AN INTRODUCTION TO COMPLEX ALGEBRAIC GEOMETRY

WITH EMPHASIS ON THE THEORY OF SURFACES

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Preface

These notes are based on courses given in the fall of 1992 at the University of Leiden and in the spring of 1993 at the University of Grenoble. These courses were meant to elucidate the Mori point of view on classification theory of algebraic surfaces as briefly alluded to in [P].
The material presented here consists of a more or less self-contained advanced course in complex algebraic geometry presupposing only some familiarity with the theory of algebraic curves or Riemann surfaces. But the goal, as in the lectures, is to understand the Enriques classification of surfaces from the point of view of Mori-theory.
In my opininion any serious student in algebraic geometry should be acquainted as soon as possible with the yoga of coherent sheaves and so, after recalling the basic concepts in algebraic geometry, I have treated sheaves and their cohomology theory. This part culminated in Serre's theorems about coherent sheaves on projective space.
Having mastered these tools, the student can really start with surface theory, in particular with intersection theory of divisors on surfaces. The treatment given is algebraic, but the relation with the topological intersection theory is commented on briefly. A fuller discussion can be found in Appendix 2. Intersection theory then is applied immediately to rational surfaces.
A basic tool from the modern point of view is Mori's rationality theorem. The treatment for surfaces is elementary and I borrowed it from [Wi]. The student doesn't need all of the material in Chapter 4 to understand it, but at some point, it is very useful to have the Stein factorisation at one's disposal. This is the main reason to insert Chapter 4 before the material on the rationality theorem.
Right from the beginning I have adopted a dual point of view. A complex projective variety can be studied both from the complex-analytic as well from the commutative algebra point of view. For instance, I have treated coherent sheaves and their cohomology from the algebraic point of view, since this is the most elementary way to do. On the other hand, sometimes it is useful to be able to look at smaller sets than just affine open sets and then the complex topology is more natural. For instance, if you have a morphism $f : X \to Y$ between say smooth complex projective varieties, $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ if and only if all fibres of f are connected, but this is hard to prove in the algebraic context (one needs the formidable theory of formal functions), but relatively elementary in the complex analytic context. It is in chapter 4 that the fruits of the dual point of view are reaped. The construction of the normalisation of a projective variety is easy from an algebraic point of view, but the proof of Zariski's main theorem etc. is much easier if you use complex topology. The subsequent treatment of Kodaira dimensions is not too hard and follows [U]. I also profited from Otto Forster's exposition on this subject in Bologna (I cherish my notes of the course he gave in Italian; I made use of the lecture delivered on 'Venerdi Santo 1980').
Of course, one must pay a price for this flexibility: the basic GAGA theorems have to be assumed so that one can switch between the two approaches at will. I have stated these theorems in an Appendix (without proofs).
Besides the rationality theorem one needs a few other specific aspects from the theory of surfaces that deal with fibrations and with families of curves. In §15 some general facts are treated and then, in the next section, an elementary treatment is given for the so-called canonical bundle formula for elliptic fibration (avoiding the use of relative duality; the latter is used for instance in [B-P-V] to arrive at the canonical bundle formula). Section 17 is the most sketchy one. The reason is that I did not have the time to treat deformation theory of curves in greater detail so that I had to invoke the local-triviality theorem of Grauert-Fischer instead. The 'Grand Final' is presented in section 18, a proof of the Enriques Classification theorem. After all the preparations

the proof becomes very short indeed.
It should be clear that most of the material presented is not very original. Chapter 3 has a large overlap with Arnaud Beauville's book [Beau]. Chapter 2 is adapted from [Ha], but I tried to simplify the treatment by restricting to projective varieties. This is rewarding, since then one does not need the abstract machinary of derived functors which, in my opininon, makes [Ha] hard to digest at times. For instance I have given a very elementary proof for the fact that the cohomology of coherent sheaves on a variety vanishes beyond its dimension. The final chapter borrows from [B-P-V], but again with simplifications as mentioned before. Needless to say I did not have time to treat the topic of surfaces exhaustively. Surfaces of general type and their geography could not be treated, nor the beautifully detailed theory of K3-surfaces and Enriques surfaces. Non algebraic surfaces all as well as phenomena particular to non zero characteristics are almost completely absent (I only give the Hopf surface as an example of a non-Kähler surface).
From the preceding description of the content of the course one might conclude that it nevertheless has been rather demanding for the pre-graduate students it was aimed at. I am glad they not only stayed untill the very end, but also contributed much to improve on this written exposition. I want to thank all of them, but in particular Robert Laterveer who very carefully read first drafts of this manuscript. I also want to thank José Bertin, Jean-Pierre Demailly and Gerardo Gonzalez-Sprinberg for useful conversations.

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Chapter 0. Introduction

This course will mainly be an introduction into the techniques of complex algebraic geometry with a focus on surfaces. Some familiarity with curves is assumed (e.g. the material presented in [G]).

> In this course a *surface* will be a *connected but not necessarily compact complex manifold of dimension* 2 and an *algebraic surface* will be a *submanifold of projective space of dimension* 2 *which is at the same time a projective variety.*

I will explain these concepts fully in section 2. For the moment let me just remark that by definition a surface is covered by open sets each of which is homeomorphic to an open set in $\mathbb{C}^2$ and that the transition functions are holomorphic maps from the open set in $\mathbb{C}^2$ where they are defined to $\mathbb{C}^2$ (for now a holomorphic map will be any C^∞-map whose coordinate functions are analytic in each variable separately). An algebraic surface in addition is a submanifold of complex projective space given as the zero locus of some polynomials.

Examples

1. Any connected open set in $\mathbb{C}^2$ is a surface.

2. If C and D are Riemann surfaces (or algebraic curves) their product $C \times D$ is a surface.

3. If $\gamma_1, \ldots, \gamma_k \in \mathbb{C}^2$ are k independent vectors (over the reals, so $k \leq 4$) the group $\Gamma = \mathbb{Z}\gamma_1 \oplus \ldots \oplus \mathbb{Z}\gamma_k$ acts on $\mathbb{C}^2$ and the quotient $\mathbb{C}^2/\Gamma$ is a complex manifold which is compact precisely when $k = 4$. In this case $\mathbb{C}^2/\Gamma$ is homeomorphic to the product of four circles or two real tori and is called a complex 2-torus.

These notes will be aiming at the so-called Enriques-Kodaira classification of surfaces which is the analogue in two dimensions of the (coarse) classification of Riemann surfaces by means of their genus. At this point it is not possible to formulate the main classification theorem. Several concepts and examples are needed which are gradually introduced. These concepts and examples in themselves are interesting and important, so stay with us!

For some of the technical details I refer to the litterature at the end the notes. Some brief comments will be given here. The reference [Beau] will be an important guide-line, which means that I mostly treat algebraic surfaces. I use [Beau] rather than [B-P-V] because results are often easier to prove in the algebraic setting. However the treatment of the classification will be based upon more modern ideas explained in [P].

Considering background the following remarks. A very general and useful book on complex algebraic geometry from the analytic point of view is [G-H] which will be used occasionally for some foundational material. For a more algebraic point of view I mention the books [Reid] (elementary, fun to read) and [Mu] (much less elementary, assumes a lot of algebra, but a very nice introduction indeed). Some background on commutative algebra is collected in Appendix A1 with [Reid] as a reference for the more elementary facts and [Ii] and [Ma] for the more advanced facts which are needed later in the course.

Sheaf theory, cohomology theories and Hodge theory will be mainly done from [Wa], a unique reference in that it collects all you ever want to know (and much more) about differentiable varieties and their cohomology theory. I will certainly not treat all proofs but formulate what is needed. Another useful reference is [Go] to which I occasionally refer.

Some background in algebraic topology is assumed such as singular (co)-homology, cap and cup products and Poincaré duality. I have given an overview of the results needed from algebraic topology in Appendix A2. Full details then can be gathered from [Gr] and [Sp]. More advanced algebraic topology will be taken from [Mu] and [Mi].

Finally, background details from complex analysis can be found in [Gu-Ro], a real classic on this subject. For another more modern treatment see [Gr-Re].

About the history of surface theory

Around 1850 an extensive study had been carried out of low degree surfaces in three dimensional projective space. It was shown that on a smooth cubic there were 27 lines. Names such as the Cayley cubic, the Kummer surface and the Steiner quartic are reminders of that period. The first generation of Italian geometers (Bertini, C. Segre, Veronese) started to look at surfaces embedded in higher dimensional projective spaces and their projections. The Veronese surface and the Del Pezzo surfaces originate from that period (1880-1890). Max Noether in Germany, using projections, established (1870-'75) an important formula for surfaces, nowadays called "Noether's Formula". The proof was not complete. Enriques, using a result of Castelnuovo, gave a correct proof in 1896. Castelnuovo and Enriques belong to the second generation of Italian geometers. From roughly 1890 to 1910 they really developed the theory of algebraic surfaces from a birational point of view, culminating in the Castelnuovo-Enriques surface classification. See the monograph [En].

The foundations of algebraic geometry were lacking in that period, many results were not clearly formulated and proofs were not always complete. These foundations were laid in the thirties and fourties by van der Waerden, Zariski and Weil. Zariski wrote a monograph [Za] about surfaces incorporating these new techniques.

The transcendental tools were developed by de Rham, Hodge and Lefschetz in the fourties and fifties. But decisive progress only came after sheaf theory had been developed and applied to algebraic geometry by Serre, Hirzebruch and Grothendieck (1955-1965). On this base Kodaira did his fundamental work on classification theory, including the non-algebraic surfaces (1960-1970). He completed the "Kodaira-Enriques classification" of surfaces. In the sixties in Moscow the Russian school of algebraic geometers (a.o. Manin, Shafarevich, Tjurin, Tjurina) did important work on the classification, see the monograph [Sh].

The Castelnuovo-Enriques classification relied on existing detailed knowledge of some classes of surfaces (rational and ruled surfaces, bi-elliptic surfaces, Enriques surfaces), but other classes were extensively studied for the purpose of this classification (K3-surfaces and elliptic surfaces) thereby gaining more detailed insight in these special classes.

Finer classification of surfaces went on in the seventies and eighties, but also some important new techniques and viewpoints from higher dimensional classification theory began to permeate surface theory. See [P] for recent developments. These new insights are incorporated in the presentation of the classification I give here.

Chapter 1. The basic notions

1. Generalities on complex and projective manifolds

I recall the basic objects and maps one works with in (complex) algebraic geometry: complex manifolds and holomorphic maps between them, projective and affine varieties and rational and regular maps between them.

First, some NOTATION.

Points in $\mathbb{C}^n$ are denoted by $z = (z_1, \ldots, z_n)$ where $z_j = x_j + iy_j$ is the standard decomposition of z_j into real and imaginary parts. Introduce

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big)$$

and either consider these as a differential operators acting on complex valued functions or as elements in the complex tangent space to any point in $\mathbb{C}^n$. They give a real basis for this complex tangent space. For the dual space, the cotangent space, the dual basis is given by

$$dz_j = dx_j + idy_j, \quad d\bar{z}_j = dx_j - idy_j.$$

With this notation one has

$$df = \underbrace{\sum_j \frac{\partial f}{\partial z_j} dz_j}_{\partial f} + \underbrace{\sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j}_{\bar{\partial} f}.$$

Definition 1. A C^∞ function $f = u + iv$ on an open set $U \in \mathbb{C}^n$ is called *holomorphic* if one of the following equivalent conditions hold:

1 The Cauchy-Riemann equations hold on U:

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}, \quad \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}.$$

2 $\bar{\partial} f = 0$ on U.

3 f admits an absolutely convergent powerseries expansion around every point of U.

For the equivalence of these definitions, see e.g. [G-H], p.2.

Remark 2. A continuous function is called analytic if it admits a convergent powerseries around each point. By Osgood's lemma [Gu-Ro, p2.] such a function is holomorphic in each variable separately and conversely. Hence a continuous function which is analytic automatically satisfies the properties 1) and 2).

Definition 3. A Hausdorff topological space M with countable basis for the topology is an *n-dimensional complex manifold* if it has a covering U_i, $i \in I$ by open sets which admit homeomorphisms $\varphi_i : U_i \to V_i \subset \mathbb{C}^n$ with V_i open and such that for all $i \in I$ and $j \in I$ the map $\varphi_i \circ \varphi_j^{-1}$ is a holomorphic map on the open set $\varphi_j(U_i \cap U_j) \subset \mathbb{C}^n$ where it is defined.

A function f on an open set $U \subset M$ is called *holomorphic*, if for all $i \in I$ the function $f \circ \varphi_i^{-1}$ is holomorphic on the open set $\varphi_i(U \cap U_i) \subset \mathbb{C}^n$. Also, a collection of functions $z = (z_1, \ldots, z_n)$ on an open subset U of M is called a *holomorphic coordinate system* if $z \circ \varphi_i^{-1}$ is a holomorphic bijection from $\varphi_i(U \cap U_i)$ to $z(U \cap U_i)$ with holomorphic inverse. The open set on which a coordinate system can be given is then called a *chart*. Finally, a map $f : M \to N$ between complex manifolds is called *holomorphic* if it is given in terms of local holomorphic coordinates on N by holomorphic functions.

Let me give some examples. The first three generalize the examples in the introduction. The fourth example is a very important basic example: complex projective space.

Examples 1. Any open subset in $\mathbb{C}^n$ is a complex manifold. More generally any open subset of a complex manifold is a complex manifold.

2. Let Γ be a discrete lattice in $\mathbb{C}^n$, i.e. the set of points $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2 + \ldots \mathbb{Z}\gamma_m$ where $\gamma_1, \ldots, \gamma_m$ are m independent points (over the reals). Then the quotient $\mathbb{C}^n/\Gamma$ is a complex manifold. If $m = 2n$, i.e. if the points $\gamma_1, \ldots, \gamma_m$ form a real basis, the manifold $\mathbb{C}^n/\Gamma$ is compact and is called a *complex torus*.

3. The *Hopf manifolds* are defined as the quotient of $\mathbb{C}^n \setminus \{0\}$ by the infinite cyclic group generated by the homothety $z \mapsto 2z$. As an exercise one may show that any Hopf manifold is homeomorphic to $S^1 \times S^{2n-1}$. If $n = 2$ this is the *Hopf surface*.

4. The set of complex lines through the origin in $\mathbb{C}^{n+1}$ forms *complex projective space* $\mathbb{P}^n$ and is a compact n-dimensional complex manifold in a natural way with $Z_0, \ldots, Z_n$ as homogeneous coordinates. A natural collection of coordinate charts is obtained by taking $U_j = \{(Z_0, \ldots, Z_n) \in \mathbb{P}^n \ ; \ Z_j \neq 0\}$ with coordinates $z^{(j)} = (Z_0/Z_j, \ldots, Z_{j-1}/Z_j, Z_{j+1}/Z_j, \ldots, Z_n/Z_j)$. These are called *affine coordinates* in U_j.

As with differentiable manifolds an important tool to produce new manifolds is the implicit function theorem, which is stated now together with the inverse function theorem. But first I recall the notion of the jacobian matrix $J(f)$ of a holomorphic map $f = (f_1, \ldots, f_m)$ defined on some open set $U \in \mathbb{C}^n$:

$$J(f) = \begin{pmatrix} \dfrac{\partial f_1}{\partial z_1} & \dfrac{\partial f_1}{\partial z_2} & \cdots & \dfrac{\partial f_1}{\partial z_n} \\ \dfrac{\partial f_2}{\partial z_1} & \dfrac{\partial f_2}{\partial z_2} & \cdots & \dfrac{\partial f_2}{\partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial z_1} & \dfrac{\partial f_m}{\partial z_2} & \cdots & \dfrac{\partial f_m}{\partial z_n} \end{pmatrix}.$$

The jacobian matrix $J(f)$ is non-singular at $a \in U$ if $m = n$ and the matrix $J(f)(a)$ is invertible.

Theorem 4. (Inverse Function Theorem) *Let U and V be open sets in $\mathbb{C}^n$ with $0 \in U$ and let $f : U \to V$ be a holomorphic map whose jacobian is non-singular at the origin. Then f is one-to-one in a neighbourhood of the origin and the inverse is holomorphic near $f(0)$.*

Theorem 5. (Implicit Function Theorem) *Given an open neighbourhood $U \subset \mathbb{C}^n$ of the origin and $f : U \to \mathbb{C}^m$ holomorphic and vanishing at the origin. Assume that the $m \times m$-matrix*

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \cdots & \frac{\partial f_1}{\partial z_m} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \cdots & \frac{\partial f_2}{\partial z_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1} & \frac{\partial f_m}{\partial z_2} & \cdots & \frac{\partial f_m}{\partial z_m} \end{pmatrix}$$

is non-singular at the origin. Then there exist open neighbourhoods of V of $0 \in \mathbb{C}^m$ and W of $0 \in \mathbb{C}^{n-m}$ with $V \times W \subset U$, and a holomorphic map $g : W \to \mathbb{C}^m$ such that $f(z_1, \ldots, z_m, z_{m+1}, \ldots, z_n) = 0$ if and only if $(z_1, \ldots, z_m) = g(z_{m+1}, \ldots, z_n)$ for $z = (z_1, \ldots, z_n) \in V \times W$.

For a proof of these theorems see Problem 2.

Note that the Inverse Function Theorem shows that the map $(g, \mathbb{1}) : W \to V \times W \cap V(f)$ has a holomorphic inverse in a neighbourhood of 0 and hence gives a local chart on

$$V(f) := f^{-1}(0).$$

If the rank of the jacobian $J(f)$ is m everywhere on points of $V(f)$, one can always reorder the coordinates and shift the origin in such a way that one can apply the implicit function theorem at any point of $J(f)$ and produce a coordinate patch at that point. Also, in the overlap the transition functions are clearly holomorphic so that $V(f)$ is a complex manifold of dimension $n - m$ in its own right.

More generally, if M is a complex manifold and a closed subset N of M is locally in coordinate patches given by a function f which always has the same rank m on $V(f)$, the set N inherits the structure of a complex manifold of dimension $n - m$ which by definition is a *complex submanifold of M*. If one drops the condition about the jacobian one has an *analytic subset* of M. It is called *irreducible* if it is not the union of non-empty smaller analytic subsets. An irreducible analytic subset is also called an *analytic subvariety* and the terms *smooth subvariety* and *non-singular subvariety* mean the same as "submanifold".

Each analytic subset is the finite irredundant union of analytic subvarieties. This is by no means trivial but it won't be made use of in these notes. The interested reader can find a proof in [Gu-Ro, Chapter IIE]. The essential ingredients are the Weierstrass Preparation Theorem and Weierstrass Division Theorem.

In the algebraic setting there is the concept of (affine or projective) algebraic variety, to be introduced now. If in the preceding set-up $U = \mathbb{C}^n$ and $f = (f_1, \ldots, f_m)$ is a polynomial mapping defined on $\mathbb{C}^n$, the zero set $V(f)$ is called an affine algebraic set. This set actually only depends on the ideal $\mathfrak{I} = (f_1, \dots, f_m)$ in $\mathbb{C}[z_1, \dots, z_n]$ generated by the f_j and therefore usually is denoted by $V(\mathfrak{I})$. If $V(\mathfrak{I})$ is *irreducible*, i.e it is not the union of non-empty smaller affine sets it is called an *affine variety.* This is for instance the case if $\mathfrak{I}$ is a prime ideal.

It is well known that each affine algebraic set is the finite irredundant union of affine varieties in a unique way. This fact won't be made use of, but for the interested reader, I remark that this follows from the fact that the ring $\mathbb{C}[Z_0, \dots, Z_n]$ is Noetherian; see [Reid, section 3].

Now, instead of holomorphic maps between affine varieties $V \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ one may consider *rational maps* i.e maps $f = (f_1, \dots, f_m)$ whose coordinates f_j are rational functions in the affine coordinates of the source space: $f_j = \dfrac{P_j}{Q_j}$, $j = 1, \dots, m$ with P_j, Q_j polynomials such that Q_j does not vanish identically on V. The rational map is not defined on the locus where some coordinate function f_j has a pole. If this is not the case, i.e. if all the f_j are polynomials one has a *regular* map.

A *Zariski-open subset* $U \subset \mathbb{C}^n$ by definition is the complement of an affine algebraic set. The Zariski-open sets form the *Zariski-topology* on $\mathbb{C}^n$. The induced topology on any affine variety V is called the Zariski-topology on V. One says that a rational function is regular on a Zariski-open subset U of an affine variety if it has no poles on U. For example, if f is any irreducible polynomial there is the basic Zariski-open set

$$U_f := \mathbb{C}^n \setminus V(f),\ f \in \mathbb{C}[X_1, \dots, X_n]$$

and any regular function on U_f is of the form $\dfrac{P}{f^k}$ with P some polynomial and $k \geq 0$.

The regular functions on U form a ring, denoted $\mathcal{O}(U)$. For instance $\mathcal{O}(U_f)$ is the localisation of the ring $\mathbb{C}[z_1, \dots, z_n]$ in the multiplicative system f^n, $n \geq 0$. See Appendix A1 for this notion.

The rational functions give the same function on $V = V(\mathfrak{I})$ if their difference is of the form $\dfrac{P}{Q}$ with $P \in \mathfrak{I}$. An equivalence class of such functions is called a *rational function on* V. The set of rational functions on V form the *function field* $\mathbb{C}(V)$ of V. It is the field of fractions of $\mathbb{C}[z_1, \dots, z_n]/\mathfrak{I}$ and in fact of any of the rings $\mathcal{O}(U)$, U Zariski-open in V.

Next, if there is given a homogeneous polynomial F in the variables $(Z_0, \dots, Z_n)$ its zero-set in a natural way defines a subset of $\mathbb{P}^n$ denoted $V(F)$. The zero locus of a set of homogeneous polynomials $F_1, \dots, F_N$ only depends on the ideal $\mathfrak{I}$ they generate and is denoted by $V(\mathfrak{I})$. These loci are called *projective algebraic sets.*

If the ideal $\mathfrak{I}$ is a prime ideal, $V(\mathfrak{I})$ is a *projective algebraic variety.* This is for instance the case, if F is irreducible.

In the projective case, rational functions on V are functions $f = \dfrac{P}{Q}$ where P and Q are homogeneous polynomials of the same degree (otherwise f is not well defined) with Q not identically vanishing on V. These form the function field $\mathbb{C}(V)$ of V. A *rational map* $f : V \dashrightarrow \mathbb{P}^n$ is defined by demanding that the homogeneous coordinates of f be rational functions. If the map f can be given by polynomials, it is a *morphism* or *regular map* and these are examples of holomorphic maps.

Of course, on projective varieties one can introduce the Zariski-topology as well and as before one can speak of the ring of regular functions on any Zariski-open subset of a projective variety. Its field of fractions again coincides with the function field of the variety.

Also, each projective algebraic set is the finite irredundant union of projective varieties in a unique way. This follows from the corresponding assertion for affine varieties. See [Mu, section 2A] for details.

A projective variety is a complex subvariety of $\mathbb{P}^n$ but in general not a submanifold because of the jacobian condition. If it is, it is a *projective manifold.* So by definition an algebraic surface is a projective manifold of dimension two.

Example 6. A hypersurface $V(F)$ where F is a homogeneous polynomial of degree d. Consider the open set U_0 and for simplicity set $z^{(0)} = z$. The inhomogeneous polynomial

$$f(z_1, \ldots, z_n) = F(1, \frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}) = \left(\frac{1}{Z_0}\right)^d F(Z_0, \ldots, Z_n)$$

vanishes in U_0 precisely where F vanishes and if at a point of $V(f)$ some partial, say $\dfrac{\partial f}{\partial z_1}$ is non-zero, the implicit function theorem implies that $z_2, \ldots, z_n$ can be taken as local coordinates on $V(F)$ and hence that $V(F)$ is a manifold locally at that point. The locus where all the partials $\dfrac{\partial f}{\partial z_j}$, $j = 1, \ldots, n$ vanish on $V(f)$ is the set of non-manifold points, the so-called *singular set* $S(V(f))$.

To treat all coordinate patches simultaneously, recall Euler's formula

$$d \cdot F = Z_0 \frac{\partial F}{\partial Z_0} + \ldots + Z_n \frac{\partial F}{\partial Z_n}.$$

It follows that the singular set $S(V(F))$ of $V(F)$ is nothing but $V(\dfrac{\partial F}{\partial Z_0}, \ldots, \dfrac{\partial F}{\partial Z_n})$ and is a proper algebraic subset of $V(F)$. The Zariski-open complement $V(F) \setminus S(V(F))$ is a manifold of dimension $n-1$.

For the general case see Problem 3.

Observe that there is no reason why a compact complex manifold should be projective or why a submanifold of $\mathbb{P}^n$ or more generally an irreducible subvariety would be projective, i.e. can be given as the zero locus of finitely many polynomials. For dimension one one has the basic

FACT *Any compact Riemann surface is projective.*

The proof uses Hodge Theory in some form. See Appendix 3.3.

In higher dimensions this is not true. The easiest example perhaps is the Hopf surface. Again see Appendix 3 for details, more particularly, see Example A3.6.

As to subvarieties of projective space, astonishingly enough, they are always projective:

Theorem 7. (Chow's Theorem) *Any subvariety of $\mathbb{P}^n$ is a projective variety.*

A rather self contained proof of Chow's Theorem can be found in [Mu]. For a considerably shorter proof see p.167 in [G-H]. This proof however uses the so called Proper Mapping Theorem, a partial proof of which is supplied in [G-H, p.395-400].

In the same vein one can show that holomorphic maps between complex projective manifolds are in fact morphisms, i.e given by rational functions. See problem 5.

Problems.

1.1. Let U be an open subset of $\mathbb{C}^n$ and let $f = (f_1, \dots, f_m)$ be a holomorphic map defined on U. Write $f_j = u_j + iv_j$ with u_j the real part and v_j the imaginary part of f_j. Recall that $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ with $z_j = x_j + iy_j$. The differentiable map $f_{\mathbb{R}} = (u, v) : U \to \mathbb{R}^{2m}$ has a jacobian $J(f_{\mathbb{R}})$ of size $2n \times 2m$. If $n = m$ show that $\det J(f_{\mathbb{R}}) = |\det J(f)|^2$ and hence is positive if and only f is invertible. Deduce that any complex manifold is oriented in a natural way.

1.2. Prove the inverse and the the implicit function theorem.
Hint: Use the previous problem to see that one can use the ordinary inverse function theorem to find a differentiable inverse g for f and then prove that this map is in fact holomorphic by differentiating the relation $g(f(z)) = z$. See [G-H], p.18. The argument for the implicit function theorem is similar. Loc. cit. p.19.

1.3. Let F_j, $j = 1, \dots, N$ be homogeneous polynomials in $(Z_0, \dots, Z_n)$ defining the algebraic set $V := V(F_1, \dots, F_N)$ in $\mathbb{P}^n$. Consider the jacobian matrix $J(F_1, \dots, F_N)$. Prove:
(i) The locus where the rank of the Jacobian is k or less is an algebraic set. It is denoted by $J_k(V)$.
(ii) There is a minimal number m such that $J_m(V) \cap V = V$. If $J_{m-1}(V) \cap V = \emptyset$, the variety V is a manifold of dimension $n - m$. (In general, the $m \times m$ subdeterminants vanish in a proper subset of V, the singularity set of V and the complement is a manifold.)

1.4. Prove that the product of two projective varieties is projective.
Hint: use the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+m+n}$. One may consult [Mu, section 2B] for details.

1.5. Prove that holomorphic maps between projective manifolds are morphisms (Consider the graph of the holomorphic map and apply the previous problem).

2. Vector bundles

Vector bundles live on manifolds, varieties etc. I recall their basic properties, discuss the principal examples such as the canonical bundle, line bundles related to divisors and the notion of an ample line bundle. Important results are the canonical bundle formula and the Bertini theorem on hyperplane sections. The first tells you how to compute the canonical bundle of a subvariety in terms of the canonical bundle of the variety and the normal bundle of the subvariety and will be used a lot to say something about the genus of curves on surfaces. Bertini's theorem will be used to construct smooth subvarieties of a given projective manifold.

Let M be a differentiable manifold. Let me recall the notion of a *differentiable vector bundle* on M. It consists of a collection of vector spaces $E_m, m \in M$ parametrized by M such that their union E, the *total space*, is a manifold and such that

1. The natural projection $p : E \to M$ which maps E_m to m is differentiable,

2. Every point $m \in M$ has an open neighbourhood U and a diffeomorphism

$$\varphi_U : p^{-1}U \to U \times T,$$

where T is some fixed vector space and where φ_U maps E_m linearly and isomorphically onto $m \times T$. If T is a complex vector space of dimension d the manifold E is called a *complex vector bundle* of rank d. For $d = 1$ it is called *line bundle.*

The vector space E_m is called the *fibre over* m and the maps φ_U are called *trivializations* and over non-empty intersections $U \cap V$ they can be compared:

$$\varphi_V^{-1}(m,t) = \varphi_U^{-1}(m, (\varphi_{UV}(m))(t)),$$

where $\varphi_{UV} : U \cap V \to GL(T)$ is differentiable and is called the *transition function.* These transition functions satisfy a certain compatibility rule

$$\varphi_{UV} \circ \varphi_{VW} \circ \varphi_{WU} = \mathbb{1}, \quad \text{(Cocycle relation)}.$$

Conversely, given some covering of M by open sets U_i, $i \in I$ and a collection of transition functions φ_{ij} for subsets U_i and U_j having a non-empty intersection, define a set E by taking the disjoint union of the $U \times T$ and identify (m,t) and $(m, (\varphi_{ij}(m))t)$ whenever $m \in U_i \cap U_j$. This yields a vector bundle precisely if the above compatibility rule is valid as one can easily verify.

A *vector bundle homomorphism* between two vector bundles $p : E \to M$ and $p' : F \to M$ consists of a differentiable map $f : E \to F$ such that

1. $p = p' \circ f$ so that fibres go to fibres,
2. $f|E_m$ is linear.

If f is an diffeomorphism you have a *vector bundle isomorphism.*

For any vector bundle homomorphism $f : E \to F$ you can form the kernel $\ker(f)$, which consists of the union of the kernels of $f|E_m$. One can easily see that the kernel forms a vector bundle. Similarly one can form $\mathrm{im}\,(f) = \bigcup_{m \in M} \mathrm{im}\,(f|E_m)$, the *image bundle.* Often *exact sequences* of vector bundles arise. A sequence of vector bundle homomorphisms

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

is called *exact* at E if $\ker(g) = \mathrm{im}\,(f)$. A sequence of vector bundles

$$\dots E_{i-1} \xrightarrow{f_{i-1}} E_i \xrightarrow{f_i} E_{i+1} \dots$$

of arbitrary length, it is called *exact* if it is exact at all E_i. Especially, a sequence

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

is exact if and only if f is injective, g is surjective and $\ker(g) = \mathrm{im}\,(f)$.

A *section* s of a vector bundle $p : E \to M$ is a differentiable map $s : M \to E$ such that $p \circ s = \mathrm{id}_M$. Sections of a vector bundle E form a vector space denoted by $\Gamma(E)$ or $H^0(M,E)$.

Examples

1. The trivial bundle $M \times T$.

2. If E is a bundle, a *subbundle* consists of a subset $F \subset E$ such that the projection and trivialisation of E gives F the structure of a bundle. For a subbundle $F \subset E$, the fibres F_m are subspaces of E_m and one can form $\bigcup_{m\in M} E_m/F_m$ which inherits the structure of a bundle E/F, the *quotient bundle*. If

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0,$$

is an exact sequence, f identifies E' with a subbundle of E and g induces an isomorphism of E/E' with E''.

3. The tangent bundle $T(M)$. Sections are *vector fields*.

4. If E is a vector bundle, any linear algebra construction done with the fibres yields a vector bundle. You already saw the examples of subbundles and quotient bundles. One can also form $E^\vee$, the dual bundle by taking $\bigcup_{m\in M} E_m^\vee$ or the exterior powers $\bigwedge^k E$ by forming $\bigcup_{m\in M} \bigwedge^k E_m$. The highest wedge with $k = \dim T$ is also called the *determinant line bundle*

$$\det(E) = \bigwedge^{\operatorname{rank} E} E.$$

Combining these operations and applying them to the previous example you get the cotangent bundle or bundle of one-forms and its k-fold exterior power, the bundle of k-forms:

$$\mathcal{E}^k(M) = \bigwedge^k T(M)^\vee.$$

Sections in the bundle of k-forms are precisely the k-forms.

5. Likewise, if E and F are two bundles, one can form their direct sum $E \oplus F$ and their tensor product $E \otimes F$ by taking it fibre wise. The collection of line bundles on a fixed manifold form a group under the operation of tensor product provided you identify isomorphic bundles. This group plays an important role for complex manifolds and holomorphic bundles. See below.

6. The tangent bundle $T(N)$ of a submanifold N of a manifold M is a subbundle of the restriction $T(M)|N$ of the tangent bundle of M to N. The quotient $(T(M)|N)/T(N)$ is called the *normal bundle* and denoted by $N(N/M)$.

7. If $\varphi : M \to N$ is a differentiable map and $p' : F \to N$ a vector bundle, there is the *pull-back bundle* φ^*F. Its total space consists of the pairs $(m, f) \in M \times F$ with $\varphi(m) = p'(f)$. Projection comes from projection onto the first factor. One may verify that the trivialization of N induces one on φ^*F.

8. Consider the subbundle of the trivial bundle with fibre $\mathbb{C}^{n+1}$ on projective space $\mathbb{P}^n$ consisting of pairs $([w], z) \in \mathbb{P}^n \times \mathbb{C}^{n+1}$ with z belonging to the line defined by $[w]$. This is a line bundle, the *tautological line bundle* and denoted by $\mathcal{O}(-1)$.

9. Given an exact sequence

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0,$$

there is an isomorphism

$$\det E' \otimes \det E'' \xrightarrow{\sim} \det E$$

(see Problem 1).

Over a complex manifold you have *holomorphic vector bundles*. In the preceding definition of a complex vector bundle one demands that E be a complex manifold and that the differentiable maps involved are actually holomorphic. All of the constructions of the previous examples do produce holomorphic bundles out of holomorphic bundles. In particular, since for any complex manifold M the tangent spaces admit a natural complex structure so does the tangent bundle. Let me denote this complex bundle by $T_{\mathbb{C}}(M) = \bigcup_{m \in M} T_m(M)$. It is in fact a holomorphic bundle. This is likewise true for the cotangent bundle and exterior wedges which now are denoted as follows:

$$\Omega^k(M) = \bigwedge^k T_{\mathbb{C}}^{\vee}(M).$$

The line bundle $\det \Omega^1(M)$ is called the *canonical line bundle* and is sometimes denoted by K_M. If N is a submanifold of M the complex normal bundle $N(N/M)$ of N in M is the quotient of $T_{\mathbb{C}}(M)|N$ by $T_{\mathbb{C}}(N)$. Applying the remark about determinant bundles from Example 9 to the exact sequence defining the normal bundle, you arrive at an important formula:

$$K_N \cong K_M \mid N \otimes \det N(N/M) \qquad \text{(Canonical Bundle Formula)}.$$

As already said before, the collection of holomorphic line bundles on a complex manifold M modulo isomorphism form a group under the tensor product. It is called the *Picard group* and denoted by $\operatorname{Pic} M$.

