)}{B^s} = +\infty$$ and the series diverges

Notation

For $s \in Pic(V) \otimes_{\mathbb{Z}} \mathbb{R}$, we define

$$a_s(W) = \overline{\lim_{B \to +\infty}} \log(\# W_{H(\cdot)(s)}) / \log(B)$$

and

$$\Sigma_W = \{ s \in Pic(V) \otimes_{\mathbb{Z}} \mathbb{R} \mid a_s(W) < 1 \}$$

Remark

$$\Sigma_W \subset \{ s \in Pic(V) \otimes_{\mathbb{Z}} \mathbb{R} \mid \zeta_W(s) \text{ converges} \}$$
$$\subset \{ s \in Pic(V) \otimes_{\mathbb{Z}} \mathbb{R} \mid a_s(W) \leq 1 \}$$

d) Properties

Definition

A line bundle L is said to be effective if it has a non-zero section : $\Gamma(V,L) \neq \{0\}$

The effective cone $C_{eff}(V) \subset Pic(V) \otimes_{\mathbb{Z}} \mathbb{R}$ is

$$C_{eff}(V) = \overline{\bigcup_{L \text{ effective}} \mathbb{R}_{\geq 0}[L] \otimes 1}$$

It is the smallest closed cone in $Pic(V) \otimes_{\mathbb{Z}} \mathbb{R}$ which contains the classes of effective divisors.

Proposition [BATYREV-MANIN]

(i) The map $s \longmapsto a_s(W)$ and therefore Σ_W does not depend on the choice of the height system

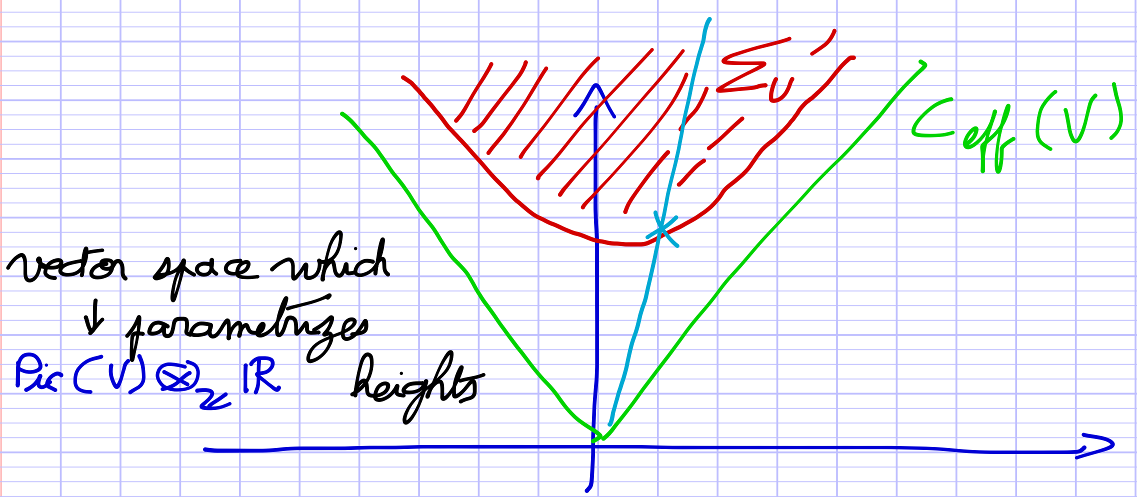
(ii) For any line bundle L such that $[L] \in \overset{\circ}{C_{eff}}(V)$ there exists an open set $U \subset V$ such that $a_U(L) < +\infty$ and $\mathbb{R}[L] \otimes 1$ meets Σ_U

(iii) For any line bundle L,

$$a_W(L^{\otimes N}) = \frac{1}{N} a_W(L)$$

In particular for any $\lambda \geq 1$ $\lambda \Sigma_U \subset \Sigma_U$

(iv) Σ_U is convex.



Remarks

a) If $W \subset W'$ then $\Sigma_{W'} \subset \Sigma_W$

b) a_W can be computed from Σ_W

For $s \in \mathrm{Pic}(V) \otimes_{\mathbb{Z}} \mathbb{R}$

(i) $a_W(s) < +\infty \Leftrightarrow \mathbb{R}_{>0}\, s \cap \Sigma_W \neq \emptyset$

(ii) in that case, $a_W(s) = \min\{\lambda > 0,\ \lambda s \in \Sigma_W\}$.

Proof

Take adelic line bundles L and M such that $L^{\otimes p} \simeq M^{\otimes q}$

Then there exists constants $c_2 > c_1 > 0$ such that

$$0 < c_1 < \frac{H_M(P)^q}{H_L(P)^p} < c_2$$

for $P \in W$

But we get

$$\begin{aligned}\# W_{H_M \leq B} &= \# \{P \in W \mid H_M(P) \leq B\} \\ &\leq \# \{P \in W \mid c_1 H_L(P)^{p/q} \leq B\} \\ &= \# W_{H_L \leq \left(\frac{B}{c_1}\right)^{q/p}}\end{aligned}$$

So we have

$$\begin{aligned}a_W(M) &= \overline{\lim}\left(\log(\# W_{H_M \leq B}) / \log(B)\right) \\ &\leq \overline{\lim}\left(\log(\# W_{H_L \leq p}) / \log(P)\right) \\ &\quad \times \underbrace{\lim_{B \to +\infty} \log\left(B/c_1\right)^{q/p} / \log(B)}_{= q/p}\end{aligned}$$

so $\frac{1}{q} a_W(M) \leq \frac{1}{p} a_W(L)$

By symmetry we have $=$ and we get (i) and (ii)

27/5/2016 To prove the second assertion, let me start with a

Lemma

Let L be an adelic line bundle which is effective as a line bundle. Then there exists an open set $U \subset V$ such that
$\forall\ x \in U(\mathbb{K}),\ H_L(x) > c.$

Proof

Take $s \in \Gamma(V, L) - \{0\}$ possible since L is effective and put $U = \{x \in V \mid s(x) \neq 0\}$ in terms of points.

For $w \in \mathcal{Pl}(\mathbb{K})$ the continuous map

$$V(\mathbb{K}_w) \longrightarrow \mathbb{R}_{\geq 0}$$
$$x \longmapsto \|s(x)\|_w$$

reaches its maximal value c_w.

Moreover s extends to $\tilde{s} : \mathcal{V} \longrightarrow \mathcal{L}$ for some models $\mathcal{V}, \mathcal{L}$ over some $\mathcal{O}_S$.

So for $w \in \mathcal{Pl}(\mathbb{K}) - S$ we may take $c_w = 1$.

For $x \in U(\mathbb{K})$

$$H_L(x) = \prod_{w \in \mathcal{Pl}(\mathbb{K})} \|s(x)\|_w^{-1} \geq \prod_{w \in \mathcal{Pl}(\mathbb{K})} c_w^{-1} \qquad \square$$

Proof of assertion (ii)

Since V is projective, V has ample line bundles. Let M be an ample line bundle on V.

Since $[L] \in \mathring{C}_{\mathrm{eff}}(V)$ there exists $N > 0$ such that

$$[L] \otimes 1 + [M] \otimes \frac{1}{N} \in C_{\mathrm{eff}}(V)$$

So $\exists\ p, q > 0$ such that $\Gamma(V, L^{\otimes p} \otimes M^{\otimes -q}) \neq \{0\}$

This implies that there exist an open set U and a constant C so that

$$\forall x \in U(\mathbb{K}),\ H_L(x)^p / H_M(x)^q > C$$

But, then,

$$\# U(\mathbb{Q})_{H_L \leq B} \leq \# U(\mathbb{Q})_{H_M \leq \frac{1}{C} B^{p/q}}$$

But M induces an embedding

$$\psi : V \hookrightarrow \mathbb{P}^N_{\mathbb{K}}$$

with $M = \psi^*(\mathcal{O}(1))$

For $\mathbb{P}^N_{\mathbb{K}}$, I will give you the proof of

<u>Theorem</u> (SCHANUEL) [To be proven]

$$\# \mathbb{P}^N(\mathbb{K})_{H_{\mathcal{O}(1)} \leq B} \sim C(\mathbb{P}^N_{\mathbb{K}})\, B^{N+1}$$

which we have already seen for $\mathbb{Q}$. So we get

$$a_L(U) \leq \frac{p}{q}(N+1).$$

It remains to prove the last statement of the theorem

<u>Proof of (iii)</u>

Take $s_1, s_2 \in \Sigma_U$ and let $s = \alpha s_1 + \beta s_2$ with $\alpha + \beta = 1$.

To prove that $s \in \Sigma_U$, we may assume $\alpha\beta \neq 0$

We may take $\eta > 0$ such that

$$a_W(s_i) < 1 - \eta$$

Then $a_W((1-\eta)s_i) > 1$

Put $s'_i = (1-\eta)s_i$ and $s' = (1-\eta)s$

the series $\displaystyle\sum_{P \in W} \frac{1}{H(x)^{s'_i}}$ converge

Put $p = \frac{1}{\alpha}$, $q = \frac{1}{q}$

By Hölder's inequality

$$\sum_{P\in W}\frac{1}{H(x)^{s'}} = \sum_{P\in W}\frac{1}{H(x)^{\frac{s_1'}{p}}}\times\frac{1}{H(x)^{\frac{s_2'}{q}}}$$

$$\leq \left(\sum_{P\in W}\frac{1}{H(x)^{s_1'}}\right)^{1/p}\times\left(\sum_{P\in W}\frac{1}{H(x)^{s_2'}}\right)^{1/q}$$

(Hölder's inequality
$$\sum_{k=1}^{n}|x_i y_i| \leq \left(\sum_{k=1}^{n}|x_i|^p\right)^{1/p}\left(\sum_{k=1}^{n}|x_i|^q\right)^{1/q}$$)

Therefore the seem converges, and therefore

$$a((1-\eta)s)\leq 1$$

So $a(s)<1$ and $s\in\Sigma_U$ as wanted. $\square$

e) Examples

Let us go again over the examples I gave at the beginning of these lectures.

α) Product of projective spaces

Let me first describe the geometry of this example

Let $n_1,\ldots,n_r>0$ and $V=\prod_{i=1}^{r}\mathbb{P}^{n_i}_{\mathbb{Q}}$

Geometrical facts

(i) The morphisms of groups

$$\mathbb{Z}^r \longrightarrow \operatorname{Pic}(V)$$
$$(a_i)_{1\leq i\leq r}\longmapsto \sum_{i=1}^{r}a_i\,\underbrace{[pr_i^*(\mathcal{O}_{\mathbb{P}^{n_i}}(1))]}_{\text{We put } e_i}$$

is an isomorphism of groups.

(ii)

$$C_{\text{eff}}(V)=\sum_{i=1}^{r}\mathbb{R}_{\geq 0}\,e_i\otimes 1\subset \operatorname{Pic}(V)\otimes_{\mathbb{Z}}\mathbb{R}.$$

(iii) $$[\omega_V^{-1}] = \sum_{i=1}^{r} (n_i+1)\, e_i$$

Hints

(i) HARTSHORNE's book Exo III.12.6
(ii) Exercise
(iii) $T(X\times Y) \simeq TX \times TY$
and we have seen the result for $\mathbb{P}^n_{\mathbb{Q}}$.

Reminder

If $L = \bigotimes_{i=1}^{r} pr_i^*(\mathcal{O}(a_i))$
then

$$H_L(x_1, \dots, x_r) = \prod_{i=1}^{r} \left(H_{\mathcal{O}_{\mathbb{P}^{n_i}}(1)}(x_i)\right)^{a_i}$$

and

$$\# V(\mathbb{Q})_{H_L \leq B} \underset{B\to+\infty}{\sim} C\, B^{a_L} \log(B)^{b_L - 1}$$

where

$$a_L = \max_{1\leq i\leq r} \left(\frac{n_i+1}{a_i}\right),$$

$$b_L = \# \left\{ i \in \{1, r\} \,\middle|\, \frac{n_i+1}{a_i} = a_L \right\} \quad \text{and } C>0.$$

Thus

$$\Sigma_V = \left\{ (a_i)_{1\leq i\leq r} \in \mathbb{R}^r_{\geq 0} \,\middle|\, \forall i \in \{1, r\}\ \frac{n_i+1}{a_i} < 1 \right\}$$

$$= [\omega_V^{-1}] + \overset{\circ}{\overline{C_{\text{eff}}(V)}}.$$

The second example I explained is

β) The plane blown up in a point

$V \overset{\iota}{\subset} \mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ defined by $yu = xv$
$[x:y:z]\ [u:v]$

$\pi : pr_1 : V \to \mathbb{P}^2_{\mathbb{Q}}$ $\quad P_0 = [0:0:1]$

$E = \pi^{-1}(P_0) \underset{pr_2}{\xrightarrow{\sim}} \mathbb{P}^1_{\mathbb{Q}}$ $\quad U = V - E$.

Geometrical facts

(i) The morphism of groups
$$\iota^* : Pic(\mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}) \to Pic(V)$$
is an isomorphism of groups

Let $e_i = \iota^*(pr_i^*(\mathcal{O}(1)))$

(ii) $C_{eff}(V) = \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\,(e_1 - e_2)$

(iii) $\omega_V^{-1} = 2e_1 + e_2$

Proof

(i) & (iii) HARTSHORNE's book exercise II 8.5

(ii) $\frac{X}{U} = \frac{Y}{V}$ defines a section of $e_1 - e_2$
(since the intersection of the open sets
$U \neq 0,\ V \neq 0$ is empty)

so $e_1 - e_2 \in C_{eff}(V)$

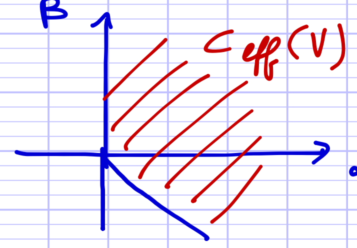
On the other hand

$\mathbb{R}_{>0}\, e_1 + \mathbb{R}_{>0}\, e_2 \subset$ ample cone $C_{am}(V)$
which is the open cone generated by ample line bundles

On a surface there is an intersection product
$$\bullet : Pic(V) \times Pic(V) \to \mathbb{Z}$$
(See HARTSHORNE's V.1)

on the basis (e_1, e_2) it is given by the matrix
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and $C_{eff}(V) \subset C_{am}(V)^{\vee}$
$$= \{ y \in Pic(V) \otimes_{\mathbb{Z}} \mathbb{R} \mid \forall x \in C_{am}(V),\ y \cdot x \geq 0 \}$$
So $\alpha e_1 + \beta e_2 \in C_{eff}(V)$
implies $\alpha \geq 0$ and $\alpha + \beta \geq 0$
Thus $C_{eff}(V) \subset \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_1 - e_2$.



$\square$

Let me now remind you of the result we had seen:

Reminder

For $[L] = a e_1 + b e_2$

$$\# E(B)_{\|\ \| \leq B} = \begin{cases} C(\mathbb{P}^1)\ B^{\frac{2}{b}} & \text{for } b > 0 \\ +\infty & \text{if } b \leq 0 \end{cases}$$

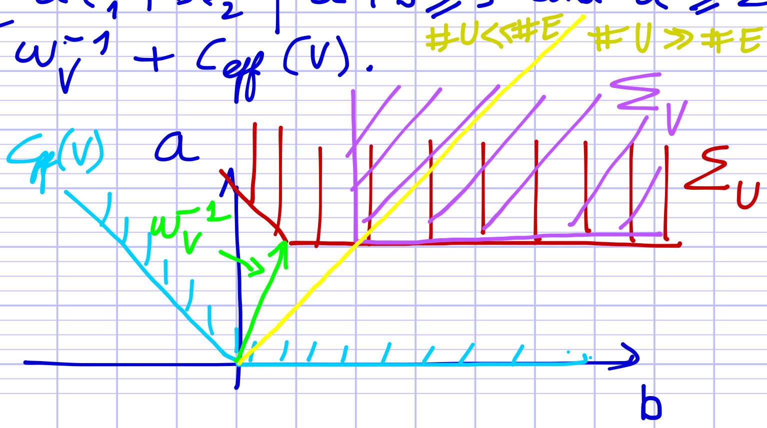
$$\# U(\mathbb{Q})_{H \leq B} = C\, B^{a_L} \log(B)^{b_L - 1}$$

$$a_L = \max\left(\frac{3}{a+b}, \frac{2}{a}\right) \qquad b_L = \begin{cases} 1 & \text{if } \frac{3}{a+b} \neq \frac{2}{a} \\ 2 & \text{otherwise.} \end{cases}$$

Thus

$$\Sigma_V = \{ a e_1 + b e_2 \mid b \geq 2,\ a \geq 2 \}$$

$$\Sigma_U = \{ a e_1 + b e_2 \mid a + b \geq 3 \text{ and } a \geq 2 \}$$
$$= \omega_V^{-1} + C_{eff}(V).$$



γ] Hyperfaces of large dimension

Let V be a smooth hypersurface in $\mathbb{P}^N_{\mathbb{Q}}$ defined by an equation $F(X_0, \dots, X_N) = 0$
with $F \in \mathbb{Z}[X_0, \dots, X_N]$ homogeneous of degree $d \geqslant 2$.
Assume

(i) $V(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$
(ii) $N+1 > 2^d(d-1)$

Geometrical Facts

(i) $\mathrm{Pic}(V) = \mathbb{Z}[\underbrace{\mathcal{O}_{\mathbb{P}^N}(1)_{|V}}_{\mathcal{O}_V(1)}]$;
(ii) $C_{\mathrm{eff}}(V) = \mathbb{R}_{\geqslant 0}\, \mathcal{O}_V(1)$;
(iii) $\omega_V^{-1} = \mathcal{O}_V(N+1-d)$.

Theorem [BIRCH thm]

$$\# V(\mathbb{Q})_{H_{\mathcal{O}(1)} \leq B} \sim C\, B^{N+1-d}$$

Proof

Let $\mathcal{D} = \prod_{i=0}^{N} [a_i, b_i]$
Put $M(B) = \#\{ P \in B\mathcal{D} \cap \mathbb{Z}^{N+1} - \{0\} \mid F(x_0, \dots, x_N) = 0\}$
Birch has proven that

$$M(B) = C B^{N+1-d} + O(B^{N+1-d-\delta})$$

$$\# V(\mathbb{Q}) = \frac{1}{2} \sum_{d \leqslant B} M\left(\frac{B}{d}\right)$$

If we take $\| X_i(x) \|_\infty = \dfrac{|x_i|}{\max\limits_{0 \leq j \leq N} |x_j|}$

and $\mathcal{D} = [-1, 1]^{N+1}$

The end of the proof is as for $\mathbb{P}^N_{\mathbb{Q}}$. □

So for $L = \mathcal{O}_V(a)$, $a_L(V) = \dfrac{N+1-d}{a}$ and $\Sigma_V = \omega_V^{-1} + C_{\mathrm{eff}}(V)$.

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5) Predictions [MANIN - BATYREV - TSCHINKEL]

In the middle a boiling cauldron. Thunder
Enter 3 witches
[...] Double, double, toil and trouble
Fire burn and cauldron bubble.

For Serre, conjectures are something you are totally sure is true but you do not know how to prove it. Not everybody agrees with this definition. For me, the worth of a conjecture can be measured by the "amount" of mathematics it generates. From this point of view the conjectures of Manin and his collaborators are very good conjectures, even though there are counter examples to some of them.

a) First level the power of B

As usual V is a nice variety / number field K.

Conjecture [MANIN]

Let $U \subset V$ an open subset
If there is an ample line bundle L such that $a_L(U) > 0$ then there is a morphism
$$\varphi : \mathbb{P}^1_K \to V$$
So that $\mathrm{Im}(\varphi) \cap U \neq \emptyset$

I do not know any counter example to that conjecture

Definition g stands for "geometric"

For all $\delta \in \overset{\circ}{\overline{C_{eff}}}(V)$, $a_g(\delta) = \inf\{\lambda \in \mathbb{R} \mid \lambda\delta \in \omega_V^{-1} + C_{eff}(V)\}$

Conjecture A [BATYREV & MANIN]

For any $\varepsilon > 0$ and any $s \in \overset{\circ}{\overline{C_{eff}(V)}}$ there exists a non-empty open set $U \subset V$ such that

$$a_s(U) \leq a_g(s) + \varepsilon$$

Remark 1

If ω_V is of general type, that is $\omega_V \in \overset{\circ}{C_{eff}(V)}$ then $a_g(s) < 0$ for any s it implies that $a_s(U) < 0$ for a small enough U which means that $U(\mathbb{K}) = \emptyset$ so it implies

LANG's conjecture

On a variety of general type, the rational points are not Zariski dense

Remark 2

I am not aware of a counter-example for conjecture A

Definition

A nice variety is Fano if ω_V^{-1} is ample.

Conjecture B [BATYREV-MANIN]

Let V be a nice Fano variety.
Then there exists a non-empty subset U and an extension $\mathbb{K}_0$ of $\mathbb{K}$ such that for any number field $\mathbb{L}/\mathbb{K}_0$, any non empty open set $W \subset U_{0\,\mathbb{L}}$, any $s \in \overset{\circ}{C_{eff}(V)}$ $a_s(W) = a_g(s)$.
In other words $\Sigma_W = \omega_V^{-1} + C_{eff}(V_{\mathbb{L}})$

Remarks

(i) No counter example is known

(ii) The condition V Fano is, in fact, probably too strong, but $\omega_V^{-1} \in C_{eff}(V)$ not strong enough

A good condition may be:

A multiple of ω_V^{-1} may be written as the sum of an ample divisor and an effective divisor with normal crossings.

Let us call this "extra-big"

Later I am going to restrict myself to that setting, that is ω_V^{-1} big so before I do that let me stress that for conjecture A there may be an infinite filtration of V by open subsets.

b) An example: K3 surfaces

Definition

A K3 surface is a nice surface S such that

(i) $\omega_S = 0$

(ii) $H^1(S, G_S)$ is trivial

Remarks

(i) Surfaces with $\omega_S = 0$ are

- K3 surfaces and
- abelian surfa

(ii) On a surface strict subvarieties are point or curves. For a curve C which has ∞ly many points there are 2 possibilities, for L ample

(i) If $g(C)=1$ $\exists\ \varphi : E \to C$, birational with E an elliptic curve $a_L(C)=0$

(ii) If $g(C)=0$ $\exists\ \varphi : \mathbb{P}^1 \to C$, birational and $(C.L)=\deg \varphi^*(L)$ so

$$a_L(C) = \frac{2}{(C,L)} > 0$$

Conjecture A in this case predicts

Let S be a K3 surface. For any $\varepsilon>0$ and any $s \in \overline{C_{eff}(S)}$ there exists a finite set T of rational curves on V such that

$$a_s(S - \bigcup_{C\in T} C) < \varepsilon .$$

Let me now give you one example of a K3 surface with an infinite number of rational curves and therefore an infinite filtration by open sets

1/16/2016 Example (I am not going to prove the details)

In $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} \supset S$
$[x:y]\ [z:t]\ [u:v]$

S defined by $P = \sum_{\substack{i_1+i_2=2\\ j_1+j_2=2\\ k_1+k_2=2}} a_{ijk}\, x^{i_1} y^{i_2} z^{j_1} t^{j_2} u^{k_1} v^{k_2} = 0$, $\in \mathbb{Z}$

In other words if we write

$$\mathcal{O}(a_1,a_2,a_3) = \bigotimes_{i=1}^{3} pr_i^*(\mathcal{O}_{\mathbb{P}^1}(a_i))$$

The above polynomial P defines a section s of $\mathcal{O}(2,2,2)$ and S is given by $s=0$

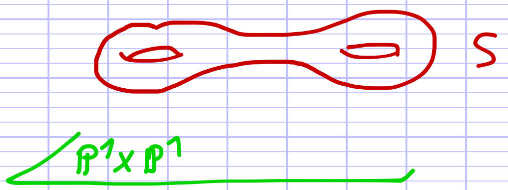
For a generic P, S is smooth and

$$\omega_S^{-1} = \mathcal{O}(2-2,2-2,2-2)_{|S} = \mathcal{O}_S$$

and S is a $K3$-surface.

For $i \in \{1,2,3\}$ the projection map
$$pr_{\hat{i}} : S \longrightarrow \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$$
obtained by taking the components other than i is dominant, of degree 2

Picture over $\mathbb{R}$



So for any point x of $\mathbb{P}^1 \times \mathbb{P}^1$ over an extension $\mathbb{L}$ of $\mathbb{Q}$ $pr_{\hat{i}}^{-1}(x) = S \times_{\mathbb{P}^1\times\mathbb{P}^1} \operatorname{Spec}(\mathbb{L})$ is the spectrum of an $\mathbb{L}$-algebra of dim 2.

that is $\mathbb{L}[X]/(P(X)) \cong \mathbb{L}[Y]/(Y^2 - a)$

and it has an involution $\sigma : \bar{Y} \mapsto -\bar{Y}$

If we apply this to the generic point we get
$$\sigma_i : S \dashrightarrow S \text{ birational}$$
and $\sigma_i^2 = Id_S$, $\sigma_i \circ pr_{\hat{i}} = pr_{\hat{i}}$

But it is defined everywhere and therefore
$$\sigma_i \in \operatorname{Aut}(S)$$

<u>Facts</u> [H. BILLARD]

(i) If we put $e_i = pr_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$, (e_1, e_2, e_3) is a basis of $\operatorname{Pic}(S)$.

(ii) σ_i^* acts on $\operatorname{Pic}(S)$ as follows

$\sigma_i^*(e_j) = e_j$ if $j \neq i$

$\sigma_i^*(e_i) = -e_i + 2e_j + 2e_k$ if $\{i,j,k\} = \{1,2,3\}$

(iii) $\sigma_1, \sigma_2, \sigma_3$ induce an isomorphism
$$\underbrace{\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}}_{\text{free product}} \cong \langle \underbrace{\sigma_1, \sigma_2, \sigma_3}_{\text{generators}},\ \underbrace{\sigma_1^2 = \sigma_2^2 = \sigma_3^2}_{\text{relations}} \rangle$$

It is a very big non commutative group
It remains to produce rational curves on S

The fibers of $pr_i : S \to \mathbb{P}_1$ are effective divisors in $\mathbb{P}^1 \times \mathbb{P}^1$ associated to $G(2,2)$

α) If it smooth then it is curve of genus 1

β) If it singular and irreducible, it is a rational curve (parametrized by divisors of $G(1,1)$ passing through a singular point) / a finite extension of $\mathbb{K}$

γ) If it is reducible it is the union of 2 rational curves / a finite extension of $\mathbb{K}$

Note

γ) Does not occur in the generic case

So after a finite field extension we get $C \subset S$ rational curve

$\mathrm{Aut}(S) \cdot C$ is Zariski dense in S

Note

(i) The conjecture is still open for these examples

(ii) Wide open (and hard!)

Is $S(\mathbb{K}) - \bigcup_{C \text{ rational curve}} C(\overline{\mathbb{Q}})$ finite?

My guess would be that it is infinite. But it is a rather wild guess. The point is that S is a fibration in curves of genus 1, which may produce many points as well.

c) The second level : the power of log(B)

Definition

Assume that there exists effective divisors $E_1, \dots, E_r$ such that
$$C_{eff}(V) = \sum_{i=1}^{r} \mathbb{R}_{\geq 0} E_i$$
then for any $L \in \overset{\circ}{C}_{eff}(V)$,
$$b_g(L) = \operatorname{codim}(\text{minimal face of } C_{eff}(V) \text{ containing } a_g(L)L - \omega_V^{-1})$$

A conjecture stays a conjecture only as long as there is no counter-example. So I am going to describe the expected value as

Definition

Assume that ω_V^{-1} is extra-big and $V(\mathbb{K}) \neq \emptyset$. We say that V satisfies the BATYREV-MANIN principle if there exists a non-empty set U in V such that for any $\Delta \in \overset{\circ}{C}_{eff}(V)$ there exists a constant $C > 0$ so that
$$\# U(\mathbb{K})_{H_\Delta \leq B} \underset{B \to +\infty}{\sim} C\, B^{a_g(\Delta)} \log(B)^{b_g(\Delta) - 1}$$

Remarks

In all the examples I know for which an estimate has been computed,
$$\# U(\mathbb{K})_{H_\Delta \leq B} \sim C\, B^{a_\Delta(U)} \log(B)^{b_\Delta(U) - 1}$$

with $a_\Delta(U) = a_g(\Delta)$ and $b_\Delta(U) \geq b_g(\Delta)$. In particular in all counter-examples known, there are too many points, not too few.

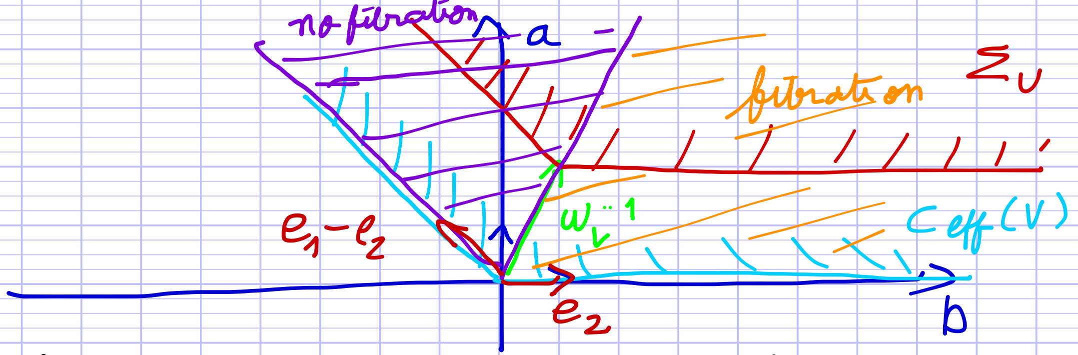
d) 3rd level : preliminary remarks about the constant

α) Back to the examples

We have seen for the product of projective spaces or the plane blown up in a point that sometimes there is a fibration where each fibre makes a non negligible contribution to the total number of points, sometimes there isn't

blowing up

$V = \mathbb{P}^2_{\mathbb{Q}}$ blown up in 1 point



Let me be more precise about this fibration it is the same for all line bundles in the orange area it corresponds to pr_2

For $L = a e_1 + b e_2$ with $\frac{3}{a+b} < \frac{2}{a}$

that is $a < 2b$

we have

$$\#U(\mathbb{Q})_{H_L \leq B} \sim \sum_{Q \in \mathbb{P}^1(\mathbb{Q})} \underbrace{C(pr_2^{-1}(Q)) B^{\frac{2}{a}}}_{S}$$

$$\overbrace{\#U(\mathbb{Q})_{H_L \leq B} \geqslant \sum_{Q \in \mathbb{P}^1(\mathbb{Q})} \# pr_2^{-1}(Q)_{H_L \leq B}}^{S}$$

and $e_2 = pr_2^*(\mathcal{O}_{\mathbb{P}^2}(1))$.

On the other side
$$\Gamma(V, e_1 - e_2) = \mathbb{Q}\, s_0$$
where s_0 is the section defined by $\frac{X}{U} = \frac{Y}{V}$
So the only map defined by $e_1 - e_2$
is
$$U - E \longrightarrow \mathbb{P}^0(\mathbb{Q}) = \{pt\}.$$
that is a constant map.

product of projective spaces

$$V = \prod_{i=1}^{r} \mathbb{P}^{n_i}_{\mathbb{Q}} \qquad L = \sum_{i=1}^{r} a_i e_i, \quad e_i = pr_i^*(\mathcal{O}_{\mathbb{P}^{n_i}_{\mathbb{Q}}}(1))$$

$$I = \left\{ i \;\middle|\; \frac{n_i+1}{a_i} = \underbrace{\max_{1\le j\le r} \frac{n_j+1}{a_j}}_{=\, a_g(L)} \right\}$$

$$I^c = \{1, \dots, r\} - I$$

The face of $\partial C_{eff}(V)$ containing $a_g(L)\, L - \omega_V^{-1}$
is given by $a_g(L)\, L - \omega_V^{-1} = \sum_{i\in I^c} \underbrace{(a_g(L)\, a_i - (n_i+1))}_{>0} e_i \in \sum_{i\in I^c} \mathbb{R}_{\geq 0} e_i$
But if you take M generic in this face, the filtration defined by $\Gamma(V, M)$ factorizes through
$$pr_{I^c} : \prod_{i=1}^{n} \mathbb{P}^{n_i}_{\mathbb{Q}} \longrightarrow \prod_{i\in I^c} \mathbb{P}^{n_i}_{\mathbb{Q}} = V_{I^c}$$

But Using the same proof as the one I gave for the product of 2 spaces it is possible to prove that

$$\#V(\mathbb{Q})_{H_L \le B} \sim \underbrace{\sum_{P\in V_{I^c}(\mathbb{Q})} \frac{C(V_I)}{H_M(P)} B^{a_g(L)} \log(B^{a_g(L)})^{\#I-1}}$$

$$\| \qquad\qquad\qquad\qquad \wr$$

$$\sum_{P\in V_{I^c}(\mathbb{Q})} pr_{I^c}^{-1}(P)_{H_L\le B} = \sum_{P\in V_{I^c}(\mathbb{Q})} \# V_I(\mathbb{Q})_{H_{\omega_{V_I}^{-1}} \le \frac{B^{a_g(L)}}{H_M(P)}}$$

$$pr_I : pr_{I^c}^{-1}(P) \xrightarrow{\sim} V_I$$

where $pr_{I^c}^*(M - \omega_{V_{I^c}}^{-1}) = a_g(L)L - \omega_V^{-1}$

Main remark

$$a_g(L)L_{|pr_I^{-1}(P)} = \omega^{-1}_{pr_I^{-1}(P)}$$

B) Reduction idea [BATYREV & TSCHINKEL]

We assume V nice, ω_V^{-1} extra-big and $C_{eff}(V)$ generated by a finite number of effective divisors

Def

Let F be the face of $\partial C_{eff}(V)$ which contains $a_g(L)L - \omega_V^{-1}$.
For $M \in F \cap \mathrm{Pic}(V)$ effective
We consider the corresponding rational map

$$\varphi_M : V \dashrightarrow \mathbb{P}(\Gamma(V, M)^\vee)$$

(given by $\psi \in \Gamma(V, M)^\vee$ belongs to $\varphi_M(x)$ if $\psi(s) = 0 \iff s(x) = 0$ in $E(x)$)
It is defined on

$$U_M = V - \bigcap_{s \in \Gamma(V,M)} s = 0$$

We pick M so that U_M and $\dim(\mathrm{Im}(\varphi_M))$ is maximal. This defines a fibration

$$V \supset U_F \overset{\varphi_F}{\twoheadrightarrow} Y_F \quad (Y_F \text{ is this image})$$

Definition

F is said to be rigid if Y_F is a single point (and φ_F is constant.)

Zariski closure

In general, for $p \in Y_F(\mathbb{Q})$ let $V_p = \overline{\varphi_F^{-1}(p)}$

Fairy land

i) For a generic $P \in Y_F(\mathbb{Q})$, V_p has mild singularities,

ii) $a_g(L)\, L_{|V_p} - \omega_{V_p}^{-1}$ belongs to a rigid face F_p of $C_{eff}(V_p)$

iii) $\#(V_p \cap U)(\mathbb{Q})_{H_L \leq B} \sim C(V_p)\, B^{a_g(L)} \log(B)^{\dim(F_p)-1}$

iv) The main term for V is obtained by suming the main terms for V_p.

Z

$\dim(F_p)$ depends on p!

and as I am going to explain this lead BATYREV & TSCHINKEL to the first counter examples of BATYREV & MANIN principle. So, eventually, we will have to leave Fairy land. But, in mathematics you can learn a lot by thinking of the question "What is the best I can hope?" So let us stay a little longer in that land. What can we say in

γ) The rigid case

Let us look once more at the plane blown up in one point.

Particular case

Take $L = a e_1$ then $H_L = (H_{6(1)} \circ pr_1)^a$

so in fact we are counting points on $\mathbb{P}^2$ (and not in V)

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and $pr_1 : V \to \mathbb{P}^2_{\mathbb{Q}}$ is the blowing <u>down</u> of E which is the unique effective divisor corresponding to $e_1 - e_2$ which is rigid.

In general, if F is rigid
$F = \sum_{i=1}^{k} \mathbb{R}_{\geq 0} [E_i]$ where E_i is an effective divisor on V
Then

for $M = \sum a_i [E_i]$, $a_i \in \mathbb{N}$
Take $s \in \Gamma(V, M) - \{0\}$ It is unique up to multiplication by a constant.
s vanishes with multiplicity a_i along E_i

<u>Fairy land</u>

(i) For <u>some</u> L such that $a_g(L) L - \omega_V^{-1} \in \overset{\circ}{F}$ relative interior
then the rational map

$$V \dashrightarrow \mathbb{P}(\Gamma(V, L^{\otimes N})^\vee) \qquad N \gg 0$$

corresponds to the blowing down of $E_1, \dots, E_k$.
Let Y be the image of V

$$\begin{array}{ccc} \psi \; V & \dashrightarrow & Y \text{ is birational} \\ \cup & & \cup \\ U & \xrightarrow{\sim} & W \end{array}$$

(ii) We reduce to count on W

$$\#\{x \in W(\mathbb{K}) \mid H_{\omega_Y^{-1}}(x) \leq f(x) B^{a_g(L)}\}$$

where on W

$$f(x) = \frac{H_{\omega_Y^{-1}}(x)}{H_L(\psi^{-1}(x))^{a_g(L)}}$$

$$= \prod_{i=1}^{r} \prod_{w \in Pl(\mathbb{K})} \|s_i(\psi^{-1}(x))\|_w^{\lambda_i}$$

where s_i is a non zero section of $L_i \Leftrightarrow E_i$
and $a_y(L)L - \psi^*(\omega_Y^{-1}) = \sum_{i=1}^{n} \lambda_i [L_i]$

NB

(i) By the product formula, since s_i is unique up to multification by a constant,
f does not depend on the choice of s_i

(ii) If you take different norms on ω_V^{-1} counting for the second norm amounts to estimate

$$\#\{x \in U(\mathbb{K}) \mid H_{\omega_V^{-1}}(x) \leq f(x) B\} \quad (*)$$

where

$f(x) = \prod_{v \in Pl(\mathbb{K})} f_v(x)$ is continuous on $V(\mathbb{A}_\mathbb{K})$

and therefore bounded. So we may see the rigid case as a more general change of height we are considering

$$f(x) = \prod_{v \in Pl(\mathbb{K})} f_v(x)$$

where f_v may have poles (or zeroes) but we control how "big" it can be in a sense which I am soon going to make clear

2 Y may be singular It is not a too serious problem but it complicates a little bit the technical details. To simplify matters I am going to restrict myself to the smooth case. So we have reached the final step: What can be the constant when we estimate $(*)$

6/6/2016

5) Constant and distribution

So, if we consider the question of interpreting the constant for different heights we are lead to the following question:

Question

Fix a norm on ω_V^{-1}, let H be the corresponding height, Assume that V satisfies BATYREV–MANIN principle for U open in V and $\mathbb{K}$

Let $f : V(\mathbb{A}_{\mathbb{K}}) \longrightarrow \mathbb{R}_{\geqslant 0}$

Does

$$\frac{\#\{x \in U(\mathbb{K}) \mid H(x) \leqslant f(x) B\}}{\# U(\mathbb{K})_{H \leqslant B}} \xrightarrow[B \to +\infty]{?} C(f)$$

and what is $C(f)$

Remarks

(i) If $f = \mathbb{1}_W$ for W borelian subset of $V(\mathbb{A}_{\mathbb{K}})$ we are looking at

$$\frac{\#(W \cap U(\mathbb{K}))_{H \leqslant B}}{\# U(\mathbb{K})_{H \leqslant B}} = \delta_{U(\mathbb{K})_{H \leqslant B}}(W)$$

Again, it is the question of the distribution of points in the adelic space

(ii) Conversely, Assume also that we know that the measures converge

$$\delta_{U(\mathbb{K})_{H \leqslant B}} \xrightarrow[B \to +\infty]{w} \omega.$$

and take I finite,

$$f = \sum_{i \in I} \lambda_i \mathbb{1}_{W_i} \text{ with } W_i \text{ a partition of } V(\mathbb{A}_{\mathbb{K}})$$

$$\Big(V(\mathbb{A}_{\mathbb{K}}) = \coprod_{i \in I} W_i\Big) \text{ with } \forall i \in I, \ \omega(\partial W_i) = 0$$

Then Since $\mathcal{G}$ assume BATYREV & MANIN principle

$$\#\{x \in U(\mathbb{K}) \mid H(x) \leq f(x) B\}$$
$$= \sum_{i \in I} \#\{x \in U(\mathbb{K}) \mid H(x) \leq \lambda_i 1\!\!1_{W_i} B\}$$
$$\underset{B \to +\infty}{\sim} \sum_{i \in I} C_{\omega_V^{-1}}(V)\, \omega(W_i)(\lambda_i B)$$
$$\| $$
$$C_{\omega_V^{-1}}(V)\left(\int_{V(\mathbb{A}_{\mathbb{K}})} f\, \omega\right) B$$

So we get

$$C(f) = \int_{V(\mathbb{A}_K)} f\, \omega$$

In particular the constant for $\frac{1}{f} H$ is expressed as an integral $\int_{V(\mathbb{A}_{\mathbb{K}})} f\, C_{\omega_V^{-1}}(V)\, \omega$

(iii) In that setting, we may consider the set $\mathcal{F}$ of functions such that

$$\frac{\#\{x \in U(\mathbb{K}) \mid H(x) \leq f(x) B\}}{\# U(\mathbb{K})_{H \leq B}} \xrightarrow[B \to +\infty]{} \int_{V(\mathbb{A}_K)} f\, \omega$$

and, again we have a sandwich principle
If $g : V(\mathbb{A}_K) \to \mathbb{R}$, is such that there exists sequences $(f_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{F}$ with

$$\forall x \in V(\mathbb{A}_K) \quad f_n(x) \leq g(x) \leq h_n(x)$$

and

$$\int_{V(\mathbb{A}_{\mathbb{K}})} (h_n - f_n)\, \omega \xrightarrow[n \to +\infty]{} 0$$ which is only a $\mathbb{L}^1$ condition!

$$\Updownarrow$$
$$\|h_n - f_n\|_{\mathbb{L}^1} \xrightarrow[n \to +\infty]{} 0$$

Then $g \in \mathcal{F}$

My last remark is due to SWINNERTON-DYER

(iv) [SWINNERTON-DYER]

Assume that V does not satisfy weak approximation that is the closure of rational points is not the adelic space

$$\overline{V(\mathbb{K})} \subsetneq V(\mathbb{A}_{\mathbb{K}})$$

$C(f)$ depends only on $f_{|\overline{V(\mathbb{K})}}$

So if $C_H(V)$ is expressed as a volume it is the volume of $\overline{V(\mathbb{K})}$ not of $V(\mathbb{A}_{\mathbb{K}})$.

The conclusion is that

To describe the constant for all possible heights we need to describe $\overline{V(\mathbb{K})} \subset V(\mathbb{A}_{\mathbb{K}})$

So now the plan for the next lecture is

<u>Plan</u>

1) Describe the expected $\overline{V(\mathbb{K})}$;
2) Define the expected constant;
3) What are the results;
4) Describe Counter-examples;
5) Upgrade the conjecture to cover all cases.

6) BRAUER-MANIN obstruction, Universal and versal torsors

<u>Hypothesis</u> (H)

V is a very nice variety $/\mathbb{K}$ number field

$\overline{\mathbb{K}}$ = algebraic closure of $\mathbb{K}$, $\overline{V} = V_{\overline{\mathbb{K}}}$

(i) ω_V^{-1} is extra-big

(ii) $H^i(V, \mathcal{O}_V) = \{0\}$ if $i \in \{1, 2\}$

(iii) $\mathrm{Pic}(V)$ is a free, finitely generated $\mathbb{Z}$-module (It has no torsion)

(iv) $C_{eff}(V)$ is generated by a finite number of effective divisors

a) (Uni)versal torsors

α) Motivation

For the Projective space or an hypersurface of large enough dimension, the first step is to lift rational solutions to integral ones using the isomorphism

$$\begin{array}{ccc} \mathbb{P}^n & \xleftarrow[\pi]{\sim} & \mathbb{A}^{n+1} - \{0\} / \mathbb{G}_m \\ \cup & & \cup \\ V & \longleftarrow & W / \mathbb{G}_m \end{array}$$

Of course, for any projective variety we can pick a very ample line bundle and embed the variety into a projective space and by taking the inverse image in the affine space we can write any variety as a quotient by $\mathbb{G}_m$. But

— Firstly if the rank of the Picard group is > is 1 which ample line bundle should we choose?

— Secondly we

Wish for W

→ The height we use can be expressed in simple terms in W;

→ The number of equations defining W in the affine space is as small as possible.