An important line bundle related to a codimension one subvariety D of a manifold M is the bundle $\mathcal{O}(D)$ on M defined by means of transition functions as follows. Choose a coordinate covering U_i, $i \in I$ of M in which D is given by the equation $f_i = 0$. In $U_j \cap U_j$ the relations $f_i = (\text{ a non-zero function } \varphi_{ij}) \cdot f_j$ enables one to form the line bundle given by the transition functions $\varphi_{ij} = f_i/f_j$. (Note that the functions φ_{ij} obviously satisfy the co-cycle relation.) Observe that the bundle $\mathcal{O}(D)$ always has a section s_D canonically defined by D. Indeed, over U_i the bundle is trivial and the function f_i defines a section over it. These patch to a section s_D of $\mathcal{O}(D)$ because $f_i = (f_i/f_j)f_j$ in $U_i \cap U_j$. Restricting the bundle $\mathcal{O}(D)$ to D itself in case D is a submanifold, you get back the normal bundle $N(D/M)$. See Problem 2. The Canonical Bundle Formula in this case reads therefore

$$K_D \cong (K_M \otimes \mathcal{O}(D))|D.$$

By definition a *divisor* is a formal linear combination $\sum_{i=1}^m n_i D_i$ with $n_i \in \mathbb{Z}$ and D_i a codimension one subvariety. If the numbers n_i are non-negative the divisor is called *effective.* Divisors on M form an abelian group $\operatorname{Div} M$. The line bundle $\mathcal{O}(D)$ is defined by setting $\mathcal{O}(D) = \mathcal{O}(D_1)^{\otimes n_1} \otimes \ldots \otimes \mathcal{O}(D_m)^{\otimes n_m}$ so that it yields a homomorphism $\operatorname{Div} M \to \operatorname{Pic} M$.

Let me next describe how divisors behave under surjective holomorphic maps $f : M' \to M$. Let g be a local defining equation for D. If the image of f avoids the support of D, the function $g \circ f$ is nowhere zero, but if f is surjective it defines a divisor on M' which is independent of the choice of the local defining equation for D. It is called *the pull-back* f^*D. It is related to the pull-back of the line bundle $\mathcal{O}_M(D)$ by means of the relation $\mathcal{O}_{M'}(f^*D) = f^*(\mathcal{O}(D))$.

In the framework of holomorphic bundles $E \to M$, the group of holomorphic sections is now denoted by $\Gamma(E)$ or $H^0(M, E)$. It is true, but by no means trivial, that for compact complex manifolds the space of sections is finite dimensional. See Appendix A3 for a treatment using Hodge theory. For projective manifolds it is easier. See Theorem 4.13

Now let me turn to projective manifolds. Note that one could have defined algebraic vector bundles using morphisms instead of holomorphic maps. Algebraic vector bundles are holomorphic. The converse is true over a projective manifold. This GAGA-principle (named after the first letters of the words in the title of the article [Se]) is considerably harder to prove than Chow's theorem and uses a lot of sheaf theory and the Kodaira embedding theorem. Let me refer to [G-H], Chapter 1 section 5 for a proof of this assertion. In a similar vein, *regular sections* of an algebraic bundle, i.e. sections which are morphisms, are holomorphic and over a projective manifold the converse is true. In Appendix A4 I collected the main results from [Se].

Since a projective manifold is compact, the space of sections of any algebraic bundle on it is finite dimensional as we have seen before. If L is a line bundle on a projective manifold M, and its space of sections is not zero, say $n+1$-dimensional with basis $x_0, \ldots, x_n$, one can define a rational map

$$\varphi_L : M \dashrightarrow \mathbb{P}^n$$

by associating to $m \in M$ the point in $\mathbb{P}^n$ with homogeneous coordinates $(x_0(m), \ldots, x_n(m))$. This map is not defined on the locus where all sections of L vanish. This locus is called the *base locus* and any point in it is called a *base point.* If φ_L is an embedding, the line bundle L is called *very ample.* If for some integer k the k-th tensor power $L^{\otimes k}$ is very ample, L is said to be *ample.*

Two numbers, generalizing the genus of a projective curve, play an important role in higher dimensions:

The dimension of the space of holomorphic m-forms is called the *geometric genus* of M and denoted by $p_g(M)$.

The dimension of the space of holomorphic 1-forms is called the *irregularity* $q(M)$ of M.

Finally, the definition of divisors and of the Picard group for projective manifolds can be modified in the obvious way by using projective codimension one subvarieties instead.

Again there is a suitable GAGA-principle.

Examples

1. The hyperplane in $\mathbb{P}^n$ defines an algebraic line bundle, the *hyperplane bundle* $\mathcal{O}(1)$. The tautological bundle is the dual of this bundle which explains the notation $\mathcal{O}(-1)$ for the tautological bundle. The line bundle $\mathcal{O}(d)$ is defined as $\mathcal{O}(1)^{\otimes d}$ for $d > 0$ and as $\mathcal{O}(-1)^{\otimes -d}$ if $d < 0$. The line bundle associated to a hypersurface of degree d is (isomorphic to) $\mathcal{O}(d)$, see Problem 5.

2. The canonical line bundle of projective space $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(-n-1)$. See Problem 4. Using the Canonical Bundle Formula you find that the canonical bundle for a smooth degree d hypersurface D in $\mathbb{P}^n$ is the restriction to D of $\mathcal{O}(d-n-1)$.

Any polynomial P which does not vanish identically on V defines a divisor (P) on V by taking $V(P) \cap V$. Any rational function $f = \dfrac{P}{Q}$ on a projective manifold V defines the divisor $(f) = (P) - (Q)$. Since one can represent rational functions on V in different ways, it is not a priori clear that this definition make sense. To see this, one has to use the fact that the ring of holomorphic functions near the origin in $\mathbb{C}^{n-1}$ is a unique factorization domain. This is a corollary of the Weierstrass preparation theorem and I won't give a proof but refer to [G-H, p.10]. One now argues as follows.

Let $f \in \mathbb{C}(M)$ be a rational function on M and D be an irreducible hypersurface. Let $p \in M$ and let $f_D = 0$ be a local equation for D at p. Since the ring $\mathcal{O}_{M,p}$ of germs of holomorphic functions at p is a unique factorization domain one can write

$$f = f_D^m \cdot (u/v)$$

with u and v not identically zero along D. It is easily verified that m does not depend on f_D and the chosen point $p \in M$ so that one can now unambiguously define m to be the *order of vanishing of f along* D denoted $\operatorname{ord}_D(f)$ and one introduces the *divisor of the rational function f* by

$$(f) = \sum_{D \text{ an irreducible hypersurface}} \operatorname{ord}_D(f) D.$$

One checks easily that this definition is the same as the previous one.

Divisors of rational functions form the subgroup of *principal divisors* of $\operatorname{Div} M$. Two divisors D and D' are said to be *linearly equivalent*, notation $D \equiv D'$ if their difference is the divisor of a rational function. Equivalent divisors define isomorphic line bundles and hence there is a well defined map

$$\operatorname{Div} M/\text{principal divisors} \longrightarrow \operatorname{Pic} M.$$

This is in fact is an isomorphism. That it is injective is not so difficult. See Problem 3. The surjectivity is not entirely trivial. See Corollary 4.21

Rational functions f with the property that $(f) + D$ is effective form a vector space traditionally denoted by $\mathcal{L}(D)$. The resulting effective divisors $(f) + D$ linearly equivalent to D form a projective space $|D|$, which is nothing but $\mathbb{P}\mathcal{L}(D)$. Any projective subspace of $|D|$ is called a *linear system of divisors*, whereas $|D|$ itself is called a *complete linear system*. Let me come back to the rational map defined by the line bundle $\mathcal{O}(D)$ associated to D. This can be generalised by taking a basis $\{s_0, \ldots, s_n\}$ for any linear subspace W of $\Gamma(\mathcal{O}(D))$ and the rational map $p \mapsto (s_0(p), \ldots, s_n(p))$ then is said to be given by the linear system $\mathbb{P}(W)$. A *fixed component* of the linear system $\mathbb{P}(W)$ is any divisor F which occurs as a component of all divisors in $\mathbb{P}(W)$. The map defined by taking away this fixed part then is the same. The resulting divisors form the *moving part* of $\mathbb{P}(W)$ and now there still can be *fixed points* which however form at most a codimension 2 subspace.

The notion of ampleness has been introduced in connection with line bundles. A divisor D is called ample if the corresponding line bundle $\mathcal{O}(D)$ is ample.

If D is a hypersurface, the line bundle $\mathcal{O}(D)$ has a section s vanishing along D and if $f \in \mathcal{L}(D)$ the product $f \cdot s$ is in a natural way a section of $\mathcal{O}(D)$ and every section can be obtained in this way (see Problem 8). So

$$\mathcal{L}(D) \xrightarrow{\otimes s} H^0(\mathcal{O}(D)) \quad \text{is an isomorphism.}$$

Let me finish this chapter with an important theorem.

Theorem 1. (Bertini) *A generic hyperplane section of a smooth projective variety is smooth.*

Proof: Let $X \subset \mathbb{P}^n$ be a smooth projective variety and let $(\mathbb{P}^n)^\vee$ be the dual projective space of hyperplanes of $\mathbb{P}^n$. Inside $X \times (\mathbb{P}^n)^\vee$ let me consider the set B consisting of pairs (x, H) such that the projectivized tangent space $T_x(X)$ to X at x and H are NOT transversal, i.e. such that $T_x(X) \subset H$. If dim $X = k$ the possible hyperplanes with this "bad" behaviour form a projective space parametrized by the $\mathbb{P}^{n-k-1}$ disjoint from $T_x(X)$. So the projection $B \to X$ realises B as a projective bundle over X and hence is a variety of dimension $k + n - 1 - k = n - 1$. Consequently, the projection of B into $(\mathbb{P}^n)^\vee$ is not surjective. The complement of this variety parametrizes the "good" hyperplanes. ∎

Problems.

2.1. Let
$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$
be an exact sequence of vector bundles over a manifold M. Introduce the subbundle F^r of $\bigwedge^k E$ whose fibre over $m \in M$ is the subspace generated by the wedges of the form $e_1 \wedge e_2 \ldots \wedge e_k$ with r of the e_j in $f(E')_m$. Prove that F^{r+1} is a subbundle of F^r and that g induces an isomorphism
$$F^r/F^{r+1} \xrightarrow{\sim} \bigwedge^r E' \otimes \bigwedge^{k-r} E''.$$
In particular, one has an isomorphism
$$\det E' \otimes \det E'' \xrightarrow{\sim} \det E.$$

2.2. Prove that the normal bundle for a smooth hypersurface D of M is isomorphic to the restriction of $\mathcal{O}(D)$ to D.

2.3. Prove that two divisors D and D' on a projective manifold give isomorphic line bundles if and only if the divisors are linearly equivalent.

2.4. Prove that the canonical bundle of $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(-n-1)$.

2.5. Prove that any hypersurface in $\mathbb{P}^n$ is linearly equivalent to dH, where d is the degree of the hypersurface and H is a hyperplane. Deduce that $\operatorname{Pic}\mathbb{P}^n \cong \mathbb{Z}$.

2.6. Prove that for a smooth hypersurface D of degree d in $\mathbb{P}^n$ the normal bundle is given by $N(D/\mathbb{P}^n) = \mathcal{O}(d)|D$.

2.7. Let $M = M_1 \times M_2$ and let $p_1 : M \to M_1$ and $p_2 : M \to M_2$ be the projections onto the factors.
i) Let $\mathcal{V}_1$ be a vector bundle on M_1 and $\mathcal{V}_2$ a vector bundle on M_2. There is a natural homomorphism

$$H^0(M_1, \mathcal{V}_1) \otimes H^0(M_2, \mathcal{V}_2) \to H^0(M_1 \times M_2, p_1^*\mathcal{V}_1 \otimes p_2^*\mathcal{V}_2).$$

Show that this is an isomorphism. Hint: restrict a section s of $p_1^*\mathcal{V}_1 \otimes p_2^*\mathcal{V}_2$ to the fibre $p_1^{-1}(x)$. This yields a section $s(x) \in H^0(M_2, \mathcal{V}_2) \otimes (\mathcal{V}_1)_x$ depending holomorphically on x.
ii) Prove that $\Omega^1(M) = p_1^*\Omega^1(M_1) \oplus p_2^*\Omega^1(M_2)$ and that $K_M = p_1^*K_{M_1} \otimes p_2^*K_{M_2}$.
iii) Prove that $q(M) = q(M_1) + q(M_2)$ and that $p_g(M) = p_g(M_1) \cdot p_g(M_2)$.

Specialize this to products of compact Riemann surfaces.

2.8. Let D be a projective hypersurface of the projective manifold M and let s_D be a regular section of $\mathcal{O}(D)$ vanishing along D. Let $f \in \mathcal{L}(D)$. Prove that $f \cdot s$ is a regular section of $\mathcal{O}(D)$ and that any regular section of $\mathcal{O}(D)$ is obtained in this way.

Chapter 2. Cohomological tools

3. Sheaves and their cohomology

A. Sheaves

As stated in the preface, sheaves form an indispensable tool for algebraic geometers. For cohomology theory fine and flasque sheaves turn out to be useful.

Let me fix a principal ideal domain R and a topological space M.

Definition 1. *A presheaf $\mathcal{F}$ of R-modules over M consists of a collection of R-modules $\mathcal{F}(U)$, one for every open set $U \subset M$, and a collection of R-module homomorphisms $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ for pairs of open sets U, V with $V \subset U$ (the restriction homomorphisms) such that:*

a. $\rho^U_U = \mathrm{Id}_{\mathcal{F}(U)}$ *for all open* $U \subset M$,

b. $\rho^V_W \circ \rho^U_V = \rho^U_W$ *for all* $W \subset V \subset U$.

If in addition the following property holds, $\mathcal{F}$ is called a *sheaf:*

c. *If U is a union of open sets $U = \bigcup_{i \in I} U_i$ then*

   1) if $f, g \in \mathcal{F}(U)$ and $\rho^U_{U_i}(f) = \rho^U_{U_i}(g)$ for all $i \in I$, then $f = g$,

   2) if $f_i \in \mathcal{F}(U_i)$ with $\rho^{U_i}_{U_i \cap U_j}(f_i) = \rho^{U_j}_{U_i \cap U_j}(f_j)$ for all $i, j \in I$, then there exists a, because of 1) unique element $f \in \mathcal{F}(U)$ with $\rho^U_{U_i}(f) = f_i$ for all $i \in I$.

An element of $\mathcal{F}(U)$ is called a *section* of $\mathcal{F}$ over U. The module $\mathcal{F}(M)$ of sections over M is also denoted by $\Gamma(M)$ or $H^0(M, \mathcal{F})$. The latter notation will be justified later.

Another useful concept is that of the *stalks*. To define it fix $m \in M$ and consider the collection of neighbourhoods of m. The stalk $\mathcal{F}_m$ at m is defined as the *direct limit*

$$\mathcal{F}_m := \operatorname*{dirlim}_{U \ni m} \mathcal{F}(U), \quad U \text{ a neighbourhood of } m,$$

which by definition is obtained by taking the disjoint union of the modules $\mathcal{F}(U)$, U a neighbourhood of m, and then identifying $m \in \mathcal{F}(U)$ with $m' \in \mathcal{F}(U')$ if there is some neighbourhood $U'' \subset U \cap U'$ of m such that $\rho^U_{U''} m = \rho^{U'}_{U''} m'$.

If $\mathcal{F}$ is a presheaf, but not a sheaf, one may enlarge it to a sheaf, the *sheaf associated to the presheaf* $\mathcal{F}$ (see [Wa, p.166]). This is sometimes useful since natural constructions which start on the level of the R-modules $\mathcal{F}(U)$ with $\mathcal{F}$ a sheaf do often give presheaves, but not always sheaves, as will be seen when homomorphisms between sheaves are treated. Although the explicit construction of the sheaf associated to a presheaf is not needed, let me give it for the sake of completeness.

So let $\mathcal{F}$ be a presheaf and define for each open set U the module $F(U)$ consisting of functions $U \ni x \mapsto s(x) \in \mathcal{F}_x$ with the property that for each point $x \in U$ there is a neigbourhood $V \subset U$ and a section $t \in \mathcal{F}(V)$ such that the germ of t at y is equal to $s(y)$ for all $y \in Y$. Then the $F(U)$ form a sheaf which by definition is the sheaf associated to $\mathcal{F}$.

It should be clear what is meant by a *sheaf homomorphism* $h : \mathcal{F} \to \mathcal{F}'$: one should have homomorphisms $\mathcal{F}(U) \to \mathcal{F}'(U)$ for each open set $U \subset M$ commuting with the restriction maps. In particular there are induced homomorphisms $h_m : \mathcal{F}_m \to \mathcal{F}'_m$ for the stalks. Observe that the modules of the kernels of $\mathcal{F}(U) \to \mathcal{F}'(U)$ do form a sheaf ker h, but the modules of the cokernels only form a presheaf. By definition coker h is the sheaf associated to this presheaf.

A sequence of sheaf homomorphisms on M:

$$\ldots \to \mathcal{F}_{i-1} \xrightarrow{f_{i-1}} \mathcal{F}_i \xrightarrow{f_i} \mathcal{F}_{i+1} \to \ldots$$

is *exact* if for all $m \in M$ the corresponding sequence of the stalks at m is exact. This does NOT mean that the corresponding sequence of the sections over all open $U \subset M$ is exact, which is the definition of an exact sequence of presheaves. The reason is that, as noticed before, the cokernels on presheaf level do not always form a sheaf.

Examples

1. Let M be any topological space. Let G be any R-module. For any open $U \subset M$ let $G(U) = \{$ locally constant functions $f : U \to G\}$. The restriction maps are the obvious ones. The properties a), b) and c) are immediate. This sheaf is called the *constant sheaf* G_M.

2. Let $f : M \to N$ be a continuous mapping between topological spaces and let $\mathcal{F}$ be a sheaf on M. The image sheaf $f_*\mathcal{F}$ is the sheaf defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$ (and the obvious restriction maps). It is easy to see that $(g \circ f)_*\mathcal{F} = g_*(f_*\mathcal{F})$ when $g : N \to P$ is a further continuous map between topological spaces. Note also that there is a canonical homomorphism $\hat{f} : (f_*\mathcal{F})_{f(m)} \to \mathcal{F}_m$ which associates to a germ $g_{f(m)} \in (f_*\mathcal{F})_{f(m)}$ represented by a section $g \in \mathcal{F}(f^{-1}U)$, U a neighbourhood of $f(m)$, the germ of g at m.

3. Let M be a differentiable manifold and for any open $U \subset M$ let $\mathcal{E}(U)$ be the ring of differentiable functions (it is a module over the real numbers) and take the usual restriction maps. Again, one verifies that this defines a sheaf, the sheaf $\mathcal{E}_M$ of differentiable functions on M. The elements of $\mathcal{E}_{M,m}$ are called *germs at $m \in M$ of differentiable functions.*

Similarly, if M is a complex manifold, there is the sheaf $\mathcal{O}_M^{\text{hol}}$ of holomorphic functions and on a projective manifold there is the sheaf $\mathcal{O}_M^{\text{alg}}$ of regular functions. If no confusion arises the same notation $\mathcal{O}_M$ for these sheaves will be used although the holomorphic sheaf is much bigger in general.

4. If M is a complex manifold, for any open $U \subset M$ one can form the ring of fractions $Q(\mathcal{O}(U))$ of the ring $\mathcal{O}(U)$ and the obvious restriction maps between them. These form only a presheaf, since Axiom C2) does not hold. To make it into a sheaf, let me define *meromorphic functions* over U so that this Axiom holds automatically. So, a meromorphic function over U should be given by a collection $\{U_i, f_i\}$ with $\{U_i\}$ an open cover of U, $f_i \in Q(\mathcal{O}(U_i))$ such that in $U_i \cap U_j$ one has $f_i = f_j$. Meromorphic functions on U form a complex vectorspace $\mathcal{M}(U)$ and in this way one does get a sheaf, the sheaf $\mathcal{M}$ of *germs*

of meromorphic functions on M. By definition, a *meromorphic function* on M is a global section of this sheaf. Denote by $\mathcal{M}_M^*$ the sheaf (of multiplicative groups) of non-zero elements in $\mathcal{M}_M$. The sheaf $\mathcal{O}_M^*$ of germs of nowhere zero holomorphic functions on M forms a subsheaf of $\mathcal{M}_M^*$. A *Cartier divisor* on M is a global section of the sheaf $\mathcal{M}_M^*/\mathcal{O}_M^*$. In concrete terms, a Cartier divisor consists of a collection of open sets $\{U_i\}$ covering M and non-zero meromorphic functions f_i on U_i such that in the overlaps $f_i = g_{ij} \cdot f_j$ with g_{ij} a nowhere vanishing holomorphic function on $U_i \cap U_j$. Two sets $\{U_i, f_i\}$ and $\{U_j', f_j'\}$ define the same Cartier divisor if in overlaps $U_i \cap U_j'$ one has $f_i/f_j' \in \mathcal{O}^*(U_i \cap U_j')$.

For a projective manifold, working with $\mathcal{O}_M^{\mathrm{alg}}$ one obtains the algebraic Cartier divisors. On a variety any rational function is completely determined by knowing it on any non-empty Zariski open subset U. So the sheaf of germs of rational functions, i.e. the sheaf of quotients of $\mathcal{O}_M^{\mathrm{alg}}$ is just the constant sheaf $\mathbb{C}(M)_M$. There is no need to do patchwork for defining rational functions. In particular the algebraic analogue of a meromorphic function on projective manifolds just is a rational function. Any irreducible hypersurface D defines a Cartier divisor by taking the local defining equations. A different choice of local defining equations yield the same Cartier divisor, more or less by definition. It follows that any divisor defines a unique Cartier divisor. Conversely, any Cartier divisor $\{U_i, f_i\}$ yields a divisor by taking $\sum_D \mathrm{ord}_D(f_i)D$. This indeed gives a well defined divisor since in $\mathcal{O}(U_i \cap U_j)$ the function f_i/f_j is nowhere vanishing and hence $\mathrm{ord}_D(f_i) = \mathrm{ord}_D(f_j)$. This shows that on a projective manifold one may identify divisors and Cartier divisors. For the case of general complex manifolds see Problem 3.

The GAGA-principle tells us that there is no difference between the group of algebraic Cartier divisors and the group of Cartier divisors.

5. A sheaf $\mathcal{F}$ of R-modules on M is a *fine sheaf*, if for every locally finite cover $\{U_i\}$ of M by open sets there are endomorphisms $h_i : \mathcal{F} \to \mathcal{F}$ with support in U_i such that $\sum_i h_i = \mathrm{Id}_{\mathcal{F}}$. Here the *support* of a homomorphism h is the closure of the points $m \in M$ where h_m is not zero. Examples include the sheaves $\mathcal{E}_M$ of differentiable functions on a differentiable manifold M, since there are partitions of unity subordinate to any locally finite open cover of M. See [Wa, p. 170].

6. A sheaf $\mathcal{F}$ is called a *flasque sheaf* if for any pair of open subsets $U \subset V$ the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective. Any sheaf $\mathcal{F}$ embeds in a flasque sheaf $\mathcal{C}^0(\mathcal{F})$, its sheaf of discontinuous sections which is defined by letting $\mathcal{C}^0(\mathcal{F})(U)$ be the set of maps $U \ni x \mapsto s(x) \in \mathcal{F}_x$.

7. Let E be a vector bundle on a manifold M. For any $U \subset M$ take $\Gamma(U, E)$ and the obvious restriction maps. This gives the *sheaf of sections associated to* E. In the differentiable setting this sheaf is denoted by $\mathcal{E}(E)$, in the holomorphic (or algebraic) setting by $\mathcal{O}(E)$. Particular cases are the sheaves $\mathcal{E}_M^p$ of differentiable p-forms on a differentiable manifold M and the sheaf Ω_M^p of holomorphic p-forms on a complex manifold. The sheaves $\mathcal{E}_M^p$ are fine. This follows with partitions of unity.

8. If M is a complex manifold, an affine or a projective variety one often uses *sheaves of* $\mathcal{O}_M$*-modules*, which by definition are sheaves $\mathcal{F}$ of complex vector spaces such that for every open $U \subset M$ the vector space $\mathcal{F}(U)$ in addition is an $\mathcal{O}(U)$-module and if $V \subset U$ is open, the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}(U) \to \mathcal{O}(V)$. It should be clear what is meant by a *morphism of*

$\mathcal{O}_M$*-modules.*

Special cases of $\mathcal{O}_M$-modules include $\oplus^n \mathcal{O}_M$, the *trivial* $\mathcal{O}_M$-module of rank n and the *locally trivial* $\mathcal{O}_M$*-modules* $\mathcal{F}$ of rank n, which by definition have the property that there is a cover of M by open sets U over which $\mathcal{F}$ is trivial of rank n. The sheaf $\mathcal{O}(E)$ of holomorphic sections of a vector bundle ise locally free and conversely. See Problem 1.

Let E be a holomorphic vector bundle on a complex manifold M and let $m \in M$. There is the following useful relation between the fibre of E and the stalk of $\mathcal{O}(E)$ at m.

$$\mathcal{O}(E)_m/\mathfrak{m}_m \cdot \mathcal{O}(E)_m \xrightarrow{\cong} E_m$$

where $\mathfrak{m}_m$ is the maximal ideal of the point m in $(\mathcal{O}_M)_m$ and where the isomorphism comes from evaluating germs of sections of E at m.

In the usual way, out of the two sheaves of $\mathcal{O}_M$-modules $\mathcal{F}$ and $\mathcal{G}$ one produces $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{G}$ by forming the presheaf given over U by $\mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$ which in fact is a sheaf. The sheaf $\mathcal{H}om_{\mathcal{O}_M}(\mathcal{F}, \mathcal{G})$ is constructed in an essentially different way by taking the $\mathcal{O}(U)$-module $\mathrm{Hom}_U(\mathcal{F}|U, \mathcal{G}|U)$ of the $\mathcal{O}_M|U$-module homomorphisms $\mathcal{F}|U \to \mathcal{G}|U$ and the obvious restrictions. One cannot take $\mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{F}(U), \mathcal{G}(U))$ since then there would be no apparent way to define the restrictions.

Finally, if $f : M \to N$ is a holomorphic map between complex manifolds (or a morphism between varieties) the image sheaf $f_*\mathcal{O}_M$ is a sheaf of $\mathcal{O}_N$-modules in a natural way (holomorphic (or regular) functions on $U \subset N$ pull back to holomorphic (or regular) functions on $f^{-1}U$). Thus you can view the image of any sheaf $\mathcal{F}$ of $\mathcal{O}_M$-modules as a sheaf of $\mathcal{O}_N$-modules. This is the *(analytic or algebraic) direct image sheaf* which is still denoted by $f_*\mathcal{F}$.

8. The exponential sequence on a complex manifold M:

$$0 \to \mathbb{Z}_M \to \mathcal{O}_M \xrightarrow{exp} \mathcal{O}_M^* \to 0,$$

where "exp" means the map $f \mapsto \exp(2\pi i f)$. This sequence is a typical example of an exact sequence of sheaves which is not exact as sequence of presheaves. See Problem 2.

B. Cohomology

Cohomology for sheaves is introduced axiomatically. Not all proofs are presented here, but the reader can find them in the references given. De Rham's theorem, Dolbeault's theorem and Leray's theorem are explained in some more detail. An important application is given: the cohomological interpretation of the Picard group.

Let me briefly and informally recall the axiomatic set-up for a cohomology theory. You start with a fixed topological manifold and a class of sheaves on the manifold. Of course, in order to ensure that the axioms that follow make sense, the manifold should have some good properties and the same holds for the sheaves on it. Let me not be precise about this now. Let it be sufficient to say that one may take for example an arbitrary topological space and sheaves of abelian groups on it. Another possibility is that you take a Hausdorff space with countable basis for the topology and any sheaf of R-modules on it, where R is a fixed principal ideal domain. Lastly, there is the most widely used example of an algebraic variety X with sheaves of $\mathcal{O}_X$-modules.

For a cohomology theory you want groups $H^q(M, \mathcal{F})$, $q \in \mathbb{Z}$, for any allowable sheaf $\mathcal{F}$ and topological space M and for any allowable sheaf homomorphism $h : \mathcal{F} \to \mathcal{F}'$ there should be induced homomorphisms $H^q(h) : H^q(M, \mathcal{F}) \to H^q(M, \mathcal{F}')$. These groups and homomorphisms should satisfy the following axioms.

A. $H^q(M, \mathcal{F}) = 0$ for $q < 0$ and there are isomorphisms $H^0(\mathcal{F}) \cong \mathcal{F}(M)$ commuting with any induced homomorphism $H^0(f)$, where $f : \mathcal{F} \to \mathcal{F}'$ is a homomorphism of sheaves of R-modules.

B. $H^q(\mathcal{F}) = 0$ for all $q > 0$ if $\mathcal{F}$ is a fine sheaf or a flasque sheaf.

C. The correspondence which associates sheaves of R-modules and homomorphisms to their q-th cohomology groups and induced homomorphisms is *functorial*:

   C1. $H^q(\mathrm{Id} : \mathcal{F} \to \mathcal{F}) = \mathrm{Id}_{H^q(\mathcal{F})}$ and

   C2. $H^q(f \circ g) = H^q(f) \circ H^q(g)$.

D. For any short exact sequence

$$0 \to \mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{j} \mathcal{F}'' \to 0$$

there exist coboundary homomorphisms $H^q(M, \mathcal{F}'') \to H^{q+1}(M, \mathcal{F}')$ so that the sequence

$$\ldots \to H^q(M, \mathcal{F}') \xrightarrow{H^q(i)} H^q(M, \mathcal{F}) \xrightarrow{H^q(j)} H^q(M, \mathcal{F}'') \longrightarrow H^{q+1}(M, \mathcal{F}') \to \ldots$$

is exact. Furthermore, any homomorphism between short exact sequences of sheaves yields a homomorphism between the corresponding long exact sequences in cohomology.

As to *existence* of cohomology theories let me only remark that on an arbitrary topological space M and any sheaf of R-modules $\mathcal{F}$ on M, one can define $H^q(M, \mathcal{F})$ as the q-th cohomology group of the complex $\Gamma(M, \mathcal{C}^\bullet(\mathcal{F}))$. Here $\mathcal{C}^0(\mathcal{F})$ is the sheaf of discontinuous sections of $\mathcal{F}$ as introduced in Example 6. Setting $\mathcal{Z}^1(\mathcal{F}) = \mathcal{C}^0(\mathcal{F})/\mathcal{F}$ one defines $\mathcal{C}^1(\mathcal{F})$ as the sheaf of discontinuous sections of the sheaf $\mathcal{Z}^1$. Next, one inductively introduces $\mathcal{Z}^q = \mathcal{C}^{q-1}(\mathcal{F})/\mathcal{Z}^{q-1}$ and $\mathcal{C}^q(\mathcal{F})$ as the sheaf of discontinuous sections of $\mathcal{Z}^q$. For the verification of the axioms see [Go]. More precisely, axiom A is clear, B is II, Théorème 4.4.3 (Fine sheaves on a Hausdorff space with countable basis for the topology satisfy (b) in this theorem by [Go II, 3.7]) , C and D are the content of [Go II, Théorème 4.4.2.]

From the preceding definition it is virtually impossible to compute cohomology groups. Now, a cohomology theory is essentially unique (I come back to this in a little while) and so one might try to find another theory which is more suitable for computations. Such a theory is Čech cohomology-theory with values in a sheaf $\mathcal{F}$ on a topological space M. Although it can be defined for any M, this does NOT yield a good cohomology theory unless M is a Hausdorff space with countable basis for the topology. The delicate point is the exactness of long exact cohomology sequences. For details of the following discussion see [Wa, p. 200-204]. Since I shall be using Čech cohomology also on algebraic varieties X with the Zariski-topology, one has to be careful with long exact sequences. I shall apply them only for sheaves of $\mathcal{O}_X$-modules and for these one can prove that there are no problems with long exact sequences. See Proposition 4.8.

To define Čech cohomology, you start with an open cover $\mathfrak{U} = \{U_i\}$ of M. A collection $(U_0, \ldots, U_q)$ of members of $\mathfrak{U}$ with non-empty intersection is called a q-simplex $\sigma = \{0, \ldots, q\}$ and its support $|\sigma|$ is by definition $U_0 \cap \ldots \cap U_q$. The i-th face of σ is the $q-1$-simplex $\sigma^i = \{0, \ldots, i-1, i+1, \ldots, q\}$. A *$q$-cochain* is a function f which assigns to any q-simplex σ an element $f(\sigma) \in \mathcal{F}(|\sigma|)$. This is the same as saying that f is an element of the *free product* of the R-modules $\mathcal{F}(|\sigma|)$ where σ runs over the q-simplices of $\mathfrak{U}$. This free product is again an R-module (with the obvious module-operations):

$$\prod_{\sigma \text{ a q-simplex of } \mathfrak{U}} \mathcal{F}(|\sigma|) = C^q(\mathfrak{U}, \mathcal{F}).$$

There is the coboundary homomorphism

$$d : C^q(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F})$$

defined by

$$df(\sigma) = \sum_{i=0}^{q+1} (-1)^i \rho_{|\sigma|}^{|\sigma^i|} f(\sigma^i)$$

which satisfies $d \circ d = 0$ and hence one obtains a cochain complex (see Appendix 2), the *Čech cochain complex* $C^\bullet(\mathfrak{U}, \mathcal{F})$and it has cohomology groups $H^q(\mathfrak{U}, \mathcal{F})$. By definition,

$$H^q(M, \mathcal{F}) := \operatorname*{dirlim}_{\mathfrak{U}} H^q(\mathfrak{U}, \mathcal{F}),$$

where the direct limit is taken over the set of coverings, partially ordered under the refinement relation. If $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$, there are indeed natural homomorphisms $H^q(\mathfrak{U}, \mathcal{F}) \to H^q(\mathfrak{U}', \mathcal{F})$ which are to be used in forming the direct limit. See Appendix 1.

Clearly one has $H^q(M, \mathcal{F}) = 0$ for $q < 0$ and $H^0(M, \mathcal{F}) = \mathcal{F}(M)$. For fine sheaves, one even has $H^q(\mathcal{F}) = 0$ for all $q > 0$ (loc. cit.).

For any sheaf homomorphism $h : \mathcal{F} \to \mathcal{F}'$ there are induced R-module homomorphisms $H^q(h) : H^q(M, \mathcal{F}) \to H^q(M, \mathcal{F}')$. Moreover, if for a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

one can define a coboundary operator $H^q(M, \mathcal{F}'') \to H^{q+1}(M, \mathcal{F}')$ which fits into a long cohomology sequence

$$\ldots \to H^{q-1}(M, \mathcal{F}'') \to H^q(M, \mathcal{F}') \to H^q(M, \mathcal{F}) \to H^q(M, \mathcal{F}'') \to H^{q+1}(M, \mathcal{F}') \to \ldots.$$

If M is Hausdorff and has a countable basis for the topology, this sequence is exact. (In taking a limit one might have problems with the exactness on more general spaces.)