Let us look at one example:

Example

For $\mathbb{P}^{n_1}_{\mathbb{Q}} \times \mathbb{P}^{n_2}_{\mathbb{Q}}$ the smallest embedding is

$$\mathbb{P}^{n_1}_{\mathbb{Q}} \times \mathbb{P}^{n_2}_{\mathbb{Q}} \hookrightarrow \mathbb{P}^{n_1 n_2 + n_1 + n_2}_{\mathbb{Q}}$$

which gives $n_1 n_2$ equations!

But $\mathbb{P}^{n_1}_{\mathbb{Q}} \times \mathbb{P}^{n_2}_{\mathbb{Q}} \simeq \mathbb{A}^{n_1+1}_{\mathbb{Q}} - \{0\} \times \mathbb{A}^{n_2+1}_{\mathbb{Q}} - \{0\} / \mathbb{G}_m \times \mathbb{G}_m$

so the idea is to consider the quotient by bigger group.

B) Topological Background

Reminder

In topology, for a topological pointed space X, x a universal covering of X at x is a covering $\pi : \tilde{X} \to X$ with a point $\tilde{x} \in \tilde{X}$, $\pi(\tilde{x}) = x$ such that for any covering $\varphi : Y \to X$ and $y \in Y$ such that $\varphi(y) = x$, there is a unique morphism

$$\psi : \tilde{X} \to Y$$

such that

$$\begin{tikzcd} \tilde{X} \arrow[r, "\psi"] \arrow[dr, "\pi"'] & Y \arrow[d, "\varphi"] \\ & X \end{tikzcd}$$

commutes and $\psi(\tilde{x}) = y$

If it exists it is unique up to a unique isomorphism.

⚠ The unicity requires working in the category of pointed sets.

Remark

Let $\mathrm{Aut}_X(\tilde{X})$ be the group of automorphisms of $\tilde{X}$ above. The definition implies that

The map $\varphi \mapsto \varphi(\tilde{x})$ is a bijection from

$\mathrm{Aut}_X(\tilde{X})$ to the fibre $X_x = \pi^{-1}(x)$
In other words $\mathrm{Aut}_X(\tilde{X})$ acts simply transitively on that fibre.
And $\tilde{X}$ equipped with $\pi, \tilde{x}$ and $\mathrm{Aut}_X(\tilde{X}) \circlearrowright \tilde{X}$ is also universal for pointed Galois coverings.

γ) Torsors

Definition
Let X be a variety over a field $\mathbb{L}$
and G be an algebraic group over $\mathbb{L}$
A G-torsor over X is a variety E over $\mathbb{L}$ equipped with

- a morphism $\pi : E \longrightarrow X$
- an action $m : G \times E \longrightarrow E$

so that

(i) π is faithfully flat (Technical condition)

(ii)
$$\begin{tikzcd} G \times E \arrow[rr] \arrow[dr, "\pi \circ pr_2"'] & & E \arrow[dl, "pr_1"] \\ & X & \end{tikzcd}$$
commutes

(iii) The map
$$G \times E \longrightarrow E \times_X E$$
$$(g, e) \longmapsto (ge, e)$$
is an isomorphism.

NB in terms of points
If A is commutative $\mathbb{L}$ algebra and $x \in X(A)$ we get a bijection
$$G(A) \times \underset{\pi^{-1}(x)}{\underset{\parallel}{E(x)}} \longrightarrow E(x) \times E(x)$$

So, if $E(x) \neq \emptyset$, $G(A)$ acts simply and transitively on $E(x)$.

In particular if we have a covering of X, for some Grothendieck topology τ, $(\varphi_i : U_i \to X)_{i \in I}$ by affine schemes so that

$$\pi_{U_i} : E \times_X U_i \to U_i$$

has a section, then $E \times_X U_i \cong G \times U_i$ we say that the covering splits E.
Since E is faithfully flat we know that at least E splits on a faithfully flat covering (Well that's precisely $E \times_X E \cong G \times E$).

Remark

The glueing data gives a 1-cocycle in Čech cohomology $\check{H}^1_\tau(X, G)$ and thus in $H^1_\tau(X, G)$ (even when G is not commutative) and $H^1_\tau(X, G)$ classifies G torsors which split in τ coverings up up to isomorphism.

Example

Take $G = \mathbb{G}_m = \mathrm{Spec}\left[\mathbb{Z}[T, T^{-1}]\right]$
Then for any line bundle we may consider
$L^\times = L -$ zero section
Then the scalar multiplication induces

$$\mathbb{G}_m \times L^\times \to L^\times$$
$$(\lambda, e) \mapsto \lambda e$$

and $L^\times$ is a $\mathbb{G}_m$-torsor over V
In fact any $\mathbb{G}_m$ torsor split in Zariski topology and one can prove

Proposition

The functor $L \mapsto L^\times$ defines an equivalence of category from the category of line bundles over V to the category of $\mathbb{G}_m$-torsors over V. In particular,

$$\mathrm{Pic}(V) \xrightarrow{\sim} H^1_{Zar}(V, \mathbb{G}_m) \xrightarrow{\sim} H^1_{et}(V, \mathbb{G}_m)$$

An inverse functor can be defined by $E \to E \times \mathbb{A}_1 / \mathbb{G}_m$

$$\lambda(e, \mu) = (\lambda e, \lambda^{-1}\mu)$$

Remarks

It is an extension of Hilbert's Theorem 90, which says that

$$H^1(k, \mathbb{G}_m) = H^1_{et}(\mathrm{Spec}(k), \mathbb{G}_m) = \{0\}.$$

In fact Hilbert's theorem 90 reduces to

Let $\mathbb{L}/\mathbb{K}$ be a Galois cyclic extension

Let σ generate $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ and let $x \in \mathbb{L}^*$

If $N_{\mathbb{L}/\mathbb{K}}(x) = 1$ then $\exists y \in \mathbb{L}^*$, $x = \sigma(y)/y$

Reference

J.-P. SERRE Corps locaux. HERMANN.

As I said we want to consider a more general class of groups. It turns out that there is a class of group which are easy to deal with because they are classified by simple objects

δ) Groups of multiplicative type

Definition

Let $\mathbb{L}$ be a field, $\overline{\mathbb{L}}$ an algebraic closure of $\mathbb{L}$ an algebraic group G is said to be

a) <u>of multiplicative type</u> if $\overline{G} = G_{\overline{\mathbb{L}}}$ is isomorphic to a subgroup of $\mathbb{G}^n_{m,\overline{\mathbb{L}}}$ for some $n \geq 0$;

b) an algebraic torus if G is isomorphic to $\mathbb{G}^n_{m,\mathbb{L}}$ for some $n \geq 0$

2 In the litterature "tori" may be used with two different meanings
- In complex algebraic geometry $\mathbb{C}^n/\Lambda$ where Λ is a lattice in $\mathbb{C}^n$ is a tori (for $n=1$ it looks like )

Here this type of algebraic group is called abelian varieti
- Algebraic tori as defined above

Terminology

If V is a variety or an algebraic group or a whatever $/\mathbb{L}$, a form of V over $\mathbb{L}$ is a variety or algebraic group or a whatever $V'/\mathbb{L}$ such that $V \cong V'$ as whatever

Example of groups of multiplicative type

(i) If $n \geq 1$, $n \neq 0$ in $\mathbb{L}$ (that the characteristic of $\mathbb{L}$ does not divide $\mathbb{L}$)

$$\mu_{n,\mathbb{L}} = \mathrm{Spec}(\mathbb{L}[T]/(T^n - 1))$$
$$\cap$$
$$\mathbb{G}_{m,\mathbb{L}} = \mathrm{Spec}(\mathbb{L}[T, T^{-1}]).$$

$\mu_{n,\mathbb{L}}$ is of multiplicative type but not an algebraic tori.

(ii) $\mathbb{S}^1_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[X,Y]/(X^2+Y^2-1))$

with $m : \mathbb{S}^1_{\mathbb{R}} \times \mathbb{S}^1_{\mathbb{R}} \longrightarrow \mathbb{S}^1_{\mathbb{R}}$ defined by

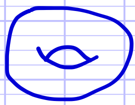
$$X \longmapsto X \otimes X + Y \otimes Y$$
$$Y \longmapsto X \otimes Y + Y \otimes X$$

Think of complex multiplication.

Let me prove that it is an algebraic torus, that is

$$\mathbb{C}[X,Y]/(X^2+Y^2-1) \xleftarrow{\sim} \mathbb{C}[T,T^{-1}]$$
$$X+iY \longleftarrow\!\shortmid \; T$$
$$X-iY \longleftarrow\!\shortmid \; T^{-1}$$

<u>Note</u>

$(\mathbb{S}^1_{\mathbb{R}})^2(\mathbb{R})$ 

② Over $\mathbb{R}$ the set of points is compact, but the variety is <u>not</u> proper and it is not an abelian group.

<u>Definition</u>

For an algebraic group G, <u>the group of of characters of G</u> is the group

$$X^*(G) = \mathrm{Hom}_{alg\text{-}gr/\mathbb{L}}(G, \mathbb{G}_{m,\mathbb{L}})$$

<u>Theorem</u>

Let $\mathbb{L}^s$ be a separable closure of $\mathbb{L}$

a) For any group of multiplicative type G there exists an embedding

$$G^s = G_{\mathbb{L}^s} \hookrightarrow \mathbb{G}^n_{m,\mathbb{L}^s} \text{ for some } n \geq 0$$

b) The contravariant functor

$$G \to X^*(G^s) = \mathrm{Hom}_{alg\text{-}gr/\mathbb{L}^s}(G^s, \mathbb{G}_{m,\mathbb{L}^s})$$

defines an equivalence of category between the opposite of the category of multiplicative type and the category of finitely generated $\mathbb{Z}$-modules equipped with an action of the Galois group

$$\mathcal{G}_{\mathbb{L}} = \mathcal{G}al(\mathbb{L}^s/\mathbb{L})$$

An inverse functor may be defined as

$$\Lambda \longmapsto \mathrm{Spec}(\mathbb{L}^s[\Lambda]^{\mathcal{G}_{\mathbb{L}}})$$

where the action of $\mathcal{G}_{\mathbb{L}}$ is given by

$$\sigma\Big(\sum_{\lambda\in\Lambda} a_\lambda \lambda\Big) = \sum_{\lambda\in\Lambda} \sigma(a_\lambda)\,\sigma(\lambda).$$

Reference

A. BOREL, linear algebraic groups, §8
Graduate texts in Math, Springer-Verlag.

E] Universal torsor [Colliot-Thélène & SANSUC]

As in topology this makes sense in the category of pointed schemes

Reminder

- A pointed scheme/A is a scheme X with a chosen point $\bullet_X \in X(A)$ (also noted $\bullet$)
  a morphism of pointed scheme is a morphism $\varphi: X \to Y$ such that $\varphi(\bullet_X) = \bullet_Y$
- A pointed G-torsor over a pointed scheme X is a torsor T with a selected point $\bullet_T$ in the fibre of $\bullet_X$

Definition

Let X be a nice pointed variety over $\mathbb{L}$,
A universal torsor on X is a pointed G-torsor $\tilde{X}$ over X, with T a group of multiplicative type
So that for any pointed G torsor E over X with G a group of multiplicative type
There exists a unique morphism $\varphi: T \to G$ and a unique morphism of pointed varieties $\psi: \tilde{X} \to E$ above X such that

$$\begin{array}{ccc} T\times\tilde{X} & \xrightarrow{m_{\tilde{X}}} & \tilde{X} \\ \downarrow(\varphi,\psi) & & \downarrow\psi \\ G\times E & \xrightarrow{m_E} & E \end{array}$$ commutes

$(\tilde{X}, T)$ is unique up to a unique isomorphism

Remark

It is the base point which makes it possible to have unicity.

Theorem

If X is a nice variety $/\mathbb{L}$, $char(\mathbb{L})=0$, such that $Pic(\bar{X})$ is finitely generated then, for any choice of a base point in $X(\mathbb{L})$ a universal torsor exists. Moreover $X^*(T)$ is canonically isomorphic to $Pic(\bar{X})$

8/6/2016 To prove the existence of universal torsors I am going to prove a statement which classifies the pointed torsors under a multiplicative group

$\mathbb{L}$ field of characteristic 0, $\bar{\mathbb{L}}$ algebraic closure
X nice variety $/\mathbb{L}$ such that
$Pic(\bar{X})$ is finitely generated, $x = \cdot_X \in X(\mathbb{L})$

We consider two categories

$\mathcal{C}^m_{X,x}$: the category of pointed torsors under multiplicative group over X:

objects: A variety E equipped with an algebraic group G of multiplicative type and the structure of pointed G-torsor over X.

morphisms: morphisms of varieties $\varphi : E \to E'$ over X such that there exists $\varphi_* : G \to G'$ so that

$$\varphi(g e) = \varphi_*(g) \cdot \varphi(e)$$

NB $G \to E(x)$ is bijective

$g \longmapsto g \cdot_E$

$$\begin{tikzcd} G \arrow[r, "\varphi_*"] \arrow[d] & G' \arrow[d] \\ E(x) \arrow[r, "\varphi"] & E'(x) \end{tikzcd}$$

determines φ_*.

$\mathcal{M}_{Pic}$: category

Objects : finitely generated $\mathbb{Z}$-modules Λ with an action of $\mathcal{G}_{\mathbb{L}} = \mathrm{Gal}(\mathbb{L}^s/\mathbb{L})$ and a morphism of $\mathbb{Z}[\mathcal{G}_{\mathbb{L}}]$-module $\varphi : \Lambda \to \mathrm{Pic}(\overline{V})$.

Let us define a functor $\mathcal{T}^m_{X,x} \longrightarrow \mathcal{M}_{Pic}$

Take E in $\mathcal{T}^m_{X,x}$ with group G, put $\Lambda = X^*(\overline{G})$

for $\lambda \in \Lambda$, λ is a morphism from $\overline{G}$ to $\mathbb{G}_{m,\overline{\mathbb{L}}}$

Lemma

a) With notation as above, let Y be a quasi-projective variety and $m : G \times Y \longrightarrow Y$ an action of G on Y of variety then the contracted product

$$E \times^G Y = E \times Y / G \qquad g(e, y) = (ge, g^{-1}y)$$

is well defined

b) for any morphism $\varphi : Y \to Y'$ compatible with the action ω^G ω^G we get $\psi_E : E \times^G Y \longrightarrow E \times^G Y'$.

c) It is compatible with products: for $\overset{G}{\underset{\circlearrowleft}{}}Y$, $\overset{G}{\underset{\circlearrowleft}{}}Y'$ define

$G \circlearrowright Y \times Y'$ $\quad E \times^G (Y \times Y') \xrightarrow{\sim} (E \times^G Y) \times_X (E \times^G Y')$

d) Moreover if $(\varphi_i : U_i \to X)$ is a splitting covering for E, then $U_i \times_X (E \times^G Y) \xrightarrow{\sim} U_i \times Y$ for $i \in I$.

Idea

d) $U_i \times_X E \xrightarrow{\sim} U_i \times G$ and $G \times^G Y \xrightarrow{\sim} Y$, $\quad \overline{(g,y)} \mapsto gy$

a) The $E \times^G Y$ is obtained by glueing the varieties described in the moreover statement. (but we need an étale glueing for which we use that Y is quasi projective). $\square$

So from $\chi : \overline{G} \to \mathbb{G}_{m,\overline{X}}$ let $F = \overline{E} \times^G \mathbb{G}_m$ it is a $\mathbb{G}_m$ torsor$/\overline{X}$ and we denote by $\rho_E(\chi)$ its class in $\mathrm{Pic}(\overline{X})$. We get the morphism

$$\rho_E : X^*(\overline{G}) \longrightarrow \mathrm{Pic}(\overline{X})$$

This construction is functorial

Theorem

The functor $E \mapsto \rho_E$ defines an equivalence of category from $(\mathcal{C}^m_{X,x})^0$ to $\mathcal{A}_{\mathrm{Pic}}$. In particular, a torsor corresponding to $\mathrm{Id}_{\mathrm{Pic}(\overline{X})}$ is universal.

Sketch of the proof

- First let us show that

If $\varphi, \varphi' : E \to E'$ satisfy $\rho_\varphi = \rho_{\varphi'} : X^*(\overline{G}') \to X^*(\overline{G})$ then $\varphi = \varphi'$.

$$\begin{tikzcd} X^*(\overline{G}') \arrow[rr] \arrow[dr, "\rho_{E'}"'] & & X^*(\overline{G}) \arrow[dl, "\rho_E"] \\ & \mathrm{Pic}(\overline{X}) & \end{tikzcd}$$

By the equivalence of categories for group

$\varphi_\cdot = \varphi'_x : G \to G'$

Then there exists a morphism $\psi : X \to G'$ so that $\forall y \in E_{\overline{L}}$, $\varphi'(y) = \psi(\pi_E(y))\, \varphi'(y)$

But on $\overline{L}$ $G \hookrightarrow \mathbb{G}^n_{m,\overline{L}}$

So $i \circ \psi : X \to \mathbb{G}^n_{m,\overline{L}}$ which has to be constant since X is projective

but $\varphi(\bullet_E) = \varphi'(\bullet_E) = \bullet_{E'}$ (here we use $\bullet_E$)

so $\psi(x) = 1_{G'}$ for all x

and $\varphi = \varphi'$.

- for $[L] \in \mathrm{Pic}(\overline{X})$ $L^\times$ is a $\mathbb{G}_m$ torsor $/\overline{X}$ corresponding to

$$\mathbb{Z} \to \mathrm{Pic}(\overline{X})$$
$$n \mapsto n[L]$$

If $n[L] = 0$ in $\mathrm{Pic}(\overline{X})$ then $L^{\otimes n} \cong \mathcal{O}_{\overline{X}}$

and

$\mu_n(L) = \{ y \in L \mid y^{\otimes n} = 1 \}$ defines a μ_n torsor corresponding to

$$\mathbb{Z}/n\mathbb{Z} \to \mathrm{Pic}(\overline{X})$$
$$\overline{1} \mapsto [L].$$

- Let E be an object of $\mathcal{M}^m_{X,x}$, G the corresponding group; for a commutative diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\varphi"] \arrow[d] & X^*(\overline{G}) \arrow[d, "\rho_E"] \\ \mathrm{Pic}(\overline{X}) \arrow[r, equal] & \mathrm{Pic}(\overline{X}) \end{tikzcd}$$

$1 \mapsto [L]$

By definition of $\rho_{E,x}$, φ defines a morphism $\overline{E} \to L^\times$

If $n\varphi(1) = 0$ in $X^*(\overline{G})$ then $n[L] = 0$

this induces a morphism $E \to \mu_n(L)$

More generally, if $X^*(\bar{G}) \xleftarrow{\varphi} \bigoplus_{i=1}^{r} \mathbb{Z} e_i \oplus \bigoplus_{i=1}^{m} \mathbb{Z}/a_i\mathbb{Z} f_i$

if choose E_i, F_i so that

$$[E_i] = \delta_E(\varphi(e_i)) \text{ and } [F_i] = \delta_E(\varphi(f_i))$$

we get an morphism

$$\varphi^* \bar{E} \to \left(\bigtimes_{i=1}^{r}{}_X E_i^\times\right) \times_X \left(\bigtimes_{i=1}^{m}{}_X \mu_{a_i}(F_i)\right)$$

which is unique ! and therefore this behaves well with composition

If φ is an isomorphism so is φ^* which proves that any object is isomorphic to one of that form / $\bar{\mathbb{L}}$!

If φ is invariant under $Gal(\bar{\mathbb{L}}/\mathbb{M})$ then φ^* is defined / $\mathbb{M}$

- Let us construct a universal torsor

Since we assumed that $Pic(\bar{X})$ is finitely generated, we may write

$$Pic(\bar{X}) = \left(\bigoplus_{i=1}^{t} \mathbb{Z}[L_i]\right) \oplus \left(\bigoplus_{i=1}^{m} \mathbb{Z}/a_i\mathbb{Z}[T_i]\right)$$

Let $\tilde{X}_{\bar{\mathbb{L}}} = \left(\bigtimes_{i=1}^{t}{}_X L_i^\times\right) \times_X \left(\bigtimes_{i=1}^{m}{}_X \mu_{a_i}(T_i)\right) \quad / \bar{X}$

it is a torsor under $T_{NS} / \bar{X}$, $X^*(T_{NS}) = Pic(\bar{X})$

The action of $Gal(\bar{\mathbb{L}}/\mathbb{L})$ on $Pic(\bar{X})$ factors through a finite quotient $Gal(\mathbb{M}/\mathbb{L}) = H$

So $\tilde{X}_{\bar{\mathbb{L}}}$ comes from $\tilde{X}_{\mathbb{M}}$ defined over $\mathbb{M}$. Moreover using the last point, we get an action $H \circlearrowright E_{\mathbb{M}}$ so that for any σ in H the following diagram commutes

$$\begin{tikzcd} \tilde{X} \arrow[r, "\sigma^*"] \arrow[d] & \tilde{X} \arrow[d] \\ V \times_{Spec(\mathbb{L})} Spec(\mathbb{M}) \arrow[r, "Id_V \times \sigma^{-1}"] & V \times_{Spec(\mathbb{L})} Spec(\mathbb{M}) \end{tikzcd}$$

Since $\tilde{X}$ is quasi projective $/\mathbb{L}$,
$\tilde{\mathcal{X}} = \tilde{X}/_H$ is defined as a variety $/\mathbb{L}$
and by the above diagram
we get $\tilde{\mathcal{X}} \to X$; Similarly we $m : T_{NS} \times \tilde{X} \to \tilde{X}$
and $\tilde{X}$ is a T_{NS} torsor and, by construction,
$\tilde{\mathcal{X}}$ is invariant under the action of G
so it is defined over $\mathbb{L}$. We get a
pointed T_{NS} torsor $\tilde{\mathcal{X}}$.

- An inverse to $E \longmapsto \rho_E$ is given by $\rho \longmapsto \rho_*(\tilde{\mathcal{X}})$. The difficulty is to prove that there is an isomorphism from E to $\rho_{E_*}(\tilde{\mathcal{X}})$ but this follows from the above construction of morphisms

Remarks

(i) The main point of the proof is solving the descent problem.

- If $\mathbb{M}/\mathbb{L}$ is a Galois extension and X is defined over $\mathbb{L}$, then you have an action of $H = Gal(\mathbb{M}/\mathbb{L})$ on $X_{\mathbb{M}}$ over $Spec\,\mathbb{L}$:

$$\begin{tikzcd} X \times_{Spec(\mathbb{K})} Spec(\mathbb{M}) \arrow[r, "Id \times \sigma^{-1}"] \arrow[d] & X \times_{Spec(\mathbb{K})} Spec(\mathbb{M}) \arrow[d] \\ Spec(\mathbb{M}) \arrow[r, "\sigma^{-1}"] \arrow[dr] & Spec(\mathbb{M}) \arrow[d] \\ & Spec(\mathbb{L}) \end{tikzcd}$$

So given E defined over $\mathbb{M}$
to find a form of E over $\mathbb{L}$,
the first thing is to construct an action
of H on E so that

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E \\ \downarrow & \sigma^{-1} & \downarrow \\ \mathrm{Spec}(\mathbb{L}) & \xrightarrow{\sigma} & \mathrm{Spec}(\mathbb{L}) \end{array}$$

commutes then a form is given by E/H.
It is the base points which ensure that
we really have an action $\tau(\sigma(e)) = (\tau\sigma)(e)$.

(ii) Over $\overline{\mathbb{L}}$, $\tilde{X} \to X$ is surjective
so for any $x' \in X$ we may choose $\tilde{x}' \in \tilde{X}(x')$
and get a universal torsor above X, x'.
By unicity of the universal torsors we
get that
All universal torsors, whatever the base point, as
T_{NS}-torsors are a form of $\tilde{X}$.

__Definition__
A versal torsor is a T_{NS}-torsor over $\mathbb{L}$
which is a form of $\tilde{X}$.
(we forget about the base-point)

__Remark__
Versal is a terminology of GROTHENDIECK
it is universal without unicity.

__Theorem__ (COLLIOT-THÉLÈNE & SANSUC)
V/ number field which satisfies $\mathcal{H}$,

北京大学

then there is a finite number of isomorphism classes of versal torsors having a rational point over $\mathbb{K}$.

NB

Let $(\tilde{V}_i)_{i\in I}$ be those torsors $\pi_i : \tilde{V}_i \to V$ the projections

By the above proof

$$V(\mathbb{K}) = \coprod_{i\in I} \pi_i(\tilde{V}_i(\mathbb{K}))$$

and

$$V(\mathbb{K}) \subset \bigcup_{i\in I} \pi_i(\tilde{V}_i(\mathbb{A}_{\mathbb{K}})) \subset V(\mathbb{A}_{\mathbb{K}})$$

Question

When do we have

$$\overline{V(\mathbb{K})} = \underbrace{\bigcup_{i\in I} \pi_i(\tilde{V}_i(\mathbb{A}_{\mathbb{K}}))}_{V(\mathbb{A}_{\mathbb{K}})^{TU}} \; ?$$

Remark

This more or less is the same as saying that smooth compactifications of the $\tilde{V}_i$ satisfy weak approximation.

Examples

- If $V \subset \mathbb{P}^N$ smooth hypersurface of dimension ≥ 3, the universal torsor is given by the one
$$W = \pi^{-1}(V) \subset \mathbb{A}^{N+1} - \{0\}$$
- If $\tilde{V} \to V$ and $\tilde{V}' \to V'$ are universal torsors (with V, V' satisfying (H)) then $\tilde{V} \times \tilde{V}' \to V \times V'$ is the universal torsor (over $(\cdot_x, \cdot_{x'})$).

3) Connexion to the Cox ring

In some sense the Cox ring is the rang of all sections of all possible line bundles on the variety.

Definition

A pointed line bundle over a pointed variety X is a line bundle L with a chosen point

$$\cdot_L \in L(\cdot_X) - \{0\}$$

and a morphism of pointed line bundles is a morphism of line bundles which maps the base point to the base point

Remarks

Given pointed line bundles L, L' / a nice X which are isomorphic as line bundles

Let $\psi : L \xrightarrow{\sim} L'$ be an isomorphism

Then $\exists\, \lambda \in \mathbb{K}^*$ such that $\psi(\cdot_L) = \lambda \cdot_{L'}$

and $\lambda^{-1}\psi$ is the unique isomorphism of pointed line bundles from L to L'

So $\mathrm{Pic}(X)$ is also the group of isomorphism classes of pointed line bundles on X (with $\cdot_{L\otimes L'} = \cdot_L \otimes \cdot_{L'}$) and for any element in $\mathrm{Pic}(X)$ there is a unique pointed line bundle representing up to unique isomorphism

Moreover $[L]+[L']$ is represented by $L\otimes L'$

This enables us to define

Definition

The Cox ring for the the pointed nice variety X is

$$C_X = \bigoplus_{[L]\in \mathrm{Pic}(X)} \Gamma(X, L)$$ with the product

pointed!

$s \in \Gamma(V, L)$, $s' \in \Gamma(V, L')$
for L'' such that $[L''] = [L] + [L']$
Take the unique isomorphism $\psi : L \otimes L' \xrightarrow{\sim} L''$
$s s' = \psi \circ (s \otimes s')$.

Connexion with the universal torsors

Assume $(\mathcal{H})$

(i) For any L pointed line bundle and $s \in \Gamma(V, L)$
s defines a morphism $L^\vee \xrightarrow{s^\vee} \mathbb{A}^1_{\mathbb{K}}$
and there exists a unique morphism
$\psi_{L^\vee} : \widetilde{\ } \longrightarrow (L^\vee)^\times$
in $\mathcal{T}^m_{V, \circ_V}$. $s^\vee \circ \psi_{L^\vee} \in \Gamma(\widetilde{V}, \mathcal{O}_{\widetilde{V}})$
we get a morphism of algebras
$\tau : \mathcal{C}_V \longrightarrow \Gamma(\widetilde{V}, \mathcal{O}_{\widetilde{V}})$

(ii) for any L, $[L]$ defines a character
$\chi_L : T_{NS} \longrightarrow \mathbb{G}_m$ and T_{NS} acts on $\Gamma(V, L)$ via χ_L
τ is compatible with the actions of T_{NS}
on both sides

Theorem [HASSETT, TSCHINKEL]
Assume $(\mathcal{H})$ and that $\mathcal{C}_V$ is finitely generated
then this gives an open equivariant embedding
of $\widetilde{X}$ in $\operatorname{Spec}(\mathcal{C}_X)$ the image of which
is the open set on which T_{NS} acts freely.

Example
$V = \prod_{i=1}^{r} \mathbb{P}^{n_i}_{\mathbb{Q}}$, $\operatorname{Cox}(V) = \mathbb{K}[X_{i,j}, 1 \leq i \leq r, 0 \leq j \leq n_i]$
and $\widetilde{V} = \prod_{i=1}^{r} \mathbb{A}^{n_i+1} - \{0\}$.

* b) Brauer-Manin obstruction

This obstruction is of cohomological nature:

Definition

The cohomological BRAUER group of a variety V is

$$Br(V) = H^2_{et}(V, \mathbb{G}_m)_{tors}$$

this is a contravariant functor in V

Reference

MILNE étale cohomology

GROTHENDIECK Dix exposés sur la cohomologie des schémas.

For a field, $Br(\mathbb{L}) = Br(Spec(\mathbb{L}))$ classifies skew algebras of finite dimension over $\mathbb{L}$ which are skew fields with center $\mathbb{L}$.

One of the deepest theorem in algebraic number theory during the 20th century is the following one

Theorem (Global class field theory)

Let $\mathbb{K}$ be a number field

For any place $w \in Pl(\mathbb{K})$ there exists a canonical injective morphism

$$inv_w : Br(\mathbb{K}_w) \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

quaternion algebra

with image $\begin{cases} \{0\} & \text{if } w \text{ is complex} \\ \{0, 1/2\} & \text{——— real } (1/2 \Leftrightarrow \mathbb{H}) \\ \mathbb{Q}/\mathbb{Z} & \text{if } w \text{ is ultra metric} \end{cases}$

So that the sequence

$$0 \longrightarrow Br(\mathbb{K}) \longrightarrow \bigoplus_{w \in Pl(\mathbb{K})} Br(\mathbb{K}_w) \xrightarrow{\sum_w inv_w} \mathbb{Q}/\mathbb{Z} \to 0$$

is exact.

Definition

We get a pairing

$$V(\mathbb{A}_{\mathbb{K}}) \times Br(V) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$((x_w)_{w\in Pl(\mathbb{K})}, A) \longmapsto \sum_{w\in Pl(\mathbb{K})} inv_w(x_w^*(A))$$

and for any $x\in V(\mathbb{A}_{\mathbb{K}})$ a morphism

$$\eta_x \in Hom_{gr}(Br(V), \mathbb{Q}/\mathbb{Z})$$

NB

If x comes from $V(\mathbb{K})$, then by the previous exact sequence, $\eta_x = 0$

Definition

η_x is called the BRAUER-MANIN obstruction to weak approximation

$$V(\mathbb{A}_K)^{Br} = \{x\in V(\mathbb{A}_K) \mid \eta_x = 0\}$$

Theorem

$$\overline{V(\mathbb{K})} \subset V(\mathbb{A}_{\mathbb{K}})^{Br} \subset V(\mathbb{A}_{\mathbb{K}})^{TU} \subset V(\mathbb{A}_{\mathbb{K}})$$

and if $Br(\overline{V}) = \{0\}$ then

$$V(\mathbb{A}_{\mathbb{K}})^{Br} = V(\mathbb{A}_{\mathbb{K}})^{TU}.$$

There are examples with $\subsetneq$ at each level.

Example from the beginning projective

$$V: Y^2+Z^2 = (3U^2-V^2)(V^2-2U^2)T^2 \subset W/\mathbb{G}_m^2$$

$$W = \mathbb{A}^3-\{0\}\times\mathbb{A}^2-\{0\}$$

$$(\lambda,\mu)(y,z,t,u,v) = (\lambda\mu^4 y, \lambda\mu^4 z, \lambda t, \mu u, \mu v)$$

V is a conic fibration over $\mathbb{P}^1$
on $\mathbb{K}(V)$ function field of V
we consider the quaternion algebra
A is generated by I_x, J_x
with relations

$$I_x J_x = -J_x I_x \quad \begin{cases} I_x^2 = -1 \\ J_x^2 = 3u^2 - v^2 \end{cases}$$

This defines an element in $Br(\mathbb{K}(V)) - \{0\}$
which comes from $Br(V) \hookrightarrow Br(\mathbb{K}(V))$.
and $V(\mathbb{A}_{\mathbb{K}})^{Br} = \emptyset$ ↖ injective! $*$

7) Metrics and measures

a) Definition of local measures

Definition

For any place w of $\mathbb{K}$
Remember that the completion is locally compact and thus $\mathbb{K}_w$ admits a Haar measure which is unique up to multiplication by a real number
→ normalize the measure by

$$\begin{cases} \int_{\mathcal{O}_w} dx_w = 1 & \text{if } w \text{ is ultrametric} \\ \int_{[0,1]} dx_w = 1 & \text{if } w \text{ is real} \\ dx_w = 2\, dx\, dy & \text{if } w \text{ is complex} \end{cases}$$

Theorem

$\prod_{w \in Pl(\mathbb{K})} dx_w$ defines a measure on $\mathbb{A}_{\mathbb{K}}$,

$\mathbb{K}$ is discrete in $\mathbb{A}_{\mathbb{K}}$ $\mathbb{A}_{\mathbb{K}}/\mathbb{K}$ is compact and

$$\mathrm{Vol}(\mathbb{A}_{\mathbb{K}}/\mathbb{K}) = \sqrt{|d_{\mathbb{K}}|}$$

for the induced measure when $d_{\mathbb{K}}$ is the discriminant of $\mathbb{K}/\mathbb{Q}$
$\left(\sqrt{|d_{\mathbb{K}}|} = \mathrm{Vol}\,(\mathbb{K}\otimes_{\mathbb{Q}}\mathbb{R}/G_{\mathbb{K}})\right)$

<u>Proposition</u> (Change of variables)
Let W, W' be open subsets in $\mathbb{K}_w^n$ and let

$$f=(f_1,\dots,f_n) : W \longrightarrow W'$$

be a diffeomorphism. Then for any integrable function $g : W' \longrightarrow \mathbb{R}$ we have

$$\int_{W'} g \; dy_{1,w}\dots dy_{n,w} = \int_W g\circ f \left| \det\left(\frac{\partial f_i}{\partial X_j}\right)_{\substack{1\le i\le n\\ 1\le j\le n}} \right|_w dx_{1,w}\dots dx_{n,w}$$

The proof, as in the real case reduces to the formula

$$dx_v(aB) = |a|_v \, dx_v(B)$$

for a borelian $B \subset \mathbb{K}_w$.

13/6/2016 <u>Reminder</u>
Let V be a nice variety $/\mathbb{K}_w$, $n = \dim(V)$
then for any point $x \in V(\mathbb{K}_w)$
there exists an open neighbourhood W of x in $V(\mathbb{K}_w)$ and rational functions $T_1,\dots,T_n$ on V, defined at x
so that $\varphi = (T_1,\dots,T_n)$ defines an homeomorphism

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from W to an open subset of $\mathbb{K}_w^n$
and for $y \in W$ the differential
$$d_y\varphi : T_y W \longrightarrow \mathbb{K}_w^n$$
is an isomorphism of $\mathbb{K}_w$ vector spaces

NB
If we take $V \subset \mathbb{P}^N_{\mathbb{K}_w}$
we may use $T_i = x_{j_i}/x_k$
for some $\{k, j_1, \dots, j_n\} \subset \{0, \dots, N\}$

Terminology
$(T_1, \dots, T_n)$ is called a system of coordinates at x and
$$\left(\frac{\partial}{\partial T_1}(y), \dots, \frac{\partial}{\partial T_n}(y)\right)$$
is the basis of $T_y V$ obtained by taking the inverse image of the usual basis of $\mathbb{K}_w^n$ by $d_y\varphi$

Proposition / Definition
Let V be a nice variety $/\mathbb{K}$
equipped with an adelic norm on ω_V^{-1}
then there exists a unique borelian measure ω_w on $V(\mathbb{K}_w)$ such for any $x \in V(\mathbb{K}_w)$, any system of coordinates $\varphi = (T_1, \dots, T_n)$ defined on an open neighborhood W of x and any continuous $f : V(\mathbb{K}_w) \to \mathbb{R}$
$$\int_W f\, \omega_w = \int_{\varphi(W)} f \circ \varphi^{-1} \left\| \frac{\partial}{\partial T_1} \wedge \dots \wedge \frac{\partial}{\partial T_n} (\varphi^{-1}(t_1, \dots, t_n)) \right\|_w dt_{1_w} \dots dt_{n_w}$$

This proposition follows from the formula for the change of variables.

Remark

In differential geometry it is well known that a non vanishing section of ω_V defines a volume form on the variety.
The norm $\|\cdot\|_\infty$ defines such a section up to sign and the proposition is a generalization of that fact.

But we want a measure on the adelic space so we want to consider the product of this measures.

Problem

In fact, $\prod_{w \in Pl(\mathbb{K})} \omega_w(V(\mathbb{K}_w))$

does not converge!
To understand that, we need to know more about this measure

b) Other descriptions of the measure

Example / Exercice

For $V = \mathbb{P}^n_{\mathbb{Q}}$ and $\|\cdot\|_\infty$ the norm on ω_V^{-1} defined by a norm $\|\cdot\|_\infty$ on $\mathbb{R}^{n+1}$ then

$$\omega_V(B) = \underset{\text{euclidean volume}}{\mathrm{Vol}}\,(\underset{\text{for } \|\cdot\|_\infty}{B(0,1)} \cap \pi^{-1}(W))$$

$$= \mathrm{Vol}\,(\{y \in \mathbb{R}^{n+1} \mid \|y\|_\infty \leq 1 \text{ \& } \pi(y) \in W\})$$

Proposition

There exists a finite set of places $S \supset Pl(\mathbb{K})_\infty$ and a projective model $\mathcal{V}$ of V over $\mathcal{O}_S$ so that for any $w \notin S$ ω_w is the unique measure on $V(\mathbb{K}_w)$ which satisfies

$$\omega_w(r_{m_w^k}^{-1}(X)) = \frac{\# X}{\# (\mathbb{F}_w)^{kn}}$$

where $r_{m_w^k} : V(\mathbb{K}_w) \to \mathcal{V}(\mathcal{O}_w/m_w^k)$

is the reduction map and X is any subset of $\mathcal{V}(\mathcal{O}_w/m_w^k)$

Corollary

For almost all places $w \in Pl(\mathbb{K})$

$$\omega_w(V(\mathbb{K}_w)) = \frac{\# \mathcal{V}(\mathbb{F}_w)}{\# \mathbb{F}_w^n}$$

so the problem reduces to understand this number of points of $\mathcal{V}$ on the finite field $\mathbb{F}_w$

Sketch of the proof of the proposition

We choose an embedding

$V \hookrightarrow \mathbb{P}^N_{\mathbb{K}}$ $f_1, \dots, f_r$ generating $I(V)$.

there is a finite set of places $S \subset Pl(\mathbb{K})_v$ so that $f_1, \dots, f_r$ defines in $\mathbb{P}^n_{\mathcal{O}_S}$

A model $\mathcal{V} / \operatorname{Spec} \mathcal{O}_S$ which is smooth and for $v \notin S$, $\|\cdot\|_v$ is defined by ω_v^{-1}

The statement is local, it suffices to prove that if $\bar{x} \in \mathcal{V}(\mathcal{O}_w/m_w^k)$ $k \geq 1$

$$\omega_w(r_{m_w^k}^{-1}(\{\bar{x}\})) = \frac{1}{\# \mathbb{F}_w^{nk}}$$

fix $x \in V(\mathbb{K}_w)$ so that $r_{\mathfrak{m}_w^k}(x) = \overline{x}$
$x = [y_0 : - : y_N]$
with $(y_0, -, y_N) \in \mathcal{O}_w^{N+1}$, $\max_{0 \le i \le N} |y_i|_w = 1$.

By a linear change of coordinates using a matrix in $GL_N(\mathcal{O}_w)$ the first column of which is $(y_0, -, y_N)$ we may assume that $x = [1 : 0 : - : 0]$
To say that V is smooth means that

$$\min_{i,j} \left(w\left(\det \left(\frac{\partial f_{i_k}}{\partial Y_{j_l}} (1, 0, -, 0) \right)_{\substack{1 \le i_1 < - < i_c \le \Pi \\ 0 \le j_1 < - < j_c \le N}} \right) \right) = 0$$

where $c = \operatorname{codim}(V) = N - n$.
Since the line supported by $(1, 0, -, 0)$ is contained in the zero locus,

$$\frac{\partial f_i}{\partial Y_0}(1, 0, -, 0) = 0 \text{ for all } i.$$

So up to permutation of the variables and f_j, we may assume that

$$\det \left(\frac{\partial f_i}{\partial Y_j}(1, 0, -, 0) \right)_{\substack{1 \le i \le c \\ 1 \le j \le c}} \in \mathcal{O}_w^\times$$

Since $f_i \in \mathcal{O}_w[X_0, -, X_n]$
for any $(y_0, -, y_n) \in (1, 0, -, 0) + \mathfrak{m}_w^{n+1}$

$$\det \left(\frac{\partial f_i}{\partial Y_j}(y_0, -, y_n) \right) \equiv \det \left(\frac{\partial f_i}{\partial Y_j}(1, 0, -, 0) \right) \in \mathcal{O}_w^\times$$

So we get a diffeomorphism

$$W = \{ x' \in V(\mathbb{K}_w) \mid x \equiv x' \, [\mathfrak{m}_w] \} \xrightarrow{\psi} \mathfrak{m}_w^n$$
$$[y_0 : - : y_N] \longmapsto (y_1/y_0, -, y_n/y_0)$$

moreover for $k \geq 1$, $x' \in V(\mathbb{K}_w)$

$$x \equiv x' \; [m_w^k] \Leftrightarrow (x' \in W \;\&\; \psi(x') \equiv \psi(x) \; [m_w^k])$$

So $r_{m_w^k}^{-1}(\bar{x}) \subset W$ and $\psi(r_{m_w^k}^{-1}(\bar{x})) = (m_w^k)^n$

ideal ↗ ↖ set

also if we take $T_1 = \frac{X_1}{X_0}, \ldots, T_n = \frac{X_n}{X_0}$ as local coordinates on W, we have

$$\left\|\frac{\partial}{\partial T_1} \wedge \cdots \wedge \frac{\partial}{\partial T_n}(x')\right\|_w = 1$$

which implies than on W

$$\int_W f \, \omega_w = \int_{m_w^n} f \circ \psi^{-1} \, dy_{1,w} \cdots dy_{n,w}$$

So

$$\omega_w(r_{m_w}^{-1}(\bar{x})) = \int_{((m_w)^k)^n} dy_{1,w} \cdots dy_{n,w}$$

$$= \frac{1}{\#(\mathcal{O}_w/m_w^k)^n} = \# \mathbb{F}_w^{-kn} \quad \square$$

So the remaining problem is:

<u>Problem</u>

How to estimate $\# V(\mathbb{F}_w)$ as w change?

For this I need one of the most important theorem in algebraic geometry in the 20th century:

c) Weil's conjecture

This were one of the most looked for conjecture in the 60's and a large part of GROTHENDIECK's work was motivated by them. In the end, they were proven by DELIGNE, a former student of GROTHENDIECK. Historically the idea comes from topology and

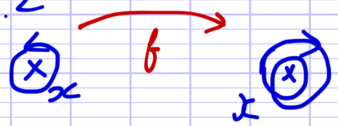
Theorem [LEFSCHETZ formula]
Let X be a triangulated compact space X and $f : X \to X$ a continuous map such that $X^f = \{x \in X \mid f(x) = x\}$ is finite, then

$$\sum_{x \in X^f} \underbrace{i_x(f)}_{\text{index of } f \text{ at } x} = \sum_{k \geq 0} (-1)^k \operatorname{Tr}(f_* \mid H_k(X, \mathbb{Q}))$$

singular homology

NB in dim. 2

$i_x(f)$



$i_x(f) = -2$

(if f is differentiable and $\det(d_x f) > 0$ then $i_x(f) = 1$)

Analogy
If $\mathbb{F}$ is a finite field of cardinal q

$$\operatorname{Gal}(\overline{\mathbb{F}}/\mathbb{F}) = \overline{\langle Fr_q \rangle} \qquad Fr_q : \overline{\mathbb{F}} \to \overline{\mathbb{F}}, \quad x \mapsto x^q$$

in particular, for a variety $X/\mathbb{F}$

$$X(\mathbb{F}) = X(\overline{\mathbb{F}})^{Fr_q}$$

Theorem (GROTHENDIECK – LEFSCHETZ formula)
If X is a nice variety $/\mathbb{F}$ $\quad n = \dim(X)$

$$\# X(\mathbb{F}) = \sum_{i=0}^{2n} (-1)^i \operatorname{Tr}(Fr_q \mid H^i_{et}(\overline{X}, \mathbb{Q}_\ell))$$

where ℓ is prime $\neq \operatorname{char}(\mathbb{F})$

$$H^i_{et}(\overline{X}, \mathbb{Q}_\ell) = \left(\varprojlim H^i_{et}(\overline{X}, \mathbb{Z}/\ell^n\mathbb{Z})\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

This only gives an estimate if we know how big these traces are which reduces to

1] know something about the dimension of these spaces

2) know something about the eigenvalues of the action of the Frobenius on these spaces.