The modules $H^q(M, \mathcal{F})$ for the various sheaves of R-modules and induced homomorphisms $H^q(f)$ taken together therefore constitute a cohomology theory.

There is essentially only one cohomology theory up to natural isomorphism (loc. cit.). From the axioms it follows then for example that one can calculate $H^q(M, \mathcal{F})$ using exact sequences of the form

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \ldots$$

with $H^q(\mathcal{F}^j) = 0$, $q > 0$, $j = 0, 1, \ldots$. These are called *cohomological resolutions*. Examples arise when $\mathcal{F}^j$ is fine, resp. flasque, since then $H^q(\mathcal{F}^j) = 0$. Such resolutions are called *fine, resp. flasque resolutions*. Observe that by functoriality the sections of $\mathcal{F}^j$ form a complex $\Gamma(\mathcal{F}^\bullet)$. The q-th cohomology group of this complex $H^q(\Gamma(\mathcal{F}^\bullet))$ is naturally isomorphic to $H^q(\mathcal{F})$. See [Wa, Theorem 5.25].

As an example, the De Rham complex

$$0 \to \mathbb{R}_M \to \mathcal{E}^0_M \to \mathcal{E}^1_M \to \ldots$$

is a fine resolution of the constant sheaf $\mathbb{R}_M$ (by the Poincaré lemma) and hence one has

Theorem 2. (De Rham) *The sheaf cohomology group $H^q(M, \mathbb{R}_M)$ is canonically isomorphic to the q-th De Rham group $H^q_{\mathrm{DR}}(M)$.*

In a similar vein one has the Dolbeault complex

$$0 \to \Omega^p_M \to \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \mathcal{E}^{p,2} \xrightarrow{\bar\partial} \cdots,$$

which is a fine resolution of Ω^p_M (by the Dolbeault lemma) and so one obtains:

Theorem 3. (Dolbeault) *The sheaf cohomology group $H^q(M, \Omega^p_M)$ is canonically isomorphic to the q-th Dolbeault group*

$$\frac{\ker\left(\mathcal{E}^{p,q} \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}\right)}{\operatorname{im}\left(\mathcal{E}^{p,q-1} \xrightarrow{\bar\partial} \mathcal{E}^{p,q}\right)}.$$

IN THE REST OF THE SECTION ČECH COHOMOLOGY IS USED FOR A CLASS OF SHEAVES FOR WHICH ČECH COHOMOLOGY IS A GOOD COHOMOLOGY THEORY.

A useful tool for computing cohomology directly from a so-called *acyclic covering* is Leray's theorem. By definition, given a sheaf $\mathcal{F}$, a covering $\mathfrak{U}$ is $\mathcal{F}$- acyclic if for every simplex σ of the covering one has $H^q(|\sigma|, \mathcal{F}) = 0$ for $q > 0$.

Theorem 4. (Leray) *Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space M and $\mathfrak{U}$ an $\mathcal{F}$-acyclic covering. Assume that either M is Hausdorff with countable basis for the topology or that M is Noetherian. The natural homomorphism $H^q(\mathfrak{U}, \mathcal{F}) \to H^q(M, \mathcal{F})$ is an isomorphism.*

Remark A topological space is said to be *Noetherian* if any descending chain of closed subsets becomes stationary, which is the case for instance for the Zariski-topology. The conditions in the theorem are used to ensure that one can interchange limits and cohomology groups. For a simple proof of this property see [Go, II,4.12]. There are other proofs valid for any topological space and a sheaf of abelian groups on it, but these make use of spectral sequences (loc. cit. II, Théorème 5.4.1, Corollaire.)

Proof: Consider the sheaf $\mathcal{C}^p(\mathfrak{U}, \mathcal{F})$ defined by $U \mapsto C^p(U \cap \mathfrak{U}, \mathcal{F})$. The reader may verify that this is indeed a sheaf. There is a natural map $j : \mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F})$ defined by sending $f \in \mathcal{F}(U)$ to the 0-cochain which associates to $U_i \in \mathfrak{U}$ the restriction of f to $U \cap U_i$. This is an embedding by the sheaf axiom C1. So you get an embedding into a complex of sheaves

$$0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$$

and I claim that this gives a cohomological resolution of $\mathcal{F}$. So I have to show that the complex is exact and that $H^q(\mathcal{C}^p(\mathfrak{U}, \mathcal{F})) = 0$ for $q > 0$.

a. Exactness at $\mathcal{C}^0(\mathfrak{U}, \mathcal{F})$ is the sheaf axiom C2. For $p > 0$ let me consider the germ at $x \in M$ of a p-cocycle. I may assume that for some open neighbourhood U of x there is a representing cocycle $\alpha \in C^p(U \cap \mathfrak{U}, \mathcal{F})$. Moreover I may assume that $U \subset U_i$ for some index i. Then, if σ is a $p-1$-simplex in $\mathfrak{U}$ one has $|i\sigma| \cap U = |\sigma| \cap U$. So one may define a $(p-1)$-cochain $\beta \in C^{p-1}(U \cap \mathfrak{U}, \mathcal{F})$ by setting

$$\beta(\sigma) = \alpha(i\sigma).$$

So then one computes for a p-simplex $\tau = \{j_0 \cdots j_p\}$ of the covering

$$d(\beta)(j_0 \cdots j_p) = \sum_{0 \le k \le p} (-1)^k \alpha(ij_0 \cdots \widehat{j_k} \cdots j_p).$$

Since $d(\alpha) = 0$ one has in $U \cap |i\tau| = U \cap |\tau|$ that

$$\alpha(j_0 \cdots j_p) - \sum_{0 \le k \le p} (-1)^k \alpha(ij_0 \cdots \widehat{j_k} \cdots j_p) = 0$$

and hence $d(\beta) = \alpha$.

b. Next I have to show that $H^q(\mathcal{C}^p(\mathfrak{U}, \mathcal{F})) = 0$. For the moment, for any covering $\mathfrak{U}$ let $N_p(\mathfrak{U})$ be the collection of its p-simplices so that

$$C^p(\mathfrak{U}, \mathcal{G}) = \prod_{\sigma \in N_p(\mathfrak{U})} \mathcal{G}(|\sigma|)$$

and so for any other covering $\mathfrak{U}'$ one has

$$C^q(\mathfrak{U}', \mathcal{C}^p(\mathfrak{U}, \mathcal{F})) = \prod_{\sigma \in N_p(\mathfrak{U})} \prod_{\sigma' \in N_q(\mathfrak{U}')} \mathcal{F}(|\sigma| \cap |\sigma'|).$$

and so

$$H^q(\mathfrak{U}', \mathcal{C}^p(\mathfrak{U}, \mathcal{F})) = H^q(\prod_{\sigma \in N_p(\mathfrak{U})} C^\bullet(\mathfrak{U}' \cap |\sigma|, \mathcal{F})) = \prod_{\sigma \in N_p(\mathfrak{U})} H^q(C^\bullet(\mathfrak{U}' \cap |\sigma|, \mathcal{F})).$$

For any p-simplex σ of $\mathfrak{U}$ one has $H^q(|\sigma|, \mathcal{F}) = 0$ if $q > 0$. So the direct limit of the groups $H^q(C^\bullet(\mathfrak{U}') \cap \sigma, \mathcal{F})$ vanishes. But then also the free product

$$\prod_{\sigma \in N_p(\mathfrak{U})} \operatorname{dirlim}_{\mathfrak{U}'} H^q(C^\bullet(\mathfrak{U}' \cap |\sigma|, \mathcal{F}))$$

vanishes. Interchanging product and limit, which is allowed thanks to the assumptions (see the remark preceding the proof), you find

$$0 = \operatorname{dirlim}_{\mathfrak{U}'} \prod_{\sigma \in N_p(\mathfrak{U})} H^q(C^\bullet(\mathfrak{U}' \cap |\sigma|, \mathcal{F})) = H^q(M, \mathcal{C}^p(\mathfrak{U}, \mathcal{F})).$$

To complete the proof I have to verify that the isomorphism $H^q(\mathfrak{U}, \mathcal{F}) \to H^q(M, \mathcal{F})$ obtained in this way from the cohomological resolution is exactly the canonical map. This I leave to the reader.

■

As an example of the use of sheaf theory, let me come back to the group $\operatorname{Pic} M$ of isomorphism classes of line bundles on a projective manifold M. Let me recall that a line bundle $\mathcal{L}$ can be given by a trivialising open cover $\mathfrak{U} = \{U_i\}$ and nowhere zero transition functions $f_{ij} \in \mathcal{O}(U_i \cap U_j)$. The collection of transition functions defines a cochain $f \in C^1(\mathfrak{U}, \mathcal{O}^*_M)$ satisfying the cocycle relation $f_{ij} \cdot f_{jk} \cdot f_{li} = 1$, (written multiplicatively) i.e $df = 0$ and hence f defines an element in $H^1(\mathfrak{U}, \mathcal{O}^*_M)$. If one chooses a different trivialisation over the same cover, the new transition functions are seen to give a cocycle differing by a coboundary from f. So the class $[f] \in H^1(\mathfrak{U}, \mathcal{O}^*_M)$ is well defined. If you look at trivialisations on a different open cover their union is a common refinement $\mathfrak{U}'$. The two cohomology classes associated to the two trivialisations coincide in $H^1(\mathfrak{U}', \mathcal{O}^*_M)$. So the isomorphism class of the line bundle $\mathcal{L}$ gives a well defined element in $H^1(M, \mathcal{O}^*_M)$. Conversely, any element in $H^1(M, \mathcal{O}^*_M)$ gives a line bundle up to isomorphism. So one has

$$\operatorname{Pic} M \cong H^1(M, \mathcal{O}^*_M).$$

Let me now come back to the exponential sequence

$$0 \to \mathbb{Z}_M \to \mathcal{O}_M \xrightarrow{exp} \mathcal{O}^*_M \to 0,$$

and look at its induced cohomology sequence

$$\ldots \to H^1(M, \mathbb{Z}_M) \to H^1(M, \mathcal{O}_M) \to H^1(M, \mathcal{O}^*_M) \xrightarrow{c_1} H^2(M, \mathbb{Z}_M) \to \ldots.$$

The coboundary map is called the *First Chern Class* map. Its kernel is a subgroup $\operatorname{Pic}^0 M \subset \operatorname{Pic} M$. The exact sequence shows that there is a natural isomorphism

$$\operatorname{Pic}^0 M \cong \frac{H^1(M, \mathcal{O}_M)}{\operatorname{im} H^1(M, \mathbb{Z})}.$$

The latter quotient group in fact is a torus, the Picard variety. See Corollary 11.2. For curves you get the *jacobian* of C in this way.

The remaining part of the Picard group, the image under c_1, by definition is the *Néron-Severi group* $\operatorname{NS} M$ of M which is the group of isomorphism classes of divisors modulo homological equivalence: two divisors are said to be homologically equivalent if they have the same first Chern class. There is an exact sequence which summarises this situation

$$0 \to \operatorname{Pic}^0 M \to \operatorname{Pic} M \xrightarrow{c_1} \operatorname{NS} M \to 0.$$

Since for a compact manifold the cohomology groups are finite-dimensional (see Appendix 3) the Néron-Severi-group must be a finitely generated group, a fact which will be used several times later on.

Problems.

3.1. Prove that for any holomorphic vector bundle E of rank d on a complex manifold M the sheaf $\mathcal{O}(E)$ is locally free of rank d and that conversely any locally free sheaf of $\mathcal{O}_M$-modules of rank d is of the form $\mathcal{O}(E)$ with E a holomorphic vector bundle of rank d.

3.2. Give an example of an open set U in $\mathbb{C}$ such that the sequence

$$0 \to \mathbb{Z}(U) \to \mathcal{O}(U) \xrightarrow{exp} \mathcal{O}^*(U) \to 0$$

is not exact.

3.3. For a meromorphic function f and an irreducible hypersurface $D \subset M$ one can define the order of vanishing of f along D $\operatorname{ord}_D(f)$ in the same way as for rational functions on a projective manifold and hence one can speak of divisors of meromorphic functions. Generalise the concepts $\operatorname{Div} M$ and $\operatorname{Pic} M$. See [G-H, Chapter 1.1].

3.4. Show that for a connected compact complex manifold M the sequence

$$0 \to \mathbb{Z}(M) \to \mathcal{O}(M) \to \mathcal{O}^*(M) \to 0$$

is exact. Deduce that $H^1(M, \mathbb{Z}_M)$ embeds naturally into $H^1(M, \mathcal{O}_M)$.

3.5. Show that $\operatorname{Pic}^0(\mathbb{P}^n) = 0$.

3.6. Show that for the quadric surface $Q \subset \mathbb{P}^3$, the Picard group is the free abelian group generated by the divisor classes corresponding to the two rulings.

3.7. Let $Q \subset \mathbb{C}^3$ be the singular quadric defined by $xy - z^2$. The x-axis gives a subvariety L of Q of codimension one, which one may consider as a divisor. Prove that locally near the vertex, L cannot be given by an equation, while $2L$ is cut out by $z = 0$.

4. Serre's Finiteness and Vanishing Theorems

In this section $\mathbb{C}^n$ and projective space equipped with the *Zariski-topology* and *algebraic* sheaves on them will be studied.

A. Coherent sheaves

Coherent and quasi-coherent sheaves form global objects which are defined algebraically over affine sets thus permitting to translate their geometric properties into algebra.

Projective varieties $M \subset \mathbb{P}^n$ are to be looked at first. Introduce the homogeneous coordinates $X_0, \ldots, X_n$ on $\mathbb{P}^n$. Let me recall that $U_j = \{X_j \neq 0\} \cong \mathbb{C}^n$ are the basic affine open sets.

By definition a sheaf $\mathcal{F}$ of $\mathcal{O}_M$-modules on a projective variety M is coherent if every point of M has a Zariski neighbourhood U over which there is an exact sequence of the form

$$\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|U \to 0.$$

In other words: there is a Zariski-open cover over which the sheaf is a quotient of a finitely generated locally free module by a finitely generated submodule. In particular, any locally free sheaf is coherent. More examples can be found upon proving:

Proposition 1. *Let U be Zariski-open in a projective variety. A sheaf $\mathcal{F}$ of $\mathcal{O}_U$-modules is coherent if and only if it is, locally in the Zariski-topology, the quotient of a free $\mathcal{O}_U$-module of finite rank.*

Proof: Any Zariski-open subset of $\mathbb{C}^n$ is the union of the basic open sets

$$U_f := \mathbb{C}^n \setminus V(f),\ f \in \mathbb{C}[X_1, \ldots, X_n]$$

and so any Zariski-open set in $\mathbb{P}^n$ can also built up from such basic open sets by restricting to any of the affine open sets $U_j = \{X_j \neq 0\}$. Let me for the moment fix such an affine open set and identify it with $\mathbb{C}^n$. Consider its intersection with M. This intersection is the zero-locus $V(\mathfrak{p})$ of some prime-ideal $\mathfrak{p} \subset \mathbb{C}[x_1, \ldots, x_n]$. The ring of regular functions on $V(\mathfrak{p})$ is just the quotient ring $\mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}$ and hence Noetherian (see Appendix A1). Hence also the ring R of regular functions on $V(\mathfrak{p}) \cap U_f$ is Noetherian since it is a localisation of a Noetherian ring. It follows that any submodule of an R-module of finite rank again is of finite rank. ■

Example 2.

The sheaf of ideals $\mathcal{I}_M$ of any projective variety $M \subset \mathbb{P}^n$ is coherent. The sheaf of ideals $\mathcal{I}_M$ is defined in the usual way by letting $\mathcal{I}_M(U)$ be the ideal of $\mathcal{O}(U)$ generated by the equations of M. Since finitely many suffice (Hilbert's Basis Theorem) you get a surjection $\mathcal{O}(U)^{\oplus n} \to \mathcal{I}_M(U)$. This even gives a surjective sheaf homomorphism, since the same equations for M are used over every open set.

Remark 3. Of course, one can likewise introduce the concept of coherent sheaf on any complex manifold. But the validity of the preceding proposition is much less trivial. This result is known as *Oka's lemma* a proof of which can be found in [Gu-Ro, Chapter IV B,C]. Note that the definition of coherent sheaf given there differs from ours and the results proved there essentially say that the definitions agree for sheaves of $\mathcal{O}_U$-modules where U is some open subset of $\mathbb{C}^n$ (in the ordinary topology).

B. Coherent sheaves on $\mathbb{C}^n$

The central result here is the vanishing of higher cohomology groups for coherent sheaves on affine varieties. This is needed in the next section

In this subsection put

$$R := \mathbb{C}[X_1, \ldots, X_n].$$

Note that the structure sheaf on $\mathbb{C}^n$ is completely determined by the modules

$$\mathcal{O}(U_f) := R_f,$$

the *localisations* of the ring R in f.

Let me now study in detail the coherent sheaves on $\mathbb{C}^n$.

Given any R-module M define a sheaf $M^\sim$ on $\mathbb{C}^n$ by

$$M^\sim(U_f) = M_f = \{m/f^n \ ; \ n \in \mathbb{Z}_{\geq 0}, m \in M\}.$$

Such a sheaf by definition is called *quasi-coherent*. This is motivated by the remark that $M^\sim$ is coherent if M is a finitely-generated R-module. Indeed, $M(U_f)$ is finitely generated and so a quotient of $R_f^{\oplus n}$ by the submodule of the relations. So $M(U_f)$ is of the desired shape and over U_f the sheaf $M^\sim$ itself then is a quotient of the free sheaf $\mathcal{O}_{U_f}^{\oplus n}$ by the subsheaf coming from the relations.

The need for quasi-coherent sheaves originates from the following example.

Example 4. Let $f : X \to Y$ be a morphism between affine varieties and let $\mathcal{F}$ be quasi-coherent. Then $f_*\mathcal{F}$ is quasi-coherent. Indeed, one has $\mathcal{F} = M^\sim$ for some $R = \mathcal{O}(X)$-module M. You can also consider M as an $S = \mathcal{O}(Y)$-module using the natural pull-back of functions. The associated sheaf is just $f_*\mathcal{F}$. Note that even if $\mathcal{F}$ is coherent $f_*\mathcal{F}$ need not be coherent.

Now the following proposition says that a coherent sheaf on $\mathbb{C}^n$ is a quasi-coherent sheaf associated to a finitely generated $\mathcal{O}(\mathbb{C}^n)$ module.

Proposition 5. *If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}$-modules on $\mathbb{C}^n$, the module of its global sections $\Gamma(\mathcal{F})$ is a finitely generated R-module. The associated sheaf $\Gamma(\mathcal{F})^\sim$ is naturally isomorphic to $\mathcal{F}$.*

Proof: Cover $\mathbb{C}^n$ by finitely many Zariski-open sets U_{f_i}, $i = 1, \ldots, N$ over which $\mathcal{F}|U_{f_i} := \mathcal{G}_i$ is the quotient of $\mathcal{O}_{U_{f_i}}^{\oplus n}$ by a free $\mathcal{O}_{U_{f_i}}$-submodule

$$\mathcal{O}_{U_{f_i}}^{\oplus m} \xrightarrow{\alpha_i} \mathcal{O}_{U_{f_i}}^{\oplus n} \to \mathcal{G}_i \to 0.$$

Observe that now for all $g \in R$ the module $\mathcal{G}_i(U_{f_i} \cap U_g)$ is the cokernel of the restriction of α_i to $U_{f_i} \cap U_g = U_{f_i g}$. Look at the diagram

$$\begin{array}{ccccccc} R_{f_i}^{\oplus m} & \xrightarrow{\alpha_i} & R_{f_i}^{\oplus n} & \to & \mathcal{G}_i(U_{f_i}) = G_i & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ R_{f_i g}^{\oplus m} & \to & R_{f_i g}^{\oplus n} & \to & \mathcal{G}_i(U_{f_i g}) & \to & 0 \end{array}$$

It follows that $\mathcal{G}_i(U_{f_i g}) = (G_i)_g$.

The natural restrictions $M = \mathcal{F}(\mathbb{C}^n) \xrightarrow{\rho} \mathcal{F}(U_g)$ induce a sheaf-homomorphism $M^\sim \to \mathcal{F}$ and I claim that it is an isomorphism. I must show that restriction

$$M^{\sim}(U_g) = M_g \xrightarrow{\rho_g} \mathcal{F}(U_g)$$

gives isomorphisms over the basis for the topology $\{U_g \ ; \ g \in R\}$.

1. ρ_g is injective. Suppose that for some $s \in M_g$ one has $\rho_g(s) = 0$. Let $s_i \in \mathcal{F}(U_{f_i}) = G_i$ be the restriction of s to U_{f_i}. Since s_i restricts to zero in $\mathcal{F}(U_g \cap U_{f_i}) = (G_i)_g$ one gets $s_i \cdot g^n = 0$. Since there are only finitely many U_{f_i} one can find an n which works simultaneously for all U_{f_i}. So $s \cdot g^n = 0$ and hence $s = 0$ since it is an element of the localisation M_g.

2. ρ_g is surjective. This can be shown in a similar fashion. One considers $t \in \mathcal{F}(U_g)$ and its restriction t_i to $U_{f_i} \cap U_g$. Since $t|U_{f_i} \in \mathcal{F}(U_g \cap U_{f_i}) = (G_i)_g$ one can write $t|U_{f_i} = s_i/g^{n_i}$ with $s_i \in G_i$. Now take $n = n_i$ independent of i. Now the sections t_i and t_j agree on $U_{f_i} \cap U_{f_j} \cap U_g$ (there they are equal to $s \cdot g^n$). So on $U_{f_i} \cap U_{f_j}$ itself you must have $g^m(t_i - t_j) = 0$. Again you can assume that m is independent of i and j. So the sections $g^{m+n}t_i$ glue to a section, say s of $\mathcal{F}(\mathbb{C}^n) = M$ with the property that $s/g^{m+n} = t|U_g$.

Finally, to complete the proof one has to see that M is a finitely generated R-module. The localisations M_{f_i} are known to be finitely generated R_{f_i} modules for a covering U_{f_i} of $\mathbb{C}^n$. The fact that this is a covering means that the f_i generate the ring R.

FACT: Let $N \subset M$ be a submodule and let $\mathrm{loc}_i : M \to M_{f_i}$ be the localisation map. Then

$$N = \bigcap_i \mathrm{loc}_i^{-1}(\mathrm{loc}_i(N) \cdot M_{f_i}).$$

Assume this fact. To show that M is finitely generated one only has to show that an increasing sequence of submodules $N_1 \subset N_2 \ldots$ becomes stationary. The submodules $\mathrm{loc}_i(N_1) \cdot M_{f_i}$, $\mathrm{loc}_i(N_2) \cdot M_{f_i}, \ldots$ become stationary in M_{f_i} since these are finitely generated. But this is true for any of the finitely many i. So there is some index independent of i beyond which the sequences become stationary. But then the fact can be applied to see that $N_1 \subset N_2 \ldots$ becomes stationary.

So it remains to establish the fact. Only the inclusion $\supset$ is non trivial. So suppose that $m \in M$ with $\mathrm{loc}_i(m) = n_i/f_i^{k_i}$ with $n_i \in N$. You may assume that $k = k_i$ independent of i and hence $f_i^k m - n_i = 0$ in M_{f_i}. So $f_i^{l_i}(f_i^k m - n_i) = 0$ in M. Again one may assume that $l_i = l$ independent of i and so

$$f_i^{k+l} m = f_i^l n_i \in N.$$

Now the f_i generate R, and hence this is true for the powers f_i^{l+k}, i.e. for some combination of R-coefficients one has $1 = \sum_i c_i f_i^{k+l}$ and it follows that

$$m = \sum_i c_i f_i^{k+l} m \in N$$

as required. ■

In the next subsection one needs that on affine varieties a short exact sequence of sheaves sometimes gives a short exact sequence for the sections.

Lemma 6. *Let $X \subset \mathbb{C}^n$ be an affine variety and let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence of $\mathcal{O}_X$-modules. Assume that $\mathcal{F}'$ is quasi-coherent. Then there is an induced exact sequence for the global sections

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0.$$

Proof: The only non-trivial point here is the fact that $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ is a surjection. So let $s \in \Gamma(X, \mathcal{F}'')$ be a global section. Lift s locally, say over an open subset U_f to a section t of $\mathcal{F}$. I CLAIM that first of all for suitable natural number N the section $f^N s$ lifts to a global section of $\mathcal{F}$. Indeed, cover X by finitely many sets $U_i = U_{f_i}$ such that s lifts over U_i to $t_i \in \Gamma(U_i, \mathcal{F})$. Over $U_f \cap U_i$ the sections t and t_i both lift s and so their difference is a section of $\mathcal{F}'$. Since $\mathcal{F}'$ is quasi-coherent, Problem 4 shows that for suitable $n \in \mathbb{N}$ the section $f^n(t - t_i)$ extends to a section $u_i \in \Gamma(U_i, \mathcal{F}')$. One can take the same n for all U_i. Then $v_i := f^n t_i + u_i \in \Gamma(U_i, \mathcal{F})$ is a lifting of $f^n s$ which coincides with $f^n t$ on $U_i \cap U_f$. On $U_i \cap U_j$ the two sections v_i and v_j both lift $f^n s$ and so $v_i - v_j \in \Gamma(U_i \cap U_j, \mathcal{F}')$. Since v_i and v_j coincide over $U_i \cap U_j \cap U_f$, again by Problem 4 for some $m \in \mathbb{N}$, which can be taken independent of i and j, you have $f^m(v_i - v_j) = 0$. Now the sections $f^m v_i \in \Gamma(U_i, \mathcal{F})$ agree on overlaps so define a global section lifting $f^{n+m} s$.

Now cover X by a finite number of open sets $U_i' = U_{g_i}$ over which $g_i^n s$ lifts to section t_i of $\mathcal{F}$. Since the sets U_i' cover X, the ideal generated by the n-th powers of g_i generate the unit ideal in the coordinate ring $\mathcal{O}(X)$ of X and one can write

$$1 = \sum_i r_i g_i^n, \quad r_i \in \mathcal{O}(X).$$

The section $t := \sum_i r_i t_i \in \Gamma(X, \mathcal{F})$ has image $\sum_i r_i g_i^n s = s$ in $\Gamma(X, \mathcal{F}'')$. ∎

Let me now prove a fundamental result which is seemingly stronger (remember, I am working with Čech cohomology for which the exactness for the cohomology sequence has not been established; In 5C it will be shown for coherent sheaves on projective varieties):

Proposition 7. *Let $X \subset \mathbb{C}^n$ be an affine variety and $\mathcal{F}$ a quasi-coherent sheaf on X. Then for the Čech groups one has $H^q(X, \mathcal{F}) = 0$ for $q \geq 1$.*

Proof: Extend $\mathcal{F}$ by zero outside of X. The resulting sheaf now is quasi-coherent as a sheaf of $\mathcal{O}_{\mathbb{C}^n}$-modules (see Problem 2.) So I may assume that $X = \mathbb{C}^n$. From the previous proposition it follows that $\mathcal{F} = M^\sim$ for some R-module M of finite rank. I show that for any finite affine covering $\mathfrak{U}$ of $\mathbb{C}^n$ given by open sets of the form U_f the groups $H^q(\mathfrak{U}, \mathcal{F})$ vanish for $q > 0$. Suppose $\mathfrak{U} = \{U_{f_i}\}$, $i = 1, \ldots, N$. Let $c \in C^q(\mathfrak{U}, \mathcal{F})$ and let $\sigma = \{i_0, \ldots, i_q\}$ be a q-simplex of $\mathfrak{U}$. Then

$$c(\sigma) = \frac{m_{i_0 \cdots i_q}}{f_{i_0}^{n_{i_0}} \cdots f_{i_q}^{n_{i_q}}}, \quad m_{i_0 \cdots i_q} \in M.$$

There are polynomials P_j such that

$$\sum_j P_j f_j^{n_j} = 1.$$

This is the case because $\mathfrak{U}$ is a covering so that the f_j and hence also the $f_j^{n_j}$ generate R. Define $g \in C^{q-1}(\mathfrak{U}, \mathcal{F})$ by setting for any $(q-1)$-simplex $\tau = \{i_0 \dots i_{q-1}\}$

$$g(\tau) = (-1)^q \sum_k P_k \frac{m_{i_0 \cdots i_k \cdots i_{q-1} k}}{f_{i_0}^{n_{i_0}} \cdots f_{i_{q-1}}^{n_{i_{q-1}}}}.$$

Suppose that $d(c) = 0$. This implies

$$\sum_{k=0}^{q+1} (-1)^k m_{i_0 \cdots \widehat{i_k} \cdots i_{q+1}} f_{i_k}^{n_{i_k}} = 0.$$

Then I have

$$\begin{aligned} d(g)(\sigma) =& (-1)^q \sum_k \sum_{l=0}^{q} \frac{(-1)^l m_{i_0 \cdots \widehat{i_l} \cdots i_q k} P_k f_{i_l}^{n_{i_l}}}{f_{i_0}^{n_{i_0}} \cdots f_{i_q}^{n_{i_q}}} \\ =& \sum_k P_k \frac{m_{i_0 \cdots i_q} f_k^{n_k}}{f_{i_0}^{n_{i_0}} \cdots f_{i_q}^{n_{i_q}}} \\ =& \frac{m_{i_0 \cdots i_q}}{f_{i_0}^{n_{i_0}} \cdots f_{i_q}^{n_{i_q}}} \\ =& c(\sigma). \end{aligned}$$

So every cocycle is a coboundary. ■

C. Coherent sheaves on $\mathbb{P}^n$

Here, the explicit description, due to Serre, of the cohomology groups of the basic coherent sheaves $\mathcal{O}(k)$ on projective space is given. Using this, Serre's finiteness and vanishing results are proven for coherent sheaves on arbitrary projective varieties. A suitable relative version allows one to deduce that the higher direct images of coherent sheaves remain coherent (under a morphism between projective varieties).

In this subsection, let me put

$$S := \mathbb{C}[X_0, \dots, X_n]$$

and consider it as a graded ring, where you grade by degree. The rings $S(d)$ are the same ring as S but you shift the grading up, i.e. the degree of a homogeneous polynomial of degree e is given degree $e - d$ in $S(d)$. It is a graded S-module.

Now follows a fundamental construction for coherent sheaves on $\mathbb{P}^n$. Let M be a graded S-module and define the associated sheaf $M^{\sim}$ by

$$M^{\sim}(U) = \{m/f \ ; \ m \in M, \ f \in S, \deg\, f = \deg\, m, \ f(x) \neq 0, \forall x \in U\}.$$

If M is finitely generated this sheaf is coherent.

Example $S(d)^{\sim} \cong \mathcal{O}(d)$. See Problem 3.

Before passing to the central result of this section let me briefly pause to look back at the Čech cohomology groups. These do not give a good cohomology theory: for a short exact sequence of sheaves, the induced sequence in cohomology need not be exact. Now I show that this is a good cohomology theory when restricted to quasi-coherent sheaves on (quasi-)projective varieties:

Proposition 8. *Let X be any (quasi-) projective variety and let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence of quasi-coherent sheaves. The associated sequence in Čech cohomology is exact.

Proof: Use Lemma 4.6 to see that the induced sequence of groups of Čech cochains

$$0 \to C^q(\mathcal{F}') \to C^q(\mathcal{F}) \to C^q(\mathcal{F}'') \to 0$$

is exact. Now these are ordinary cochain complexes and thus there is a long exact sequence for the associated cohomology. Then one passes to the direct limit. ■

In view of the remarks made in §4 on the axiomatic aspect of cohomology theory, one can now unambiguously speak of THE cohomology groups for quasi-coherent sheaves on (quasi-)projective varieties. In particular one can apply Leray's theorem.

Proposition 9. *Let X be a projective variety, $\mathfrak{U}$ an open affine cover and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The natural map*

$$H^p(\mathfrak{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$$

is an isomorphism.

Proof: The intersection of two affine open sets is again affine. By Proposition 4.7, the higher cohomology groups of a coherent sheaf vanish on any affine set. One can then apply Leray's result, Theorem 3.4. ■

Concerning the sheaves $\mathcal{O}(d)$ there is the following fundamental result due to Serre:

Theorem 10.

(i). The natural map $S \to \bigoplus_{n\in\mathbb{Z}} H^0(\mathcal{O}(n))$ is a graded isomorphism.

(ii). $H^i(\mathcal{O}(k)) = 0$ for $0 < i < n$.

(iii). $H^n(\mathcal{O}(-n-1)) \cong \mathbb{C}$ and $H^n(\mathcal{O}(k)) = 0$ for $k > -n-1$.

(iv). For $k \geq 0$ the natural map

$$H^0(\mathcal{O}(k)) \times H^n(\mathcal{O}(-k-n-1)) \longrightarrow H^n(\mathcal{O}(-n-1) \cong \mathbb{C}$$

is a perfect pairing.

Proof: Introduce the sheaf $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}(n)$ and let $\mathfrak{U}$ be the standard cover of $\mathbb{P}^n$ by affines $X_j \neq 0$, $j = 0, \dots, n$. Although this sheaf is not coherent, it is the direct sum of coherent sheaves and by the previous theorem $H^q(|\sigma|, \mathcal{F}) = 0$ for $q > 0$ and any simplex σ of $\mathfrak{U}$. So Leray's theorem (see Theorem 3.4) then shows that the cohomology of $\mathcal{F}$ on $\mathbb{P}^n$ can be computed as the Čech -cohomology with respect to $\mathfrak{U}$. First consider

$$\mathcal{F}(|\sigma|) = \{\frac{F}{G}\ ;\ F, G \in S,\ G \neq 0 \text{ on } |\sigma|\}.$$

Now $G \neq 0$ on $|\sigma|$ with $\sigma = \{i_0, \dots, i_p\}$ means that G is a polynomial in $X_{i_0}, \dots, X_{i_p}$ only. To compute H^0 one considers quotients $F_i/X_i^{d_i}$, $i = 0, \dots, n$ so that $F_i/X_i^{d_i} = F_j/X_j^{d_j}$ in the overlaps. But then $F_i = FX_i^{d_i}$ for some $F \in S$. Hence (i). follows.

Next, note that

$$\begin{aligned} \mathcal{F}(U_0 \cap U_1 \cap \ldots \cap U_n) &= \bigoplus_{d_i\in\mathbb{Z}} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n}, \\ \mathcal{F}(U_0 \ldots \cap \widehat{U_j} \cap \ldots \cap U_n) &= \bigoplus_{d_i\in\mathbb{Z},\, d_j\geq 0} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n} \end{aligned}$$

and the Čech -coboundary $C^{n-1}(\mathfrak{U}, \mathcal{F}) \to C^n(\mathfrak{U}, \mathcal{F})$ is the natural inclusion

$$\bigoplus_j \bigoplus_{d_i\in\mathbb{Z},\, d_j\geq 0} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n} \to \bigoplus_{d_i\in\mathbb{Z}} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n}.$$

It follows that $H^n(\mathfrak{U}, \mathcal{F}) = \bigoplus_{d_i<0} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n}$ and the part in degree $-n-1$ which computes $H^n(\mathfrak{U}, \mathcal{O}(-n-1))$ is one-dimensional with basis $\mathbb{C}X_0^{-1} \cdots X_n^{-1}$. This proves the first part of (iii). Furthermore, observe that there is nothing in the cokernel of degree $> -n-1$, proving the remaining assertion of (iii).