Theorem [DELIGNE]

For a nice variety X over $\mathbb{F}$, the eigenvalues of Fr_q acting on $H^i_{et}(\bar{X}, \mathbb{Q}_\ell)$ are algebraic integers ξ such that

$$|\xi| = q^{i/2}$$

Remark

In particular,

$$Tr(Fr_q \mid H^i_{et}(\bar{X}, \mathbb{Q}_\ell)) \leq q^{\frac{i}{2}} \dim_{\mathbb{Q}_\ell}(H^i_{et}(\bar{X}, \mathbb{Q}_\ell))$$

d) Estimates for $a_w(V(\mathbb{K}_w))$

Assume V satisfies $(\mathcal{H})$;

$\mathcal{V}$ is a smooth & projective model of V / G_S

Fact

For almost all prime $\mathfrak{p} \in Pl(\mathbb{K}_f)$,

$$\dim(H^i_{et}(\mathcal{V}, \mathbb{Q}_\ell)) = \dim(H^i_{et}(\mathcal{V}_{\bar{\mathbb{F}}_\mathfrak{p}}, \mathbb{Q}_\ell))$$

Combining this with Weil's conjecture we get an estimate of the number of points on the residue field. But we need some extra information on the groups of the highest degree

we can twist the cohomology groups in the following way:

$$\mu_{\ell^k}(\bar{\mathbb{F}}) - \ell^n \text{ roots of } 1 \text{ in } \bar{\mathbb{F}}$$

$$\circlearrowleft \quad \xi \longmapsto \xi^q$$

$$Fr_q$$

$$\mu_{\ell^k}(\bar{\mathbb{F}})^{\otimes i} = \begin{cases} i\text{-th tensor product of the } \mathbb{Z}\text{-module } \mu_{\ell^k}(\bar{\mathbb{F}}) \text{ if } i \geq 0 \\ \mathrm{Hom}(\mu_{\ell^k}^{\otimes i}, \mathbb{Z}/\ell^k\mathbb{Z}) \text{ if } i < 0 \end{cases}$$

I additive notations, Fr_q acts via $\lambda \mapsto q^i\lambda$ on $\mu_{\ell^k}(\bar{\mathbb{F}})^{\otimes i}$

$$H^i_{et}(\bar{V}, \mathbb{Q}_\ell(j)) = \left(\varprojlim_k H^i_{et}(\bar{V}, \mu_{\ell^k}(\bar{\mathbb{F}})^{\otimes j})\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

2 On $H^i_{et}(\bar{V}, \mathbb{Q}_\ell(j))$ we use the «geometric» Frobenius coming from its action on $V/\mathbb{F}$

So it is a contravariant action

as a group $H^i_{et}(\bar{V}, \mathbb{Q}_\ell(j))$ is isomorphic to $H^i_{et}(\bar{V}, \mathbb{Q}_\ell)$, but the action is given by q^{-j} the action of Fr_q on $H^i_{et}(V, \mathbb{Q}_\ell)$.

POINCARÉ's duality theorem

(i) $H^{2n}_{et}(\bar{X}, \mathbb{Q}_\ell(n)) \xrightarrow{\sim} \mathbb{Q}_\ell$ and

(ii) $\cup\ H^i_{et}(\bar{X}, \mathbb{Q}_\ell(j)) \otimes H^{2n-i}_{et}(\bar{X}, \mathbb{Q}_\ell(n-j)) \longrightarrow H^{2n}_{et}(\bar{X}, \mathbb{Q}_\ell(n))$

defines an isomorphism $H^i_{et}(\bar{X}, \mathbb{Q}_\ell(j)) \xrightarrow{\sim} H^{2n-i}_{et}(\bar{X}, \mathbb{Q}_\ell(n-j))^\vee$

Then we consider the following sequence

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \to 1$$

Using $H^0_{et}(\bar{X}, \mathbb{G}_m) = \bar{\mathbb{F}}^*$

$H^1_{et}(\bar{X}, \mathbb{G}_m) = \mathrm{Pic}(\bar{X})$ and $H^2_{et}(\bar{X}, \mathbb{G}_m)_{tors} = \mathrm{Br}(\bar{X})$

we get

$$0 \to H^1_{et}(\bar{X}, \mu_{\ell^n}) \to \mathrm{Pic}(\bar{X})[\ell^n]$$

and

$$0 \to \mathrm{Pic}(\bar{X})/\ell^n \to H^2_{et}(\bar{X}, \mu_{\ell^n}) \to \mathrm{Br}(\bar{X})[\ell^n] \to 0$$

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But for almost all $\mathfrak{p}$,

$$\begin{cases} Pic(V_{\overline{\mathbb{F}_p}}) \xrightarrow{\sim} Pic(\overline{V}) \\ Br(V_{\overline{\mathbb{F}_p}}) \xrightarrow{\sim} Br(\overline{V}) \end{cases}$$ which is finite under (H)

Thus by taking the projective limit,

for almost all $\mathfrak{p}$

$$H^1_{et}(V_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell) = 0 = H^{2n-1}_{et}(V_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell)$$

and

$$Pic(\overline{V}) \xrightarrow{\sim} H^2_{et}(V_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} H^{2n}_{et}(V_{\overline{\mathbb{F}_p}}, \mathbb{Q}(2n-1))$$

Combining all this, we get

for almost all $\mathfrak{p}$

$$\omega_{\mathfrak{p}}(V(\mathbb{K}_{\mathfrak{p}})) = \frac{\# V(\mathbb{F}_p)}{(\# \mathbb{F}_p)^n}$$

$$= 1 + \frac{1}{\# \mathbb{F}_p} Tr(Fr_{\mathfrak{p}} \mid \underbrace{Pic(\overline{V}) \otimes_{\mathbb{Z}} \mathbb{Q}}_{Pic(\overline{V})_{\mathbb{Q}}}) + O\left(\frac{1}{(\# \mathbb{F}_p)^{3/2}}\right)$$

But if $\varphi \in End(E)$

$$Det(1 - T\varphi \mid E) = 1 - T\, Tr(\varphi) + T^2 Q(T)$$

So

$$\omega_{\mathfrak{p}}(V(\mathbb{F}_p)) = Det(1 - \# \mathbb{F}_p^{-1} Fr_{\mathfrak{p}} \mid Pic(\overline{V})_{\mathbb{Q}})^{-1} (1 + O(\# \mathbb{F}_p^{-\frac{3}{2}}))$$

e) The constant

Definition

For $\mathfrak{p} \in \mathcal{P}\ell(\mathbb{K})_f - S$

$$L_{\mathfrak{p}}(s, Pic(\overline{V})) = \frac{1}{Det(1 - \# \mathbb{F}_p^{-s} Fr_{\mathfrak{p}} \mid Pic(\overline{V})_{\mathbb{Q}})}$$

$$L_S(s, Pic(\overline{V})) = \prod_{\mathfrak{p} \notin S} L_{\mathfrak{p}}(s, Pic(\overline{V}))$$

Theorem [Consequence of a theorem of ARTIN]

$L_S(s, Pic(\overline{V}))$ converges for $Re(s) > 1$ and has a pole of order $t = rk(Pic(V))$ at $s = 1$.

Remark

From the fact that
$$\zeta_{\mathbb{K}}(s)^{-1} = \prod_{\mathfrak{p} \in \mathcal{P}l(\mathbb{K})_f} \left(1 - \#\mathbb{F}_{\mathfrak{p}}^{-s}\right) \text{ converges}$$
for $\mathrm{Re}(s) > 1$ (which might be seen as a particular case of ARTIN's theorem as well),
$$\prod_{\mathfrak{p} \notin S} \left(1 - 6\left(\#\mathbb{F}_{\mathfrak{p}}^{-3/2}\right)\right) \text{ converges}$$

Definition

Let $\lambda_w = \begin{cases} 1 & \text{if } w \in S \\ L_w(1, \mathrm{Pic}(\overline{V}))^{-1} & \text{if } w \notin S \end{cases}$

We define a measure ω on $V(\mathbb{A}_{\mathbb{K}})$ by
$$\omega = \frac{1}{\sqrt{|d_{\mathbb{K}}|}^{\,n}} \times \lim_{s \to 1} (s-1)^t L_S(s, \mathrm{Pic}(\overline{V})) \prod_{w \in \mathcal{P}l(\mathbb{K})} \lambda_w \omega_w$$
and the induced probability measure on $V(\mathbb{A}_{\mathbb{K}})^{Br}$:
$$\mu(W) = \frac{\omega\left(W \cap V(\mathbb{A}_{\mathbb{K}}^{Br})\right)}{\omega\left(V(\mathbb{A}_{\mathbb{K}}^{Br})\right)}$$

It turns out it almost gives the constant but not quite,
$$\alpha(V) = \frac{1}{(t-1)!} \int_{C_{eff}(V)^\vee} e^{-\langle \omega_V^{-1}, y \rangle}\, dy \in \mathbb{Q}$$
$\quad$ under (H)

where
$$C_{eff}(V)^\vee = \left\{ y \in \left(\mathrm{Pic}(V)_{\mathbb{R}}\right)^\vee \mid \forall x \in C_{eff}(V),\ \langle x, y \rangle \geq 0 \right\}$$

$$\beta(V) = \#\, Br(V) \quad \text{finite}$$

The empirical constant for V is

$$C(V) = C_H(V) = \alpha(V)\,\beta(V)\,\omega\left(V(\mathbb{A}_{\mathbb{K}})^{Br}\right)$$

Remark

ω does not depend on the choice of S and therefore $C(V)$ depends only on the choice of the adelic norm on ω_V^{-1}.

We then consider

The empirical formula

rank of the Picard group

$$(F) \qquad \# U(\mathbb{K})_{H \leq B} \sim C(V)\, B \log(B)^{t-1}$$

The empirical distribution

$$(E) \qquad \delta_{U(\mathbb{K})_{H\leq B}} \xrightarrow[B \to +\infty]{w} \mathbb{N}.$$

15/6/2016 f) Connection between (E) and (F)

Theorem

The following three statements are equivalent

(i) (F) is true for any choice of the adelic norm on ω_V^{-1};

(ii) (E) and (F) are true for at least one choice of the norm on ω_V^{-1};

(iii) (E) and (F) are true for any choice of the adelic norm on ω_V^{-1}.

Sketch of the proof

We have to prove

$(ii) \Rightarrow (i) \Rightarrow (E)$ for any norm

Fix a norm $(\|\cdot\|_w)_{w \in Pl(\mathbb{K})}$ on ω_V^{-1}

assume $(E) + (F)$

Let $(\|\cdot\|'_w)_{w \in Pl(\mathbb{K})}$ be any adelic norm

we have

$$f(x)\, H'(x) = H(x)$$

where $f : V(\mathbb{A}_K) \to \mathbb{R}_{>0}$ is continuous. Let $\varepsilon > 0$

Using the fact that the measure ω is locally given by the Haar measure times a continuous density, we may construct

$(U_i)_{i \in I}$ finite partition of $V(\mathbb{A}_K)$

with U_i borelian for $i \in I$, $\omega(\partial U_i) = 0$

and $(\lambda_i)_{i \in I} \in \mathbb{R}_{>0}^I$ so that

$$\forall x \in V(\mathbb{A}_K) \quad |f(x) - \underbrace{\sum_{i \in I} \lambda_i \mathbb{1}_{U_i}(x)}_{g}| < \varepsilon$$

By assuming ε small enough,

we may assume $\lambda_i > \varepsilon$ for $i \in I$.

Using (ii) and the argument given one week ago

$$\# \{ x \in V(\mathbb{K}) \mid H(x) \leq g(x) \pm \varepsilon \}$$

$$\sim \underbrace{\underbrace{\int_{V(\mathbb{A}_K)^{Br}} g \pm \varepsilon \ \mu \quad C_H(V)}\, B \log(B)^{r-1}}$$

Let us now prove that

$$\xrightarrow[\varepsilon \to 0]{} \underbrace{\int_{V(\mathbb{A}_K)^{Br}} f \mu \ C_H(V)}_{\substack{\| \\ C_{H'}(V)}}$$

$(i) \Rightarrow (E)$ for H

Probability theory tells us that

For a Borelian W, $\omega(\partial W) = 0$

if and only if there exists continous functions $f, g : V(\mathbb{A}_K) \to \mathbb{R}_{\geq 0}$ such that

$$f \leq \mathbb{1}_W \leq g \text{ and } \int_{V(\mathbb{A}_K)} g - f \ \omega < \varepsilon$$

Using the heights $\frac{1}{f+\varepsilon} H$ and $\frac{1}{g+\varepsilon} H$

we get that

$$\#(W \cap U(\mathbb{K}))_{H\leq B} \underset{B\to+\infty}{\sim} \omega(V(\mathbb{A}_{\mathbb{K}})^{Br} \cap W) B \log(B)^{t-1}$$

So $\delta_{U(\mathbb{K})_{H\leq B}}(W) \longrightarrow \mu(W)$. $\square$

So the formula can not be true if we do not have equidistribution!

g) A few consequences of distribution

Let me finish with a few remark about equidistribution:

α) Let $F \subsetneq V$ closed subvariety.

$$\omega(F(\mathbb{A}_{\mathbb{K}})) = 0$$

So if (E) is valid

$$\#(F \cap U(\mathbb{K}))_{H\leq B} = o(\# U(\mathbb{K})_{H\leq B})$$

and

$\forall\, U' \subset U$ not empty

$$\# U'(\mathbb{K})_{H\leq B} \sim \# U(\mathbb{K})_{H\leq B}$$

and the formula (F) is also valid for any non empty open set in U ("small enough")

β) For almost all w, ω_w is characterized by $\omega_w(\pi^{-1}_{\mathfrak{m}_w^k}(X)) = \frac{\# X}{\# \mathbb{F}_w^{nk}}$

So if V satisfies E and $X \subset V(\mathcal{O}_w/\mathfrak{m}_w^k)$, $w \notin S$ then

$$\frac{\#\{x \in U(\mathbb{K}) \mid H(x) \leq B,\ \pi_{\mathfrak{m}_w^k}(x) \in X\}}{\# U(\mathbb{K})_{H\leq B}} \xrightarrow[B\to+\infty]{} \frac{\# X}{\# V(\mathcal{O}_w/\mathfrak{m}_w^k)}$$

h) Expression of the constant in terms of versal torsors

I assume (H) + $Br(V) = \{0\}$.

d) Geometric properties of versal torsors

The main idea is that versal torsors are geometrically and arithmetically more simple.

Proposition

Let E be a versal torsor above V

Then

a) $\Gamma(\overline{E}, \mathbb{G}_m) = \overline{\mathbb{K}}^*$

b) $Pic(\overline{E}) = \{0\}$

c) $Br(Y) = \{0\}$ for any smooth compactification Y of $\overline{E}$ ($\overline{E}$ open in Y smooth and

In fact this statement is all about the $\mathbb{G}_m$-cohomology in low degree. This follows from a more general

Theorem [SANSUC]

Let $\mathbb{L}$ be a perfect field, G a smooth connected linear algebraic group on $\mathbb{L}$, X a smooth variety $/\mathbb{L}$ and E a G-torsor over X, $\pi : E \to X$

then there exists a natural exact sequence

$$0 \to \mathbb{L}[X]^* \xrightarrow{\pi^*} \mathbb{L}[E]^* \to X^*(G) \to Pic(X)$$
$$\xrightarrow{\pi^*} Pic(E) \to Pic(G) \to Br(E) \xrightarrow{\pi^*} Br(X)$$

Proof of Theorem $\Rightarrow$ Proposition

In our case, we have

X projective, so $\overline{\mathbb{K}}[\overline{X}]^* = \overline{\mathbb{K}}^*$

$X^*(\overline{T}_{NS}) \xrightarrow{\sim} Pic(\overline{X})$

$Pic(\overline{T}_{NS}) = Pic(\mathbb{G}^n_{m,\overline{\mathbb{K}}}) = \{0\}$

The Brauer group is a stably birational invariant for smooth projective varieties which means that X, Y are nice varieties and $\exists\, m, n$ so that $X \times \mathbb{P}^m \overset{!}{\dashrightarrow} Y \times \mathbb{P}^n$ birational then $Br(X) \underset{pr_1^*}{\xrightarrow{\sim}} Br(X \times \mathbb{P}^m) \xrightarrow{\sim} Br(Y \times \mathbb{P}^n) \xrightarrow{\sim} Br(Y)$.

But over $\overline{\mathbb{L}}$ E splits for Zariski topology and is stably birational to V so $Br(\overline{E}) = \{0\}$. $\square$

<u>Corollary of the proposition</u>

Up to multiplication by a constant, there exists a unique section s of $\omega_{\overline{E}}^{-1}$ such $s(x) \neq 0$ for any closed point x of E.

<u>Lemma</u>

X nice $/\mathbb{L}$, char $\mathbb{L} = 0$, $Pic(X) \hookrightarrow Pic(\overline{X})^{Gal(\overline{\mathbb{L}}/\mathbb{L})}$ and it is an isomorphism if $X(\mathbb{L}) \neq \emptyset$

<u>Proof</u>

We have two exact sequences

$$0 \to \overline{\mathbb{L}}^* \to \overline{\mathbb{L}}(\overline{V})^* \to \overline{\mathbb{L}}(\overline{V})^*/\overline{\mathbb{L}}^* \to 0$$

and

$$0 \to \overline{\mathbb{L}}(\overline{V})^*/\overline{\mathbb{L}}^* \to Div(\overline{V}) \to Pic(\overline{V}) \to 0$$

If we define the Galois cohomology as the right derived functor of the left exact functor from the Category of $\mathbb{Z}$-modules with an action of $\underbrace{Gal(\overline{\mathbb{L}}/\mathbb{L})}_{\mathcal{G}}$ to the category of $\mathbb{Z}$-modules

$$M \mapsto M^{\mathcal{G}}$$

then $\underbrace{H^i(\mathbb{L}, M)}_{\text{group for this cohomology}} = H^i_{et}(Spec(\mathbb{L}), M)$

we have two exact sequences for X nice / $\mathbb{L}$

$$1 \to \bar{\mathbb{L}}^* \to \bar{\mathbb{L}}(\bar{X})^* \to \bar{\mathbb{L}}(\bar{X})^*/\bar{\mathbb{L}}^* \to 1$$

and $1 \to \bar{\mathbb{L}}(\bar{X})^*/\bar{\mathbb{L}}^* \to \mathrm{Div}(\bar{X}) \to \mathrm{Pic}(\bar{X}) \to 0$

Taking the corresponding long exact sequence in cohomology we get

$$1 \to \underbrace{\mathbb{L}^*}_{\bar{\mathbb{L}}^{*\mathcal{G}}} \to \mathbb{L}(X)^* \to \left(\bar{\mathbb{L}}(\bar{X})^*/\bar{\mathbb{L}}^*\right)^{\mathcal{G}} \to \overline{H^1(\mathbb{L}, \bar{\mathbb{L}}^*)}$$

$\{0\} = \mathrm{Pic}(\mathrm{Spec}(\mathbb{L})) = H^1_{et}(\mathrm{Spec}(\mathbb{L}), \mathbb{G}_m)$

In fact by HILBERT's theorem 90 $\|$ 0

$\{0\}$ $\|$ $\leftarrow$ gives directly

$$\to H^1(\mathbb{L}, \bar{\mathbb{L}}(\bar{X})^*) \to H^1(\mathbb{L}, \bar{\mathbb{L}}(\bar{X})^*/\bar{\mathbb{L}}^*) \to \mathrm{Br}(\mathbb{L}) \to H^2(\mathbb{L}, \bar{\mathbb{L}}(\bar{X})^*)$$

ev_x $\subset\downarrow$ $\downarrow$

if $x \in X(\mathbb{L})$ $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(\mathbb{L}(X))$

and

$$\begin{array}{ccccc} \mathbb{L}(X)^*/\mathbb{L}^* & & \mathrm{Div}(X) & & \\ \downarrow\wr & & \| & & \\ 1 \to \left(\bar{\mathbb{L}}(\bar{X})^*/\bar{\mathbb{L}}^*\right)^{\mathcal{G}} & \to & \mathrm{Div}(\bar{X})^{\mathcal{G}} & \to & \left(\mathrm{Pic}(\bar{X})\right)^{\mathcal{G}} \end{array}$$

$$\to H^1(\mathbb{L}, \bar{\mathbb{L}}(\bar{X})^*/\bar{\mathbb{L}}^*) \to \underbrace{H^1(\mathbb{L}, \mathrm{Div}(\bar{X}))}_{\{0\}}$$

From the definition of Picard group,

$1 \to \mathbb{L}^* \to \mathbb{L}(X)^* \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0$ is exact

and $0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\bar{X})^{\mathcal{G}} \to \mathrm{Br}(\mathbb{L})$ is exact

$\subset$ 0 if $X(\mathbb{L}) \neq \emptyset$ □

Proof of Corollary

Since $\mathrm{Pic}(\bar{E}) = \{0\}$, $\omega_E^{-1} \xrightarrow{\sim} \mathcal{O}_E$

and such an isomorphism gives a non vanishing section s

If we have two sections we have two isomorphisms

$$\begin{tikzcd} & \mathcal{O}_E \\ \omega_E^{-1} \arrow[ur, "\sim"] & \\ & \mathcal{O}_E \arrow[ul, "\sim"] \arrow[uu, "\wr\,\psi"'] \end{tikzcd}$$

But ψ corresponds to section of $\mathbb{G}_m$ an therefore is the multiplication by a constant. □

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Construction

Let $s \in \Gamma(E, \omega_E^{-1})$ be a section as above.
For $w \in Pl(\mathbb{K})$, on ω_E^{-1} there is a unique norm $\|\cdot\|_w$ such that $\|s(x)\|_w = 1$ for any $x \in V(\mathbb{K}_w)$.
It defines a measure ω_w on $E(\mathbb{K}_w)$

2. E is not projective

Proposition

For almost all $w \in Pl(\mathbb{K})$
$$\omega_w(E(\mathcal{O}_w)) = L_p(1, Pic(\bar{V}))^{-1} \times \omega_w(V(\mathbb{K}_w))$$

Main ideas of the proof

The torsor splits locally for w-adic topology and this gives, for the places where the metric is given by a smooth projective model $\mathcal{V}$ and where E has a model $/\mathcal{V}$
$$\omega_w(E(\mathcal{O}_w)) = \underbrace{Vol(T_{NS}(\mathcal{O}_w))}_{\substack{\text{for some Haar measure} \\ \text{given by a model of } T_{NS}}} \times \omega_w(V(\mathbb{K}_w))$$

And for almost all places
$$Vol(T_{NS}(\mathcal{O}_w)) = L_w(1, Pic\,\bar{V})^{-1} \quad \text{[ONO, 1961]} \quad = \frac{\# T_{NS}(\mathbb{F}_w)}{\# \mathbb{F}_w^{t}}$$

Assume that $Pic(\bar{V})$ splits over a finite unramified extension of $\mathbb{K}_w$

Remark The formula on the right is easy to check if $Pic(\bar{V})$ has a basis globally invariant under the action of the Galois group. $\square$

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Reminder

$$E(\mathbb{A}_{\mathbb{K}}) = \bigcup_{\substack{S_0 \subset S \subset Pl(\mathbb{K}) \\ \text{finite}}} \left(\prod_{w \in S} E(\mathbb{K}_w) \right) \times \left(\prod_{w \notin S} E(\mathcal{O}_w) \right)$$

Corollary

$$\omega = \frac{1}{\sqrt{|d_{\mathbb{K}}|}^{n+t}} \prod_{w \in Pl(w)} \omega_w$$ defines a measure on $E(\mathbb{A}_{\mathbb{K}})$

Remark

By the product formula, since the section s of ω_E^{-1} is unique upto multiplication by a constant, the measure ω on $E(\mathbb{A}_{\mathbb{K}})$ does not depend on the choice of s

Conclusion

If (H) and $B_r(V) = 0$, the adelic space $E(\mathbb{A}_{\mathbb{K}})$, for E versal Torsor over V is equipped with a <u>canonical</u> measure

Remark

This measure is compatible with the action of T_{NS} in the following sense

If $t = (t_w)_{w \in Pl(\mathbb{K})} \in T_{NS}(\mathbb{A}_{\mathbb{K}})$

$$\|t\| = \prod_{w \in Pl(\mathbb{K})} |\underbrace{\omega_V^{-1}(t_w)}_{\in \mathbb{K}_w^*}|_w$$

(Remember $[\omega_V^{-1}] \in Pic(V) = X^*(T_{NS})$)

$$\omega(tB) = \|t\| \quad \omega(B)$$

if $t \in T(\mathbb{K})$ $\|t\| = 1$ and $\omega(tB) = \omega(B)$

Descent method [P. SALBERGER + ...]

Let $(E_i)_{i\in I}$ be representants of the isomorphisms classes of versal torsors having a rational point of $\mathbb{K}$. For each $i \in I$ and each $B \in \mathbb{R}_{\geq 1}$ there exists a domain $\mathcal{D}_i(B) \subset E_i(\mathbb{A}_{\mathbb{K}})$

(i) For any $x \in V(\mathbb{K})$, let i be the unique element of I such that $E_i(x) \neq \emptyset$, then

$$\#\left(\pi_i^{-1}(x) \cap \mathcal{D}_i(B)\right) = \begin{cases} 0 \text{ if } H(x) > B \\ \#\left(T_{NS}(\mathbb{K})_{tors}\right) \text{ if } H(x) \leq B \end{cases}$$

(ii)

$$\sum_{i\in I} \omega_i(\mathcal{D}_i(B)) \sim C_H(V)\, B \log(B)^{t-1}$$

Conclusion

So the formula (F) reduces to

$$\#\left(E_i(\mathbb{K}) \cap \mathcal{D}_i(B)\right) \sim \omega(\mathcal{D}_i(B)) \; ?$$

which gives a strong theoretical evidence for the value of the constant

Next week

I shall explain this method for $\mathbb{P}^n_{\mathbb{K}}$ it is exactly the method used by SCHANUEL.

VI Examples

First, I would like to stress that the formula (F) has been proven for many examples of many kind. So even if it is not always true it is has a large domain of validity.

1) A list of results (without proofs)

α) Flag varieties [LANGLANDS, FRANKE, MANIN, P.]

For $\mathbb{P}^N_{\mathbb{K}}$, SCHANUEL's theorem implies (F) for a particular height. This theorem generalizes as follows

Definition

A linear algebraic group$/\mathbb{L}$ is a subgroup of $GL_{n,\mathbb{L}}$, ($GL_n = \mathrm{Spec}(\mathbb{Z}[T_{ij}, 1 \leq i,j \leq n]\left[\frac{1}{\mathrm{Det}(T_{ij})}\right])$
It is said to be affine if the scheme is affine
I am going to need some notions about algebraic groups

Reference

A. BOREL, linear algebraic groups, Graduate Texts in Math, Springer

So I am going to give the definition in parallel with examples

Definitions	Example
G affine linear algebraic group $/ \mathbb{L} = \mathbb{L}$	$G = GL_{n+1,\mathbb{L}} = \mathrm{Spec}\left(\mathbb{L}[X_{ij}]\left[\frac{1}{\mathrm{Det}(X_{ij})}\right]\right)$
Derived group	
$\mathcal{D}^0 G = G,$	

$\mathcal{D}^{n+1}G = [\mathcal{D}^n G, \mathcal{D}^n G]$
they are closed and
thus algebraic subgroup

G is solvable if
$\exists\, n \,/\, \mathcal{D}^n G = \{e\}$
$B \subset G$ Borel subgroup
is a maximal
connected solvable
subgroup of G

A parabolic subgroup
of G is a subgroup
P such that G/P
is a compact variety

$\mathcal{D}^1 GL_{n+1,\mathbb{K}} = SL_{n+1,\mathbb{K}}$

$= \mathrm{Spec}\left(\mathbb{K}[X_{ij}]/(\mathrm{Det}(X_{ij})-1)\right)$

upper triangular matrices

$B = T_n = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$

$P = \begin{pmatrix} \overbrace{*}^{m} & * \\ 0 & * \end{pmatrix}$

$G/P = Gr(m, n+1)$
points are subspaces of
dimension m

Borel subgroups exist since an increasing
sequence of closed irreducible varieties is
finite

Facts

a) All Borel subgroups of G are conjugate
b) Un closed subgroup P is parabolic if
and only if it contains a Borel subgroup.

When $\mathbb{K}$ is not algebraically closed, we use
the same terminology (P parabolic) if
the group obtained by extension of scalars to $\overline{\mathbb{K}}$
satisfies the condition. (eg $\overline{P}$ parabolic)

Definition

A generalized flag variety is a variety V equipped with an action of a linear algebraic group G, which is a form of G/P (equipped with the natural action of G)

Remark

$V(\mathbb{L}) \neq \emptyset \Rightarrow V \simeq G/P$ over $\mathbb{L}$

(choose $x \in V(\mathbb{L})$, let P be the stabilizer of x

$$G/P \xrightarrow{\sim} V$$
$$\bar{g} \longmapsto gx \;)$$

Since we are looking at varieties with rational points, in our setting $V = G/P$

Theorem [LANGLANDS, MANIN, FRANKE, ...]

Let $V = G/P$ be a generalized flag variety with a rational point $/\mathbb{K}$ number field then

V satisfies (F) and (E) with $U = V$

Examples

(i) This implies the result of SCHANUEL for $\mathbb{P}^n_{\mathbb{K}}$.

(ii) Grassmannians $Gr(m, n+1)$

(iii) Complete flag variety $V = GL_n / B$

$GL_n / B = \{ (F_0, \dots, F_n), \{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = \mathbb{K}^n \}$
F_i subspace of $\mathbb{K}^n$

(iv) Any quadric

$Q = O(q) / P \qquad O(q) = \{ M \mid {}^tM\, Q\, M = I_n \}$ (Q: matrix of q)

<u>Method</u>: We shall see that there are essentially two types of methods. Here it goes is the class of Adelic harmonic analysis: the height zeta functions are particular Eisenstein series and satisfy some functional equation.

β) Complete intersections of large dimension

<u>Theorem</u> [∈ BIRCH]
Let $V \subset \mathbb{P}^N_{\mathbb{Q}}$ be defined by $f_1 = \cdots = f_c = 0$
where $c = N - n$ is the codimension of V
with $d = \deg(f_1) = \cdots = \deg(f_c)$
Assume
(i) V smooth
(ii) $V(\mathbb{R}_{\mathbb{Q}}) \neq \emptyset$ and
(iii) $N > 2^{d-1}(d-1)\, m\,(m+1)$
then
V satisfies (F) and (E) for $U = V$.

<u>Method</u>
very easy! (Descent method
(here $E = W \subset \mathbb{A}^{N+1} - \{0\}$ defined by $f_1 = \cdots = f_c = 0$)
hard (and circle method. □

20/6/2016 γ) <u>Toric varieties</u>

<u>Definition</u>
Let T be an algebraic torus $/\mathbb{K}$
A toric variety over $\mathbb{K}$ is an algebraic

variety V equipped with an action of T so that there exists an open orbit U on which T acts faithfully

Remark

In other words U is a principal homogeneous space under T or U is a T torsor over $\mathrm{Spec}(\mathbb{K})$

If $x \in U(\mathbb{K})$ $\quad T \longrightarrow U$ is an isomorphism
$\qquad t \longmapsto tx$

But $U(\mathbb{K})$ may be empty

Theorem [BATYREV-TSCHINKEL]

Let V be a nice toric variety $/\mathbb{K}$ with $U(\mathbb{K}) \neq \emptyset$
Then V satisfies (F) for $U =$ open orbit
for at least one choice of the norm on ω_V^{-1}

Method

Adelic harmonic analysis using the action of T

It should be possible to prove (E) with these methods, but it was never published. $\square$

Examples

(i) Again the projective space is a particular case
$$\mathbb{G}_m^n \curvearrowright \mathbb{P}^n_{\mathbb{K}} \quad \text{via}$$
$$(t_1, \ldots, t_n) \cdot [x_0 : \cdots : x_n] = [x_0 : t_1 x_1 : \cdots : t_n x_n]$$

(ii) If V is a toric variety and $F \subset V$ an irreducible subvariety such that
$$T \cdot F \subset F$$

(ie the closure of an orbit)
Then $Bl_F V$ is still a toric variety
In particular, the blowing of $\mathbb{P}^2_{\mathbb{Q}}$ in one point is a toric variety.
(iii) The product of two toric varieties is a toric variety
So this result covers all the elementary examples I gave at the very beginning

δ) Compactification of affine space

Let $\mathbb{G}_a = \operatorname{Spec}(\mathbb{K}[T])$ be the additive group
$\mathbb{G}_a^n \circlearrowright \mathbb{A}^n_{\mathbb{K}}$ it is the additive action by translation
An equivariant compactification of $\mathbb{A}^n_{\mathbb{K}}$ is a variety V with an action of $\mathbb{G}_a^n$ such that there is an open orbit isomorphic to $\mathbb{A}^n_{\mathbb{K}}$

Theorem [CHAMBERT-LOIR – TSCHINKEL]
Let V be a smooth equivariant compactification of the affine space / number field $\mathbb{K}$.
Then V satisfies (F) for $U =$ open orbit for at least one choice of the norm on ω_V^{-1}

Method
Adelic harmonic analysis using the action of T

Examples
Again the projective space is a particular case

$$\mathbb{G}_a^n \times \mathbb{P}^n_{\mathbb{K}} \longrightarrow \mathbb{P}^n_{\mathbb{K}}$$

$$(u_1, \dots, u_n), [X_0 : \dots : X_n] \longmapsto [X_0 : X_1 + u_1 X_0 : \dots : X_n + u_n X_0]$$

Note that the action is trivial on the hyperplane at ∞ trivial action on $H_\infty : X_0 = 0$

Thus If $Y \subset H_\infty$

$Bl_Y \mathbb{P}^n_{\mathbb{K}}$ is again an equivariant compactification of the affine space (eg the blowing up of $\mathbb{P}^2_{\mathbb{K}}$ in N aligned points)

E) Smooth Del Pezzo Surfaces

A lot of work has been invested in the case of surfaces. The point is that (F) and (E) are expected to be valid for open subsets on surfaces. First of all, there is a classification of surfaces with an anticanonical line bundle which is ample

Definition

A Del Pezzo surface is a surface S with ω_S^{-1} ample

References

- MANIN: Cubic Forms, North Holland
- BROWNING: Quantitative arithmetic of projective varieties

Classification

Given $\mathbb{C}$ or $\overline{\mathbb{Q}}$, a smooth Del Pezzo surface is isomorphic to one of the following surfaces

(i) $\mathbb{P}^1 \times \mathbb{P}^1$ or

(ii) $\mathbb{P}^2$ blown up in k points in general position, $0 \leq k \leq 8$

Remarks

1) General position means the following
 (i) they are distinct
 (ii) 3 of these points are not on a same line
 (iii) There exists a unique conic going through 5 points in general position, 6 of these points are not on a conic

2) On $\overline{\mathbb{Q}}$ 4 points in general position can be sent to
$$[1:0:0],\ [0:1:0],\ [0:0:1],\ [1:1:1]$$
So for $k \leq 4$ the isomorphism class is determined by k. For $k \geq 5$ this is not true anymore, the moduli space is not trivial

3) Quadric surfaces$/\overline{\mathbb{Q}}$ are isomorphic to $\mathbb{P}^1_{\overline{\mathbb{Q}}} \times \mathbb{P}^1_{\overline{\mathbb{Q}}}$
Cubic surfaces $/\overline{\mathbb{Q}}$ are isomorphic to the plane blown up in 6 points.

4) for $k \leq 3$ or $\mathbb{P}^1_K \times \mathbb{P}^1_K$ the surfaces are toric varieties so the formula (F) follows from BATYREV & TSCHINKEL. Although the result was in fact known before them in particular cases.

Theorem [R. DE LA BRETÈCHE]
 Let V be the split Del Pezzo surface $/\mathbb{Q}$ and $U = V - 10$ exceptional lines, (F) is valid for (U)

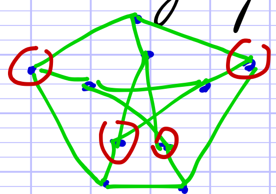
Explanations

- split means it is isomorphic to the blowing up of 4 rational points $/\mathbb{Q}$ (In general it is only so over a finite extension of $\mathbb{Q}$)
- The exceptional lines are rational curves $C \subset S$ such that the intersection product $(C.C) < 0$

More explicitely, in this particular case, they are given by

- the inverse image of the 4 points blown up,
- the strict lifting of the 6 lines through 2 of these points (ie $\overline{\pi^{-1}(\overline{(AB)} - \{A,B\})}$)

The intersection graph is the PETERSEN GRAPH



- vertices : exceptional line
- edge between E & E' if $E \cap E' \neq \emptyset$

Any family of 4 points not connected by edges correspond to a morphism to $\mathbb{P}^2$ which is the blowing up of 4 points.

Method of proof

- Descent method actually the versal torsor over S has a very nice description let me explain it

In my notes, I described the Cox ring. Here the Cox ring has 10 generators corresponding to the unique (up to multiplication) sections of the line bundles corresponding to the exceptional curves.

$P_1 = [1:0:0]$, $P_2 = [0:1:0]$, $P_3 = [0:0:1]$, $P_4 = [1:1:1]$

$X_{i,j}$ corresponds to the strict lifting $E_{i,j}$ of the line through $(P_k P_l)$; $\{i,j,k,l\} = \{1,\dots,4\}$

$X_{5,i}$ corresponds to the inverse image $E_{5,i}$ of P_i

<u>NB</u> $E_{i,j} \cap E_{k,l} = \emptyset \Leftrightarrow \{i,j\} \cap \{k,l\} \neq \emptyset$

The lifting of rational points to the torsor is done as follows

Start with $[x:y:z] \in \mathbb{P}^2(\mathbb{Q}) - \bigcup_{i \neq j} (P_i P_j)$

(x,y,z) primitive in $\mathbb{Z}^3$

$$
\begin{aligned}
x_{1,5} &= \gcd(y,z) & x_{1,4} &= x/(x_{2,5}\ x_{3,5}) \\
x_{2,5} &= \gcd(x,z) & x_{2,4} &= y/(x_{1,5}\ x_{3,5}) \\
x_{3,5} &= \gcd(x,y) & x_{3,4} &= z/(x_{1,5}\ x_{2,5}) \\
x_{4,5} &= \gcd(x-y, y-z) & x_{2,3} &= (y-z)/(x_{1,5}\ x_{4,5}) \\
& & x_{3,1} &= (z-x)/(x_{2,5}\, x_{4,5}) \\
& & x_{1,2} &= (x-y)/(x_{3,5}\, x_{4,5})
\end{aligned}
$$

These integers satisfy

(1) $x_{ij} x_{kl} + x_{ik} x_{lj} + x_{il} x_{jk} = 0$

for $\# \{i,j,k,l\} = 4$ where $x_{ij} = -x_{j,i}$

for example for 1, 2, 3, 4, we get

$$(x-y)z - (x-z)y + x(y-z) = 0$$

and for 2, 3, 4 5

$$(y-z) - y + z = 0$$

and

(2) $\gcd(x_{ij}, x_{k,l}) = 1$ if $\{i,j\} \cap \{k,l\} \neq \emptyset$

which is equivalent to $E_{ij} \cap E_{kl} = \emptyset$

(1) are the Plücker equations for the Grassmannian of planes in a space of dimension 5 : $E = \mathbb{Q}^5$

$$\{(u,v)\in E\times E,\ u,v \text{ not colinear}\} \longrightarrow \mathbb{P}^2(\Lambda^2 E)$$
$$(u,v) \longmapsto u\wedge v$$

gives en embedding $Gr(2,5) \hookrightarrow \mathbb{P}(\Lambda^2 E)$ the image of which is given by the equation

$$a \wedge b = 0$$

which in the basis $(e_i\wedge e_j)_{i<j}$ are given by (1)

So let $W\subset \Lambda^2 E - \{0\}$ be the cone above the Grassmannian

$$\mathbb{G}_m^5 = T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \curvearrowright E$$

and let $U \subset W$ be the open subset defined by $(x_{ij}, x_{ik}) \neq 0$ for $\#\{i,j,k\} = 3$

then U is stable under the action of T and

$$U/\mathbb{G}_m^5 \xrightarrow{\sim} V$$

Moreover

$$U(\mathbb{Z}) = \left\{ (x_{ij})_{1\le i<j\le 5} \in \mathbb{Z}^{10} \mid (1)\ \&\ (2) \right\}$$

we get a map

$$\pi : U(\mathbb{Z}) \longrightarrow V(\mathbb{Q})$$

so that $\forall P\in V(\mathbb{Q})$ $\quad \#\pi^{-1}(P) = 2^5$

So the problem reduces to count points which satisfy (1) & (2)

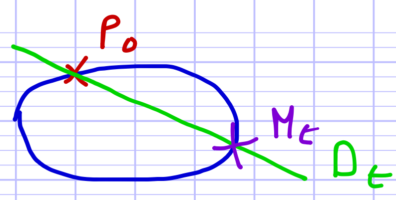
- Generalization of the Möbius enables one to change (2) in a condition $d_{ij} \mid x_{ij}$
- Clever analytic number theory (hard part).

So that settles the case $k=4$. Let us turn to $k=5$

Theorem [D. LA BRETÈCHE, T. BROWNING]
There exists a Del Pezzo surface S with $k=5$ which satisfies (F) for a choice of the height and the complement of the 16 exceptional curves.

Remark
16 exceptional curves:
- Inverse image of the points blown up (5)
- strict lifting of lines through 2 points (10)
- " " of the conic through the 5 points (1)

Method of the proof:
- fibration method:
  V is given as the smooth complete intersection of two quadrics in $\mathbb{P}^4$. They take
$$\begin{cases} X_0X_1 - X_2X_3 = 0 \\ X_0^2 + X_1^2 + X_2^2 - X_3^2 - 2X_4^2 = 0 \end{cases}$$
There are two morphisms
$$f_i : V \to \mathbb{P}^1_{\mathbb{Q}} \qquad i \in \{1, 2\}$$
Indeed $\det\begin{pmatrix} x_0 & x_3 \\ x_2 & x_1 \end{pmatrix} = 0$ on V
so $f_1([x_0 : x_1 : x_2 : x_3]) = \mathrm{Vect}\left(\begin{pmatrix} x_0 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ x_1 \end{pmatrix}\right)$
$f_2(\underline{\qquad\qquad}) = \mathrm{Vect}\left(\begin{pmatrix} x_0 \\ x_3 \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}\right)$
The fibers of f_1 and f_2 are conics:
(we speak of conic bundles)
$$C_{a,b} : (a^2 - b^2)x^2 + (a^2 + b^2)y^2 = 2z^2$$
But now, as babylonians, we can parametrize the conics: There is an obvious solution $(1, 1, a)$ and we use the lines through this point:



gives explicit $\varphi_{a,b} : \mathbb{P}_1 \xrightarrow{\sim} C_{a,b}$

But on $\mathbb{P}^1$ we can estimate the number of points for any choice of the height!

$$\# {C_{a,b}}_{H \leq B} = \frac{g(|a^4 - b^4|)}{\max |a|, |b|^2} B + O_{a,b}(\sqrt{B}^{1+\varepsilon})$$

where g is an arithmetic multiplicative function

Moreover there is a finite group acting on V which exchanges f_1 and f_2

Enough to take the sum over $[a : b] \in \mathbb{P}^1(\mathbb{Q})_{H_1 < \sqrt{B}}$

The problem is to prove that the sum of the error terms is really negligible. $\square$

<u>Still open</u>

$k = 6$ (ie smooth cubic surfaces)

(they contain 27 lines) (convincing numerical tests have been made on computers $H(P) \leq 10^6$)

$k = 7, 8$

As usual in mathematics, when you are stuck on a problem, you try another one

7) <u>Singular Del Pezzo surfaces</u>

<u>Remark</u>

Up to now I have only considered smooth varieties. But It was noticed by BATYREV & TSCHINKEL that in fact you can also consider the problem on singular projective variety and that it reduces to the smooth case

Indeed let V be a singular projective variety and $H : V \to \mathbb{R}_{\geq 0}$ be a height defined by a line bundle L/V.
Then HIRONAKA's theorem tells us that V admits a desingularization, that is a morphism from a smooth projective variety $\tilde{V}$

$$f : \tilde{V} \to V$$

which is birational (In fact the method consists in using a stratification of the singular locus

$$V \supset F_1 \supset F_2 \supset \cdots \supset F_k$$

with $F_i - F_{i+1}$ smooth and blowing up F_k and then F_i as many times as needed. The problem is to show that the process stops.
Then let $U \subset V$ be the open subset on which f is an isomorphism, $H \circ f$ is a height relative to $f^*(L)$ and

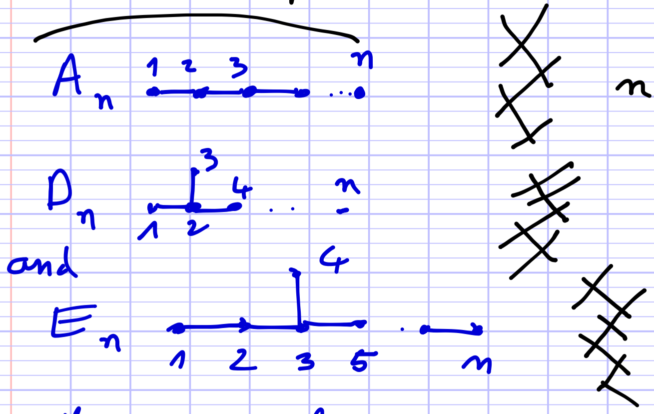
$$\# U(\mathbb{K})_{H \leq B} = \# f^{-1}(U)(\mathbb{K})_{H \circ f \leq B} .$$

22/6/2016 Remarks

For normal surfaces, the singularities are points and there is a minimal desingularization

$$\tilde{S} \to S .$$

where the inverse image of the singular points is a union of rational curves. For each point we can make the intersection graph of these curves. For singular Del Pezzo the only possibilities are multiple copies when there is more than one singular point.



n curves for $1 \leq n \leq 8$

for $4 \leq n \leq 8$

for $6 \leq n \leq 8$

Moreover this resolution is crepant:

$$\omega_{\tilde{S}}^{-1} = \pi^*(\omega_S^{-1})$$

Thus the expected behaviour is

$$\# U(\mathbb{K})_{H \leq B} \sim C_H(\tilde{V})\, B \log(B)^{rk(Pic(\tilde{V}))-1}$$

Degree 3

There are 20 possible type of singular Del Pezzo surface of degree 3 / $\mathbb{Q}$

A_1, $2A_1$, A_2, $3A_1$, $A_1 + A_2$, A_3, $4A_1$, $2A_1 + A_2$
$A_1 + A_3$, $2A_2$, A_4, D_4, $2A_1 + A_3$, $A_1 + 2A_2$, $A_1 + A_4$,
A_5, D_5, $3A_2$, $A_1 + A_5$, E_6

Example

$X^2Z + YZ^2 + T^3 = 0$ is of type E_6.