Now, continuing with the previous computation, the part in degree $-k-n-1$ is the $\mathbb{C}$-vector space with basis consisting of the 'monomials' of the form $X_0^{d_0} \cdots X_n^{d_n}$ with all degrees d_j negative and with total degree $-k-n-1$. Consider the multiplication

$$H^0(\mathcal{O}(k)) \times H^n(\mathcal{O}(-k-n-1)) \longrightarrow H^n(\mathcal{O}(-n-1))$$

which translates into the natural multiplication

$$\bigoplus_{\substack{d_i\geq 0\\ \sum d_i=k}} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n} \times \bigoplus_{\substack{d_i'<0\\ \sum d_i'=-k-n-1}} \mathbb{C}X_0^{d_0'} \cdots X_n^{d_n'} \to \mathbb{C}X_0^{-1} \cdots X_n^{-1}.$$

The product of $X_0^{d_0} \cdots X_n^{d_n}$ with $X_0^{d_0'} \cdots X_n^{d_n'}$ is zero in $H^n(\mathfrak{U}, \mathcal{O}(-n-1))$ if any $d_i + d_i' \geq 0$. So one only gets a non-zero element if $d_i' = -d_i - 1$ for all $i = 0, \ldots n$. So the pairing is perfect since the basis dual to the basis $\{X_0^{d_0} \cdots X_n^{d_n}\ ;\ d_i \geq 0; \sum d_i = k\}$ is the basis $\{X_0^{-d_0-1} \cdots X_n^{-d_n-1}\}$. This proves (iv).

I prove (ii) by induction on n. This is done in two steps:

Step 1. I show that multiplication by X_n induces a bijection on $H^k(\mathcal{F})$.

Consider the exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{\cdot X_n} \to \mathcal{F} \to \mathcal{F}|\{X_n = 0\} \to 0.$$

Let me put $H = \{X_n = 0\}$. Part of the long exact sequence in cohomology reads as follows

$$H^{i-1}(\mathcal{F}|H) \to H^i(\mathcal{F}(-1)) \xrightarrow{\cdot X_n} H^i(\mathcal{F}) \to H^i(\mathcal{F}|H).$$

Note that $\mathcal{F}|H = \bigoplus_{k\in\mathbb{Z}} \mathcal{O}_H(k)$ and so induction shows that for $i = 2, \ldots, n-2$ multiplication by X_n gives an isomorphism. When $i = 1$ you have a surjection and when $i = n-1$ you have an injection. Applying (i) one obtains an exact sequence

$$0 \to H^0(\mathcal{F}(-1)) \to H^0(\mathcal{F}) \to H^0(\mathcal{F}|H) \to 0$$

which implies that the next map $H^1(\mathcal{F}(-1)) \xrightarrow{X_n} H^1(\mathcal{F})$ in the sequence is injective in addition to being surjective.

Similarly, applying (iii) one finds an exact sequence

$$0 \to H^{n-1}(\mathcal{F}|H) \to H^n(\mathcal{F}(-1)) \to H^n(\mathcal{F}) \to 0$$

and so multiplication by X_n is surjective on $H^{n-1}(\mathcal{F})$ in addition to being injective.

Step 2. I show that for a given $u \in H^k(\mathcal{F})$ one has $X_n^s u = 0$ for a suitable non-negative power of X_n.

Note that U_n is affine and so $0 = H^k(U_n, \mathcal{F}) = H^k(\mathfrak{U} \cap U_n, \mathcal{F}) = H^k(C^\bullet(\mathfrak{U} \cap U_n, \mathcal{F}))$. But the module $C^q(\mathfrak{U}\cap U_n, \mathcal{F})$ is nothing but the localisation $C^q(\mathfrak{U}, \mathcal{F})_{X_n}$ and so $H^k(C^\bullet(\mathfrak{U}\cap U_n, \mathcal{F})) = H^k(C^\bullet(\mathfrak{U}, \mathcal{F}))_{X_n}$. This localisation vanishes precisely when for all $u \in H^k(\mathcal{F})$ some power of X_n kills u. ■

Let me derive an important consequence of this computation. First you need to know that any coherent sheaf on $\mathbb{P}^n$ is the quotient of a direct sum of line bundles.

Proposition 11. *There is a short exact sequence*

$$\bigoplus_{j=1}^{k} \mathcal{O}(n_j) \to \mathcal{F} \to 0.$$

Proof: I'll show that in fact for large enough N there is a surjection of the trivial sheaf $\mathcal{O}^{\oplus m}$ onto $\mathcal{G} = \mathcal{F}(N)$. This means that $\mathcal{G}$ is generated by sections, i.e. there are sections s_j, $j = 1, \ldots m$ such that every stalk $\mathcal{G}_x$ is generated by the $s_j(x)$. Indeed the standard generators of $\mathcal{O}^{\oplus m}$ map to generators of $\mathcal{G}$.

To prove this, consider the standard affine cover U_i, $i = 0, \ldots, n$. Now $\mathcal{F}|U_i = M_i^{\sim}$ for some R_i-module M_i, where $R_i = \mathbb{C}[X_0/X_i, \ldots, X_n/X_i]$. I shall make use of the following lemma.

Lemma 12. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and let f be a global section of $\mathcal{O}(d)$, i.e. a homogeneous polynomial of degree d. Let $U_f = \{x \in \mathbb{P}^n \; ; \; f(x) \neq 0\}$ and suppose that one has $t \in \Gamma(U_f, \mathcal{F})$. Then for some $N \in \mathbb{N}$ the section $f^N t$ of $\mathcal{F}(Nd)$ over U_f extends over $\mathbb{P}^n$.*

For a proof I refer to Problem 4.

View X_i as a section of $\mathcal{O}(1)$. It follows that for any $s \in M_i$ for large enough N the section $X_i^N s \in M_i\tilde{}(N)$ extends as a section of $\mathcal{F}(N)$ over $\mathbb{P}^n$. Let me take N large enough so that I can use it for all $i = 0, \ldots, n$. Let me apply this simultaneously to the finitely many generators $\{s_{ij}\}$ of the module M_i. Now multiplication by x_i^N induces an isomorphism $M_i\tilde{} \to \mathcal{F}(N)|U_i$ and so the sections $s_{ij}x_i^N$ generate the latter sheaf. But these sections extend to sections of $\mathcal{F}$, so together they generate $\mathcal{F}(N)$ everywhere.

■

Theorem 13. (Serre's Finiteness and Vanishing Theorem) *Let $X \subset \mathbb{P}^n$ be a projective variety and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then*

(a) for each $i \geq 0$ the vector space $H^i(X, \mathcal{F})$ is finite-dimensional.

(b) there is an integer n_0 depending only on $\mathcal{F}$ so that $H^i(\mathcal{F}(n)) = 0$ for all $i > 0$ and all $n \geq n_0$.

Proof: One can reduce to the case $X = \mathbb{P}^n$ since the sheaf obtained from $\mathcal{F}$ by extending it by zero on the complement of X is a coherent sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules (Problem 1.) It follows immediately from Theorem 4.10 that the theorem holds for any sheaf which is a direct sum of sheaves of the form $\mathcal{O}(n)$. Now by the previous lemma, $\mathcal{F}$ is a quotient of such a direct sum $\mathcal{E}$ and so there is an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{F} \to 0$$

with $\mathcal{E}'$ again coherent. From the resulting exact sequence one gets

$$\ldots \to H^i(\mathcal{E}) \to H^i(\mathcal{F}) \to H^{i+1}(\mathcal{E}') \to \ldots.$$

The vector space on the left is finite-dimensional by the previous theorem.

To prove the theorem I now use descending induction on i. For $i > n$ one has $H^i(\mathbb{P}^n, \mathcal{F}) = 0$ since the standard affine covering of $\mathbb{P}^n$, which computes the cohomology, consists of $n+1$ elements. It follows that I may assume that $H^{i+1}(\mathcal{E}') = 0$ and so $H^i(\mathcal{F})$ is finite-dimensional. This proves (a).

To prove (b) let me twist the preceding sequence and consider the following piece of the resulting long exact sequence

$$\ldots \to H^i(\mathcal{E}(n)) \to H^i(\mathcal{F}(n)) \to H^{i+1}(\mathcal{E}'(n)) \to \ldots.$$

Now again, by the previous theorem the vector space on the left vanishes for all n larger than a certain number m_0 which works for all i. By the induction hypothesis $H^{i+1}(\mathcal{E}'(n)) = 0$ for $n \geq m_1$ independent of i. Now take $n_0 = \max(m_0, m_1)$. ■

Inspecting the proof of this theorem more closely one derives the following proposition.

Proposition 14. *Let X be a projective variety of dimension n and $\mathcal{F}$ a coherent sheaf on X. Then $H^q(X, \mathcal{F}) = 0$ if $q > n$.*

Proof: If $X \subset \mathbb{P}^N$ a general linear subspace of codimension $n+1$ is disjoint from X so that projecting from it yields a morphism of X onto $\mathbb{P}^n$. Now, the standard affine covering of $\mathbb{P}^n$ consists of $n+1$ elements. Since q is a projection, the inverse by q of an affine open set on $\mathbb{P}^n$ gives an affine open set on X and so you get an acyclic cover of X by $n+1$ open sets. Leray's theorem then implies that $H^q(X, \mathcal{F}) = 0$ for $q > n$. ■

Note that the preceding finiteness theorem can be formulated in relative form.

Proposition 15. *Let $U \subset \mathbb{C}^m$ be affine and let X be any irreducible Zariski-closed subset in $\mathbb{P}^n \times U$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is a finitely generated $\mathcal{O}(U)$-module.*

For the proof let me refer to Problem 5.

Corollary 16. *Let $f : X \to Y$ be a morphism between projective varieties and $\mathcal{F}$ any coherent sheaf of $\mathcal{O}_X$-modules. The direct image sheaf $f_*\mathcal{F}$ is coherent.*

Proof: Since the question is local one may assume that Y is affine. Take a covering of X by a finite number of affines U_i and let $V \subset Y$ be affine. Giving a section of $\mathcal{F}$ over $f^{-1}V$ is the same as giving sections over $U_i \cap f^{-1}V$ which patch over the intersections, i.e there is an exact sequence

$$0 \to f_*\mathcal{F} \to \bigoplus_i f_*(\mathcal{F}|U_i) \to \bigoplus_{i,j} f_*(\mathcal{F}|U_i \cap U_j),$$

and since the last two terms are quasi-coherent (Example 4.4) the first sheaf is quasi-coherent as well (Problem 1). So it is the sheaf associated to $f_*\mathcal{F}(Y) = \mathcal{F}(X) = H^0(X, \mathcal{F})$ which however is a finitely generated $\mathcal{O}(Y)$-module by the previous Proposition.

■

Next, let me introduce higher direct images.

Definition 17. *For any continuous map $f : X \to Y$ between topological spaces and any sheaf $\mathcal{F}$ of abelian groups on X let the q-the direct image sheaf $R^q f_*\mathcal{F}$ be the sheaf associated to the presheaf*

$$V \mapsto H^q(f^{-1}(V), \mathcal{F}|f^{-1}V).$$

To compute higher direct images in the case of morphisms between projective manifolds let me first consider the case where the target space is an affine variety.

Lemma 18. *let X be a (quasi-)projective variety, Y affine, $\varphi : X \to Y$ a morphism and $\mathcal{F}$ a quasi-coherent sheaf on X. The higher direct image $R^q\varphi_*\mathcal{F}$ is the sheaf on Y associated to the module $H^q(X, \mathcal{F})$.*

Proof: Choose an affine open cover $\mathfrak{U}$ of X. By proposition 4.7 this is an acyclic cover. Recall that in the proof of Leray's theorem, Theorem 3.4, the sheaf $\mathcal{C}^p(\mathfrak{U}, \mathcal{F})$ given by $U \mapsto C^p(U \cap \mathfrak{U}, \mathcal{F})$ has been introduced. The complex $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ has been shown to be exact and to give a cohomological resolution for $\mathcal{F}$ since the covering is acyclic. So, if f is in the coordinate ring $R(Y)$ of Y defining the open set U_f, one has $H^q(\varphi^{-1}U_f, \mathcal{F}) = H^q(\Gamma(\mathcal{C}^\bullet(\mathfrak{U} \cap U_f, \mathcal{F})))$. Now $\mathcal{C}^q(\mathfrak{U}, \mathcal{F})$ is the quasi-coherent sheaf associated to the $R(Y)$-module $C^q(\mathfrak{U}, \mathcal{F}) = \prod_\sigma \mathcal{F}(|\sigma|)$ where you take the product over all q-simplices. So one can write

$$H^q(\varphi^{-1}U_f, \mathcal{F}) = H^q(C^\bullet(\mathfrak{U}, \mathcal{F})\tilde{}(U_f)) = H^q(C^\bullet(\mathfrak{U}, \mathcal{F})\tilde{}(U_f),$$

and by Leray's theorem, this last module is isomorphic to $H^q(X, \mathcal{F})\tilde{}(U_f)$. ■

Corollary 19. *Let $f : X \to Y$ be a morphism between projective varieties and $\mathcal{F}$ any coherent sheaf of $\mathcal{O}_X$-modules. The higher direct image sheaves $R^q f_* \mathcal{F}$ are coherent.*

Proof: The assertion is local on Y and so one can assume that Y is affine and then one can apply the previous result. Now, by Proposition 4.15, $H^q(X, \mathcal{F})$ is a module of finite rank over the affine coordinate ring of Y and so the sheaf $R^q f_* \mathcal{F}$ is not only quasi-coherent but even coherent. ■

D. Applications to very ampleness.

I derive an ampleness criterion for line bundles which play a central role in the proof of Nakai's ampleness criterion for divisors on surfaces, to be treated later.

Next, let me study line bundles L on compact complex manifolds X. To prove that L is very ample one has to show:

1. The map φ_L must be everywhere defined. So, for every point $x \in X$ there is a section of L which is non-zero in x. Let $\mathfrak{m}_x$ be the maximal ideal in the ring $\mathcal{O}_x$, i.e. the set of germs of functions vanishing at x and let $\mathcal{L} = \mathcal{O}(L)$. The exact sequence

$$0 \to \mathfrak{m}_x \cdot \mathcal{L} \to \mathcal{L} \to L_x \to 0$$

shows that it is sufficient to prove that $H^1(\mathfrak{m}_x \cdot \mathcal{L}) = 0$.

2. The map φ_L must be injective, i.e. sections must separate pairs of points. So, for every two points $x, y \in X$ there must be sections s, s' of L with $s(x) = 0$, $s(y) \neq 0$ and $s'(x) \neq 0, s'(y) = 0$. The exact sequence

$$0 \to \mathfrak{m}_x \cdot \mathfrak{m}_y \cdot \mathcal{L} \to \mathcal{L} \to L_x \oplus L_y \to 0$$

shows that it is enough to show that $H^1(\mathfrak{m}_x \cdot \mathfrak{m}_y \cdot \mathcal{L}) = 0$.

3. Sections must separate tangent directions. This means the following. Locally sections of L are holomorphic functions on an open subset U of X and if $x \in U$ there must be enough sections vanishing at x so that their differentials span $T_x(X)$. So, for every cotangent direction $v^* \in T_x^\vee X$ there must be a section s of L vanishing at x and with

$ds(x) = v^*$. This can be formulated more intrinsically as follows. There is a well-defined sheaf map

$$d_x : \mathfrak{m}_x \cdot \mathcal{L} \to L_x \otimes T_x^\vee$$

and it should be surjective on the level of sections. The exact sequence

$$0 \to \mathfrak{m}_x^2 \cdot \mathcal{L} \to \mathfrak{m}_x \cdot \mathcal{L} \xrightarrow{d_x} L_x \otimes T_x^\vee \to 0$$

shows that it is enough to show the vanishing of $H^1(\mathfrak{m}_x^2 \cdot \mathcal{L})$.

Let me collect the results:

Proposition 20. *Let L be a line bundle on a compact complex manifold X. Let $\mathcal{L} = \mathcal{O}_X(L)$ be the corresponding locally free sheaf. The map φ_L is defined at x if $H^1(X, \mathfrak{m}_x \cdot \mathcal{L}) = 0$. It separates x from y if $H^1(X, \mathfrak{m}_x \cdot \mathfrak{m}_y \cdot \mathcal{L}) = 0$ and it separates tangents at x if $H^1(X, \mathfrak{m}_x^2 \cdot \mathcal{L}) = 0$.*

Now, if φ_L is defined at x this is true in a Zariski-open neigbourhood if x. Similarly, if φ_L separates x and y it will separate points in a neigbourhood of x from points in a neighbourhood of y and if φ_L is an immersion at x it will be so in a neighbourhood. By compactness, the previous remarks show that it sufficient to prove vanishing of $H^1(\mathfrak{m}_x \cdot \mathcal{L})$, $H^1(\mathfrak{m}_x \cdot \mathfrak{m}_y \cdot \mathcal{L})$ and $H^1(\mathfrak{m}_x^2 \cdot \mathcal{L})$ for a certain finite number of points x and y. Since this involves a finite number of coherent sheaves on X, by the previous theorem one can find some large integer N so that the desired groups vanish provided you replace $\mathcal{L}$ by $\mathcal{L}(N)$. In other words, $\mathcal{L}(N)$ will be very ample. So

Corollary 21. *Any line bundle L on a projective manifold is of the form $L' \otimes L''^{-1}$ with L' and L'' very ample. In particular, every line bundle on X is of the form $\mathcal{O}_X(D)$ for some divisor D.*

From this Corollary it follows that the map

$$\operatorname{Div}(X) \to \operatorname{Pic} X,$$

introduced in Chapter 3, is surjective.

Remark 22. Since a generic hyperplane section of a projective manifold is smooth (by Bertini, see 2.1) and connected by A2.21, it follows that one can assume $D = D_1 - D_2$ with D_1 and D_2 smooth and connected.

Let me finish by proving a very useful criterion for ampleness which is used when proving the Nakai Ampleness Criterion 11.14 for surfaces.

Proposition 23. (Criterion for Ampleness) *Let M be a projective manifold and L a line bundle on M. The following are equivalent.*

1. *L is ample,*
2. *$H^p(\mathcal{F} \otimes \mathcal{O}_M(L^{\otimes n})) = 0$, $p > 0$, for all coherent sheaves $\mathcal{F}$ on M and $n > n(\mathcal{F})$.*
3. *$\mathcal{F} \otimes \mathcal{O}_M(L^{\otimes n})$ is spanned by its sections for all coherent sheaves $\mathcal{F}$ on M and $n > m(\mathcal{F})$.*

Proof:
1. $\Longrightarrow$ 2. For very ample L this is Serre's Theorem. 4.13. Otherwise, if $L^{\otimes m}$ is very ample, one has by loc. cit. $H^p(\mathcal{F} \otimes \mathcal{O}_M(L^{\otimes r} \otimes L^{\otimes ms}) = 0$, $p > 0$ for $s > n_r$, $r = 0, \ldots, m-1$. Take $n(\mathcal{F}) = m \max n_r$. Then, writing $n = ms + r$ with $0 \le r < m$ one has for $n > n(\mathcal{F})$ that $H^p(\mathcal{F} \otimes L^{\otimes n}) = 0$, $p > 0$.

2. $\Longrightarrow$ 3. To prove that $\mathcal{F} \otimes \mathcal{O}_M(L^{\otimes n})$ is spanned by sections at $x \in M$ it is sufficient to show that $H^1(\mathfrak{m}_x \cdot (\mathcal{F} \otimes \mathcal{O}_M(L^{\otimes n})) = 0$ which by assumption is the case for $n > n_x$. Spannedness then holds in a Zariski-open neighbourhood of x, say U_x. By compactness finitely many such sets cover M, say U_{x_i}, $i = 1, \ldots, N$. Now take $n(\mathcal{F}) = \max(n_{x_i})$.

3. $\Longrightarrow$ 1. Start with an affine neighbourhood U of x and let $N \subset M$ be the complement. Then for some $n > 0$ the sheaf $\mathcal{I}_N \otimes \mathcal{O}_M(L^{\otimes n})$ is generated by sections and in particular there is a section s of $L^{\otimes n}$ vanishing at x. By construction $U_s = \{y \in M \ ; \ s(y) \neq 0\} \subset U$ and hence is an affine neighbourhood of x over which $L^{\otimes n}$ is trivial. Finitely many such sets $U_{s_1}, \ldots, U_{s_k}$ cover M, since M is compact. Let $X_1^j, \ldots, X_n^j$ be affine coordinates in U_{s_j}. By Problem 4 there is an integer m such that all of the functions $s_j^m X_k^j$ extend to global sections t_j^k of $L^{\otimes mn}$ and one replaces $L^{\otimes n}$ by $L^{\otimes mn}$ and s_j by s_j^m. Already the sections s_j^m generate the bundle $L^{\otimes mn}$ globally and so these sections together with the t_j^k define a morphism $X \to \mathbb{P}^N$ which is constructed in such a way that it gives an embedding when restricted to each U_{s_i}. But points on U_{s_i} separate from points in the complement, since s_i^m is not zero on U_{s_i} while this section vanishes on the complement. ∎

Corollary 24. *Let $f : X \to Y$ a morphism between projective varieties with finite fibres and let L be ample on Y. Then f^*L is ample on X.*

Proof: The sheaf $f_*\mathcal{F}$ is coherent on Y by 4.16. Furthermore, one has $f_*(\mathcal{F} \otimes \mathcal{O}_X(f^*L^{\otimes n})) = f_*\mathcal{F} \otimes \mathcal{O}_Y(L^{\otimes n})$ (see Problem 8). Now, since f has finite fibres, $H^p(f_*(\mathcal{O}_X(f_*\mathcal{F} \otimes \mathcal{O}_Y(L^{\otimes n})) = H^p(f_*\mathcal{F} \otimes \mathcal{O}_Y((L^{\otimes n}))$ by Problem 7. The result follows from the previous criterion. ∎

Problems.

4.1. Let $\mathcal{F} \to \mathcal{G}$ be a homomorphism between (quasi-) coherent sheaves on a Zariski-open subset of a projective variety. Show that the kernel, the cokernel and the image are (quasi-) coherent. Next, let
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
be an exact sequence of sheaves. Show that if any two of the preceding sheaves is (quasi-) coherent then so is the third.

4.2. Let U be a Zariski-open subset of $\mathbb{P}^n$ and let $X \subset \mathbb{P}^n$ be a projective variety. If $\mathcal{F}$ is coherent on $U \cap X$ show that the sheaf $\mathcal{F}$ considered as a sheaf of $\mathcal{O}_U$-modules is also coherent.

4.3. Prove that that $S(d)^{\sim} \cong \mathcal{O}(d)$.

4.4. Let X be a projective variety and $\mathcal{F}$ a quasi-coherent sheaf on X. Let L a line bundle on X, s a section of L and set $U_s = \{y \in X \ ; \ s(y) \neq 0\}$.

a. If t is a global section of $\mathcal{F}$ restricting to zero on U_s, there exists $n > 0$ such that $f^n s = 0$.

b. Suppose that t now is a section of $\mathcal{F}$ over U_s. Prove that there is an integer $n > 0$ such that $s^n t$ extends to a global section of $\mathcal{F} \otimes \mathcal{O}_X(L)$.

4.5. Prove Lemma 4.12. Hint: use the previous Problem.

4.6. Prove the relative version of Serre's finiteness theorem (Theorem 4.15).

4.7. Let $f : X \to Y$ be a continuous map between topological spaces and $\mathcal{F}$ a sheaf on X with the property that $R^p f_* \mathcal{F} = 0$, $p > 0$. Prove that $H^q(X, \mathcal{F}) \cong H^q(Y, p_* \mathcal{F})$, $q \geq 0$. Show that this can be applied to morphisms between projective varieties with finite fibres. (This is a special case of Leray's Spectral Sequence)

4.8. Let $f : X \to Y$ be a morphism between projective varieties. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and E a locally free $\mathcal{O}_X$-module of finite rank. Prove the *projection formula*

$$R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* E) \cong R^p f_* \mathcal{F} \otimes_{\mathcal{O}_Y} E.$$

Chapter 3. The first steps in surface theory

5. Intersection Theory on Surfaces

I introduce the intersection product for divisors following [Beau]. This is the same as the topological intersection product as given in the Appendix. Riemann-Roch for divisors is proven and as a consequence, the genus formula for an irreducible curve. This will be used a lot in what follows.

In this section let S be an algebraic surface. For any line bundle $\mathcal{L}$ on S set

$$h^i(\mathcal{L}) = \dim H^i(S, \mathcal{L})$$

and

$$\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L}) \quad \text{(Euler-Poincaré characteristic of } \mathcal{L}\text{)}.$$

For a divisor D set

$$h^i(D) = h^i(\mathcal{O}(D)).$$

The topological Euler-Poincaré characteristic is the alternating sum

$$e(S) = \sum_j (-1)^j \dim H^j(\mathbb{C}_S).$$

Recall (see Appendix A2.7), that $H^i(S, \mathbb{C}_S)$ is finite-dimensional and zero for $i > 4$.

Let $C \subset S$ be an irreducible curve. Recall the notion of *normalisation* or *desingularisation*

$$\nu : \tilde{C} \longrightarrow C.$$

Here $\tilde{C}$ is a non-singular curve (projective manifold of dimension 1) and ν is a finite map which is an isomorphism outside the singularities of C. For existence and uniqueness let me refer to [G-H, p. 498] or [G, Chapter II]. One only needs to know that the construction is done locally (in the ordinary topology) and one can form the normalisation of any part of a curve in an open subset of S. Suppose now that two distinct irreducible curves C and C' meet in a point x. Suppose that at x one has local equations $f = 0$ for C and $g = 0$ for C', where $f, g \in \mathcal{O}_x$ the local ring of germs of holomorphic functions at x. Of course f nor g need to be irreducible. Let me recall (see e.g. [G, p.83]) how one may define the *intersection multiplicity of* C *and* C' *at* x denoted $(C, C')_x$. One first assumes that f is irreducible in the coordinate neighourhood U of x where it is defined. Choose coordinates in U such that $x = (0, 0)$. Let $\Delta \subset \mathbb{C}$ be a suitable small disk centered at 0 in the t-plane and let $\nu : \Delta \to U$ be a local normalisation for $U \cap C$ and suppose that

$$g \circ \nu(t) = a_l t^l + \text{higher order terms}$$

with $a_l \neq 0$. This number l by definition is $(C, C')_x$. In general one can write $f = \prod_j f_j^{m_j}$ with f_j irreducible and defining locally an irreducible curve C_j. Let me extend the definition by linearity: $(C, C')_x = \sum_j m_j (C_j, C)_x$.

Remark 1. The intersection multiplicity of C and C' at x can also be defined as the dimension of the $\mathbb{C}$-vector-space $\mathcal{O}_{S,x}/(f,g)$. It is not difficult to show that this is the same as the preceding definition. See Problem 1.

From the preceding remark it is clear that $(C,C')_x = (C',C)_x$ so, defining the intersection number

$$(C,C') = \sum_x (C,C')_x$$

one obtains a symmetric pairing on the set of irreducible curves. I want to extend this definition to all divisors. This is not at all obvious, since one doesn't know what for instance (C,C) should be. The idea now is that in forming the intersection product one should be allowed to move a curve in its linear equivalence class. Since in particular, any very ample divisor can be moved at will in its linear equivalence class, one can define (C,C) for these divisors. Then, remembering that any divisor is the difference of two very ample divisors, one can define the intersection product for any two divisors. But of course, there are many ways to write a divisor as a difference of two very ample divisors and it is not clear that this yields a well-defined intersection product. Although one in principle can carry out this program, it is a little faster to follow Beauville's route [Beau, Chapt. 1].

Theorem 2. *For any two divisors D, D' one poses*

$$(D,D') = \chi(\mathcal{O}_S) - \chi(\mathcal{O}(-D)) - \chi(\mathcal{O}(-D')) + \chi(\mathcal{O}(-D-D')).$$

This defines a symmetric bilinear product on $\operatorname{Pic} S$ *extending the intersection product on irreducible curves. This product is the unique product on* $\operatorname{Pic} S$ *which satisfies the following properties;*

1. *If C and C' are two smooth curves which intersect transversally, (C,C') is exactly the number of intersection points,*
2. *it is symmetric: $(D,D') = (D',D)$,*
3. *it is linear: $(D+D',D'') = (D,D'') + (D',D'')$.*
4. *it depends on the linear equivalence classes of the divisors only.*

Note that this theorem implies that the intersection product only depends on the sheaves defined by the divisors, i.e. one can move a divisor in its linear equivalence class without changing intersection products.

Proof:

Step 1. Uniqueness. Given two divisors C and D, one can write them as differences of very ample divisors (Serre's Theorem), say $C = C_1 - C_2$ and $D = D_1 - D_2$. One can choose smooth curves C'_k in $|C_k|$, $D'_k \in |D_k|$ $(k = 1,2)$ such that C'_k meets D'_j transversally $j,k = 1,2$. This follows from Bertini. The four properties then totally determine (C,D), since $(C,D) = (C'_1,D'_1) - (C'_1,D'_2) - (C'_2,D'_1) + (C'_2,D'_2)$ and each of the four terms is equal to the number of intersection points of the curves involved.

Step 2. Let me prove that the definition for distinct irreducible curves C and C' with coincides with (C,C') as defined by means of local intersection numbers.

Choose a non-trivial section s, resp s' of the line bundle $\mathcal{O}(C)$, resp. $\mathcal{O}(C')$ vanishing on C, resp. C'. The sequence

$$0 \to \mathcal{O}_S(-C-C') \xrightarrow{(s',-s)} \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C') \xrightarrow{\begin{pmatrix} s \\ s' \end{pmatrix}} \mathcal{O}_S \to \mathcal{O}_{C\cap C'} \to 0$$

then is exact. This simply is a translation of the exactness of the *Koszul sequence* at any point x of S:

$$0 \to \mathcal{O}_x \xrightarrow{(g,-f)} \mathcal{O}_x^{\oplus 2} \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} \mathcal{O}_x \to \mathcal{O}_x/(f,g) \to 0.$$

I leave it to the reader that this sequence is indeed exact. See Problem 2. From this exact sequence one immediately verifies the desired equality.

Step 3. Let me show that for any irreducible smooth curve C on S and every divisor D on S one has

$$(C,D) = \deg(D|C).$$

To show this you may employ the exact sequences

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

and

$$0 \to \mathcal{O}_S(-D-C) \to \mathcal{O}_S(-D) \to \mathcal{O}_C(-D) \to 0.$$

One finds that

$$\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) = \chi(\mathcal{O}_C)$$
$$-\chi(\mathcal{O}_S(-D)) + \chi(\mathcal{O}(-C-D)) = -\chi(\mathcal{O}_C(-D))$$

and so $(C,D) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_C(-D)) = -\deg(-D|C) = \deg(D|C)$ by the usual Riemann-Roch theorem on the smooth curve C.

Step 4. Completion of proof

Let me introduce for any three divisors D_1, D_2, D_3 the expression

$$s(D_1, D_2, D_3) = (D_1, D_2 + D_3) - (D_1, D_2) - (D_1, D_3).$$

This is symmetric in the arguments as one readily verifies. It vanishes if D_1 is a non-singular curve (by Step 3.) and hence it also vanishes if D_2 or D_3 is a non-singular curve. One needs to see that it always vanishes. But as remarked before, one can write any divisor and hence also D_2 as a difference $D_2 = A - B$ with A and B very ample. Moreover, by Remark 4.22 one can suppose that A and B are smooth and connected. Now you write out $0 = s(D_1, D_2, B) = (D_1, A) - (D_1, D_2) - (D_1, B)$. From this it follows immediately that the product is bilinear. ■

Remark 3. Note that the latter coincides with the topological intersection product since this product also enjoys the preceding properties. This one sees as follows. Any divisor

$D = \sum n_i C_i$ yields a cohomology class $c(D) = \sum n_i c(C_i)$ where the class $c(C_i) \in H^2(S, \mathbb{Z})$ is defined as the Poincaré-dual of the fundamental class $h(C_i) \in H_2(S, \mathbb{Z})$ of C_i. Likewise one has $h(D) = \sum_i n_i h(C_i) \in H_2(M, \mathbb{Z})$, a class which is Poincaré dual to $c(D)$. For any two homology classes $c, c' \in H_2(S, \mathbb{Z})$ the intersection product (c, c') is defined in Appendix A.2 where it is shown that this product is the same as evaluating the cup product of the Poincaré dual classes on the orientation class $o_S \in H_4(S, \mathbb{Z})$. Clearly, one gets a pairing on divisors by setting $(D, D') = (h(D), h(D'))$. This pairing only depends on the linear equivalence class. Indeed, by Proposition A3.8 the first Chern class of a divisor $c_1(D)$ coincides with the fundamental cohomology class $c(D)$ and the first Chern class depends only on the isomorphism class $\mathcal{O}_M(D)$ of the divisor D. This shows that the fourth property holds. The first property is Claim A2.17. The remaining ones are trivial.

Examples

1. $S = \mathbb{P}^2$. Any divisor D is linearly equivalent to dL, with L a line. So, if $D \equiv dL$ and $D' \equiv d'L$ one finds that $(D, D') = dd'$ which is Bezout's theorem.

2. $S = \mathbb{P}^1 \times \mathbb{P}^1$. One has the two fibres $F = \mathbb{P}^1 \times a$ and $F' = b \times \mathbb{P}^1$ and leaving these away one gets $\mathbb{C}^2$ on which every divisor is the divisor of some rational function. It follows that all divisors on S are linearly equivalent to $mF + m'F'$. Clearly $(F, F') = 1$ and since on $\mathbb{P}^1$ any two points are linearly equivalent any two fibres of the form $\mathbb{P}^1 \times a$ are linearly equivalent and so $(F, F) = 0 = (F', F')$. This completely determines the intersection pairing.

Let me now show that one can derive a weak form of the Riemann-Roch theorem, using Serre-duality A3.3.

Theorem 4. (Riemann-Roch) *For any divisor D on S one has*

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}((D, D) - (D, K_S)).$$

Proof: By definition $(-D, D - K_S) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}(D)) - \chi(\mathcal{O}_S(K_S - D)) + \chi(\mathcal{O}(K_S))$. By Serre duality $\chi(\mathcal{O}(K_S)) = \chi(\mathcal{O}_S)$ and $\chi(\mathcal{O}_S(K_S - D)) = \chi(\mathcal{O}_S(D))$. So $-(D, D) + (D, K_S) = (-D, D - K_S) = 2(\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(D))$. ■

Remark 5. Using Serre duality once more one can rewrite the Riemann-Roch theorem as follows.

$$h^0(D) - h^1(D) + h^0(K_S - D) = \chi(\mathcal{O}_S) + \frac{1}{2}((D, D) - (D, K_S)).$$

From this way of writing Riemann-Roch one derives an inequality which will be used a lot in the sequel

$$h^0(D) + h^0(K_S - D) \geq \chi(\mathcal{O}_S) + \frac{1}{2}((D, D) - (D, K_S)).$$

Remark 6. The strong form of the Riemann-Roch theorem, also called Hirzebruch-Riemann-Roch theorem, expresses $\chi(\mathcal{O}_S)$ in topological invariants of S. For algebraic surfaces this goes back to Noether and therefore is called the Noether formula. It reads as follows

$$\chi(\mathcal{O}_S) = \frac{1}{12}((K_S, K_S) + e(S)).$$

That indeed the self intersection of K_S is a topological invariant follows since it equals the self-intersection of the first Chern class of the surface, which is a topological invariant. Let me refer to [G-H, Chapter 4.6] for a geometric proof of the Noether formula. For the general Riemann-Roch theorem for projective manifolds I refer to [Hir].