Note

For curves, singular points lower the genus and the corresponding curve are arithmetically more simple. Similarly, in some sense, singular surfaces are more easy to deal with. So it is possible to prove the formula for singular cubic surface but, up to now, not for smooth ones

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Results

[BATYREV & TSCHINKEL, ...]

$3A_2$ $XYZ+T^3=0$ is toric (F) for U open

[JOYCE, DE LA BRETÈCHE, BROWNING, DERENTHAL]

E_6 $X^2Z+YZ^2+T^3=0$ (F)

[BROWNING, DERENTHAL]

D_5 $XZ^2+X^2T+Y^2Z=0$ (F)

[HEATH-BROWN] Cayley cubic surface

$4A_1$ $XYZ+YZT+ZTX+TXY=0$

$$B\log(B)^{t-1} \ll \# U(\mathbb{Q})_{H\leq B} \ll B\log(B)^{t-1}$$

right order of growth

Singular Del Pezzo surfaces of degree 4

Complete intersection of 2 quadrics in $\mathbb{P}^4$

15 possible type of singularities

[BATYREV, TSCHINKEL, DE LA BRETÈCHE, BROWNING, DERENTHAL,

(F) has been obtained for $4A_1$, $2A_1+A_2$, A_1+A_3, A_4, D_4, $2A_1+A_3$, D_5, ...

The type $2A_1$ is of particular interest because it is the only one in which

$$S(\mathbb{A}_\mathbb{Q})^{Br} \neq S(\mathbb{A}_\mathbb{Q})$$

This are the Châtelet surfaces we have already met

(*) W: $X^2+Y^2=P(U,V)T^2$ in $\mathbb{A}^3-\{0\}\times\mathbb{A}^2-\{0\}$

where $P\in\mathbb{Z}[U,V]$ is homogeneous of degree 4 with distinct roots $/\overline{\mathbb{Q}}$

$\mathbb{G}_m^2 \circlearrowright W$ $(\lambda,\mu)((x,y,t),(u,v)) = ((\lambda\mu^2 x, \lambda\mu^2 y, \lambda t),(\mu u, \mu v))$

$\tilde{S} = W/\mathbb{G}_m^2$ is a minimal desingularization of the corresponding singular Del Pezzo surface.

<u>Theorem</u> [R. DE LA BRETÈCHE, T. BROWNING, –, G. TENENBAUM, K. DESTAGNOL] Various families of Châtelet surfaces of the form $(*)$ satisfy (F) with $U = \tilde{S}$ – exceptional divisors

<u>Method of the proof</u>

- The rank of the Picard group depends on the degrees of the polynomials in the decomposition of P in irreducible factors. Here I am going to explain one case:
  Assume P is the product of 4 linear forms.
$$P(U,V) = \prod_{i=1}^{4} L_i(U,V)$$
- Geometry & Reduction
  We have a morphism $\pi : S \to \mathbb{P}^1_{\mathbb{Q}}$ induced by the projection
$$(x, y, t, u, v) \longrightarrow (u, v)$$
which is compatible with the actions $\circlearrowleft \quad \circlearrowleft$
$$\mathbb{G}_m^2 \longrightarrow \mathbb{G}_m$$
The fiber over $[u : v]$ is the circle
$$X^2 + Y^2 = P(u, v) T^2$$
which is a degenerate (ie non irreducible $/\bar{\mathbb{Q}}$) fibre whenever $L_i(u, v) = 0$
Write $L_i(U,V) = a_i U + b_i V$,
The degenerate fibres correspond to the 4 points
$$P_i = [-b_i : a_i]$$
Over $\mathbb{Q}(i)$ these Châtelet surfaces are classified

by the cross-ratio

$$\alpha = \frac{\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \Big/ \begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix}}{\begin{vmatrix} a_4 & a_1 \\ b_4 & b_1 \end{vmatrix} \Big/ \begin{vmatrix} a_4 & a_2 \\ b_4 & b_2 \end{vmatrix}}$$

and we reduce to the case

$P_1 = [0:1]$, $P_2 = [1:0]$, $P_3 = [1:1]$ and $P_4 = [1:\alpha]$

and we are reduced to an equation of the form

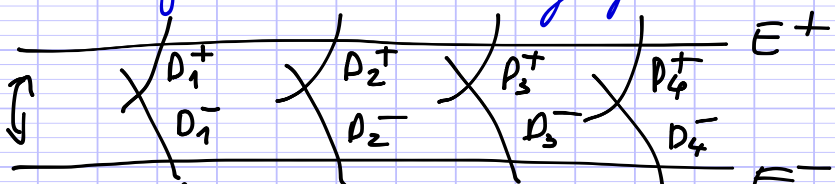
$$X^2 + Y^2 = UV(U-V)(aU+bV)T^2$$

We assume pgcd $(a,b) = 1$

— The exceptional divisors, the Picard group

- We have two sections of π / $\mathbb{Q}(i)$ corresponding to $[X:Y:t] = [1:i:0]$ and $[X:Y:t] = [1:-i:0]$
- The degenerate fibers are 2 conjugate lines

$\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ $\updownarrow$



(10 exc lines as for the corresponding smooth Del Pezzo surface)

class of a fiber

relations $[D_i^+] + [D_i^-] = [D_j^+] + [D_j^-] = [F]$

and, using $(X+iY)/(T L_i(U,V) L_j(U,V))$,

$$[E^+] + [D_i^+] + [D_j^+] = [E^-] + [D_k^-] + [D_l^-]$$

$$\mathrm{Pic}(V) = \mathrm{Pic}(\overline{V})^{g} = \mathbb{Z}[F] \oplus \mathbb{Z}[\omega_S^{-1}], \quad g = \mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$$

$$\omega_S^{-1} = 2[E^+] + \sum_{i=1}^{4} [D_i^+] = 2[E^-] + \sum_{i=1}^{4} [D_i^-]$$

— Again this case is based upon the descent method

So let us describe the versal torsors the interesting point is that

There might be several isomorphism classes of versal torsors having a rational point. Let me explain the spirit:

Reminder

$u \in \{-1, 1\}$

$n = u \prod_{p \in \mathcal{P}} p^{v_p(n)} \in \mathbb{Z}$ can be written as the sum of $2\,\square$ if and only if

$$\begin{cases} u = 1, \\ p \equiv 3\ [4] \Rightarrow v_p(n) \text{ even.} \end{cases}$$

A solution of the equation

(1) $x^2 + y^2 = \left(\prod_{i=1}^{4} L_i(u,v)\right) t^2$

implies that

(2) $\prod_{i=1}^{4} L_i(u,v) = \square + \square$

But since $\gcd(u,v) = 1$

$\gcd(L_i(u,v), L_j(u,v)) \mid \Delta_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$

So if $p \equiv 3\ [4]$ and $p \nmid \prod_{i,j} \Delta_{i,j}$

(Here $\Delta_{ij} = 1$ except for $\Delta_{1,4} = a$, $\Delta_{2,b} = b$ $\Delta_{3,4} = a-b$)

Then (2) $\Rightarrow \forall i$ $v_p(L_i(u,v))$ even

Also if $p \equiv 3\ [4]$ and $p \mid t$ then $p \mid x$ and $p \mid y$ absurd!

so $|t| = \square + \square$

Thus let $\Delta = \prod_{\substack{p \mid \prod_{i \neq j} \Delta_{ij} \\ p \equiv 3\ (4)}} p$

For any solution of (1) there exists $\underline{m} = (m_i)_{1 \leq i \leq 4} \in \mathbb{Z}^4$ such that

(i) $m_1 > 0$

(ii) $m_i \mid \Delta$

(iii) $\prod_{i=1}^{4} m_i = \square = S_m^2$ square

(iv) $m_i L_i(u,v) = \square + \square$

$$\mathcal{T}_{\underline{m}} \subset \operatorname{Spec}(\mathbb{Q}[X_i, Y_i,\quad 0 \leq i \leq 4])$$

You may note that, again the number of variables is the number of exceptional divisors. It's not a coincidence.

Equations of $E_{\underline{m}}$

(3) $\Delta_{i,j}\, n_k (X_k^2+Y_k^2) + \Delta_{j,k}\, n_i (X_i^2+Y_i^2) + \Delta_{k,i}\, n_l (X_k^2+Y_k^2) = 0$ for $1 \leq i < j < k \leq 4$.

and $(x_i, y_i, x_j, y_j) \neq 0$ if $1 \leq i < j \leq 4$.

$\pi : E_{\underline{m}} \longrightarrow \tilde{S}$ ← gives gcd condition.

$\exists!\, (u,v),\ L_i(u,v) = m_i (X_i^2+Y_i^2)$ for $1 \leq i \leq 4$

and $x + iy = \delta_m (X_0 + iY_0)^2 \prod_{i=1}^{4} (X_i + iY_i)$ } gives x, y, t.

$t = x_0^2 + y_0^2$.

So the problem is reduced to counting solutions of (3)

- Use Moebius inversion to remove gcd condition
- Analytic number theory:

Define $\tau(n) = \frac{1}{4}\{\#(x,y) \in \mathbb{Z}^2 \mid x^2+y^2 = n\}$

$4 = \#\mu(\mathbb{Q}(i))$

Then τ is multiplicative and

$$\tau(p^k) = \begin{cases} 0 & \text{if } p \equiv 3\ (4),\ k \text{ odd} \\ 1 & \phantom{\text{if } p \equiv 3\ (4),}\ k \text{ even} \\ k+1 & \text{if } p \equiv 1\ (4) \end{cases}$$

We get sums of the form

$\sum_{u,v} \prod_{i=1}^{4} \tau\left(\frac{L_i(u,v)}{e_i}\right)$ that one has to

estimate using methods of analytic number theory □

This concludes the list of explicit examples I wanted to describe. There are two more positive results I want to mention, which gives still more examples.

2) Compatibilities

a) The product of varieties

Proposition [FRANKE, MANIN, TSCHINKEL]

W_1, W_2 sets with maps $H_i : W_i \longrightarrow \mathbb{R}_{>0}$ so that

(i) $(W_i)_{H_i \leq B} = \{P \in W_i \mid H_i(P) \leq B\}$ is finite for any B

(ii) $\#(W_i)_{H_i \leq B} = C_i B \log(B)^{t_i - 1} + O\left(B \log(B)^{t_i - 2}\right)$

for $i \in \{1, 2\}$

on $W = W_1 \times W_2$ define $H(P, Q) = H_1(P) H_2(Q)$

Then

$$\# W_{H \leq B} = \frac{(t_1-1)!\,(t_2-1)!}{(t_1+t_2-1)!} C_1 C_2 B \log(B)^{t_1+t_2-1} + O\left(B \log(B)^{t_1+t_2-2}\right)$$

Idea of the proof

Same as the one I gave for $\mathbb{P}^1 \times \mathbb{P}^1(\mathbb{Q})$

$$\# W_{H \leq B} = \sum_{P \in (W_1)_{H_1 \leq B}} \#(W_2)_{H_2 \leq \frac{B}{H_1(P)}}$$

$$= \sum_{P \in (W_1)_{H_1 \leq B}} \left[C_2 \frac{B}{H_1(P)} \log\left(\frac{B}{H_1(P)}\right)^{t_2-1} + O\left(\frac{B}{H_1(P)} \log\left(\frac{B}{H_1(P)}\right)^{t_2-1}\right) \right]$$

the error term has a form similar to the main term so it is enough to compute the sum for the main term.

We put $f(t) = C_2 \frac{B}{t} \log\left(\frac{B}{t}\right)^{t_2-1}$ $\quad g(t) = \#(W_1)_{H_1 \leq t}$

We get

$$\int_1^B f(t)\, dg(t) = \underbrace{[fg]_1^B}_{O\left(B \log(B)^{t_1+t_2-2}\right)} + \int_1^B f'(t) g(t)\, dt$$

Since $g(t) = C_1 \, t \log(t)^{t_1 - 1} + O\left(t \log(t)^{t_1 - 2}\right)$
we get, up to $O\left(B \log(B)^{t_1+t_2-2}\right)$

$$C_1 C_2 \, B \log(B)^{t_1+t_2-1} \int_1^B \left(\frac{\log(t)}{\log(B)}\right)^{t_1-1} \left(1 - \frac{\log(t)}{\log(B)}\right)^{t_2-1} d\left(\frac{\log(t)}{\log(B)}\right)$$

$$= C_1 C_2 \, B \log(B)^{t_1+t_2-1} \underbrace{\int_0^1 u^{t_1-1} (1-u)^{t_2-1} \, du}_{\frac{(t_1-1)!\,(t_2-1)!}{(t_1+t_2-1)!}}.$$

Lemma

If V_1 and V_2 satisfy $\mathcal{H}$, then

a)
$$\mathrm{Pic}(V_1) \times \mathrm{Pic}(V_2) \longrightarrow \mathrm{Pic}(V)$$
$$([L_1], [L_2]) \longrightarrow pr_1^*([L_1]) + pr_2^*([L_2])$$
$$\hphantom{([L_1], [L_2]) \longrightarrow pr_1^*(} \shortparallel$$
$$\hphantom{([L_1], [L_2]) \longrightarrow pr_1^*(} L_1 \boxtimes L_2$$

is an isomorphism

b) $T(V_1 \times V_2) \xrightarrow{\sim} TV_1 \times TV_2$

gives $\omega^{-1}_{V_1 \times V_2} \xrightarrow{\sim} \omega^{-1}_{V_1} \boxtimes \omega^{-1}_{V_2}$

and we can equip $\omega^{-1}_{V_1 \times V_2}$ with the tensor product of the pull backs of the norms

c) The corresponding height is given by
$$H(P, Q) = H_1(P) H_2(Q)$$
and the measure on $V(\mathbb{A}_K) = V_1(\mathbb{A}_K) \times V_2(\mathbb{A}_K)$
$$\omega = \omega_1 \times \omega_2$$

d)
$$\alpha(V_1 \times V_2) = \frac{(t_1-1)!\,(t_2-1)!}{(t_1+t_2-1)!} \, \alpha(V_1) \, \alpha(V_2)$$
$$\beta(V_1 \times V_2) = \beta(V_1) \, \beta(V_2)$$

Conclusion

(F) or (E) with an error term $\ll \frac{1}{\log(B)}$ is compatible with product of varieties

B) Compatibility with Weil's restriction

Definition

Let A be a commutative ring

B ——— A-algebra

X be a scheme $/B$

The Weil restriction of X to A (if it exists) is a scheme $R_{B/A}X$ over A which represents the functor which sends a commutative A-algebra C to $X(B\otimes_A C) = \mathrm{Hom}_B(B\otimes_A C, X)$.

More generally

$$\mathrm{Hom}_A(Y, R_{B/A}X) = \mathrm{Hom}_B(Y_B, X).$$

($R_{B/A}$ is a "righ adjunct" to $- \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$)

Theorem

If $\mathbb{L}/\mathbb{K}$ is a finite separable field extension and X is a quasiprojective variety $/\mathbb{L}$ Then $R_{\mathbb{L}/\mathbb{K}}X$ exists

Idea of the proof

If $\sigma : \mathbb{L} \to \overline{\mathbb{K}}$ embedding $/\mathbb{K}$

Define $X^\sigma = X \times_{\mathrm{Spec}(\mathbb{L})} \mathrm{Spec}(\overline{\mathbb{K}})$ for $\sigma : \mathrm{Spec}(\overline{\mathbb{K}}) \to \mathrm{Spec}(\mathbb{L})$

Then if $\Sigma(\mathbb{L}/\mathbb{K})$ is the set of embeddings of $\mathbb{L}$ in $\overline{\mathbb{K}}$

$\# \Sigma(\mathbb{L}/\mathbb{K}) = [\mathbb{L} : \mathbb{K}]_s = [\mathbb{L}/\mathbb{K}]$

$\prod_\sigma X^\sigma$ has an action of $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ and it descends to a variety $R_{\mathbb{L}/\mathbb{K}}V$ on $\mathrm{Spec}(\mathbb{K})$ which is the one we were looking for $\square$

Remark

$$\dim(R_{\mathbb{L}/\mathbb{K}} X) = [\mathbb{L}:\mathbb{K}] \dim(X)$$

Example

$$X = \mathrm{Spec}\left(\mathbb{L}[X_0, \dots, X_N]/(f_1, \dots, f_r)\right)$$

$$X^\sigma = \mathrm{Spec}\left(\overline{\mathbb{K}}[X_0, \dots, X_N]/(\sigma(f_1), \dots, \sigma(f_r))\right)$$

$$R_{\mathbb{L}/\mathbb{K}} X = \mathrm{Spec}\left(\overline{\mathbb{K}}[X_{i,\sigma}, 0 \leq i \leq N, \sigma \in \Sigma(\mathbb{L}/\mathbb{K})]/(\sigma f_j)\right)^{\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})}$$

2. $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ acts on the coefficients and the variables.

Example (exercise)

Riemann sphere

$$R_{\mathbb{Q}(i)/\mathbb{Q}} \mathbb{P}^1_{\mathbb{Q}(i)} \xrightarrow{\sim} S^2 \subset \mathbb{P}^3_{\mathbb{Q}}$$
$$X^2+Y^2+Z^2 = T^2$$

Theorem [D. LOUGHRAN]

$\mathbb{L}/\mathbb{K}$ extension of number fields
$V/\mathbb{L}$ satisfy $\mathcal{H}$.
(F) (or (E)) are true for a non empty open set $U \subset V/\mathbb{L}$
if and only if they are true for
$$R_{\mathbb{L}/\mathbb{K}} U \subset R_{\mathbb{L}/\mathbb{K}} V$$

So, in some sense, it is enough to consider the problem over $\mathbb{Q}$.
To finish this chapter on positive results, I would like to fulfill a promise I made about projective spaces

3) Schanuel's proof for $\mathbb{P}^N_{\mathbb{K}}$

In fact it illustrates some of the techniques for descent over number fields

Notation

$h = \# \mathcal{C}\ell(\mathcal{O}_{\mathbb{K}})$

$r_1 = \#$ real places

$r_2 = \#$ complex places $\quad r = r_1 + r_2 - 1$

$R =$ regulator of $\mathbb{K}$

covolume of $\mathrm{im}\,(\mathcal{O}_{\mathbb{K}}^* \longrightarrow H = \mathrm{Ker}(\mathbb{R}^{r_1+r_2} \xrightarrow{\Sigma} \mathbb{R}))$

$x \longmapsto (\log |x|_w)_{w \in \mathcal{M}(\mathbb{K})_\infty}$

$\sqrt{|d|} = \mathrm{covol}(\mathcal{O}_{\mathbb{K}})$ in $\mathbb{K} \otimes_{\mathbb{Q}} \mathbb{R}$

$w = \# \mu_\infty(\mathbb{K})$

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a} \in \mathcal{I}(\mathcal{O}_{\mathbb{K}})} \frac{1}{N(\mathfrak{a})^s} \qquad N(\mathfrak{a}) = \# \mathcal{O}_{\mathbb{K}}/\mathfrak{a}$$

Theorem [Schanuel]

For $H = \prod_{w \in \mathcal{M}(\mathbb{K})} \max_{0 \le i \le n} (|x_i|_w)^{n+1}$

$$\# \mathbb{P}_n(\mathbb{K})_{H \le B} \underset{B \to +\infty}{\sim} C\, B$$

$$C = \frac{h}{\zeta_{\mathbb{K}}(n+1)} \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{d}}\right)^{n+1} (n+1)^{r_1+r_2-1}\, \frac{R}{w}$$

Proposition

$$C = c_H(\mathbb{P}^n_{\mathbb{K}})$$

Proof of the theorem

There are two difficulties when one tries to generalize the elementary proof which works $/\mathbb{Q}$

Problems

1) $(\mathbb{A}^{n+1}-\{0\})(\mathcal{O}_{\mathbb{K}}) = \{\text{primitive elements in } \mathcal{O}_{\mathbb{K}}^{n+1}\}$
$\pi \downarrow$ is not surjective
$\mathbb{P}^n(\mathbb{K})$

2) for x in its image
$\pi^{-1}(x)$ is an $\mathcal{O}_{\mathbb{K}}^*$ orbit
and is not finite (if $r>0$).