The Riemann-Roch theorem is also valid for line bundles $\mathcal{L}$ on any surface, even if these are not of the form $\mathcal{O}(D)$. Clearly, even to make sense of the Riemann-Roch formula, one needs to use here the topological definition of the intersection product.

Next, let me give a formula for the genus of an irreducible curve on a surface.

Lemma 7.

1. *For any effective divisor D on a surface S one has*

$$-\chi(\mathcal{O}_D) = \frac{1}{2}((D, D) + (D, K_S)).$$

2. (Genus formula) *For an irreducible curve C with genus $g = \dim H^1(\mathcal{O}_C)$ one has*

$$2g - 2 = (K_S, C) + (C, C).$$

Proof: There is an exact sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0,$$

which implies $-\chi(\mathcal{O}_D) = -\chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(-D)) = \frac{1}{2}((D, D) + (D, K_S))$ by Riemann-Roch. If D is irreducible, $h^0(\mathcal{O}_D) = 1$ and hence $2g(D) - 2 = -2\chi(\mathcal{O}_D)$. ■

Remark 8. The genus g as defined above for a singular curve C is related to the genus $\tilde{g}$ of its normalisation $\nu : \tilde{C} \to C$ as follows. There is an exact sequence

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}} \to \Delta \to 0,$$

where $\Delta = \oplus \Delta_x$ is a sky-skraper sheaf concentrated in the singular points x of C. Taking the Euler-characteristics one gets

$$g(C) = g(\tilde{C}) + \sum_x \dim \Delta_x.$$

The important consequence is that

$$g(C) \geq g(\tilde{C}) \text{ with equality if and only if } C \text{ is smooth.}$$

Let me finish this section by saying a few words about the *dualising sheaf* of an effective divisor D on a surface. One defines it by

$$\omega_D = \mathcal{O}_D(K_S) \otimes \mathcal{O}_D(D)$$

which entails an exact sequence

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S) \otimes \mathcal{O}_S(D) \xrightarrow{res} \omega_D \to 0.$$

For a divisor $D = \sum C_i$ where D is a curve with ordinary double points, this sequence leads to an explicit description of the dualising sheaf.

Proposition 9. *Let C be a curve with only ordinary double points $\{p_1, \ldots, p_d\}$, let $\nu : \tilde{C} \to C$ be its normalisation and $\nu^{-1}(p_i) = \{p_i', p_i''\}$. The sheaf $\nu^{-1}\omega_C$ consists of the germs of meromorphic differential forms α on $\tilde{C}$ having at most poles of order 1 at the points $\{p_i', p_i''\}$, $i = 1, \ldots, d$ and such that $res_{p_i'}\alpha + res_{p_i''}\alpha = 0$.*

Proof: The residue homomorphism res in the preceding exact sequence can be computed as follows. Let $\{u, v\}$ be local coordinates around a point of C and let f be a local equation for C such that $\dfrac{\partial f}{\partial v}$ is not identically zero on C. If

$$\varphi = \frac{h}{f} du \wedge dv$$

is a local holomorphic section of $\mathcal{O}_S(K_S) \otimes \mathcal{O}_S(C)$ (so h is holomorphic). Then $res(h) = (hdu/\dfrac{\partial f}{\partial v})|C$ as is easily verified. Now around an ordinary double point one can take $f = uv$ so that $res(h) = h\dfrac{du}{u}|\{v = 0\} = -h\dfrac{dv}{v}|\{u = 0\}$. Taking the normalisation separates the two branches and $\nu^*(h)$ becomes meromorphic on each branch with pole of order at most one and with opposite residues. ■

Problems.

5.1. Prove the equivalence of the two definitions of intersection index. (See Remark 1). Hint: Interpret the number l as the dimension of the vector space $W := \mathcal{O}(\Delta)_0/(g \circ \nu(t))$ and construct a surjection $\mathcal{O}_x \to W$ whose kernel is (f, g).

5.2. Prove that the Koszul sequence is exact.

5.3. Let $C \subset \mathbb{P}^2$ be a smooth curve of degree d. Using the adjunction formula, show that the genus of C is equal to $\frac{1}{2}(d-1)(d-2)$.

5.4. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of bidegree (a, b), i.e. $(C, F') = a$ and $(C, F) = b$. Derive a formula for the genus of C in terms of a and b.

6. Birational geometry of surfaces

Birational maps between surfaces can be described totally by blowing ups or sigma-processes. The resulting exceptional curves can be recognised by Castelnuovo's contraction criterion. As a result, minimal surfaces are the same as surfaces without -1-curves. I finish this section with a reformulation of some of the results in 'Mori-terminology'.

I start by recalling the notion of the σ-process. Let S be a surface and $x \in S$ which serves as the origin in a coordinate chart (z, w) in a neighbourhood U of x. Now define

$$\tilde{U} = \{((z, w), (Z, W)) \in U \times \mathbb{P}^1 \; ; \; zW = wZ\}$$

where (Z, W) are the homogeneous coordinates of $\mathbb{P}^1$. Projection onto the first factor defines a holomorphic map

$$p : \tilde{U} \to U$$

which maps the curve $E = p^{-1}(x) = x \times \mathbb{P}^1$ onto x and $\tilde{U} \setminus E$ biholomorphically onto $U \setminus x$. Note that on $\tilde{U} \cap U \times \{Z \neq 0\}$ the coordinates $u = z, v = W/Z$ can be used and the map p is given by $p(u, v) = (u, uv)$. A similar remark applies to $\tilde{U} \cap U \times \{W \neq 0\}$.

Now glue S and $\tilde{U}$ over the set $\tilde{U} \setminus E = U \setminus x$. This yields a new smooth surface $\tilde{S}$ and a holomorphic map $p : \tilde{S} \to S$ which maps $E \cong \mathbb{P}^1$ to x and $p : \tilde{S} \setminus E \to S \setminus x$ is biholomorphic. This is *the σ-process* at x. The curve E is called the *exceptional curve.* The surface $\tilde{S}$ is called the *blow up* of S at x.

If S is algebraic then so is $\tilde{S}$. See Problem 1.

If C is any curve through x one can form the closure $\tilde{C}$ of the set $p^{-1}(C \setminus x) \cong C \setminus x$ in $\tilde{S}$. This curve is called the *strict transform* of the curve C.

Lemma 1. *If C is an irreducible curve passing with multiplicity m through x the total transform $p^*(C)$ is related to the strict transform by*

$$p^*(C) = \tilde{C} + mE.$$

Proof: Clearly $p^*(C) = \tilde{C} + kE$ with some integer k which one can determine by a local computation around x. Use coordinates (z, w) around x and (u, v) on part of $\tilde{U}$ as before. The local equation at x for C can be written as

$$f = f_m(z, w) + \text{higher order terms}$$

with f_m homogeneous of degree m. So in (u, v)-coordinates one has

$$f \circ p = f(u, uv) = u^m(f_m(1, v) + \ldots)$$

and so one must have $k = m$. ■

Proposition 2. *Let S be an algebraic surface, $p : \tilde{S} \to S$ the blow up at x and let E be the exceptional curve.*

i. The homomorphism $\operatorname{Pic} S \oplus \mathbb{Z} \to \operatorname{Pic} \tilde{S}$ *defined by* $(D, n) \mapsto p^*D + nE$ *is an isomorphism. A similar assertion holds for the Néron-Severi groups.*

ii. For any two divisors D, D' *on* S *one has* $(p^*D, p^*D') = (D, D')$, $(E, p^*D) = 0$ *and* $(E, E) = -1$

iii. One has $K_{\tilde{S}} = p^* + E$.

Proof: One can replace divisors by linearly equivalent divisors for which the components do not pass through x. Then the first two formulas in (ii). follow. Now choose a curve C passing through x with multiplicity 1 so that the strict transform $\tilde{C}$ meets the exceptional curve transversally. So $1 = (\tilde{C}, E) = (p^*C - E, E) = -(E, E)$. This completes the proof of (ii).

To show (i), note that the given map is clearly surjective (every irreducible curve distinct from E on $\tilde{S}$ is the strict transform of its image on S). To show injectivity, suppose that $p^*D + nE$ is linearly equivalent to 0. Intersecting with E and applying (ii) one finds that $n = 0$, but then D also must be linearly equivalent to zero. Replacing 'linearly equivalent' by 'homologically equivalent' yields an isomorphism on the level of the Néron-Severi groups.

To prove (iii), observe that p^*K_S and $K_{\tilde{S}}$ coincide outside E. So $K_{\tilde{S}} = K_S + mE$ for some integer m. The adjunction formula shows that $-2 = (K_{\tilde{S}}, E) + (E, E) = -m - 1$ and hence $m = 1$. ■

Let me now show how to 'eliminate the points of indeterminacy' of a rational map $f : S \dashrightarrow \mathbb{P}^n$. One has $f = (f_0, \ldots, f_n)$ with f_i polynomials, the map f is not defined at the set where all the f_i simultaneously vanish on S. Since one can assume that the f_i have no common factor this set F must be finite. Let $f(S)$ be the Zariski-closure of $f(S \setminus F)$ in $\mathbb{P}^n$.

Proposition 3. *Let* $f : S \dashrightarrow \mathbb{P}^n$ *be a rational map. There is a sequence of blowings up* $S_m \xrightarrow{\sigma_m} S_{m-1} \xrightarrow{\sigma_{m-1}} \ldots \xrightarrow{\sigma_1} S_0 = S$ *such that the rational map* $f \circ \sigma_1 \circ \ldots \circ \sigma_m$ *is everywhere defined.*

Proof: One may assume that $f(S)$ is not contained in any hyperplane. But then the system of hyperplanes yields a linear system $|D|$ of divisors which have the points in F as base points. If $F = \emptyset$ you are ready. Otherwise, you blow up S at a point of F. Say $\sigma_1 : S_1 \to S$. Then you can write $\sigma_1^*D = D_1 + m_1E$ with a certain multiplicity m_1 which can be chosen in such a way that $|D_1|$ does not have E in its base locus. If $|D_1|$ does not have a base locus you are ready. Otherwise you can blow up in one of the base points of the new linear system. One must see that this process stops and it is here that one makes essential use of the intersection theory. Indeed, at the k-th step one finds

$$(D_k, D_k) = (D_{k-1}, D_{k-1}) - m_k^2 < (D_{k-1}, D_{k-1})$$

but since $|D_k|$ has no curve in its base locus one has $(D_k, D_k) \geq 0$. So the self intersection numbers of the divisors in the linear systems constructed in this way must stabilise for some $k \leq (D, D)$ and then there are no more base points left. ■

Examples 1. Let $S \subset \mathbb{P}^n$ be a surface and p a point of S. Projection from p is a rational map $S \dashrightarrow \mathbb{P}^{n-1}$ which is defined everywhere except at p. By blowing up S at p one obtains a morphism $\tilde{S} \to \mathbb{P}^{n-1}$, where $\tilde{S}$ is the blow up of S at p.

2. Let $Q \subset \mathbb{P}^3$ be a smooth quadric and let $Q \dashrightarrow \mathbb{P}^2$ be projection from a point on the quadric maps. The two lines on Q passing through p are mapped to two distinct points, say $\bar{p}$ and $\bar{q}$. Blowing up Q at p gives a surface $\tilde{Q}$ and a morphism $\tilde{Q} \to \mathbb{P}^2$ which is the blow up of $\mathbb{P}^2$ at the points $\bar{p}$ and $\bar{q}$.

In the proof of the preceding proposition I used that the points at which a birational map is not defined is finite. This implies that for a birational morphism the number of curves which map to points must be finite since at these points the inverse birational map is not defined. Let me prove that these are the only points at which the inverse is not defined.

Lemma 4. *Let S be a projective variety of dimension two and S' a projective surface. If a birational morphism $f : S \to S'$ has the property that f^{-1} is not defined at a certain point p' , the inverse image $f^{-1}(p')$ is a curve.*

Proof: One may replace S by an affine variety, say $j : S \subset \mathbb{C}^n$ (so that still $f^{-1}(p') \neq \emptyset$). The rational map $j \circ f^{-1} : S' \dashrightarrow \mathbb{C}^n$ is given by n rational functions of which the first may be assumed to be not defined at p', is of the form $\dfrac{P}{Q}$ with Q a non-constant polynomial with $Q(p') = 0$. One may assume that P and Q have no common factors. Let D be the curve defined by $f^*Q = 0$. On S one has

$$f^*P = x_1 \cdot f^*Q,$$

with x_1 the first coordinate function on $\mathbb{C}^n$. So on D both functions f^*P and f^*Q are zero and $D = f^{-1}D'$ with $D' = \{P = Q = 0\}$. Since P and Q have no common factor this must be a finite set containing p'. Restricting to a smaller Zariski-open neighbourhood of p' one may therefore assume that $D' = \{p'\}$ and so $f^{-1}(p') = D$, a curve. ■

Corollary 5. *If a birational map of surfaces is not defined at a certain point, the inverse map contracts a curve onto that point.*

Proof: Let $f : S \dashrightarrow S'$ be a birational map which is defined on the Zariski-open $U \subset S$ and let $\Gamma \subset U \times S'$ be its graph and let S_1 be its closure in $S \times S'$. This is a variety (of dimension 2) possibly with singularities. Let $q : S_1 \to S$ and $q' : S_1 \to S'$ be the two projections. Suppose that f is not defined at p. Then q^{-1} is likewise not defined at p. By the Lemma, q contracts a curve $\tilde{C}$ onto p. But, since $S_1 \subset S \times S'$, the image of $\tilde{C}$ in S by assumption is a curve which maps to p by f^{-1}. ■

Proposition 6. (Universal property of the blowing up) *Let $f : S' \to S$ be a birational morphism between surfaces. Suppose that f^{-1} is not defined at $p \in S$. Then f factors as $f = \sigma \circ g$ with g a birational morphism and σ the blow up at p.*

Proof: Let $g = \sigma^{-1} \circ f$ and $h = g^{-1}$. One hase to show that g is a morphism. Let me assume that this is not the case. Then the previous lemma shows that $h(C) = p'$ for some curve $C \subset \tilde{S}$ and some point $p' \in S'$. Since then $\sigma(C) = f(p')$ the curve C must be the exceptional curve E and $f(p') = p$. Let u be any local coordinate at p (this means that the curve $\{u = 0\}$ passes simply through p. I claim that f^*u must be a local coordinate at p'. If not, it would be in $\mathfrak{m}_{p'}^2$ and hence also $h^*f^*u = \sigma^*u$ would be in $\mathfrak{m}_e^2$ for any $e \in E$ where h is defined, which is at all but finitely many points. But the blowing up has the property that any coordinate function on S at p lifts to a coordinate function at all points of E but one. On the other hand, there does exist u with $f^*u \in \mathfrak{m}_{p'}^2$. To see this one chooses local coordinates x, y at p and considers f^*y. If this is in $\mathfrak{m}_{p'}^2$ one sets $u = y$. Otherwise f^*y is a coordinate near p' and it vanishes with multiplicity one along $f^{-1}p$ in a neighbourhood of p'. So f^*y gives a local equation for $f^{-1}p$ and then $f^*x = v \cdot f^*y$ for some $v \in \mathcal{O}_{p'}$. So then $u = x - v(p') \cdot y$ has the property that $f^*u \in \mathfrak{m}_{p'}^2$ which completes the contradiction.
■

Remark 7. There is also a complementary universal property for the blowing up: if $h : \tilde{S} \to \mathbb{P}^n$ is a morphism wich contracts E, it factors as $g \circ \sigma$ with a morphism $g : S \to \mathbb{P}^n$.

This is much easier to see. One may write $h = (h_0, \dots, h_n)$ and consider h_i/h_j which is a regular function on $S_j = \tilde{S} \cap \{h_j \neq 0\}$ and, by assumption, on $\sigma(S_j)$. So h also defines a morphism $g : S \to \mathbb{P}^n$ with $h = g \circ \sigma$.

Theorem 8. *Any birational morphism between surfaces is the composition of a sequence of blowings up and isomorphisms.*

Proof: Let $f : S \to S'$ be a birational morphism which is not an isomorphism. Then f^{-1} is not defined at some point p_1 and by the previous proposition, $f = \sigma_1 \circ f_1$ with σ_1 the blowing up at p_1 and f_1 a birational morphism. Observe that f contracts the curves which f_1 contracts but also at least one more curve, namely any curve which by f_1 is mapped to the exceptional curve for σ. So the number of curves contracted by f_1 is strictly less than the number of curves contracted by f. If no curves are contracted by f_1 the inverse map is a morphism and so f_1 is an isomorphism. Otherwise one can continue and write $f_1 = \sigma_2 \circ f_2$ with σ_2 a blowing up and f_2 a birational morphism contracting fewer curves than f_1. After a finite number of steps this process terminates. ■

Corollary 9. *Every birational map $S \dashrightarrow S'$ between surfaces fits into a commutative diagram*

$$\begin{tikzcd} & S'' \arrow[dl, "h"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed] & & S' \end{tikzcd}$$

with h and g a composition of blowings up and isomorphisms.

Proof: This follows immediately from Proposition 3 and Theorem 8. ∎

For the purpose of reducing the birational classification to a biregular classification, the previous theorem is important. One introduces the following basic definition, which underlines this.

Definition 10. *A surface S is minimal if every birational morphism $S \to S'$ is an isomorphism*

The previous theorem then shows that every surface can be mapped to a miminal surface by a birational morphism. Indeed, if S is not minimal, there is some surface S' and a birational morphism $S \to S'$ which, by the previous theorem is a sequence of blowings up and isomorphisms. Since under a blowing up the rank of the Néron-Severi-group increases by one, this process must terminate. It follows moreover, that on a non-minimal surface there must be exceptional curves for some σ-process. These are smooth rational curves with self intersection (-1). Let me call such curves (-1)-curves. These are always exceptional curves for a blowing up by the following theorem.

Theorem 11. (Castelnuovo's contraction criterion). *A smooth rational curve E on a surface S with $(E, E) = -1$ is the exceptional curve for a σ-process $S \to S'$.*

Proof: Choose a very ample divisor H on S such that $H^1(S, \mathcal{O}(H)) = 0$, which is possible by Serre's Theorem (Theorem 4.13). Now E has a certain degree $d = (H, E)$ with respect to the embedding given by $|H|$ and $H' = H + dE$ now has the property that $(H', E) = 0$. I want to show that in fact $|H'|$ gives a morphism $\sigma : S \to S'$ of S onto a smooth surface S'. The fact that $(H', E) = 0$ then implies that H' is trivial on E and so this morphism contracts E to a point p. I shall show that σ is an isomorphism from $S \setminus E$ to $S' \setminus p$. It then follows from Remark 7 that this morphism is the σ-process at p.

Let me construct a special basis for the sections of $\mathcal{O}(H')$. The long exact sequence associated to the sequences

$$0 \to \mathcal{O}(H + (i-1)E) \to \mathcal{O}(H + iE) \to \mathcal{O}_E(d-i) \to 0, \quad i = 1, \ldots, d,$$

successively shows that $H^1(\mathcal{O}_S(H + iE)) = 0$ and that $H^0(\mathcal{O}_S(H + iE))$ surjects onto $H^0(\mathcal{O}_E(d-i))$. So I can take a basis for $H^0(\mathcal{O}(H+dE))$ by first taking a basis $\{s_0, \ldots, s_n\}$ for the sections of $\mathcal{O}(H)$ and adding sections which come from sections $\{s_{i,0}, \ldots, s_{i,d-i}\}$ of $\mathcal{O}(H + iE)$ which restrict to a basis for $H^0(\mathcal{O}_E(d-i))$. So, if s is a section of $\mathcal{O}_S(E)$ defining E I get the following basis for $H^0(\mathcal{O}(H'))$:

$$\{s^d s_0, \ldots, s^d s_n, s^{d-1} s_{1,0}, \ldots, s^{d-1} s_{1,d-1}, \ldots, s s_{d-1,0}, s s_{d-1,1}, s_{d,0}.\}$$

The rational map given by this basis is everywhere defined outside E and E is mapped entirely onto the point $p = (0, \ldots, 1)$. Clearly it gives a biregular morphism of $S \setminus E$ onto $S' \setminus p$. One only needs to show that p is smooth on S'.

Let $U \subset S$ be an open neighbourhood of E where $s_{d,0} \neq 0$. Define the following sections of $\mathcal{O}_U(-E)$:

$$x = \frac{s_{d-1,0}}{s_{d,0}}, \quad y = \frac{s_{d-1,1}}{s_{d,0}}.$$

These restrict to a basis for $H^0(\mathcal{O}_E(1))$ and so I may assume that U is small enough so that x and y are not simultaneously zero on U and so one gets a morphism $z_2 : U \to \mathbb{P}^1$. The functions sx, sy define a morphism $z_1 : U \to \mathbb{C}^2$ and together with z_2 even a morphism $(z_1, z_2) : U \to \mathbb{C}^2 \times \mathbb{P}^1$ which in fact maps to the blow up $\widetilde{\mathbb{C}^2}$ of $\mathbb{C}^2$ considered as submanifold of $\mathbb{C}^2 \times \mathbb{P}^1$. Summarising, one gets a morphism

$$z : U \longrightarrow \widetilde{\mathbb{C}^2},$$

which by construction, fits into the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{z} & \widetilde{\mathbb{C}^2} \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} \\ \sigma(U) & \xrightarrow{\tilde{z}} & \mathbb{C}^2 \end{array}$$

where τ is the blowing up of $\mathbb{C}^2$ in the origin. Note that $\sigma(U)$ is open in S'.

Let me show that, replacing U by a smaller open neighbourhood of E (in the complex topology) the map $\tilde{z}$ is an isomorphism from $\sigma(U)$ onto its image which is an open neighbourhood of the origin in $\mathbb{C}^2$ and thus p is a smooth point.

First of all, by construction $z|E$ maps E isomorphically on E. Furthermore, z is a local isomorphism around each point of E (in the complex topology). Indeed, let (u, v) be coordinates on $\mathbb{C}^2$ and let (U, V) be homogeneous coordinates on $\mathbb{P}^1$. Then $\widetilde{\mathbb{C}^2}$ is the submanifold given by $uV = vU$. If $q \in E$, one may assume that $z(q) = ((0,1),(0,1)) \in \mathbb{C}^2 \times \mathbb{P}^1$. Now at $z(q)$ the functions $v, U/V$ are local coordinates on $\widetilde{\mathbb{C}}^2$. One has

$$z^*v = sy, \quad z^*(U/V) = x/y.$$

The first function vanishes with multiplicity 1 along E while the second function gives a local coordinate at q. This implies that in a neighbourhood of q where both functions are defined the map z is a local isomorphism.

Now one invokes an easy lemma from topology.

Lemma *Let $f : X \to Y$ be a continuous map between Hausdorff spaces which restricts to a homeomorphism $f|K \to f(K)$ on a compact set K and which is a local homeomorphism in the neighbourhood of each point of K. Then there exists an open neighbourhood of K which is mapped homeomorphically by f onto an open subset of Y.*

The proof of this lemma is left to the reader.

From the lemma one finds an open neigbourhood U of E which maps isomorphically to $W \subset \widetilde{\mathbb{C}^n}$. Consider the morphism $\sigma \circ z^{-1}|W$. It contracts the exceptional curve in W and hence, by Remark 7 it factors as $g \circ \tau$, where g is a morphism which must be the inverse of $\tilde{z}$. So $\sigma(U)$ is isomorphic to the open neighbourhood $\tau(W)$ of the origin in $\mathbb{C}^2$.

∎

Corollary 12. *A surface is minimal if and only if it does not contain (-1)-curves.*

I end this section with a few remarks which are intended to illustrate the point of view of birational geometry since Mori theory came into existence.

As demonstrated previously, for surfaces there always exists some minimal model in the birational equivalence class of a given surface. In principle there could be many minimal models. It turns out that, with the exception of the ruled surfaces there is a unique minimal model up to isomorphism. By definition a *ruled surface* is any surface which admits a birational map onto $C \times \mathbb{P}^1$ with C a curve, so these are known from a birational point of view. However one still needs to know the distinct minimal ruled surfaces. This shall be done in the next sections.

In higher dimensions there need not exist a smooth minimal model. When the concept of minimal model is suitably modified, in order to have such a model one necessarily has to allow singularities in codimension ≥ 3. It turns out that you can only expect a minimal model if K, the canonical divisor is *nef* which means that K intersects non-negatively with any curve. Mori theory also shows that there is a basic distinction between the case K nef and K not nef. I shall illustrate this for surfaces.

Proposition 13. *If there exists a curve C on S with $(K_S, C) < 0$ and $(C, C) \geq 0$, all plurigenera of S are zero. If S is a surface with at least one non-vanishing plurigenus and C is a curve on S with $(K_S, C) < 0$, the curve C is an exceptional curve of the first kind, i.e. C is a smooth rational curve with $(C, C) = -1$.*

Proof: Let D be an effective pluricanonical divisor. Write it like $D = aC + R$. Since $(D, C) < 0$ the divisor D actually contains C, i.e. $a > 0$. Then $0 > m(K_S, C) = (D, C) = a(C, C) + (R, C) \geq a(C, C)$. Since this is ≥ 0 in the first case, one arrives at a contradiction: the plurigenera must all vanish. In the second case, if $(K_S, C) \leq -2$ the adjunction formula gives $(C, C) \geq 0$ and we again have a contradiction. So $(K_S, C) = -1$ and the adjunction formula shows that C is an exceptional curve of the first kind. ■

Recall that the Kodaira-dimension $\kappa(S)$ of S is equal to $-\infty$ means that all plurigenera of S vanish. This is for instance the case for rational and ruled surfaces as will be shown in the next section. So using the notion of nef-ness and Kodaira-dimension there is a reformulation à la Mori for the previous Proposition.

Reformulation 14. *Suppose S is a surface whose canonical bundle is not nef. Then either S is not minimal or $\kappa(S) = -\infty$.*

Let me give a second illustration of the Mori-point of view with regards to the question of uniqueness of the minimal model.

Proposition 15. *Let S and S' be two surfaces and let $f : S \dashrightarrow S'$ be a birational map. If $K_{S'}$ is nef, f is a morphism. If moreover K_S is nef, f is an isomorphism.*

Proof: Let $\sigma : \tilde{X} \to X$ be the blow up of any surface with exceptional curve E and let $\tilde{C} \subset \tilde{X}$ an irreducible curve such that $C := \sigma(\tilde{C})$ is again a curve, one has

$$(K_{\tilde{X}}, \tilde{C}) = (\sigma^* K_X + E, \sigma^* C - mE) = (K_X, C) + m \geq (K_X, C).$$

So if K_X is nef there can be no curve $\tilde{C}$ on $\tilde{X}$ mapping to a curve on X and for which $(K_{\tilde{X}}, \tilde{C}) \leq -1$. Since any morphism is composed of blowings up this then also holds for an arbitrary morphism $X' \to X$.

Let me apply this in the present situation with $X = S'$. Resolve the points of indeterminacy of f. Choose a resolution where you need the minimal number of blowings up. One may suppose that one needs at least one blow up. Then the image $C = f(E)$ of the exceptional curve E of the last blow up must be a curve, which contradicts the preceding since $(K, E) = -1$ on the last blown up surface. So f is a morphism. Similarly, if K_S is nef, f^{-1} is a morphism and so f is an isomorphism. ■

Problems.

6.1. Prove that the blow up of an algebraic surface is again algebraic. Hint: define the blow-up $\widetilde{\mathbb{P}^n}$ of the point $p = (0, \ldots, 1)$ in $\mathbb{P}^n$ as a subvariety of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ given by the bihomogeneous equations $X_i Y_j - Y_j X_i = 0$, $i, j = 0, \ldots, n-1$. Next one shows that the Segre-embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ identifies $\mathbb{P}^n \times \mathbb{P}^m$ with a projective submanifold of $\mathbb{P}^{nm+n+m}$. So $\mathbb{P}^n \times \mathbb{P}^m$ is a projective manifold. Next, every subset of $\mathbb{P}^n \times \mathbb{P}^m$ given by bihomogeneous equations can be seen to correspond to a projective subvariety of $\mathbb{P}^{nm+n+m}$. Show that the blow-up of a surface $S \subset \mathbb{P}^n$ in p is the Zariski-closure of $S \setminus \{p\}$ in $\widetilde{\mathbb{P}^n}$.

7. Ruled and rational surfaces

The minimal models of non-rational ruled surfaces are geometrically ruled. These are always projective bundles associated to rank 2 vector bundles. The minimal models for rational surfaces are the Hirzebruch surfaces $\mathbb{F}_n$, $n \neq 1$.

Definition 1.

1. *A surface S is called a ruled surface if it is birationally isomorphic to $C \times \mathbb{P}^1$ where C is a smooth curve. If $C = \mathbb{P}^1$ one calls S rational.*

2. *A surface S is called geometrically ruled if there is a morphism $p : S \to C$ of maximal rank onto a smooth curve with fibres $\mathbb{P}^1$.*

Two remarks are in order. First, a surface is rational if and only if it is isomorphic to $\mathbb{P}^2$ since, as shown in section 7, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birationally isomorphic. Secondly, it is by no means clear that a geometrically ruled surface is actually ruled. This however is the case. In fact a little more is true.

Proposition 2. *If $f : S \to C$ is a surjective morphism of a surface S onto a curve C for which $c \in C$ is a regular value and whose fibre at c is isomorphic to $\mathbb{P}^1$, then there is a Zariski-open neighbourhood U of c in C such that $f^{-1}U$ is isomorphic to $U \times \mathbb{P}^1$ in a fibre preserving manner.*

Proof:
Step 1. $H^2(S, \mathcal{O}) = 0$.

Serre-duality implies that $H^2(S, \mathcal{O})$ is dual to $H^0(S, K_S)$. Suppose that this would be non-zero, i.e. that there would be an effective canonical divisor K. Let F be the fibre over c. Since $(F, F) = 0$ (a nearby fibre F' is linearly equivalent to F and so $(F, F) = (F, F') = 0$) the genus formula yields

$$-2 = (K, F) + (F, F) = (K, F).$$

If however K is effective you can write it as $K = nF + G$ where $n \geq 0$ and G is disjoint from F and so $(K, F) = (G, F) \geq 0$.

Step 2. Construction of a divisor H with $(H, F) = 1$.

Now observe that the exponential sequence and Step 1. yields a surjection

$$\mathrm{Pic}(S) \xrightarrow{c_1} H^2(S, \mathbb{Z})$$

and therefore it is sufficient to produce a cohomology class h with $h \cdot f = 1$. Here $f = c_1(F)$ and the product on $H^2(S, \mathbb{Z})$ is the one coming from cup-product. (Remark 5.3). Now by Poincaré-duality the cup-product pairing is a perfect pairing on $\mathbb{H} := H^2(S, \mathbb{Z})/\mathrm{Torsion}$. This means that the natural map

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathrm{Hom}(\mathbb{H}, \mathbb{Z}), \\ x &\longmapsto \{y \mapsto y \cdot x\} \end{aligned}$$

is an isomorphism. The numbers $x \cdot f$ form an ideal in $\mathbb{Z}$, say (d) and so the linear functional $x \mapsto 1/d(x \cdot f)$ must be of the form $x \mapsto (x \cdot f')$ for some $f' \in \mathbb{H}$ for which one then has $f = df'$. (The element f' is called *primitive* and this shows that if one writes $x = nx'$ with $x' \in \mathbb{H}$ and n as large as possible, the resulting x' is primitive).

I claim that in our case $d = 1$. Look at $k = c_1(K)$. Since $f \cdot k = -2$ as shown before, one must have $f' \cdot k = -2/d$. Now by the genus formula $f(x) := x \cdot x + x \cdot k \pmod 2 = 0$ for x the class of an irreducible curve and hence, since $f(x)$ is linear, this is true for all of $\mathbb{H}$. In particular $-2/d$ must be even and so $d = 1$. But now f is primitive and so there exists some $h \in \mathbb{H}$ with $h \cdot f = 1$.

Step 3. End of proof.

Consider the exact sequence

$$0 \to \mathcal{O}_S(H + (r-1)F) \to \mathcal{O}_S(H + rF) \to \mathcal{O}_F(1) \to 0,$$

which in cohomology gives

$$H^0(\mathcal{O}_S(H + rF)) \xrightarrow{a_r} H^0(\mathcal{O}_F(1)) \to H^1(\mathcal{O}_S(H + (r-1)F)) \xrightarrow{b_r} H^1(\mathcal{O}_S(H + rF)) \to 0.$$

The map b_r being surjective for all r means that the dimensions of $H^1(\mathcal{O}_S(H + rF))$ form a decreasing sequence which eventually must become stable and then b_r becomes an isomorphism and hence a_r will be a surjection. For such an r pick a plane inside $H^0(\mathcal{O}_S(H+rF))$ which is mapped isomorphically onto $H^0(F, \mathcal{O}(1))$ by a_r and let P be the corresponding pencil. This pencil will separate points on F and so the possible fixed locus of P consists of points in fibres distinct from F or of curves in fibres disjoint from F. Let me take away all these fibres as well as the reducible fibres and look at the restriction P' of the pencil on this part of the surface, which is of the form $f^{-1}U$ with $U \subset C$ Zariski open. Now a generic member C_t of the moving part P' of P meets F in exactly one point, so if it would be reducible, it would contain some fibers. This however is not possible since then C_s with $s \neq t$ would meet C_t in the intersection points of these fibres and so they would be base points. It follows that P' is a pencil entirely consisting of sections of the restricted fibration. The pencil P' defines a morphism $g : f^{-1}U \to \mathbb{P}^1$ with fibres C_t meeting the fibres of $f^{-1}U \to U$ in exactly one point and so $(f, g) : f^{-1}U \to U \times \mathbb{P}^1$ is the desired fibre preserving isomorphism. ■

Let me next relate the geometrically ruled surfaces $S \to C$ to rank two vector bundles on C. If E is such a bundle you can replace every fibre E_x over $x \in C$ by the corresponding projective line. In this way you obtain $\mathbb{P}(E)$, a $\mathbb{P}^1$-bundle over C.