And these problems also occur in general for the descent method

For 1) we can be more precise

Define $\varphi : \mathbb{P}^n(\mathbb{K}) \longrightarrow Cl(\mathcal{O}_{\mathbb{K}})$
$[x_0 : \dots : x_n] \qquad [(x_0, \dots, x_n)]$
class of the ideal generated by $(x_0, \dots, x_n)$.

We are going to estimate
for $c \in Cl(\mathcal{O}_{\mathbb{K}})$
$\#\{x \in \mathbb{P}^n(\mathbb{K}) \mid H(x) \le B \text{ and } \varphi(x) = c\}$
and we are going to show that asymptotically it does not depend on the class c.

Fix $c \in Cl(\mathcal{O}_{\mathbb{K}})$ choose $\mathfrak{a}$ such that $[\mathfrak{a}] = c$
If $\varphi([x_0 : \dots : x_n]) = c$ then $\exists\, (y_0, \dots, y_n) \in \mathbb{K}^n$
such that $[x_0 : \dots : x_n] = [y_0 : \dots : y_n]$
and $(y_0, \dots, y_n) = \mathfrak{a}$

Moreover if $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$
then for any prime ideal $\mathfrak{p}$

$$\max_{0 \le i \le n} (|y_i|_{\mathfrak{p}}) = N(\mathfrak{p})^{-v_{\mathfrak{p}}(\mathfrak{a})}$$

So $H([y_0 : \dots : y_n]) = \dfrac{1}{N(\mathfrak{a})} \prod_{w|\infty} \max_{0 \le i \le n} (|y_i|_w)$

Now let us deal with problem 2)

Let $\log : \mathbb{K}^n \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \prod_{w|\infty} (\mathbb{R} \cup \{-\infty\})$ be defined by

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$$\begin{array}{ccc} \times_{w|\infty} (\mathbb{K}_w^n - \{0\}) & \longrightarrow & \prod_{w|\infty} \mathbb{R} \\ \cap & & \cap \\ \mathbb{K}^n \otimes_{\mathbb{Q}} \mathbb{R} \cong \bigoplus_{w|\infty} (\mathbb{K}_w^n) & \longrightarrow & \prod_{w|v} (\mathbb{R} \cup \{-\infty\}) \\ (y_w)_{w|\infty} & \longmapsto & (\log \|y_w\|_\infty)_{w|\infty} \end{array}$$

and for $\lambda \in G_{\mathbb{K}}^*$ we have

$$\underline{\log}(\lambda y) = \underline{\log}(\lambda) + \underline{\log}(y)$$

Choose a basis $(e_1, \ldots, e_\pi)$ for $\Lambda \subset H$ and put $F = \{\sum_{i=1}^{\pi} t_i e_i, \ t_i \in [0,1[\}$

and let $pr : \prod_{w|\infty} \mathbb{R} \longrightarrow H$ be the orthogonal projection

$$(x_w)_{w|\infty} \longmapsto (x_w - \sum_{w|\infty} x_w)_{w|\infty}$$

$\mathcal{D} = \underline{\log}^{-1}(pr^{-1}(F)) \subset \prod_{w|\infty} \mathbb{K}_w^n - \{0\}$ is a fundamental domain for $G_{\mathbb{K}}^*$ modulo $\mu_\infty(\mathbb{K})$:

(i) $\prod_{w|\infty} \mathbb{K}_n - \{0\} = \bigcup_{\lambda \in G_{\mathbb{K}}^*} \lambda \mathcal{D}$

(ii) $\lambda \mathcal{D} \cap \mathcal{D} = \begin{cases} \emptyset & \text{if } \lambda \notin \mu_\infty(\mathbb{K}) \\ \mathcal{D} & \text{if } \lambda \in \mu_\infty(\mathbb{K}) \end{cases}$

So we get :

$$\# \{P \in \mathbb{P}^n(\mathbb{K}) \mid H(P) \leq B \ \& \ \varphi(P) = c\}$$

kind of primality condition

$$= \frac{1}{w} \# \left\{ (y_0, \ldots, y_n) \in \mathfrak{a}^{n+1} \;\middle|\; \begin{array}{l} (y_0, \ldots, y_n) = \mathfrak{a} \\ (y_0, \ldots, y_n) \in \mathcal{D} \\ z(\underline{\log}(y_0, \ldots, y_n)) \leq \log\left(\frac{B^{\frac{1}{n+1}}}{N(\mathfrak{a})}\right) \end{array} \right\}$$

Define $\mu : \mathcal{I}(G_{\mathbb{K}}) \longrightarrow \{-1, 0, 1\}$ multiplicative

$$\prod_{\mathfrak{p}} \mathfrak{p}^{m_\mathfrak{p}} \longmapsto \prod_{\mathfrak{p}} \mu(\mathfrak{p}^{m_\mathfrak{p}})$$

and $\mu(p^k) = \begin{cases} 1 \text{ if } k=0 \\ -1 \text{ if } k=1 \\ 0 \text{ otherwise} \end{cases}$

The cardinal we are interested in is given by

$$\sum_{\mathfrak{b} \subset \mathfrak{a}} \mu(\mathfrak{b}/\mathfrak{a}) \; \# \left(\mathfrak{b}^{n+1} \cap \frac{B^{1/(n+1)}}{N(\mathfrak{a})} D_1 \right)$$

where $D_1 = D \cap \{ y \mid \sigma(\log(y)) \leq 0 \}$

We apply Masser and Vaaler to get that this is equivalent to

$$\left(\sum_{\mathfrak{b} \subset \mathfrak{a}} \mu(\mathfrak{b}/\mathfrak{a}) \left(\frac{N(\mathfrak{b})}{N(\mathfrak{a})} \right)^{n+1} \right) B \operatorname{Vol}(D_1)$$

- But D_1 is the union of $(n+1)^{r_1+r_2}$ domains given by $\max(|y_i|_w) = |y_{i_w}|$ and using a change of variables,

$$\operatorname{Vol}(D_1) = \left(2^{r_1} (2\pi)^{r_2} \right)^{n+1} \operatorname{Vol}(F) (n+1)^{r_1+r_2} \underbrace{\int_0^1 t^n \, du}_{1/n+1}$$

- $$\sum_{\mathfrak{b} \subset \mathfrak{a}} \mu(\mathfrak{b}/\mathfrak{a}) \left(\frac{N(\mathfrak{b})}{N(\mathfrak{a})} \right)^{n+1} = \prod_{p} \left(1 - \frac{1}{N(p)^{n+1}} \right) = \zeta_K(n+1)^{-1}$$

Summing over the classes in the ideal class group, we are done. $\square$

The main tool to prove the proposition is the following theorem of number theory

<u>Theorem</u>

$$\lim_{s \to 1} (s-1) \zeta_K(s) = h \, \frac{2^{r_1} (2\pi)^{r_2}}{\sqrt{|d|}} \, \frac{R}{w}$$

Proof of the proposition

- $L_p(s, Pic(\bar{V})) = \left(1 - \dfrac{1}{N(p)^s}\right)^{-1}$

  and $L(s, Pic(\bar{V})) = \zeta_{K}(s)$

  $$\lim_{s \to 1} (s-1) L(s, Pic(\bar{V})) = \lim_{s\to 1} (s-1)\zeta_K(s)$$
  $$= h \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}} \frac{R}{w}$$

- $L_p(1, Pic(\bar{V}))^{-1} \omega_p(\mathbb{P}^n(K_p))$

  $$= \left(1 - \frac{1}{N(p)}\right) \frac{\# \mathbb{P}^n(\mathbb{F}_p)}{\# \mathbb{F}_p^{\,n}}$$
  $$= \left(1 - \frac{1}{N(p)^{n+1}}\right)$$

  So $\displaystyle\prod_p L_p(1, Pic(\bar{V}))^{-1} \omega_p(\mathbb{P}^n(K_p)) = \frac{1}{\zeta_K(n+1)}$

- $$\omega_w(V(K_w)) = \begin{cases} (n+1)\, 2^n & \text{if } w \text{ real} \\ (n+1)(2\pi)^n & \text{if } w \text{ real} \end{cases}$$

  $$\omega(\mathbb{P}^n(\mathbb{A}_K)) = \frac{h}{\zeta_K(n+1)} \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{d}}\right)^{n+1} (n+1)^{r_1+1} \frac{R}{w}$$

  $$C = \frac{1}{n+1} \omega(\mathbb{P}^n(\mathbb{A}_K))$$

  $$\alpha(\mathbb{P}^n_K) = \frac{1}{n+1}. \quad \square$$

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The upgraded version of BATYREV & MANIN program

Before I go to the upgraded version let me explain the

I Spirit of the BATYREV-MANIN principle

1) A formula

For simplicity let us assume
V is a smooth projective, geometrically integral variety$/\mathbb{Q}$
such that ω_V^{-1} is very ample
Let $\varphi : V \longrightarrow \mathbb{P}^N_{\mathbb{Q}}$ be a corresponding embedding
with $\varphi^*(\mathcal{O}_{\mathbb{P}^N_{\mathbb{Q}}}(1)) = \omega_V^{-1}$
On $\mathbb{P}^N_{\mathbb{Q}}$ define $H([y_0 : \dots : y_n]) = \|(y_0, \dots, y_N)\|_\infty$
where $(y_0, \dots, y_N) \in \mathbb{Z}^{N+1}$, $\gcd(y_0, \dots, y_N)$
$\|\cdot\|_\infty : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}_{\geq 0}$ is a norm
Then one wants to study

$$V(\mathbb{Q})_{H \leq B} = \{ P \in V(\mathbb{Q}) \mid H(P) \leq B \}$$

Naive formula

Assume that $V(\mathbb{Q})$ is Zariski dense

$$(F_V) \quad \# V(\mathbb{Q})_{H \leq B} \underset{B \to +\infty}{\sim} \underbrace{C_H(V)}_{\text{explicit}} B \log(B)^{t-1}$$

where $t = \mathrm{rk}(\mathrm{Pic}(V))$
If it is true for any choice of the height this implies an equidistribution principle

Naive equidistribution

$\exists N_0 \ \forall N \ \gcd(N, N_0)$
V has good reduction at N and for $P_0 \in V(\mathbb{Z}/N\mathbb{Z})$

$$(E_V) \quad \frac{\# \{ P \in V(\mathbb{Q}) \mid \mathrm{red}_N(P) = P_0 \}_{H \leq B}}{\# V(\mathbb{Q})_{H \leq B}} \underset{B \to +\infty}{\longrightarrow} 0$$

for $F \subsetneq V$ closed $\quad \dfrac{\# F(\mathbb{Z}/M\mathbb{Z})}{\# V(\mathbb{Z}/M\mathbb{Z})} \xrightarrow[M \to +\infty]{} 0$

So (E_V) implie

$$\# F(\mathbb{Q})_{H \leq B} = o\left(\# V(\mathbb{Q})_{H \leq B}\right).$$

2) Examples

Theorem ($\Leftarrow$ BIRCH)

V smooth hypersurface of degree d in $\mathbb{P}^N_{\mathbb{Q}}$ such that $V(\mathbb{R}) \neq \emptyset$ and $V(\mathbb{Z}/M\mathbb{Z}) \neq \emptyset$ for any $M > 0$ then

(F_V) and (E_V).

Theorem

$V = G/P$ $\quad G$ linear connected algebraic group

P parabolic subgroup

(F_V) and (E_V)

In particular, it is true for any quadric.

II Counter examples

1) The plane blown up in a point

$V \subset \mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} \quad : \quad xv = yu$

$\pi \downarrow \quad (x:y:z) \quad (u:v)$

$E = \pi^{-1}(0:0:1) \leftarrow P_0$

$\mathbb{P}^2_{\mathbb{Q}} \quad U = V - E \quad \pi_{|U} : U \to \pi(U)$ isomorphism

blowing of P_0

$\omega_V^{-1} = \mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$

$H(P,Q) = H_2(P)^2 \, H_1(Q)$

On E $\quad H(P_0, Q) = H(Q)$

Get $\quad \#(E(\mathbb{Q}))_{H \leq B} = C(\mathbb{P}^1) B^2$

So (F_V) and (E_V) can not be true!

But $\# U(\mathbb{Q})_{H \leq B} \sim c_H(V)\, B \log(B)$
and $U(\mathbb{Q})_{H \leq B}$ satisfies equidistribution!
It was Batyrev and Manin who suggested to remove a closed subset:
BATYREV & MANIN principle (refined)
$\exists\, U$ so that (F_U) and thus (E_U).

2) accumulating thin subset
BATYREV & TSCHINKEL
$V \subset \mathbb{P}^3_{\mathbb{Q}} \times \mathbb{P}^3_{\mathbb{Q}}$ $\quad \sum_{i=0}^{3} y_i x_i^3 = 0$
$H(x,y) = H_3(x) H_3(y)^3$ since $\omega_V^{-1} = \mathcal{O}(1,3)_{|V}$
$\pi = pr_2 : V \longrightarrow \mathbb{P}^3_{\mathbb{Q}}$ for $y \in \mathbb{P}^3(\mathbb{Q})$, $V_y = \pi^{-1}(y)$
$V_y : \sum y_i x_i^3 = 0$
smooth cubic surface if $\prod_{i=0}^{3} y_i \neq 0$.
For the fiber
$$\# V_y(\mathbb{Q})_{H \leq B} \sim c_H(V_x)\, B \log(B)^{t_x - 1}$$
where $t_x = rk(Pic(V_x))$.
For cubic surface, the Picard group is generated by the lines contained in the surface
Here we have diagonal cubic surfaces and it is possible to prove that
$$1 \leq t_4 \leq 4 \text{ and } t_x = 4$$
if and only if x_i / x_j is a cube for all i, j.
But we can apply Lefschetz theorem and therefore the restriction gives an isomorphism from the Picard group of $\mathbb{P}^3_{\mathbb{Q}} \times \mathbb{P}^3_{\mathbb{Q}}$ to the Picard group of V thus $rk(Pic(V)) = 2$ and the expected formula for V is for $U \neq \emptyset$, small enough
$$\# U(\mathbb{Q})_{H \leq B} \underset{B \to +\infty}{\sim} c_H(V)\, B \log(B)$$

But the problem is that the points in $\mathbb{P}^3(\mathbb{Q})$ which satisfy the condition that the quotients of the coordinates are $\boxed{}$ are Zariski dense

So for any non-empty open U in V $\exists\, [x_0 : \dots : x_3]$ with $\prod_{i=0}^{3} x_i \neq 0$ and x_i/x_j cube such that $U \cap V_x \neq \phi$

So there is a contradiction between the conjecture for the fibers and the expected formula for V

In fact, BATYREV, TSCHINKEL & STYMIUS have proven that there are too many points on V

On the other hand, one can say that most of the points in $\mathbb{P}^3(\mathbb{Q})$ do not satisfy the cube condition. More precisely:

$$\frac{\#\{[x_0 : \dots : x_3] \in \mathbb{P}^3(\mathbb{Q}) \mid \forall i,j\ x_i/x_j \text{ is a cube}\}_{H \leq B}}{\#\mathbb{P}^3(\mathbb{Q})_{H \leq B}} = O(B^{-\frac{2}{3}})$$

So there is a natural question: What happens on the complement? First it is not enough to remove the fibers with $t_x = 4$ one has to to remove all fibers with $t_x > 1$

Put $T = \bigcup_{x \mid \prod x_i = 0} V_x \cup \bigcup_{i=2}^{4} \bigcup_{x \mid t_x > 1} V_x$

T is a thin subset in the following sense

Definition

A thin subset T of $V(\mathbb{Q})$ is a subset such that there exists a morphism $\varphi : X \to V$ which satisfies

(i) φ is generically finite;

(ii) φ has no rational section;
(iii) $T \subset \varphi(X(\mathbb{Q}))$.

So, in our case, I may state the question as

Question

Do we have the expected behaviour on $V(\mathbb{Q}) - T$?

Theorem [C. LE RUDULIER] proved that for $V = \mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ there exists a thin subset T such that

(i) $\forall \emptyset \neq U \subset V \quad \#(U \cap T)_{H \leq B} \gg B \log(B)^t$
(ii) $\#(V(\mathbb{Q}) - T)_{H < B} \sim C_H(V)\, B \log(B)^{t-1}$

Challenge

Which points should we remove?

III Freeness of a point

1) Geometric analog

This notion comes from the analogy with rational curves. Take

$\varphi : \mathbb{P}^1 \longrightarrow V$ a morphism

$\underbrace{\varphi^*(TV)}_{\text{vector bundle on } \mathbb{P}^1} \xrightarrow{\sim} \bigoplus_{i=1}^{n} \mathcal{O}(a_i)$ with $a_1 \geq \cdots \geq a_n$

Deformations of φ cover V if $a_n \geq 0$

You can even prescribe extra conditions like $\varphi(t_i) = p_i$ if a_n is big enough.

$\deg_{\omega_V^{-1}} \varphi = \sum_{i=1}^{n} a_i \Leftrightarrow \log \circ H$

The arithmetic analog is provided by the notion of slopes introduced by J.-B. Bost.

2) Slopes of arithmetic modules

Definition

Let E vector space$/\mathbb{Q}$ of dim n equiped with

(i) $\|\cdot\|_\infty$ euclidean on $E_\mathbb{R} = E \otimes_\mathbb{Q} \mathbb{R}$

(ii) $\Lambda \subset E$ lattice

then $\widehat{\deg}\, E = -\log(\mathrm{Vol}(\Lambda/E))$

corresponding to the euclidean structure

Any vector subspace F can be equiped with

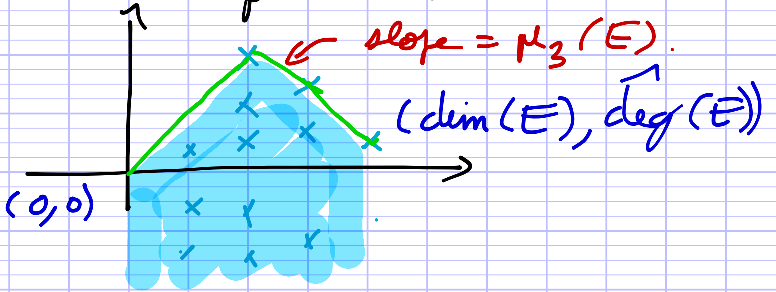
$\Lambda_F = \Lambda \cap F$ and $\|\cdot\|_{\infty\, |F}$

and we can consider its degree

The Newton polygon associated to E is

$P(E) = $ convex hull $(\{(\dim(F), \widehat{\deg}(F)\})$

This domain is bounded from above. it looks as follows:



$m_E : [0, \dim(E)] \to \mathbb{R}$ the maximum

$m_E(t) = \max\{y \in \mathbb{R} \mid (t,y) \in P(E)\}$

By construction this function is piecewise affine. And the successive slopes of its graph are $\mu_i(E) = m_E(i) - m_E(i-1)$ for $i \in \{1, \dots, \dim(E)\}$.

By definition we have, with $n = \dim(E)$

$$\begin{cases} \mu_1(E) \geq \mu_2(E) \dots \geq \mu_n(E); \\ \sum_{i=1}^{n} \mu_i(E) = n. \end{cases}$$

3 Slopes of a rational point

To do this we need some extra data

Let E be a vector bundle $/V$ $\mathrm{Pl}(\mathbb{Q}) = \{\infty\} \cup \{p \text{ prime}\}$

$(\|\cdot\|_w)_{w \in \mathrm{Pl}(\mathbb{Q})}$ adelic norm on E:

(i) for any $w \in \mathrm{Pl}(\mathbb{Q})$
$$\|\cdot\|_w : E(\mathbb{Q}_w) \longrightarrow \mathbb{R}_{\geq 0}$$
is continuous

(ii) $\forall w \in \mathrm{Pl}(\mathbb{Q})$, $\forall x \in V(\mathbb{Q}_w)$,

$\|\cdot\|_w \,\big|\, \underbrace{E(x)}_{\mathbb{Q}_w \text{ vector space}}$ is a w-adic norm, euclidean if $w = \infty$

(iii) For almost all w $\|\cdot\|_w$ is defined by a model $\mathcal{E}$ of E

Remark

1) This is Arakelov's geometry point of view to define heights: take a line bundle L with an adelic norm $(\|\cdot\|_w)_{w \in \mathrm{Pl}(\mathbb{Q})}$
$$H(x) = \prod_{w \in \mathrm{Pl}(\mathbb{Q})} \|y\|_w^{-1}$$
where $y \in L(x) - \{0\}$. By the product formula, it is independent of the choice of y.

2) If we have a norm on E, we can define one on $\det(E) = \Lambda^{\mathrm{rk}(E)} E$

Definition

- Adelic metric on V is an adelic norm on TV
- It defines a norm on $\omega_V^{-1} = \det(TV)$ and therefore a height H. $h = \log \circ H$.

- For $x \in V(\mathbb{Q})$ define on $T_x V$
  - $\Lambda_x = \{ y \in T_x V \mid \forall p, \ \|y\|_p \leq 1 \}$ lattice in $T_x V$
  - $\|\cdot\|_x$ euclidean norm on $T_x V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$
- $\mu_i(x) = \mu_i(T_x V)$ slopes of x

__Remark__

One has $\begin{cases} \mu_1(x) \geq \cdots \geq \mu_n(x) \\ \sum_{i=1}^{n} \mu_i(x) = \widehat{\deg}(T_x V) = h(x) \end{cases}$ and

So $h(x)/n$ is the mean of the slopes

__Déf__

$$\ell(x) = \frac{\mu_{min}(T_x V)}{h(x)/n} \text{ if } \mu_{min}(x) \geq 0, \ 0 \text{ otherwise}$$

__Remark__

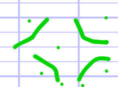
$\ell(x) \in [0,1]$

__Hope__

Get equidistribution for
$$V(\mathbb{Q})^{\varepsilon - \ell}_{H \leq B} = \{ P \in V(\mathbb{Q}) \mid H(P) \leq B, \ \ell(P) \geq \varepsilon \}$$

OK for very simple examples, product and seems to remove the bad points in the previous examples.

__Particular case__

$\mathbb{P}^1 \times \mathbb{P}^1$, $\ell(x,y) = \dfrac{\min(h(x), h(y))}{h(x) + h(y)}$ $\mathbb{P}^1 \times \mathbb{P}^1$ 

$h(x) = \log(H(x))$

On $\mathbb{P}^1 \times \mathbb{P}^1$ $\ell(x) > \varepsilon$ removes a >0 proportion of points

Take $\ell(x) > \varepsilon(B)$ with $\varepsilon(B) \to 0$ instead?