Proposition 3. *Every geometrically ruled surface $S \to C$ is isomorphic to the $\mathbb{P}^1$-bundle associated to some rank two vector bundle E on C. Two bundles $\mathbb{P}(E)$ and $\mathbb{P}(E')$ are isomorphic if and only if $E' \cong E \otimes L$ for some line bundle L on C.*

Proof: Recall that isomorphism classes of line bundles on C are classified by the set $H^1(\mathcal{O}^*)$. One can introduce the sheaf $Gl(2, \mathcal{O}_C)$ of invertible 2×2-matrices with coefficients in $\mathcal{O}_C$ and the quotient sheaf $PGl(2, \mathcal{O}_C)$. The set $H^1(Gl(2, \mathcal{O}_C))$ classifies the rank 2-bundles on C and $H^1(PGl(2, \mathcal{O}_C))$ the $\mathbb{P}^1$-bundles. The defining exact sequence

$$1 \to \mathcal{O}_C^* \to Gl(2, \mathcal{O}_C) \to PGl(2, \mathcal{O}_C) \to 1$$

in cohomology gives an 'exact sequence of sets'.

$$H^1(\mathcal{O}_C^*) \xrightarrow{a} H^1(Gl(2, \mathcal{O}_C)) \xrightarrow{p} H^1(PGl(2, \mathcal{O}_C)) \to H^2(\mathcal{O}_C^*).$$

This means that $p(e) = p(e')$ if and only if $a(l) \cdot e = e'$ for some $l \in H^1(\mathcal{O}_C^*)$, where $H^1(\mathcal{O}_C^*)$ acts on the set of rank two bundles by the tensor product. The result follows if one can show that $H^2(\mathcal{O}^*) = 0$.This however follows immediately from the exponential sequence on C. ■

Let me now show that the minimal models of the non-rational ruled surfaces are exactly the geometrically ruled surfaces. First note a simple lemma from topology whose proof is left to the reader.

Lemma 4. *Let $f : S \to C$ be a surjective morphism of a surface onto a smooth curve. Assume that the smooth fibres are all connected. Then all fibres are connected.*

Proposition 5. *Let S be without (-1)-curves and let $f : S \to C$ be a morphism onto a smooth curve C such that the generic curve of f is $\mathbb{P}^1$. Then $f : S \to C$ gives S the structure of a geometrically ruled surface.*

Proof: As in previous arguments, for every fibre F one has $(F,K) = -2$ and hence no fibre can be multiple, i.e. $c_1(F)$ is primitive. So all irreducible fibres have genus 0 which, by Lemma 5.7 implies that they are all $\mathbb{P}^1$. All you have to do now is to to rule out the possibility of reducible fibres, since then Proposition 2 can be applied. So suppose that $F = \sum_i n_i C_i$ is a reducible fibre. Now compute $n_i C_i^2 = (C_i, F - \sum_{j\neq i} n_j C_j) = -\sum_{j\neq i} n_j (C_j, C_i) < 0$ since C_i meets at least one C_j (the fibre F is connected by the previous lemma). But then C_i has negative self intersection and since $(K, C_i) + (C_i, C_i) = 2g(C_i) - 2$ one concludes that $(K, C_i) \geq -1$ with equality if and only if C_i is a smooth rational curve with self intersection -1, i.e. a (-1)-curve. But these don't exist on S and so $(K, C_i) \geq 0$ and $(K, F) \geq 0$ whereas $(K, F) = -2$. This contradiction shows that there are no reducible fibres present and therefore the proof is complete. ■

Corollary 6. *A minimal model of a non-rational ruled surface is geometrically ruled.*

Proof: Let S be minimal and let $S \dashrightarrow C \times \mathbb{P}^1$ be birational and consider the resulting rational map $S \to C$. It necessarily is a morphism, because otherwise one would have to blow up at least once to eliminate points of indeterminacy and such an exceptional curve would have to be mapped to a point on C (since C is not rational). But by the 'easy' universal property for blowing up (Remark 6.7) one can 'factor out' the σ-process for the exceptional curve without creating points of indeterminacy.

Since the generic fibre of $S \to C$ is $\mathbb{P}^1$ and since S does not contain (-1)-curves, the result follows from the preceding Proposition.

■

Next topic: the rational geometrically ruled surfaces. For this, one needs Grothendieck's result on the splitting of vector bundles on $\mathbb{P}^1$.

Lemma 7. *Every vector bundle on $\mathbb{P}^1$ is the direct sum of line bundles.*

Proof: Let me first consider the question: 'when does an exact sequence of vector bundles (on any manifold) split?'. So let

$$0 \to V' \xrightarrow{a} V \xrightarrow{b} V'' \to 0$$

be an exact sequence of vector bundles. It splits by definition, if there is a subbundle of V which by b is mapped isomorphically onto V''. Equivalently, there should exist a homomorphism $c : V'' \to V$ such that $b \circ c = \mathrm{Id}_{V''}$. To put this into the language of exact sequences, note that applying $\mathrm{Hom}(V'', -)$ to the preceding exact sequence yields an exact sequence of vector bundles

$$0 \to \mathrm{Hom}(V'', V') \xrightarrow{a^*} \mathrm{Hom}(V'', V) \xrightarrow{b^*} \mathrm{Hom}(V'', V'') \to 0,$$

with e.g. $b^*(c) = b \cdot c$. Now $\mathrm{Id}_{V''}$ is a global section of $\mathrm{Hom}(V'', V'')$ and the splitting is equivalent to the existence of a global section c of $\mathrm{Hom}(V'', V)$ with $b^*(c) = \mathrm{Id}_{V''}$. Looking at the exact sequence in cohomology, one sees that it suffices that $H^1(\mathrm{Hom}(V'', V')) = 0$.

In this case one applies this to the following situation. Consider a vector bundle E and fix $k \in \mathbb{Z}$ such that the bundle $E(k')$ has no sections for $k' < k$ but does have a section for $k' = k$. This section has no zeroes, otherwise some $E(k')$ with $k' < k$ would have had a section. But then the section defines a trivial sub line bundle of $E(k)$ and hence an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to E(k) \to F \to 0.$$

Now you twist this sequence by $\mathcal{O}(-1)$ and consider the resulting exact sequence. Since $H^0(E(k-1)) = 0$, the space $H^0(F(-1))$ goes injectively in $H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. So $H^0(F(-1))$ and hence also $H^0(F(-2))$ must vanish. Serre duality then gives that $0 = H^1(F^\vee) = H^1(\mathrm{Hom}(F, \mathcal{O}))$. By the previous considerations, the preceding sequence splits and so by induction E splits as a direct sum of line bundles.

■

By twisting by $\mathcal{O}_{\mathbb{P}^1}(k)$ one can always normalize a $\mathbb{P}^1$-bundle on $\mathbb{P}^1$, say $\mathbb{P}(F' \oplus F'')$ in such a way that F' becomes trivial. Then, upon writing $F'' = \mathcal{O}_{\mathbb{P}^1}(n)$ you arrive at the definition of the Hirzebruch surfaces

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)).$$

(Miles Reid suggests that the 'F' might stand for 'Fritz', Hirzebruch's first name).

Let me summarise the discussion so far in the form of a proposition.

Proposition 8. *A geometrically ruled surface over $\mathbb{P}^1$ is a Hirzebruch surface.*

Let me conclude this chapter by computing the invariants for the geometrically ruled surfaces. Let me first recall that $\mathbb{P}(E)$ always admits a divisor H with $(H, F) = 1$ (cf. proof of Proposition 2). Let us set $h = c_1(H)$.

Proposition 9. *Let $\varphi : S \to C$ be a geometrically ruled surface and let g be the genus of C. Then*

1. *$H^2(S, \mathbb{Z})$ is generated by the class f of a fibre and the class h. In fact* $\mathrm{Pic}(S) = f^* \mathrm{Pic}(S) \oplus \mathbb{Z} \cdot H$.
2. *$c_1(K_S) = -2h + (2g - 2 + d)f$, where $d = h \cdot h$. In particular $(K, K) = 8(1 - g)$.*

Proof:
1. First look at the Picard group. It suffices to show that divisors D on S with $(D, F) = 0$ are of the form $\varphi^* d$ with $d \in \mathrm{Pic}\, C$. If D would be effective this follows immediately, so it suffices to show that $D_n := D + nF$ is effective for n sufficiently large. Now $(D_n, D_n) = (D, D)$ while $(D_n, K_S) = (D, K) - 2n$, so $h^0(\mathcal{O}_S(D_n)) + h^0(\mathcal{O}_S(K_S - D_n)) \geq n + \text{constant}$ by the Riemann-Roch inequality. Since $h^0(\mathcal{O}_S(K_S - D_n)) = 0$ for large n (look at the degree of the divisor $K_S - D_n$ with respect to some very ample divisor on S) one indeed gets a section in $h^0(\mathcal{O}_S(D_n))$ for n large enough.

For the cohomology group, simply look at the exponential sequence by which $H^2(S, \mathbb{Z})$ is a quotient of $\mathrm{Pic}(S)$ and so is generated by f (all fibres are cohomologically equivalent) and h which are independent since $h \cdot f = 1$.

2. Write $c_1(K_S) = af + bh$ and intersect with f to get $b = -2$. Then one finds $(K_S, H) = a - 2d$ while the genus formula gives $2g - 2 = ((K_S, H)) + (H, H) = a - d$. ■

To compute the other invariants one needs

Proposition 10. *q, p_g and $P_n := \dim H^0(K_S^{\otimes n})$ are birational invariants.*

Proof: Let me give the proof for holomorphic 2-forms. The other cases are similar. So let $f : S \dashrightarrow S'$ be a birational map. f is a morphism outside a finite set F. So, if α is a holomorphic 2-form, $f^*\alpha$ is a rational 2-form and regular on $S \setminus F$. But then it is regular on S, since $f^*\alpha$ has poles in divisors at most. It follows that pulling back gives an injection

$$H^0(S', K_{S'}) \hookrightarrow H^0(S, K_S).$$

The inverse of f then yields an inverse to this map and so this is an isomorphism. ■

It follows that one can use the model $C \times \mathbb{P}^1$ to compute the invariants q, p_g and P_n. One has $\Omega^1_{C\times\mathbb{P}^1} = p^*\Omega^1_C \oplus q^*\Omega^1_{\mathbb{P}^1}$ (here p and q denote the projections onto the factors) and hence $K_{C\times\mathbb{P}^1} \cong p^*K_C \otimes q^*K_{\mathbb{P}^1}$. It then follows (see Problem 3.7) that $h^0(\Omega^1_{C\times\mathbb{P}^1}) = h^0(\Omega^1_C) + h^0(\Omega^1_{\mathbb{P}^1}) = g$, the genus of C and that $h^0(K_{C\times\mathbb{P}^1}) = h^0(K_C) \cdot h^0(K_{\mathbb{P}^1}) = 0$ and similarly one finds that $P_n = 0$. Summarising, you get

Lemma 11. *For a ruled surface S birationally isomorphic to $C \times \mathbb{P}^1$ one has $p_g = P_n = 0$ and $q = g(C)$.*

Next, let me turn to the invariants of the Hirzebruch surfaces. Observe that there are two natural types of sections of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, the unique section $\hat{S}$ which in each fibre assumes the value $(0, 1)$ and the sections $S = S_p$ of the form $s(x) = (1, p(x))$ where p is a homogeneous polynomial of degree n. Clearly $(S, S) = n$ and one would like to compute $\hat{S}$ in terms of S and F. One hase $\hat{S} = S + aF$ since $\hat{S}$ is a section. Moreover $(S, \hat{S}) = 0$ since the two sections never meet. So $0 = (S, \hat{S}) = n + a$ and hence $a = -n$. It follows that $(\hat{S}, \hat{S}) = -n$. I claim that $\hat{S}$ is the unique curve with strictly negative self intersection. Indeed, if $cF + dS \equiv D \neq \hat{S}$ one hase $c = (D, \hat{S}) \geq 0$ and also $d = (D, F) \geq 0$ and so $(D, D) = d^2n + 2cd \geq 0$. It follows that the $\mathbb{F}_n$ with $n > 0$ are mutually non-isomorphic; they are distinguished by the self intersection number of the unique curve on them with negative self intersection. Since $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ any curve on this surface has non-negative self intersection and so $\mathbb{F}_0$ is not isomorphic to $\mathbb{F}_n$ with $n > 0$. Finally, $\mathbb{F}_1$ is non-minimal, since the section with negative self intersection is an exceptional curve. The other $\mathbb{F}_n$ are minimal since they do not contain an exceptional curve. Summarizing one has

Proposition 12. *The Hirzebruch surfaces $\mathbb{F}_n$, $n \neq 1$ are minimal and pairwise non-isomorphic.*

Using Proposition 7.8 one finds

Corollary 13. *The Hirzebruch surfaces $\mathbb{F}_n$, $n \neq 1$ are precisely the minimal geometrically ruled surfaces with $q = 0$.*

Problems.

7.1. Prove that $\mathbb{F}_1$ is the projective plane blown up in a point.

7.2. Give a direct proof for the fact that every algebraic line bundle on an affine curve is trivial.

7.3. The Hirzebruch surface $\mathbb{F}_n$ is a non-singular model of the cone on the rational normal curve in $\mathbb{P}^{n+1}$. Prove this by considering the linear system $|s| = |f^*\mathcal{O}_{\mathbb{P}^1}(n)|$ on $\mathbb{F}_n$.

7.4. With the linear system $|S+kF|$ with $k \geq 1$ one embeds $\mathbb{F}_n$ in $\mathbb{P}^{n+2k+1}$ as a surface of degree $n+2k$. The fibres map to straight lines, the unique section with negative self intersection to a rational normal curve in $\mathbb{P}^k \subset \mathbb{P}^{n+2k+1}$ and the sections S_p are mapped to rational normal curves in a linear subspace (depending on p) of dimension $n+k$.
Conversely, if one starts with two disjunct subspaces of $\mathbb{P}^{n+2k+1}$ of dimensions k and $n+k$, take rational normal curves C and C' in these spaces, choose an isomorphism $u : C \to C'$ and joint $u \in C$ and $u' \in C$ by a straight line. The resulting surface is isomorphic to $\mathbb{F}_n$ embedded by means of $|S+kF|$.

Chapter 4. More advanced tools from algebraic geometry

8. Normalisation and Stein factorisation

Normalisation is a cruder process than desingularisation and is easy to describe algebraically. Zariski's main theorem is proved and, finally, the Stein factorisation theorem for projective morphisms. If the target space is normal much more can be said and this will be used later on.

In Appendix A1.3 I have gathered some fundamental properties of normal rings which are used freely in what follows.

A variety X is *normal* at $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is normal. X is called normal if it is normal at every point.

Example 1.

1. Any smooth point is normal. If one uses the fact that the local ring is a unique factorisation domain (in the analytic case, see [Gr-Re,Chapter 2§2], in the algebraic case see [Mu, §1B.]) this is easy. Suppose that a rational function P/Q satisfies an equation
$$(P/Q)^n + A_{n-1}(P/Q)^{n-1} + \ldots + A_0 = 0, \quad A_i \in \mathcal{O}_{X,x}$$
you multiply with P^n to see that Q divides P^n. Since you may assume that P and Q have no common factor in $\mathcal{O}_{X,x}$ it follows that Q in fact must be a unit in $\mathcal{O}_{X,x}$ and so $P/Q \in \mathcal{O}_{X,x}$.
2. If X is a reducible hypersurface at x, the point x is not normal. Indeed, if f and g are local equations of two distinct hypersurfaces at x making up X, one can introduce the function $h := f/(f+g)$. It cannot be holomorphic along X since it would be identically zero along one component and identically 1 along the other component. By assumption $fg = 0$ along X and so
$$h^2 - h = \frac{f^2 - f(f+g)}{(f+g)^2} = 0.$$
This shows that the meromorphic function h is integral over $\mathcal{O}_{X,x}$.
3. A curve is normal if and only if it is smooth. In the algebraic setting a short proof can be found in [Ii, §2.7]. In the complex-analytic setting this follows from a more general fact, namely that the set of normal points form a subvariety of codimension two or more [Gr-Re, Chapter 6 §5].

Definition 2. *Let X be a variety. A pair (X', f) consisting of a normal variety X' and a morphism $f : X' \to X$ is called a normalisation of X if f is finite and birational.*

Theorem 3. *For any affine resp. projective algebraic variety X the normalisation exists as an affine resp. projective algebraic variety. It is unique in the following sence. If $f'' : X'' \to X$ is another normalisation, there exists an isomorphism $\iota : X' \to X''$ with $f'' \circ \iota = f'$.*

Proof:
Step 1. The affine case.

Since the coordinate ring $R(X)$ of X is Noetherian its integral closure R' in the field of rational functions on X is a finitely generated $R(X)$-module, and so $R' = \mathbb{C}[X_1, \ldots, X_n]/I'$. The ideal I' defines a variety X' and the embedding $R[X] \hookrightarrow R'$ defines a morphism $X' \to X$. The map is finite by definition and clearly birational since $R[X]$ and R' have the same quotient field.

The uniqueness is obvious: both X' and X'' correspond to the integral closure of $R[X]$ in the function field $\mathbb{C}[X]$. Here you use that X'' is normal if and only if its coordinate ring is normal. This folows from the fact that an integral domain is normal if and only if the localisations in all maximal ideals are normal.

Step 2. X is a projective subvariety of $\mathbb{P}^n$.

Now one uses the homogeneous coordinate ring $R[X]$ of X and forms its integral closure R' in $\mathbb{C}(X)$ which is of the form $\mathbb{C}[X_0, \ldots, X_n]/I'$ where I' is a homogeneous ideal which defines a normal projective variety X'. Again, the inclusion $R[X] \hookrightarrow R'$ defines a morphism $X' \to X$. I claim that in the standard affine pieces $U_i = \{(X_0, \ldots, X_n)\ ;\ X_i \neq 0\}$ this variety is just the normalisation constructed in Step 1. To see this, let $X_i := X \cap U_i$ with coordinate ring R_i and similarly we define $X'_i := X' \cap U_i$ with ring R'_i. The localisation of $R[X]$ in X_i is isomorphic to R_i ('making inhomogeneous'). Since localisation and normalisation commute, the ring R'_i is just the normalisation of the ring R_i. By uniqueness of the normalisation in the affine case it then follows that X'_i is the normalisation of X_i. In particular, the map $X' \to X$ is finite as in the affine case. The uniqueness also follows from the fact that X' is a union of affine normalisations for which one has uniqueness.
■

Next, I want to give a simple proof of Zariski's Main Theorem, using however the complex topology. First I need

Proposition 4. *Let $f : X \to Y$ be a morphism between projective varieties such that the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. Then the fibres of f are connected and non-empty. Conversely, if f is surjective, Y is normal and the fibres are connected, one has an isomorphism $\mathcal{O}_Y \xrightarrow{\sim} f_*\mathcal{O}_X$.*

Proof: Assume that the fibre of f above $y \in Y$ is not connected. Since f is proper you can find a neighbourhood V of y (in the complex topology) such that $f^{-1}V$ is not connected. To see this, suppose that a fiber F is disconnected, say $F = A \cup B$, where A is connected, A and B disjoint. Then, since A and B are compact, there is a neighborhood of F of the form $W = U \cup V$, U neigborhood of A and V of B, whilst U and V are still disjoint. Again by compactness, all points in X sufficiently close to F belong to the given neighborhood W. Now, since both X and Y are compact metric spaces, distances between points on X and their images can be compared uniformly. So the inverse image of a sufficiently small ball centered at $y = f(F)$ must be contained in W, and so cannot be connected. But then the canonical map

$$\mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y$$

cannot be surjective. If the fibre at y would be empty, this map would not be injective.

For the converse, let $V \ni y$ be open and connected in the complex topology and let $g \in \mathcal{O}(f^{-1}V)$ be bounded. The function g has the same value on each smooth fibre $f^{-1}y$, since such a fibre is connected. So there is a bounded continuous function h' on an open dense subset V' of V such that $g = h' \circ f$ on $f^{-1}V'$. You can take V' to be the set of the smooth points of V over which f has maximal rank. This means that every point in $f^{-1}V'$ has a neighbourhood of the form $U \times V''$ with f the projection onto the second factor. But then, in V'' the function h' is holomorphic and so h is holomorphic in V' entirely. In view of the normality of V, one can extend the bounded function h' to a holomorphic function h on V (see [Gr-Re, Chapter 7 §4.2]) and so the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism at $y \in Y$. ∎

Corollary 5. *Suppose that $f : X \to Y$ is a surjective morphism between projective varieties, that Y is normal and that the fibres of f are connected. For any line bundle $\mathcal{L}$ on Y there is a natural isomorphism*

$$f^* : \Gamma(Y, \mathcal{L}) \longrightarrow \Gamma(X, f^*\mathcal{L})$$

given by $f^*(t) = t \circ f$.

Proof: The map f^* as given above is clearly injective. To see that it is surjective, invoke the following special case of the Projection Formula.

$$f_*f^*\mathcal{L} \cong \mathcal{L} \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X.$$

Since $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, there is a canonical isomorphism

$$f_*f^*\mathcal{L} \cong \mathcal{L}.$$

This holds in particular for the global sections so that

$$\dim \Gamma(X, f^*\mathcal{L}) = \dim \Gamma(Y, f_*f^*\mathcal{L}) = \dim \Gamma(Y, \mathcal{L}).$$

It follows that f^* must be surjective. ∎

Corollary 6. (Zariski's Main Theorem) *Let Y be a normal projective variety and let $f : X \to Y$ be a birational morphism. Then f has connected fibres.*

Proof: By the preceding proposition, one only has to verify that $f_*\mathcal{O}_X = \mathcal{O}_Y$. The question is local and so one may assume that Y is affine with coordinate ring $A := R[Y]$ and $B := \Gamma(f_*\mathcal{O}_X)$ is a finitely generated A-module (since $f_*\mathcal{O}_X$ is coherent). Both A and B are integral domains with the same field of fractions $\mathbb{C}(Y)$ and A is integrally closed. So $A = B$ and thus $f_*\mathcal{O}_X = \mathcal{O}_Y$. ∎

Let me apply this to the situation of general rational map between projective varieties $f : X \dashrightarrow Y$. Let $\Gamma_f \subset X \times Y$ be the closure of the graph and let $p : \Gamma_f \to X$ and $q : \Gamma_f \to Y$ be the two projections.

Corollary 7. *Let $f : X \dashrightarrow Y$ be a rational map between projective varieties. If X is normal, for every $x \in X$ the set $f(x) := q(p^{-1}x)$ is connected.*

Proof: The projection p is a birational morphism to which Zariski's main theorem can be applied. So $p^{-1}(x)$ and hence its image $f(x)$ is connected. ■

Corollary 8. *Let $f : X \to Y$ be a morphism between projective varieties and let (X', i), resp. (Y', j) be the normalisation of X, resp. Y. There exists a morphism $f' : X' \to Y'$ such that $j \circ f' = f \circ i$.*

Proof: The morphism j is birational, let j^{-1} be its inverse and define $f' = j^{-1} \circ f \circ i$. Clearly, $j^{-1}(f \circ i)(x)$ is a finite set of points and so $f'(x) \subset j^{-1}(f \circ i)(x)$ is finite as well. But it is connected by the previous Corollary and so it consists of one point, i.e. f' is a morphism. ■

Theorem 9. (Stein factorisation) *Let $f : X \to Y$ be a surjective morphism between projective varieties. There exists a variety Y', a finite surjective morphism $g : Y' \to Y$ and a morphism $f' : X \to Y'$ with connected fibres such that the following diagram is commutative.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f'"'] & & Y \\ & Y'. \arrow[ur, "g"'] & \end{tikzcd}$$

If X is normal, then so is Y'.

Proof:

Let Y be projective and let $L := \mathbb{C}(X)$, a finite extension of $\mathbb{C}(Y)$. As in the proof of the existence of a normalisation, one finds a projective variety Y' with $\mathbb{C}(Y')$ equal to the algebraic closure of $\mathbb{C}(Y)$ in L and a finite morphism $Y' \to Y$. The variety Y' is in fact constructed from open affine pieces V' with coordinate ring equal to the integral closure of $\mathcal{O}(f^{-1}V)$ in $\mathbb{C}(Y)$ where V is an open affine piece of Y. In particular, $\mathcal{O}(Y')$ is canonically isomorphic to $f_*\mathcal{O}_X$. This is a coherent sheaf of $\mathcal{O}_Y$-algebras and so the natural morphism $g : Y' \to Y$ is finite. By construction $f = g \circ f'$ and $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$. So the fibres of f are connected by the previous proposition. Finally, if X is normal, then so is Y'. ■

Corollary 10. *Let X and Y be projective varieties and $f : X \to Y$ a morphism. If Y is normal and the generic fibre of f is connected, then so is every fibre.*

Proof: The map g appearing in the Stein normalisation must have degree one in this case and hence must be birational. By Zariski's Main Theorem all fibres must consist of one point and so g is an isomorphism and the map f must have connected fibres. ■

9. Kodaira-dimensions

Any coarse classification proceeds according to the Kodaira-dimension. Its characterisation by a certain growth-behaviour is essential as well as the fact that the Kodaira dimension does not change under finite unramified coverings. Full proofs of these facts are given,

In this section X is a normal projective variety and $\mathcal{L} = \mathcal{O}_X(D)$ with D an effective divisor. Associated to D there is the ring

$$R(X,D) := \oplus_{k \geq 0} H^0(X, \mathcal{L}^{\otimes k})$$

and its homogeneous field of fractions

$$Q(X,D) := \{\frac{s}{t}\ ;\ s,t \in H^0(X, \mathcal{L}^{\otimes k}) \quad k \geq 0\}.$$

If D is the hyperplane divisor, one has $Q(X,D) = \mathbb{C}(X)$. Its transcendence degree is precisely the dimension of X (see [Reid, §9]). In general one has:

Proposition 1. *$Q(X,D)$ is algebraically closed in $\mathbb{C}(X)$. In particular, its transcendence degree is finite and at most equal to* $\dim X$.

Proof: Assume that $f \in \mathbb{C}(X)$ satisfies an equation

$$f^r + a_1 f^{r-1} + \ldots + a_r = 0, \quad a_i = \frac{s_i}{t}, \quad s_i, t \in H^0(X, \mathcal{L}^{\otimes k}).$$

Then $h := f \cdot t$ is a meromorphic section of $\mathcal{L}^{\otimes k}$ which satisfies

$$h^r + s_1 h^{r-1} + \ldots + s_r t^{r-1} = 0$$

with holomorphic coefficients. Now X being normal implies that locally at every point $x \in X$ the section h is in $\mathcal{O}_{X,x}$, i.e. h is a regular section of $\mathcal{L}^{\otimes k}$ and so $f = \dfrac{h}{t}$ belongs to $Q(X,D)$. ■

Definition 2. *The D-dimension $\kappa(D)$ is the transcendence degree of the field $Q(X,D)$. In the special case when D is a canonical divisor, $\kappa(D)$ is called the Kodaira dimension of X and denoted $\kappa(X)$.*

Remark 3. This definition can be extended to cases where D is not effective. If $H^0(X, \mathcal{L}^{\otimes k}) = 0$ for all $k \geq 0$ you simply set

$$\kappa(X,D) := -\infty.$$

Otherwise you introduce the set $\mathbb{N}(D) \subset \mathbb{N}$ of natural numbers k for which $\mathcal{L}^{\otimes k}$ does have sections and restrict the preceding discussion to the rational maps $\varphi_{\mathcal{L}^{\otimes k}}$ for $k \in \mathbb{N}(D)$. The definition is then easily modified.

Let me now relate the field $Q(X,D)$ to the geometry of the rational maps

$$\varphi_{\mathcal{L}^{\otimes k}} : X \dashrightarrow \mathbb{P}^{N_k}, \quad N_k + 1 = \dim\, H^0(\mathcal{L}^{\otimes k}).$$

Let W_k be the image of $\varphi_{\mathcal{L}^{\otimes k}}$. Recall that this is the closure in $\mathbb{P}^{N_k}$ of the image of the maximal subset of X on which $\varphi_{\mathcal{L}^{\otimes k}}$ is a morphism. In terms of a basis $\{s_0, \ldots, s_{N_k}\}$ for the sections of $H^0(X, \mathcal{L}^{\otimes k})$ the function field of W_k is given by

$$\mathbb{C}(W_k) = \mathbb{C}(s_1/s_0, s_2/s_0, \ldots, s_{N_k}/s_0).$$

So the union of these fields is $Q(X,D)$.

Lemma 4. *There is a natural number k_0 with $\mathbb{C}(W_k) = Q(X,L)$ for all $k \geq k_0$.*

Proof: There are natural embeddings $H^0(X, \mathcal{L}^k) \subset H^0(X, \mathcal{L}^{k+1})$ which induce embeddings $\mathbb{C}(W_k) \subset \mathbb{C}(W_{k+1})$. Their union is $Q(X,D)$ as observed before. Now the overfield $\mathbb{C}(X)$ is finitely generated over $\mathbb{C}$ (this follows from the interpretation of $\dim_{\mathbb{C}} X$ as the transcendence degree of the field extension $\mathbb{C}(X)/\mathbb{C})$) and hence $Q(X,D)$ is finitely generated over $\mathbb{C}$. So from some k_0 on the sequence of inclusions $\mathbb{C}(W_k) \subset \mathbb{C}(W_{k+1})$ stabilise and then $\mathbb{C}(W_k) = Q(X,D)$. ∎

Corollary 5. $\kappa(X,D) = \max \dim W_k$.

Proof: $\kappa(X,D)$ is the transcendence degree of the field extension $Q(X,D)/\mathbb{C}$. By the previous Lemma $Q(X,D) = \mathbb{C}(W_k)$ for all $k \geq k_0$ and so $\kappa(X,D) = \dim\, W_k$ for all $k \geq k_0$. ∎

The fact that $Q(X,D)$ is algebraically closed in $\mathbb{C}(X)$ can be translated in terms of the rational maps $\varphi_{\mathcal{L}^{\otimes k}}$ as follows. Let me first assume that the latter map is actually a morphism.

Proposition 6. *If for some $k \geq k_0$ the rational map $\varphi_{\mathcal{L}^{\otimes k}}$ is a morphism, its generic fibre is connected.*

Proof: Consider the Stein factorisation of $f := \varphi_{\mathcal{L}^{\otimes k}} : X \to W_k$, say $f = g \circ f'$ with $g : Y \to W_k$ finite and $f' : X \to Y$ connected. There is the inclusion of fields

$$\mathbb{C}(X) \supset \mathbb{C}(Y) \supset \mathbb{C}(W_k) = \mathbb{Q}(X,D)$$

and since the last field extension is finite algebraic and $Q(X,D)$ is algebraically closed in $\mathbb{C}(X)$, one must have $\mathbb{C}(Y) = \mathbb{C}(W_k)$ and so g must in fact be birational. So the fibres of f and f' are the same generically and in particular, the generic fibre of f is connected. ∎

Remark 7. Since Y is normal and since g is also finite, it follows that $g : Y \to W_k$ is the normalisation of W_k.

The general case needs a little elaboration on the elimination of points of indeterminacy of rational maps given by linear systems. This is a generalisation of the surface case. I won't give all the details, but refer to [Ha, Example 7.17.3] for them. Briefly, if $s_0, \dots, s_n$ forms a basis for the sections of a line bundle $\mathcal{M}$ on any projective variety X, you consider the subsheaf of $\mathcal{M}$ generated by these. Only at points where all sections vanish you'll end up in the maximal ideal at that point (after choosing a local trivialisation of $\mathcal{M}$ at that point). Now you look at the algebraic set defined by the simultaneous vanishing of all of these sections (more precisely, you look at the 'scheme' defined by them, but you just can think of the equations). This is the base-locus of the linear system defined by $\mathcal{M}$. Now you blow up X in this base locus. As in the surface case one shows that one obtains a projective variety $\tilde{X}$ plus morphisms $\sigma : \tilde{X} \to X$ and $\tilde{f} : \tilde{X} \to Y$ fitting into a commutative diagram

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\sigma"'] \arrow[dr, "\tilde{f}"] & \\ X \arrow[rr, dashed, "f"'] & & Y. \end{tikzcd}$$

Next, suppose that X is normal. Then by Theorem 8.6 the fibres of σ are all connected and hence, by Corollary 8.5 for any line bundle $\mathcal{M}$ on X one has natural isomorphisms

$$f^* : H^0(X, \mathcal{M}) \xrightarrow{\sim} H^0(\tilde{X}, \sigma^*\mathcal{M}).$$

Unfortunately, it is not automatically true that $\tilde{X}$ is normal again, but you can replace $\tilde{X}$ by its normalisation. Let me assume this and consider now the case at hand with $\mathcal{M} = \mathcal{L}^{\otimes k}$. It follows that I can replace X by another normal variety X' such that $\varphi_{\mathcal{L}^{\otimes k}}$ lifts to a morphism f'_k just by lifting the sections. So the preceding proposition just applies to X', f'_k and so I may assume that the rational map $\varphi_{\mathcal{L}\otimes k}$ simply is a morphism.

I admit the following simple theorem about the existence of the Hilbert polynomial, see [Ha, Proposition 7.5].

Theorem *Let X be a projective variety of dimension n. There exists a polynomial $P_X(t)$ of degree n such that for all sufficienty large k one has*

$$P_X(k) = \dim H^0(X, \mathcal{O}(k))$$

.

Next I can state and prove the main result of this section.

Theorem 8. (Characterisation of the D- dimension) *Let X be a normal projective variety and let D be an effective divisor on X with D-dimension equal to κ. There exist positive numbers α and β such that for all sufficiently large k one has*

$$\alpha k^\kappa \le \dim H^0(X, \mathcal{O}_X(kD)) \le \beta k^\kappa.$$

Proof: As explained before, I may, on replacing D by a suitable multiple of D, assume that the linear system $|D|$ defines a morphism f of X onto a variety W of dimension $\kappa(D)$. So, if F is the fixed part of this system, one has $D = f^*H + F$ with H a hyperplane section of W. Hence

$$\dim\ H^0(kD) = \dim\ H^0(f^*(kH)) \geq \dim\ H^0(W, \mathcal{O}(k)) \geq \alpha m^\kappa,$$

by the result on Hilbert polynomials quoted before.

Now I need to prove the other inequality. This is a bit more subtle. The subtlety lies in the fact that the fixed part F of $|D|$ might contain contain components mapping surjectively onto W by f. Let me first assume that tere are no such components. then F is entirely contained in the pull back of some divisor G on W and so

$$\dim\ H^0(\mathcal{O}(kD)) \leq \dim\ H^0(f^*\mathcal{O}(kH + kG).$$

Now one may, if neccessary, add a very ample divisor to G so that $H + G$ is very ample and then the result on Hilbert polynomials gives a bound

$$\dim\ H^0(f^*\mathcal{O}(kH + kG) \leq \beta k^\kappa.$$

The only case left is the case where $F = F' + F''$ with F' the non-empty maximal divisor in F with $f(F') = W$. Obviously $|D| = |D - F'| + F'$ and I claim that also $|kD| = |kD - kF'| + kF'$. If this can be shown, you simply replace D by $D - F'$ in the preceding argument and you are done.

Now assume that there is some $E \in |kD| \setminus (|kD - kF'| + kF')$. Let G be the maximal divisor such that kF' is not contained in G with the property that $E \in |kD - G| + G$. So $E = E' + G$ and E' passes through some but not all points of F'. Now, since F' is mapped onto W by f, the fibres of f all meet F'. Any point x on E' not on F' such that the fibre of f through x meets F' in y will then have the property that $f_k(x) \neq f_k(y)$. Here f_k is the morphism defined by the system $|kD|$

The generic fibre of f as well as of f_k (which is obviously contained in the latter) is connected. So, since f_k is not constant on the generic fibre of f_k, the latter must be a strictly lower dimensional subvariety of the generic fibre of f. Now, recall the dimension formula

$$\dim\ W_k + \dim\ (\text{generic fibre of } f_k) \ = \dim\ X.$$

This formula then implies that dim $W_k >$ dim W which is impossible, since W already had maximal dimension κ. ■

The final result treated in this section will be the behaviour of the Kodaira dimension under finite unramified coverings.

Proposition 9. *Let $f : X \to Y$ be an unramified covering between smooth projective varieties. Then $\kappa(X) = \kappa(Y)$.*

Proof: Since f is unramified, one has $K_X = f^*K_Y$. The result now follows from a more general result, namely that $\kappa(f^*D) = \kappa(D)$ for any divisor D on Y. Since f is surjective, the induced map

$$f^* : H^0(Y, \mathcal{O}_Y(kD)) \to H^0(X, \mathcal{O}_X(f^*(kD)))$$

is injective and one has the inequality

$$\kappa(f^*D) \geq \kappa(D).$$

I need to show the reverse inequality. First I reduce to the case that f is a Galois covering. From the theory of covering spaces one knows that $f_* : \pi_1(Y) \to \pi_1(X)$ is injective and there is a normal subgroup N of $\pi_1(X)$ contained in $f_*(\pi_1(Y))$ such that the quotient $\pi_1(X)/N$ is a finite group occurring as a group of deck transformations of a Galois covering $f' : X' \to Y$ which factors over $f : X \to Y$. By the preceding inequality one easily reduces to the case of a Galois cover $f : X \to Y$, say with group G.

To handle this case, you first observe that the case $\kappa(f^*D) = -\infty$ is treated by the already known inequality and so you may assume that this D-dimension is nonnegative and you choose a basis $\{s_0, \ldots, s_n\}$ for $H^0(X, f^*(kD))$ with k large enough so that

$$\kappa(f^*D) = \text{transc. deg } \mathbb{C}(s_1/s_0, \ldots, s_n/s_0).$$

Now you let the group G (of order m) act on the latter field L and consider the G-invariant subfield K. I claim that K can be considered as a subfield of the function field of W_{km}. From this the desired inequality easily follows:

$$\kappa(f^*D) = \text{transc. deg } L \leq \text{transc. deg } K \leq \text{transc. deg } \mathbb{C}(W_{km}) = \kappa(D).$$

To prove the claim, consider

$$\prod_{g \in G} (X - g^*(s_i/s_0)) = X^m + a_1(s_i/s_0)X^{m-1} + \ldots + a_m(s_i/s_0), \quad i = 1, \ldots, n.$$

Then K is generated over $\mathbb{C}$ by $a_l(s_i/s_0)$ $l = 1, \ldots, m$, $i = 1, \ldots, n$. Furthermore, $t_0 = \prod_{g \in G} g^*(s_0)$ and $t_0 a_l(s_i/s_0)$ define G-invariant holomorphic sections of $\mathcal{O}_X(f^*(kmD))$ and hence define sections of $\mathcal{O}_W(km)$. So $a_l(s_i/s_0)$ is a quotient of two sections of $\mathcal{O}_X(f^*(kmD))$ and hence gives a rational function on W_{km}. In this way you get a natural embedding of K into the function field of W_{km} as asserted. ■

10. The Albanese torus

The Albanese torus and the Albanese map are universal for maps of a projective manifold to a torus, hence the importance of the Albanese. If the image of the Albanese map is a curve in the Albanese much more can be said and this will be used in the sequel.

Before introducing the Albanese, let me recall briefly a few facts about g-tori $T := V/\Gamma$, where V is any g-dimensional complex vector space and $\Gamma \subset V$ is a lattice which is of maximal rank (over the reals). The following lemma should be obvious and its proof is left to the reader.

Lemma 1. *The homomorphism*

$$t_T : \Gamma \to H_1(T, \mathbb{Z})$$

defined by assigning to $\gamma \in \Gamma$ the homology class of the 1-cycle on T defined by the straight line segment from 0 to γ is an isomorphism. Also the map

$$\tau_T : V^* \to H^0(T, \Omega^1_T)$$

which assigns to the functional f on V the one-form df on T is an isomorphism. Moreover

$$\int_{t_T(\gamma)} \tau_T(f) = f(\gamma). \tag{2}$$

.

Let X be any projective manifold (or, more generally any compact Kähler manifold). The Albanese torus $\mathrm{Alb}(X)$ can be defined very concretely as follows. Let $b = b_1(X)$, the first Betti-number of X. The Hodge decomposition for H^1 reads as follows: $H^1(X, \mathbb{C}) = H^0(\Omega^1_X) \oplus \overline{H^0(\Omega^1_X)}$, so that $b = 2g$ with $g = \dim H^0(\Omega^1_X)$. Choose a basis for the space of holomorphic 1-forms $\{\omega_1, \ldots, \omega_g\}$ and choose a basis $\{\gamma_1, \ldots, \gamma_{2g}\}$ for $H_1(X, \mathbb{Z})$ modulo torsion. The $2g \times g$-matrix

$$\begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_{2g}} \omega_1 \\ \vdots & \ddots & \vdots \\ \int_{\gamma_1} \omega_g & \cdots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}$$

is called the *period matrix* with respect to the one-forms. The $2g$ columns are independent over the reals. Indeed, any linear relation between the columms with real coefficients a_i, $i = 1, \ldots, 2g$ implies that $\sum_i a_i \int_{\gamma_i} \omega = 0$ for all $\omega \in H^0(\Omega^1_X)$ and hence, $\sum_i a_i \int_{\gamma_i} \overline{\omega} = 0$ and the Hodge-decomposition then implies that the $\mathbb{C}$-linear functional $\sum_i a_i \int_{\gamma_i}$ is zero on $H^1(X, \mathbb{C})$. The Kronecker pairing (see Appendix A2.3) between H^1 and H_1 with complex coefficients being perfect, this implies that $\sum_i a_i \gamma_i = 0$ and hence $a_i = 0$ for all $i = 1, \ldots, 2g$.

It follows that the columns of the period matrix span a lattice in $\mathbb{C}^g$ and you can form the quotient g-torus, which by definition is the *Albanese torus.* More invariantly

$$\mathrm{Alb}(X) = H^0(\Omega^1_X)^* / \operatorname{im} H_1(X, \mathbb{Z}),$$

where $\gamma \in H_1(X, \mathbb{Z})$ is mapped to the functional on $H^0(\Omega^1_X)$ given by integration over γ. Fixing a point $x_0 \in X$ and choosing any path from x_0 to x, integration along this path gives a well defined element $\alpha(x) \in \mathrm{Alb}(X)$. This gives then a map, the *Albanese map*

$$\alpha : X \to \mathrm{Alb}(X).$$

This map is holomorphic, as can be seen as follows. Since this is a local matter, one may fix $x \in X$, a path γ from x_0 to x and compute α in a coordinate ball U about x by integrating over γ followed by a straight line segment from x to $y \in U$. This gives a well defined map

$$a : U \to H^0(\Omega^1_X)^*,$$

which clearly is holomorphic and hence $\alpha : U \to \operatorname{Alb} X$ is holomorphic, since $\alpha = q \circ a$, with $q : H^0(\Omega^1_X)^* \to \operatorname{Alb}(X)$ the projection. Note moreover that

$$\tau^{-1}_{\operatorname{Alb} X} = \alpha^* : H^0(\Omega^1_{\operatorname{Alb} X}) \xrightarrow{\;\cong\;} H^0(\Omega^1_X).$$

You see this as follows. Since $\tau_{\operatorname{Alb} X}$ is an isomorphism, it suffices to prove that $\alpha^*(\tau_{\operatorname{Alb} X}(\omega)) = \omega$ for all holomorphic 1-forms ω on X. Pick $x \in U$, where U is as before and evaluate the 1-form $\alpha^* \circ \tau_{\operatorname{Alb} X}\omega$ at $x \in U$ and get $a^* \circ q^*(\tau_{\operatorname{Alb} X}(\omega))(x) = a^* d(\langle \omega, - \rangle)(x) = d(\langle \omega, a(x) \rangle) = d(\int_{x_0}^x \omega) = \omega(x)$ and so $\alpha^*(\tau_{\operatorname{Alb} X}\omega) = \omega$. Here $\langle\, , \rangle$ denotes the pairing between $H^0(\Omega^1_X)$ and its dual.

The pair $(\operatorname{Alb} X, \alpha)$ satisfies a universal property with respect to maps $f : X \to T$ of X to a complex torus: such a map factors uniquely over α, i.e there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & T \\ & \operatorname{Alb}(X). \arrow[ur, "\tilde f"'] & \end{tikzcd}$$

and $\tilde f$ is uniquely determined by commutativity of the diagram.

To show this, first look at morphisms between any two tori $T = V/\Gamma$ and $T' = V'/\Gamma'$. A linear map $H : V \to V'$ which sends the lattice Γ to Γ' induces a morphism $h : T \to T'$. Any other morphism is obtained by composing such a map with a translation, as shown by the next lemma.

Lemma 3. *Any morphism $h : T = V/\Gamma \to T' = V'/\Gamma'$ is induced by an affine-linear map $V \to V'$ which is the composition of a translation by a linear map $H : V \to V'$ with $H(\Gamma) \subset \Gamma'$. Invariantly, the transpose H^T of H fits into the commutative diagram*

$$\begin{tikzcd} V'^* \arrow[r, "H^T"] \arrow[d, "\tau_{T'}"] & V^* \arrow[d, "\tau_T"] \\ H^0(T', \Omega^1_{T'}) \arrow[r, "h^*"] & H^0(T, \Omega^1_T). \end{tikzcd}$$

Proof: By the lifting properties of the universal coverings V and V' there exists some holomorphic map $\bar h : V \to V'$ such that $\bar h(v + \gamma) - \bar h(v) \in \Gamma'$ for all $v \in V$ and $\gamma \in \Gamma$. By continuity, $\bar h(v + \gamma) - \bar h(v)$ is independent of v and so all the partial derivatives of $\bar h$ are invariant under translation by Γ and hence define holomorphic functions on T. So they must be constant and $\bar h(v) = H(v) +$ a constant vector (defining a translation) and H a linear map with $H(\Gamma) \subset \Gamma'$. The last assertion should be obvious. ∎

Apply this lemma in the situation of the Albanese torus. In this situation, since α^* is an isomorphism with inverse $\tau_{\mathrm{Alb}\,X}$, the linear map $g = \tau_{\mathrm{Alb}\,X} \circ f^* : H^0(T, \Omega^1_T) \to H^0(\mathrm{Alb}\,X, \Omega^1_{\mathrm{Alb}\,X})$ makes the diagram

$$\begin{tikzcd} H^0(X,\Omega^1_X) & & H^0(T,\Omega^1_T) \arrow[ll, "f^*"'] \arrow[dl, "g"] \\ & H^0(\mathrm{Alb}\,X, \Omega^1_{\mathrm{Alb}\,X}). \arrow[ul, "\alpha^*"] & \end{tikzcd}$$

commutative. Now let $\tilde{F} : H^0(\Omega^1_X)^* \to V$ be defined by the requirement $\tau_{\mathrm{Alb}\,X} \circ \tilde{F}^T = g \circ \tau_T$ as suggested by the commutative diagram of the previous lemma. Note that by construction

$$\tilde{F}^T = f^* \circ \tau_T. \tag{4}$$

If $\tilde{F}$ induces a morphism $\tilde{f} : \mathrm{Alb}\,X \to T$ between the corresponding tori, by the previous lemma it is unique up to translation, but since $\tilde{f}(0) = f(x_0)$ it is then is completely determined. Again by the lemma, to prove existence, one only needs to see that $\tilde{F}$ preserves the lattices. This turns out to be the case, since $\tilde{F}$ restricted to the image of $H_1(X,\mathbb{Z})$ in $H^0(\Omega^1_X)^*$ coincides with the homomorphism induced by f on the first homology group as the following computation shows. Fix $\gamma \in H_1(X,\mathbb{Z})$ (which is identified with its image in $H^0(\Omega^1_X)^*$) and $h \in V^*$. One has $\langle h, \tilde{F}\gamma\rangle = \langle \tilde{F}^T h, \gamma\rangle = \langle f^*\tau_T h, \gamma\rangle$ (by (4)) $= \int_\gamma f^*\tau_T h$ (by (2)) $= \int_{f_*\gamma} \tau_T h = \langle h, t_T^{-1} f_*\gamma\rangle$ (again by (2)) and so $t_T\tilde{F} = f_*$ and hence lattices are preserved.

Example 5. The Albanese of a curve is its Jacobian and the Albanese map is the Abel-Jacobi map. Recall from the theory of Riemann surfaces [G, Chapter V] that the period matrix can always be normalised by taking a symplectic basis for $H_1(C,\mathbb{Z})$ (see Appendix A2, Example A2.16 1.) and a suitable basis for $H^0(\Omega^1_C)$ such that the period matrix reads as follows

$$\begin{pmatrix} \mathbb{1}_g & Z_g \end{pmatrix},$$

where Z is a symmetric matrix with positive definite imaginary part. Recall also that these matrices form the so-called Siegel upper halfspace $\mathfrak{h}_g$.

Let me now formulate a few useful consequences and additions

1. If $q(X) = 0$ the Albanese reduces to 0 and the universal property of the Albanese implies that any map $X \to T$ must be constant. This applies e.g. to $X = \mathbb{P}^n$.

2. $\mathrm{Alb}\,X$ is the smallest subtorus generated by the image of the Albanese map. Indeed, let $i : A \subset \mathrm{Alb}\,X$ be this torus, the universal property shows that there is a morphism $a : \mathrm{Alb}\,X \to A$ such that $i \circ a = \mathrm{Id}$ and from this it follows that $A = \mathrm{Alb}\,X$.

3. Functoriality: any morphism $f : X \to Y$ induces a morphism between the Albanese tori $a(f) : \mathrm{Alb}\,X \to \mathrm{Alb}\,Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_X"] & Y \arrow[d, "\alpha_Y"] \\ \mathrm{Alb}\,X \arrow[r, "a(f)"] & \mathrm{Alb}\,Y \end{tikzcd}$$

is commutative. This should be obvious. Note that in view of the previous remark, $a(f)$ is surjective if f is surjective.

4. Special case: the image of the Albanese map is a curve.

Lemma 6. *If the image of the Albanese map $X \to \mathrm{Alb}\, X$ is a curve C, the fibres are connected. Moreover, C is smooth and has genus $q(S)$.*

Proof: Consider the Stein factorisation (see §9) for the Albanese map.

$$\begin{tikzcd}
X \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & C \subset \mathrm{Alb}\, X \\
& Y. \arrow[ur, "g"'] &
\end{tikzcd}$$

Since X is normal, Y is a normal and hence smooth curve. I want to show that the finite map g in fact gives an immersion into the Albanese. The map f fits into a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, "\alpha'"] \arrow[dr, "g"] & \\
\mathrm{Alb}\, X \arrow[r, "a(f)"] & \mathrm{Jac}\, Y \arrow[r, "a(g)"] & \mathrm{Alb}\, X
\end{tikzcd}$$

and since f is surjective, $a(f)$ must be surjective, as remarked before. Moreover since $a(g) \circ a(f) \circ \alpha = g \circ f = \alpha$, the universal property of α implies that $a(g) \circ a(f) = \mathrm{Id}$ and so, since I already know that $a(f)$ is surjective, it must be an isomorphism with inverse $a(g)$ and since the Abel-Jacobi map is an embedding, this then follows for $g = a(g) \circ \alpha'$. ■

Chapter 5. Divisors on surfaces

11. The Picard variety and the Néron-Severi group

I prove the Lefschetz theorem on (1,1)-classes, give a Hodge theoretic proof of the Index Theorem for surfaces, prove Nakai's ampleness criterion and rephrase it in terms of properties of the nef-cone.

Let me recall that by means of the exponential sequence on any projective manifold you get an isomorphism

$$\mathrm{Pic}^0\, M \cong \frac{H^1(M, \mathcal{O}_M)}{\mathrm{im}\ H^1(M, \mathbb{Z})},$$

where the map $H^1(M,\mathbb{Z}) \to H^1(M, \mathcal{O}_M$ is induced from the inclusion $\mathbb{Z}_M \to \mathcal{O}_M$. I want to show how Hodge theory can be used to show that the Picard variety $\mathrm{Pic}^0(M)$ is a torus.

Indeed, the Hodge decomposition (Appendix A3) theorem says that De Rham group $H^k(M,\mathbb{C})$ decomposes into a direct sum $H^{k,0}(M) \oplus \cdots \oplus H^{0,k}(M)$, where $H^{p,q}(M)$ denotes the group spanned by classes represented by closed forms of type (p,q). Furthermore, the groups $H^{p,q}(M)$ can be computed as the cohomology of the complex of global sections of the Dolbeault resolution of Ω^p

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \cdots .$$

The complex De Rham cohomology can be computed using the De Rham resolution $\mathcal{E}^\bullet_M$ of $\mathbb{C}_M$. Sending a complex valued form to its $(0,q)$-component defines a homomorphism $\mathcal{E}^\bullet \to \mathcal{E}^{0,\bullet}$ extending the inclusion $i : \mathbb{C}_M \to \mathcal{O}_M$. This just means that the $(0,q+1)$-component of $d\alpha$ is equal to $\bar\partial$ of the $(0,q)$-component of α. Passing to global sections and taking cohomology, one gets

Lemma 1. *The inclusion $j : \mathbb{C}_M \to \mathcal{O}_M$ induces the projection $H^k(M,\mathbb{C}) \to H^{0,k}(M)$ onto the Hodge $(0,k)$-component.*

Corollary 2. *The Picard variety $\mathrm{Pic}^0(M)$ is a torus.*

Proof: Let $[\alpha] \in H^1(M,\mathbb{C})$ be a class of a $(1,0)$-form α and assume that $[\alpha]$ is the image of the class of a real form. Then $\overline{[\alpha]} = [\alpha]$ and so if such classes are independent over the complex numbers they are also independent over the reals. This holds in particular for the images of a basis for $H^1(M,\mathbb{Z})$ in $H^{1,0}$ and so these form a lattice of maximal rank (since $\mathrm{rank}\, H^1(M,\mathbb{Z}) = \dim_{\mathbb{R}} H^{1,0}$). ■

Remark 3. The torus $\mathrm{Pic}^0(M)$ is an algebraic torus. This is a deeper fact which follows from Lefschetz' theory of primitive cohomology. See [G-H, Chapter 2 §6].

The Néron-Severi group $\mathrm{NS}\, M$ of a smooth projective manifold of M is the group of isomorphism classes of divisors modulo homological equivalence, where two divisors are said to be homologically equivalent if they have the same first Chern class. The exponential sequence yields an exact sequence

$$0 \to \mathrm{NS}\, M \to H^2(M,\mathbb{Z}) \xrightarrow{k^*} H^2(\mathcal{O}_M).$$

Now look at the chain of inclusions $\mathbb{Z}_M \xrightarrow{i} \mathbb{C}_M \xrightarrow{j} \mathcal{O}_M$ inducing the triangle

$$\begin{tikzcd} H^2(M,\mathbb{Z}) \arrow[rr, "k^*"] \arrow[dr, "i^*"'] & & H^2(\mathcal{O}_M) \\ & H^2(M,\mathbb{C}) \arrow[ur, "j^*"'] & \end{tikzcd}$$

Every De Rham class $[\alpha]$ in the image of i^* is a class of a real form and hence, if the $(0,2)$-component vanishes, the $(2,0)$-component is zero as well. This observation in conjunction with the previous lemma shows that the kernel of k^* consists precisely of the integral classes having type $(1,1)$ in the de Rham group $H^2_{DR}(M,\mathbb{C})$. This is the content of the following theorem which says that the Hodge Conjecture is true for divisors.

Theorem 4. (Lefschetz' Theorem on (1,1)-classes) *The Néron-Severi group of a projective manifold consists precisely of the integral classes of Hodge type* $(1,1)$.

The next topic is the intersection form on the Néron-Severi group of a surface S. I first prove a simple instance of the Hodge-Riemann bilinear relations. Assume that $S \subset \mathbb{P}^n$ and let ω the metric form belonging to the Fubini-Study metric. See Appendix A3. It is a $(1,1)$)-form which is pointwise positive definite .

Define

$$H^2_{\mathrm{prim}}(S,\mathbb{Q}) := \{[\alpha] \; ; \; [\alpha \wedge \omega] = 0\} = [\omega]^\perp$$

leading to the orthogonal direct sum decomposition

$$H^2(S,\mathbb{Q}) = \mathbb{Q} \cdot [\omega] \oplus H^2_{\mathrm{prim}}(S,\mathbb{Q}).$$

Theorem 5. *The intersection product is negative definite on* $H^2_{\mathrm{prim}}(S,\mathbb{R}) \cap H^{1,1}$.

Proof: As explained in appendix A.3.2, the Kähler identities imply that wedging with ω preserves the primitive forms and so, in the following computation, the use of forms instead of cohomology classes is allowed.

I CLAIM that

for any real $(1,1)$ form α with $\alpha \wedge \omega = 0$ one has $\alpha \wedge \alpha \leq 0$ with equality if and only if $\alpha = 0$.

The theorem then follows from the compatibility of the intersection product and the wedge-product:

$$\int_S \alpha \wedge \beta = [\alpha] \cdot [\beta] \quad \text{for all closed 2-forms } \alpha,\, \beta.$$

To prove the claim let me choose a local C^∞-trivialisation of the holomorphic cotangent bundle by two 1-forms β^1, β^2 which are everywhere orthonormal with respect to the Kähler form ω. Thus

$$\omega = \frac{i}{2}(\beta^1 \wedge \overline{\beta^1} + \beta^2 \wedge \overline{\beta^2}).$$

Set

$$\alpha = \alpha_{1\bar{1}}\beta^1 \wedge \overline{\beta^1} + \alpha_{1\bar{2}}\beta^1 \wedge \overline{\beta^2} + \alpha_{2\bar{1}}\beta^2 \wedge \overline{\beta^1} + \alpha_{2\bar{2}}\beta^2 \wedge \overline{\beta^2}.$$

The condition that α is real implies that $\alpha_{1\bar{1}}$ and $\alpha_{2\bar{2}}$ are purely imaginary and that $\alpha_{1\bar{2}} + \overline{\alpha_{2\bar{1}}} = 0$. The condition that $\alpha \wedge \omega = 0$ yields $\alpha_{1\bar{1}} + \alpha_{2\bar{2}} = 0$. So

$$\frac{1}{2}\alpha \wedge \alpha = (|\alpha_{1\bar{1}}|^2 + |\alpha_{1\bar{2}}|^2)\beta^1 \wedge \overline{\beta^1} \wedge \beta^2 \wedge \overline{\beta^2}$$

which is a non-positive multiple of the volume form, and zero precisely when $\alpha = 0$. This proves the claim. ■

Corollary 6. (Algebraic Index Theorem) *The intersection pairing restricts non-degenerately on* $\mathrm{NS}\, S$ mod torsion *and has signature* $(1, \rho - 1)$, *where* $\rho = \operatorname{rank} \mathrm{NS}\, S$ *is the Picard number.*

Proof: Note that $[\omega] \cdot [\omega] > 0$. Since by the theorem the intersection product is negative on $[\omega]^\perp$, the primitive part of the cohomology, the signature is $(1, h^{1,1} - 1)$ on $H^{1,1}$. So it either restricts non-degenerately with the stated signature or it is semi-negative (with rank one annihilator) on the Néron-Severi group. Since the latter always contains the class of an ample divisor this last possibility is excluded. ■

Remark 7. It follows that two divisors D and D' are *torsion equivalent*, i.e. homologically equivalent up to torsion if and only if they are *numerically equivalent*, i.e. $c_1(D) = c_1(D')$ mod torsion if and only if $(D, E) = (D', E)$ for all divisors E.

Remark 8. A very useful alternative formulation of the Algebraic Index Theorem runs as follows

> If D is a divisor with $(D, D) > 0$ and $(C, D) = 0$ then $(C, C) \leq 0$ with equality if and only if C is numerically equivalent to zero.

Remark 9. The preceding theorem is just a special case of the Lefschetz-decomposition theorem valid for the cohomology of any Kähler manifold. See [We, Chapt. V, sect. 6].

From the Algebraic Index Theorem it follows that the intersection pairing on the real vector space $N_{\mathbb{R}}(S) := \mathrm{NS}\, S \otimes \mathbb{R}$ has signature $(1, \rho - 1)$. Such quadratic forms have special properties. There is the *light cone* $x \cdot x = 0$ with disconnected interior $C^+(S) \amalg -C^+(S) = \{x \in N_{\mathbb{R}}(S) \ ; \ x \cdot x > 0\}$. Each connected part is convex.

Recall that the dual of a cone C in a real vector space V with non-degenerate product is the cone

$$C^\vee := \{y \in V \ ; \ y \cdot x \geq 0 \quad \text{for all } x \in C\}.$$

If $x \neq 0$ is on the light cone and in the closure of $C^+(S)$, the dual of the half-ray $\mathbb{R}_{\geq 0} \cdot x$ is the half-space bounded by the hyperplane through this ray, tangent to the light cone and containing $C^+(S)$. The intersection of all such half spaces is the closure of $C^+(S)$. Using convexity it follows that the closed cone $\overline{C^+(S)}$ is self dual.

To study divisors inside the light cone, one uses Riemann-Roch.

Proposition 10. *If for a divisor D on a surface one has $(D, D) > 0$, then $(D, H) \neq 0$ for any ample divisor H. If $(D, H) > 0$ some positive multiple of D is effective and if $(D, H) < 0$, some negative multiple of D is effective.*

Proof: The first assertion follows from the Algebraic Index Theorem.

The Riemann-Roch inequality shows that $h^0(mD) + h^0(-mD + K_S) \geq \frac{1}{2}m^2(D, D) +$ linear term in m. If $(D, H) > 0$, there can be no divisor in $|-mD + K_S|$ for m large and so $|mD|$ must contain effective divisors for m large enough. The proof of the second assertion is similar. ∎

Since the effective divisors are all on the same side of the hyperplane defined by an ample divisor it follows from the preceding proposition that only one component of the interior of the light cone can contain effective divisors. Let me once and for all choose it to be $C^+(S)$. Let me also speak of $\mathbb{Q}$-*divisors* as a formal linear combination of irreducible curves with rational coefficients. Similarly one can speak of $\mathbb{Q}$-divisor classes, the rational points in $\mathrm{NS}(S) \otimes \mathbb{R}$. Such a class is called *effective* if a positive multiple can be represented by an effective divisor. Explicitly, a $\mathbb{Q}$-divisor class $[D]$ is effective if and only if there is an integer $n > 0$ such that there is an effective divisor numerically equivalent to nD. From the preceding Proposition it follows that for divisors with positive self-intersection in this definition one can replace "numerically equivalent" by "linearly equivalent", i.e. effectivity is a numerical property for divisors with positive self-intersection.

The preceding theorem now can be conveniently reformulated as follows.

Corollary 11. *The rational points in $C^+(S)$ are effective $\mathbb{Q}$-divisors.*

In general, there are more effective divisors in $\mathrm{NS}\, S$ spanning a convex cone $\mathrm{Ef}\, S$ in the real vector space spanned by divisors.

Let me consider the dual cone

$$\mathrm{Nef}\, S := \mathrm{Ef}\, S_\vee = \{x \in \mathbb{N}_{\mathbb{R}}(S) \ ; \ x \cdot e \geq 0 \quad \forall e \in \mathrm{Ef}\, S\}.$$

Its rational points are the classes of what are called nef-divisors ("numerically effective divisors").

Definition *A divisor D is nef if $(D, C) \geq 0$ for all irreducible curves C.*

The cone $\operatorname{Nef} S$ therefore is called the *nef-cone.*

Observation 12. *If for a divisor D one has $(D, C) \geq 0$ for all irreducible curves C then $(D, D) \geq 0$.*

Proof: One has $\operatorname{Nef} S = \operatorname{Ef} S_\vee \subset C^+(S)_\vee = \overline{C^+(S)}$. So $(D, D) \geq 0$ as desired. ■

Next, let me study the ample divisors. The following technical lemma plays an important role in the proof of the Nakai-moishezon Criterion, which will be treated shortly.

Lemma 13. *Suppose that $C = A + B$ is the sum of two effective divisors on a manifold M. There is an exact sequence of coherent sheaves on M:*

$$0 \to \mathcal{O}_A(-B) \to \mathcal{O}_C \xrightarrow{\text{restriction}} \mathcal{O}_B \to 0.$$

For a proof see Problem 1.

Theorem 14. (Nakai-Moishezon) *A divisor D on a surface S is ample if and only if $(D, D) > 0$ and $(D, C) > 0$ for all irreducible curves C.*

Proof: Let H be a very ample line bundle. Since $(D, H) > 0$ and $(D, D) > 0$ by Proposition 11.10 a multiple of D is effective. By replacing D by this multiple one may assume that D is effective. Let me now show that by induction $H^1(D', \mathcal{O}(nD)) = 0$ for all divisors D' supported on D and all n sufficiently large. If D' is irreducible and $\nu : D'' \to D'$ its normalisation, one has $H^1(D', \mathcal{O}(nD)) = H^1(D'', \nu^*(\mathcal{O}(nD))$. Since $\deg \nu^*(\mathcal{O}(nD) = n(D, D')$ and $(D', D) > 0$ by assumption, for all n large enough you have indeed $H^1(D', \mathcal{O}(nD)) = 0$. If D is irreducible one is ready. Otherwise, you write $D' = D'' + R$ for some effective divisor R and irreducible D''. Consider the cohomology sequence associated to the sequence of the previous lemma:

$$\ldots \to H^1(\mathcal{O}_{D''}(nD - R)) \to H^1(\mathcal{O}_{D'} D(nD)) \to H^1(\mathcal{O}_R(nD)) \to \ldots.$$

The first term is zero for $n \geq n_0$ by a similar argument as the argument for irreducible D' while the third term is zero for $n > n_1$ by induction on the number of components (counted with multiplicity) in D'. So the middle term is zero for $n > \max\{n_0, n_1\}$. Now one considers the exact cohomology sequence associated to

$$0 \to \mathcal{O}_S((n-1)D) \to \mathcal{O}_S(nD) \to \mathcal{O}_D(nD) \to 0.$$

Let me look at the portion

$$\begin{aligned} H^0(S, \mathcal{O}(nD)) \to & H^0(D, \mathcal{O}(nD)) \to H^1(S, \mathcal{O}((n-1)D)) \to \\ \to & H^1(S, \mathcal{O}(nD)) \to H^1(D, \mathcal{O}_D(nD)) \to \ldots \end{aligned}$$

By the previous vanishing result, one finds for all large enough n:

$$\dim H^1(S, \mathcal{O}(nD)) \leq \dim H^1(S, \mathcal{O}((n-1)D)).$$

But since all these spaces are finite dimensional, their dimensions must eventually stabilise and then the map

$$H^0(S, \mathcal{O}(nD)) \to H^0(D, \mathcal{O}(nD))$$

becomes surjective. Now one can show, again by induction on the number of components of D', that $\mathcal{O}_{D'}(nD)$ is globally generated by its sections if n is large enough. Surjectivity of the preceding map then implies that $\mathcal{O}(nD)$ is generated by its sections along points of D. Now, since D is effective, $\mathcal{O}_S(D)$ has a section vanishing exactly along D and so $\mathcal{O}_S(nD)$ is also generated by sections away from D.

It follows that $\mathcal{O}(nD)$ defines a morphism

$$f : S \to \mathbb{P}^n$$

and I claim that f is a finite morphism. Indeed, if C is a curve which is mapped to a point, you take a hyperplane L in $\mathbb{P}^n$ missing this point and so $(C, L) = (C, nD) = 0$ contradicting our assumptions. Now let me recall Lemma 4.24 which implies that $f^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_S(nD)$ is ample.

■

Proposition 15. *L is ample if and only if $(c_1(L), c) > 0$ for all $c \in \overline{\mathrm{Ef}\, S} \setminus \{0\}$*

Proof: If L is ample, $(L, D) > 0$ for all effective divisors D and so $(c_1(L), c) \geq 0$ for all c in the closure of the effective cone. If $(c_1(L), c) = 0$ for some c in this closure and $c \neq 0$ one can find an effective $C' \in \mathrm{Pic}(S)$ with $(c, c_1(C')) < 0$ and then $(c_1(L^{\otimes n} \otimes C'), c) < 0$. On the other hand $L^{\otimes n} \otimes C'$ will be ample for n large enough by Nakai-Moishezon (for at worst finitely many of components D of C' you will have $(D, C') < 0$ and these can be taken care of by making n large enough). This is a contradiction and so $(c_1 L), c) > 0$.

Conversely, by the Nakai-Moishezon criterion, one only has to show that $(L, L) > 0$. Fix some ample line bundle H and consider the function $f(c) = (c_1(L), c)/(H, c)$ which is constant under homotheties and so to study its values one can restrict to the (compact) closure of $\mathrm{Ef}\, S$ in the unit ball with respect to some metric on the real vector space $N_{\mathbb{R}}(S)$. It has a positive (rational) maximum ϵ and so $(L - \frac{1}{2}\epsilon H, c) > 0$ for all $c \in \mathrm{Ef}\, S$ and in particular $L - \frac{1}{2}\epsilon H$ is nef and so has non-negative selfintersection. But then $(L, L) = (L - \frac{1}{2}\epsilon H, L - \frac{1}{2}\epsilon H) + \epsilon(H, L - \frac{1}{2}\epsilon H) + \frac{1}{4}\epsilon^2(H, H) > 0$. ■

Corollary 16. *The cone consisting of ample $\mathbb{Q}$-divisors forms an open subset in $\mathbb{NS}\, S \otimes \mathbb{Q}$ and its closure is the nef-cone.*

Proof: If H is ample and D any divisor $(H + tD, c) > 0$ for c in the closure $\mathcal{C}$ of $\mathrm{Ef}\, S$ in the unit ball in some metric on $N_{\mathbb{R}}(S)$ and for $|t| < t_0$ with t_0 the smaller of the minima of the two functions $f(c) = (-D, c)/(H, c)$ on $\mathcal{C} \cap \{(D, c) \leq 0\}$ and $g(c) = (H, c)/(D, c)$ on $\mathcal{C} \cap \{(D, c) \geq 0\}$. By the proposition $H + tD$ is ample for these values of t.

Conversely, by the Proposition, one has $(a, c') \geq 0$ for all $c' \in \overline{\mathrm{Ef}\, S}$. But this is the case precisely when $(a, c) \geq 0$ for all $c \in \mathrm{Ef}\, S$, i.e. when a is nef. ■

Problems.

11.1. Let M be a projective manifold and $C = A + B$ the sum of two effective divisors. Show that the inclusion $C \subset B$ induces an exact sequence

$$0 \to \mathcal{I}_B/\mathcal{I}_C \to \mathcal{O}_C \to \mathcal{O}_B \to 0.$$

Show that there is a canonical isomorphism

$$\mathcal{I}_B/\mathcal{I}_C \cong \mathcal{O}_A(-B).$$

12. Rationality theorem and applications

I state and prove Mori's rationality theorem following the sketch in [Wi] and give applications which are to be considered as the first steps in classification theory, e.g. Castelnuovo's Rationality Criterion.

Let me recall that the Néron-Severi group $\mathrm{NS}\, S$ is the group of divisor classes modulo homological equivalence on S. The cup product on the real vector space $\mathrm{NS}\, S \otimes \mathbb{R}$ makes it into a self dual vector space. So you may view a divisor either as giving a class in $\mathrm{NS}\, S$ or as giving a hyperplane in $\mathrm{NS}\, S \otimes \mathbb{R}$. One has the real cone $\mathrm{Ef}\, S$ of effective divisors (with real coefficients) whose dual is called the cone of nef-divisors and denoted by $\mathrm{Nef}\, S$. So a divisor D is nef if and only if the cone $\mathrm{Ef}\, S$ is on the non-negative side of the hyperplane which D defines. The cone Nef is a closed cone whose integral points in the interior are the classes of the ample divisors. So H is ample if and only if $(H, H) > 0$ and $\mathrm{Ef}\, S \setminus 0$ is on the positive side of the hyperplane defined by H. If some D is not nef the hyperplane it defines will have some part of $\mathrm{Ef}\, S$ on its negative side and in the pencil $H + sD$ there will be a smallest value for wich the resulting hyperplane no longer has Ef on the positive side. The rationality theorem says that for $D = K_S$ this happens for a rational value. This theorem has surprizingly many consequences for the classification of surfaces as you will see.

Theorem 1. (Rationality theorem) *Let S be a surface and let H be very ample on S. Assume that K_S is not nef. Then there is a rational number b such that the hyperplane corresponding to $H + bK_S$ touches the cone* $\mathrm{Ef}\, S$.

Proof: Introduce

$$b := \sup\{t \in \mathbb{R}\ ;\ H_t = H + tK_S \text{ is nef }\}.$$

Set

$$P(v, u) := \chi(vH + uK_S).$$

By Riemann-Roch this is a quadratic polynomial in v, u. If u and v are positive integers with $(u-1)/v < b$ the divisor $vH + (u-1)K_S$ is ample and so by Kodaira Vanishing (Appendix A3) $H^i(vH + uK_S) = 0$ for $i = 1, 2$. It follows that $P(v, u) \geq 0$.

Assume now that b is irrational. Number theory ([HW, Theorem 167]) implies that b can be approximated by rational numbers of the form p/q, p and q arbitrarily large integers in such a way that

$$p/q - 1/(3q) < b < p/q.$$

The polynomial $P(kq, kp)$ is quadratic in k. If it is identically zero, $P(v, u)$ must be divisible by $(vp - uq)$. Taking p and q sufficiently large one may assume that this is not the case. For $k = 1, 2, 3$ the numbers $v = kq$ and $u = kp$ satisfy $(u - 1)/v < b$ and hence $P(kq, kp) \geq 0$ for these three values of k. Since a quadratic polynomial has at most two zeroes, it follows that for at least one pair of positive integers (v, u) with $t_0 := u/v > b$ one has $\dim H^0(vH + uK_S) > 0$. So there is an effective divisor (with coefficients in $\mathbb{Q}$) $L := H_{t_0} = \sum a_j\Gamma_j$, $a_j > 0$. Now H_{t_0} is not nef. Since L is effective, it can only be negative on the Γ_j. But then one can subtract off a rational multiple of K_S from H_{t_0} to get H_b and so b would be rational contradicting our assumption. ■

Let me give a first application.

Proposition 2. *A minimal algebraic surface with K not nef is either a geometrically ruled surface or $\mathbb{P}^2$.*

Proof: Let me first look at the positive half ray in $\mathrm{NS}\, S \otimes \mathbb{Q}$ spanned by $-K_S$. There are two possibilities. The first possibility is that all ample classes of S are on this line and hence $-K_S$ is ample and $\operatorname{Pic} S$ has rank 1. Kodaira-Vanishing implies that $h^0(K_S) = h^1(K_S) = 0$ and so $p_g = q = 0$. It follows that $\operatorname{Pic} S \xrightarrow{\cong} H^2(S, \mathbb{Z})$ has rank one. Moreover $b_2 = 1$ and $b_1 = 0$ imply that $e(S) = 3$ and by Noether's Formula one has $(K, K) + 3 = 12(1 - q + p_g) = 12$ and so $(K, K) = 9$. Next, take an ample generator H of $\operatorname{Pic} S$ mod torsion and apply Riemann-Roch to H. Note that since $H - K_S$ is ample, Kodaira-Vanishing gives that $h^1(H) = 0 = h^2(H)$ and one finds $h^0(H) = \frac{1}{2}(H, H - K_S) + 1 = 3$. Indeed, since $(K, K) = 9$, K must be numerically equivalent to $-3H$. One gets a dominant (i.e. the closure of the image is the entire target space) rational map $f : S \dashrightarrow \mathbb{P}^2$ which maps H to the class of a line. Now $(H, H) = \frac{1}{9}(K_S, K_S) = 1$ implies that $|H|$ can have no fixed points and that f is birational (why?). Now f cannot contract any curves to points, since $\operatorname{Pic} S$ has rank 1. From the discussion about birational geometry it follows that f must be biregular and so S is isomorphic to $\mathbb{P}^2$.

So one may assume that there exists an ample H such that its class in $\mathrm{NS}\, S \otimes \mathbb{Q}$ does not belong to the positive half-ray spanned by $-K_S$. Now apply the rationality theorem to H and K_S.

Clearing denominators one finds a divisor

$$L = vH + uK_S,\ b = u/v = \sup\{t \in \mathbb{R} \ ;\ H_t = H + tK_S \text{ is nef }\}.$$

Now L belongs to the closure of the nef-cone, which- as shown before- is itself closed. So L is a nef divisor and so in particular, $(L, L) \geq 0$ (see Observation 11.12). If you subtract any positive rational multiple of K_S from L you come into the interior of the nef-cone, which is the ample cone. So $mL - K_S$ is ample for all $m \geq 1$. Serre duality implies that $\dim H^2(mL) = \dim H^0(-(mL - K_S)) = 0$ and so by Riemann-Roch

$$\dim H^0(mL) \geq \chi(mL) = \chi(S) + \frac{1}{2}(mL, mL - K_S).$$

One can distinguish two cases, namely $(L,L) > 0$ or $(L,L) = 0$.

i) $(L,L) > 0$. Since L is nef, for any effective divisor, one has $(L,D) \geq 0$. The equality sign can be excluded as follows. Any irreducible curve D for which $(L,D) = 0$ must be an exceptional curve of the first kind. Indeed, from the definition of L one sees that $(K_S, D) < 0$, while the Algebraic Index Theorem applied to L and D shows that $(D,D) < 0$. In combination with the adjunction formula this shows that D has to be an exceptional curve of the first kind. By assumption these don't exist and so $(L,D) > 0$ for all curves D and so, by the Nakai-Moishezon criterion, L is ample, which is impossible by construction (L is on the boundary of the nef-cone).

ii) $(L,L) = 0$. Since L is nef one has $(L,H) \geq 0$, and if $(L,H) = 0$ an application of the Algebraic Index Theorem shows that L is numerically trivial. In this last case, the class of H in $\mathbb{NS}\, S \otimes \mathbb{Q}$ would be on the positive half-ray spanned by the class of $-K_S$, which has been excluded. So $(L,H) > 0$. From $0 = 1/v(L,L) = (L, H + bK_S)$ one infers that $(L, K_S) < 0$ and so dim $H^0(mL)$ grows like a linear function of m. You may replace L by mL and assume that dim $|L| \geq 1$. Now write $L = L' + L_{\text{fixed}}$, where L_{fixed} is the fixed part of $|L|$. I claim that L' is still nef and that still $(L', L') = 0$. The first is clear since L' moves in a linear system. So $(L', L) \geq 0$ and $(L', L_{\text{fixed}}) \geq 0$. From

$$0 = (L,L) = (L', L) + (L_{\text{fixed}}, L) \geq 0$$

one infers $(L', L) = (L_{\text{fixed}}, L) = 0$, while

$$0 = (L', L) = (L', L') + (L', L_{\text{fixed}}) \geq 0$$

implies that $(L', L') = 0$. Moreover, for every irreducible component D of $|L'|$ the equality $(L, L') = 0$ implies that $(L, D) = 0$ and since $(L', L_{\text{fixed}}) = 0$ one also has $(D, L_{\text{fixed}}) = 0$. So $(L', D) = 0$ and from this you easily see that $(D,D) \leq 0$. By definition of L from the equality $(L,D) = 0$ one concludes that $(D, K_S) < 0$. The Adjunction Formula then implies that D is a smooth rational curve with $(D,D) = 0$.

The same reasoning applies to *any* linear subsystem of $|L|$ which has no fixed part. You can for instance take a one-dimensional subsystem of $|L|$, take off the fixed part and end up with a pencil $\mathbb{P}$ without fixed components and with $(F,F) = 0$ for every $F \in \mathbb{P}$. By the preceding discussion every irreducible component of a member of $|F|$ is a smooth rational curve.

Since $(F,F) = 0$ there can be no fixed points and so one gets a morphism $f : S \to \mathbb{P}^1$. Now by taking the Stein factorization of f (see §9) one obtains a fibration $f' : S \to C$ of S onto a curve whose fibres are smooth rational curves. So S is a geometrically ruled surface. ■

Corollary 3. (Uniqueness of Minimal Model) *If S, S' are two minimal surfaces which are not ruled then any birational map $f : S' \to S$ is an isomorphism. In particular, any surface which is not ruled or rational has a unique minimal model.*

Proof: This follows from the previous theorem and Proposition 6.15 ■

Let me give an application of which the full strength will be shown in the next sections.

Proposition 4. *Let K_S be nef. There are the following possibilities for S.*

1. $(K_S, K_S) > 0$. *Then* $P_m \geq \frac{1}{2}m(m-1)(K_S, K_S) + 1 - q + p_g$ *for* $m \geq 2$ *and always* $P_2 > 0$.
2. $(K_S, K_S) = 0$, $q = 0$. *Then* $P_2 > 0$.
3. $(K_S, K_S) = 0$, $p_g > 0$ *and* $q > 0$.
4. $(K_S, K_S) = 0$, $p_g = 0$, $q = 1$ *and* $b_2 = 2$.

Proof: Observe that nefness of K_S implies that $(K_S, K_S) \geq 0$. Now you only have to prove the following three assertions:

i. If $(K, K) > 0$ the stated bound for P_m is valid and $P_m > 0$.

ii. If $(K, K) = 0$, $p_g = q = 0$ implies $P_2 > 0$.

iii. If $(K, K) = 0$, $p_g = 0$ and $q > 0$ one has $b_2 = 2$.

If $p_g > 0$, clearly $P_m > 0$ for all $m \geq 1$ so to, prove that $P_2 > 0$ it suffices to look at the case $p_g = 0$.

So let me first consider the case $p_g = 0$. Noether's formula in this case reads

$$12(1-q) = (2 - 4q + b_2) + (K_S, K_S).$$

So $b_2 = 10 - 8q - (K_S, K_S) \geq 1$ implies that $q \leq 1$.

If $q = 1$ and $(K_S, K_S) = 0$ one must have $b_2 = 2$ and this is case 4. This already proves iii.

In the remaining cases one either has $q = 1$, $(K, K) \geq 1$ or $q = 0$ which makes the right hand side of the Riemann-Roch inequality for mK_S positive in all cases:

$$h^0(mK_S) + h^0(-(m-1)K_S) \geq \frac{1}{2}m(m-1)(K_S, K_S) + 1 - q + p_g.$$

In particular, $P_m \geq \frac{1}{2}m(m-1)(K_S, K_S) + 1 - q + p_g$ as soon as $H^0(-((m-1)K_S)) = 0$. Therefore, to prove i. and ii. I only need to see that $H^0(-((m-1)K_S)) = 0$ if $m \geq 2$. This is an immediate consequence of the following Lemma.

Lemma 5. *Let L be a nef line bundle on a surface S such that L^{-1} has a section. Then L is trivial.*

Proof: Suppose L (and hence L^{-1}) is not trivial. Then there would exist a section of L^{-1} vanishing along a divisor and any curve C transversal to this divisor would satisfy $-(L, C) > 0$ which contradicts the nefness of L. ■

■

Corollary 6. (Castelnuovo's Rationality Criterion) *A surface is rational if and only if* $P_2 = q = 0$.

Problems.

12.1. Prove that any surface with S minimal and $-K_S$ ample is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

12.2. Let S_r be the projective plane blown up in r points in general position. Suppose that $r \leq 6$.

1. Show that the linear system of cubics passing through these points corresponds to the linear system $|-K_{S_r}|$ which gives an embedding of S_r in $\mathbb{P}^{9-r}$ as a degree $9-r$ surface. Such a surface is called a *Del Pezzo surface*.
2. Prove that a surface $S \subset \mathbb{P}^N$ for which $|-K_S|$ is very ample is a Del Pezzo surface or the quadric embedded in $\mathbb{P}^8$.

Chapter 6. The Enriques Classification

13. Statement of the Enriques Classification Theorem

I introduce the classes of surfaces comprising the Enriques classification and give examples of each of these classes before I stae the classification theorem.

So I first introduce a number of useful concepts related to the classification theorem and illustrate these by examples.

Definition 1. *Let S be a surface and C a smooth (projective) curve. A morphism $f : S \to C$ is called a fibration if f is surjective and has connected fibres. If the generic fibre (which is a smooth projective curve) has genus g, the fibration f is called a genus-g fibration. A genus-1 fibration is also called an elliptic fibration. Any surface admitting an elliptic fibration is called an elliptic surface.*

Before giving some examples let me summarise the basic properties of the Kodaira-dimension as treated in §10.

Definition-Proposition 2. *The Kodaira-dimension $\kappa(X)$ of a projective manifold X is the maximal dimension of a pluricanonical image. Equivalently $\kappa(X) = k$ if and only if there are two positive numbers A, B with $A < \dfrac{P_m}{m^k} < B$ for all $m \gg 0$ for which $P_m(X) \neq 0$. If all plurigenera of X vanish, one sets $\kappa(X) = -\infty$.*

Examples 3.

1. The easiest example is of course a product $C \times F$ of two smooth curves, which is a fibration in two ways. Let me calculate the invariants. It was shown before that $q(C \times F) = g(C) + g(F)$, the sum of the genera of the factors and that $p_g(C \times F) = g(C) \cdot g(F)$. In a similar way one finds $P_n(C \times F) = P_n(C) \cdot P_n(F)$. So this gives examples of Kodaira-dimensions $-\infty$ (one of the factors $\mathbb{P}^1$), 0 (both factors elliptic), 1 (one factor elliptic and one of genus ≥ 2) or 2 (both factors of genus ≥ 2).

2. Another type are the fibre bundles over a smooth curve C with fibre F and structure group $\operatorname{Aut} F$, the group of biholomorphic automorphisms of F. You construct them by covering C by Zariski-open sets U_j and glueing the union $U_j \times F$ by means of transition functions $U_i \cap U_j \to \operatorname{Aut} F$.

Concrete examples are the quotients of a product of two curves, $C' \times F'$ by a finite group G, where G is a group of automorphisms of C' and F'. Even more concretely, one may take $G = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ as a subgroup of translations of some elliptic curve F' and construct a ramified Galois-cover $C' \to \mathbb{P}^1$ ramified in three points with covering group G. See Problem 1 for the construction of such coverings. This yields $C' \times F'/G$, at the same time a fibration over $\mathbb{P}^1$ with fibre F' and a fibration over F'/G with fibres C'.

To compute the invariants of $S = C' \times F'/G$ note that

$$\begin{aligned} H^0(S,\Omega^1) =& H^0(C'\times F',\Omega^1)^G = H^0(C',\Omega^1)^G \oplus H^0(F',\Omega^1)^G \\ H^0(S,K_S^{\otimes m}) =& H^0(C'\times F', K_{C\times F'}^{\otimes m})^G = H^0(C',K_{C'}^{\otimes m})^G \otimes H^0(F',K_{F'}^{\otimes m})^G. \end{aligned}$$

Now one uses a special case of Hurwitz formula for mappings $f : C' \to C$ between curves. Recall that locally f is given by $z \mapsto w = z^e$ and e is the ramification index and it equal to 1 except for finitely many points, the ramification points R_j with corresponding e_j. The divisor $R = \sum_j (e_j - 1)R_j$ is called the ramification-divisor and Hurwitz formula states

$$K_{C'} = f^* K_C \otimes \mathcal{O}(R).$$

This is most easily seen by observing that $dw = ez^{e-1}dz$ so that the divisor of any meromorphic 1-form on C' is $R + f^*($ the divisor of a meromorphic 1-form on C).

If C is the quotient of C' by a group G acting on C', the group-action forces the ramification to be the same on all points of a fibre of $C' \to C = C'/G$. So $R = \sum(e_k - 1)f^{-1}(Q_k)$, where $f^{-1}Q_k$ is a complete fibre above Q_j. Now $f^*Q_k = e_k f^{-1}Q_k$, and hence $R = \sum_k (1 - \dfrac{1}{e_k})Q_k$. It follows that

$$K_{C'}^{\otimes m} = f^*\Big(K_C^{\otimes m} \otimes \sum_k (1 - \frac{1}{e_k}) \cdot mQ_k\Big).$$

To compute $H^0(C', K_C^{\otimes m})^G$ note that any G-invariant m-canonical holomorphic form comes from an m-canonical meromorphic form on C and the preceding formula then shows that

$$H^0(C', K_{C'}^{\otimes m})^G = H^0(K_C^{\otimes m} \otimes \sum_k [(1 - \frac{1}{e_k}) \cdot m]Q_k),$$

where $[s]$ means the integral part of the number s. For simplicity, let us write

$$R_m(C',G) = \sum_k [(1 - \frac{1}{e_k}) \cdot m]Q_k).$$

Combining the formulas yields

$$\begin{aligned} q(S) =& g(C'/G) + g(F'/G) \\ p_g(S) =& g(C'/G) \cdot g(F'/G) \\ P_m(S) =& h^0(C'/G, K_{C'/G}^{\otimes m} \otimes \mathcal{O}(R_m(C',G))) \cdot h^0(F'/G, K_{F'/G}^{\otimes m} \otimes \mathcal{O}(R_m(F',G))). \end{aligned}$$

3. A special case of the previous case form the so-called bi-elliptic surfaces.

Definition 4. *A surface $S = E \times F/G$, where E and F are elliptic curves, G a group of translations of E acting on F in such a way that $p_g(S) = 0$ is called bi-elliptic.*

By the previous calculation $p_g(S) = 0$ if and only if F/G is a rational curve. It is relatively simple to classify the possibilities for G and F (any E will work). Since G is a translation subgroup of E it must be abelian and as a transformation group of F it then is the direct product $T \times A$ of its subgroup T of translations and the subgroup A consisting of automorphisms of F preserving the origin. Since the product is direct, the points of T must be invariant under A, which strongly restricts the possible T. Furthermore, since F/G is rational, G cannot consist of translations of F only, and so A must be cyclic of order 2, 3, 4 or 6. Since G is a group of translations of E it is either cyclic or a direct sum of two cyclic groups. From these remarks the following list of possibilities is almost immediate:

1a. $G = \mathbb{Z}/2\mathbb{Z}$ with generator acting as the canonical involution $x \mapsto -x$ on F.

1b. $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with one generator acting as in 1 a., while the other generator acts as translation over a point of order 2.

2a. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ and $G = \mathbb{Z}/4\mathbb{Z}$, the generator acting as multiplication by i.

2b. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ and $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, one generator acting as before, the other by translation over a point of order 2.

3a. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$, $\rho = e^{2\pi i/3}$ and $G = \mathbb{Z}/3\mathbb{Z}$, the generator acting as multiplication by ρ.

3b. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$, $\rho = e^{2\pi i/3}$ and $G = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, one generator acting as multiplication by ρ, the other by translation over $(1-\rho)/3$.

4. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$, $\rho = e^{2\pi i/3}$ and $G = \mathbb{Z}/6\mathbb{Z}$, with generator acting as multiplication by $-\rho$.

The formulas established in Example 2 can be used to show that first of all $p_g = 0$ and $q = 1$. Then one sees that $P_m \leq 1$ and that $K^{\otimes m} = \mathcal{O}$ where $m = 2$ (in case 1a,b), $m = 4$ (in case 2a,b), $m = 3$ (in case 3a,b), $m = 6$ (in case 4). So the Kodaira-dimension is 0 in all cases.

4. Take a fixed point free linear system $|D|$ on a curve C. Let p and q be the two projections of $C \times \mathbb{P}^2$ onto C and $\mathbb{P}^2$ and consider a generic member S of the the linear system $|p^*D \otimes q^*(3H)|$. By Bertini, S will be smooth. The projection p induces a fibration $S \to C$ with fibres plane cubic curves, i.e. this is an elliptic fibration. The canonical bundle formula shows that $K_S = p^*(K_C \otimes D)$ and hence the Kodaira dimension is 1 whenever $\deg D > -\deg K_C = 2 - 2g(C)$.

Definition 5.

1. *A surface with $q = 0$ and trivial canonical bundle is called a K3-surface.*
2. *A surface with $q = 0, p_g = 0$ and $K^{\otimes 2}$ trivial is called an Enriques surface,*
3. *A complex 2-torus which admits an embedding into a projective space is called an Abelian surface,*
4. *A surface is a surface of general type if its Kodaira-dimension is 2.*

Examples 6.

1. Let us consider complete intersections S of multidegrees $d_1, d_2, \ldots, d_n$ in $\mathbb{P}^{n+2}$. The canonical bundle formula shows that $K_S = \mathcal{O}_S(d_1+d_2+\ldots+d_n-n-3)$. It is easy to see that the only combinations giving trivial K_S are $d_1 = 4$, $(d_1, d_2) = (2, 3)$, $(d_1, d_2, d_3) = (2, 2, 2)$. An application of Lefschetz theorem (A2.21) shows that S is simply connected so that $q(S) = 0$. So there are three types of K3-surfaces which are complete intersections.

2. To find an Enriques surface S one first observes that K_S gives an element of order exactly 2 in the Picard group (since $p_g = 0$, the canonical bundle cannot be trivial) and so there exists an unramified Galois cover $\tilde{S}$ of degree 2 with $K_{\tilde{S}}$ trivial. Indeed, one may take

$$\tilde{S} = \{s \in \mathcal{K}_S \ ; \ s^{\otimes 2} = 1\},$$

where one considers the total space of K_S and 1 is the global section corresponding with the constant section 1 of the trivial bundle. Noether's Formula expresses $\chi(\mathcal{O}(S))$ as a linear combination of $e(S)$ and (K_S, K_S). By Corollary A2.8 the Euler number gets multiplied by the degree of the cover, while the selfintersection number of K_S of course also gets multiplied by the degree. So $\chi(\mathcal{O}_{\tilde{S}}) = 2\chi(\mathcal{O}_S) = 2$ and hence $q(\tilde{S}) = 0$. Conversely, any K3-surface $\tilde{S}$ with a fixed point-free involution i yields an Enriques surface. This one sees as follows. Let $q : \tilde{S} \to \tilde{S}/i = S$ be the natural degree 2 cover. Then for any divisor D on S one has $q_*q^*D = 2D$. This is clear for irreducible curves and it then follows by linearity. In particular $q_*q^*K_S = 2K_S$, but $q^*K_S = K_{\tilde{S}}$ as locally any holomorphic 2-form on $\tilde{S}$ is a lift of a holomorphic 2-form on S and so $2K_S$ is trivial.

Now you construct a fixed point free-involution on a suitable K3 which is an intersection of three quadrics in $\mathbb{P}^5$. Let $X_0, \ldots, X_5$ be homogeneous coordinates on $\mathbb{P}^5$ and consider the intersection $\tilde{S}$ of three quadrics $Q'_j(X_0, X_1, X_2) + Q''_j(X_3, X_4, X_5)$, $j = 1, 2, 3$. For generic choices of Q'_j and Q''_j this intersection $\tilde{S}$ will be a smooth surface. The involution ι given by $(X_0, X_1, X_2, X_3, X_4, X_5) \mapsto (X_0, X_1, X_2, -X_3, -X_4, -X_5)$ has two planes of fixed points: the planes $P_1 = \{X_0 = X_1 = X_2 = 0\}$ and $P_2 = \{X_3 = X_4 = X_5 = 0\}$. They miss $\tilde{S}$ precisely if the three quadrics Q'_1, Q'_2, Q'_3, resp. Q''_1, Q''_2, Q''_3 (considered as quadrics in P_2, resp. P_1) have no point in common. For generic choices of Q'_j and Q''_j this will be the case. So then $\tilde{S}/\iota$ will be an Enriques surface.

3. A classical construction of a K3-surface is the Kummer surface. One starts out with an Abelian surface A (for example the product of two elliptic curves), blows up the surface A in the sixteen points of order 2, obtaining $\sigma : \tilde{A} \to A$. The canonical involution ι (sending x to $-x$) extends to an involution $\tilde{\iota}$ on $\tilde{A}$ and the quotient $S = \tilde{A}/\tilde{\iota}$ is a K3-surface. To show this let me calculate the invariants.

Since $\tilde{\iota}$ has only fixed points along the exceptional curves of $\tilde{A}$, the possible singular points of the quotient are among these. Choose local coordinates (x, y) in a neighbourhood U of a point of order two such that ι is given by $(x, y) \mapsto (-x, -y)$. Locally in one of the two standard coordinate patches of $\sigma^{-1}U$ there are coordinates (u, v) such that $\sigma(u, v) = (uv, v)$ and $\tilde{\iota}(u, v) = (u, -v)$. In particular, the quotient by the involution is smooth, and (u, v^2) can be taken as local coordinates on the quotient. If ω is a non-zero holomorphic 2-form on A, in these local coordinates, after multiplying with a non-zero constant, you find

$$\sigma^*\omega = d(uv) \wedge dv = \frac{1}{2} du \wedge d(v^2).$$

Now let $q : \tilde{A} \to S := \tilde{A}/\tilde{\iota}$ be the natural projection. The preceding formula then shows that $\sigma^*\omega = q^*\alpha$ for some nowhere zero 2-form α on S. In particular K_S is trivial. Furthermore, any non-trivial holomorphic 1-form on S would lift to a non-trivial $\tilde{\iota}$-invariant holomorphic 1-form on $\tilde{A}$ and such a form corresponds to a non-trivial holomorphic one-form on A which is invariant under the natural involution. These don't exist and hence $q(S) = 0$.

After these preparations let me state the Enriques Classification theorem.

Theorem 7. (Enriques Classification) *Let S be a minimal algebraic surface. Then S belongs to one of the following non-overlapping classes:*

1. *($\kappa = -\infty$, $q = 0$) $S = \mathbb{P}^2$, $S = \mathbb{F}_n$, ($n = 0, 2, 3, \ldots$).*
2. *($\kappa = -\infty$, $q > 0$) S a geometrically ruled, surface over a curve of genus > 0.*
3. *($\kappa = 0$, $q = 2, p_g = 1$) S is an Abelian surface,*
4. *($\kappa = 0$, $q = 1, p_g = 0$) S is bi-elliptic,*
5. *($\kappa = 0$, $q = 0, p_g = 1$) S is K3,*
6. *($\kappa = 0$, $q = 0, p_g = 0$) S is Enriques,*
7. *($\kappa = 1$) S is minimal elliptic but NOT $\kappa = 0$ or $\kappa = -\infty$,*
8. *($\kappa = 2$) S is of general type.*

Problems.

13.1. Let C be a compact Riemann surface and let $T \subset C$ be a finite set of points. Let G be a finite quotient of the fundamental group of $C \setminus T$ and let $f' : D' \to C \setminus T$ be the finite unramified covering with covering group G. Show that there exists a compact Riemann surface $D \supset D'$ and a holomorphic map $f : D \to C$ extending f'. Apply this to the case $C = \mathbb{P}^1$, and T consisting of three points to construct a covering of $\mathbb{P}^1$ with group the direct product of two finite cyclic groups.

13.2. Let X and Y be compact complex manifolds of the same dimension and let $f : X \to Y$ be a finite surjective map of degree d. The ramification locus is defined as the locus $R_f \subset Y$ of points where *df* does not have maximal rank. Show that R_f is a divisor and that $K_Y = f^*K_X \otimes \mathcal{O}(R_f)$. Derive the Hurwitz formulas from it in case X and Y are curves.

14. The Enriques Classification: First reduction

An important part of the classification theorem rests on the following proposition which deals with Case 4. of Proposition 12.4. The proof of this proposition is very technical and will be dealt with in the following sections. In this section, the proof of the Kodaira classification will be reduced it.

Proposition 1. *Suppose S is a surface with K_S nef and $(K_S, K_S) = 0$, $q = 1$ and $p_g = 0$. Then $\kappa(S) = 0$ or 1 and $\kappa(S) = 0$ if and only if S is bi-elliptic.*

Let me give two immediate consequences of this proposition: the characterisation of $\kappa = -\infty$-surfaces and the characterisation of the minimal rational surfaces.

Corollary 2. *For a minimal surface K_S is nef if and only if $\kappa \geq 0$. In particular a surface is ruled (or rational) if and only if $\kappa = -\infty$.*

Proof: If K_S is nef Proposition 12.4 combined with the previous proposition shows that either $P_2 > 0$ so that $\kappa \geq 0$ or the surface is bielliptic and then also $\kappa \geq 0$. Conversely, if K_S is nef, the surface must be minimal by Reformulation 6.14.

This proves the first part of the corollary.

For the second part, one may assume that S is minimal and from the first part one may assume that K_S is not nef. But then S is geometrically ruled or $S = \mathbb{P}_2$. ■

Corollary 3. *Let S be a minimal rational surface. Then $S \cong \mathbb{P}^2$ or $S \cong \mathbb{F}_n$, $n \neq 1$.*

Proof: Since S is rational, $\kappa(S) = -\infty$. So, by Proposition 12.2 S is the projective plane or S is geometrically ruled. In the latter case, since $q(S) = 0$, by Corollary 7.13 the surface S must be a Hirzebruch surface. ■

Next, let me continue the proof of the classification theorem by considering the case of an elliptic fibration.

Theorem 4. *Suppose that S is a surface with K_S nef and $(K_S, K_S) = 0$. Then $\kappa(S) = 0$ or 1. In the last case S admits the structure of an elliptic fibration.*

Proof: By Proposition 14.2 one has $\kappa(S) \geq 0$. Assume that $\kappa(S) \geq 1$. Then for n large enough $|nK_S|$ is at least 1-dimensional. Let D_{f}, resp $|D|$ be the fixed part, resp. the variable part of this linear system.

Claim $(D, D) = (K_S, D) = 0$,

Proof: (of Claim) One has $0 = n(K_S, K_S) = (D_{\mathrm{f}}, K_S) + (D, K_S)$ and since each term is ≥ 0 by nefness of K_S these must vanish. Now $0 = n(D, K_S) = (D, D) + (D, D_{\mathrm{f}})$ and again, each term is non-negative, since D moves and so $(D, D) = 0 = (D, D_{\mathrm{f}})$. ■

The claim implies that the rational map $f = \varphi_{nK_S}$ is a morphism and that $f : S \to C$ maps every divisor $D \in |D|$ to a point and so C is a curve. This is true for all n large enough so that $|nK_S|$ is at most 1-dimensional and hence $\kappa(S) = 1$ in this case. If D is a smooth fibre of f, the adjunction formula says that the connected components are elliptic curves and so, taking the Stein factorisation of f, one obtains an elliptic fibration. ■

Finally, consider the case of Kodaira dimension 0.

Proposition 5. *Suppose that K_S is nef, that $(K_S, K_S) = 0$ and that $\kappa(S) = 0$. Then S is bi-elliptic, an abelian surface, a K3-surface or an Enriques surface.*

Proof: By Proposition 12.4 and Proposition 14.1 you only have to consider the cases $p_g(S) = 0 = q(S)$ and the case $p_g(S) > 0$ (and hence $p_g = 1$). Moreover, if $p_g = 0$ one must have $P_2 = 1$, again by 12.4. Let me first deal with this case. I claim that $P_3 = 0$. If not, then $P_3 = 1$. Let $D_2 \in |2K_S|$ and $D_3 \in |3K_S|$. So $3D_2$ and $2D_3$ are both divisors in $|6K_S|$. Since $P_6 \leq 1$ you must have $3D_2 = 2D_3$ and there must be an effective divisor D with $D_2 = 2D$ and $D_3 = 3D$. Necessarily $D = D_3 - D_2 \in |K_S|$, but $p_g = 0$. So indeed $P_3 = 0$.

Now apply the Riemann-Roch inequality to $3K_S$. One has

$$h^0(3K_S) + h^0(-2K_S) \geq 1$$

and hence $h^0(-2K_S) \geq 1$. Since $P_2 = h^0(2K_S) = 1$ this is only possible if $2K_S$ is trivial. It follows that S is an Enriques surface.

I next suppose that $p_g = 1$. Consider the Noether formula in this case. It reads as follows.

$$12(2 - q(S)) = e(S) = 2 - 4q(S) + b_2$$

and hence $b_2 = 22 - 8q(S)$. So $q(S) = 0$, 1, 2.

In the first case you have a K3-surface. Indeed the Riemann-Roch inequality applied to $2K_S$ yields $h^0(2K_S) + h^0(-K_S) \geq 2$. In a similar way as in the previous case, I infer from this that K_S is trivial.

I shall exclude the possibility $q(S) = 1$ and show that S is a torus in the remaining case.

Since $q(S) > 0$, you can find a non-trivial line bundle $\mathcal{O}_S(\tau)$ with $\mathcal{O}_S(2\tau) = \mathcal{O}_S$ (any non-trivial 2-torsion point of the torus $\mathrm{Pic}^0(S)$ gives such a line bundle). If $q = 1$, the Riemann-Roch inequality reads

$$h^0(\mathcal{O}_S(\tau)) + h^0(\mathcal{O}_S(K_S - \tau)) \geq 1$$

and hence $h^0(\mathcal{O}_S(K_S - \tau) \geq 1$. Take $D \in |K_S - \tau|$ and let K be any canonical divisor. One has $2D = 2K$ since $P_2 = 1$ and hence $D = K$, contradicting the fact that $\mathcal{O}_S(\tau) \not\cong \mathcal{O}_S$.

In the second case, you first look at the possible components of the canonical divisor $K = \sum_j m_j C_j$. Since K_S is nef and $(K_S, K_S) = 0$ you find $(K_S, C_i) = 0$. Writing down

$$0 = (K, C_j) = m_j(C_j, C_j) + \sum_{i \neq j} m_i(C_i, C_j)$$

you see that either $(C_j, C_j) = -2$ and C_j is a smooth rational curve, or you have $(C_j, C_j) = 0$ but also $(C_j, C_i) = 0$ for all $i \neq j$. So you can partition the connected components of $\cup C_i$ into two types: unions of smooth rational curves on the one hand and smooth elliptic curves or rational curves with one node on the other hand.

I consider the Albanese map $S \to \mathrm{Alb}\, S$. It either maps to a curve $C \subset \mathrm{Alb}\, S$ or it maps onto the (two-dimensional) Albanese. In the first case, since $q(S) = 2$ the curve C is a genus 2 curve by Proposition 10.6. Let $f : S \to C$ be the resulting fibration. By the preceding analysis, every connected component D of the canonical divisor K is either rational or elliptic. Since such curves cannot map onto a curve of genus 2 these must be contained in some fibre F of f, say over $c \in C$.

Now I need Zariski's lemma:

Lemma 6. (Zariski's Lemma) *Let $f : S \to C$ be a fibration of a surface S to a curve C and let $F = \sum_i m_i C_i$ be a fibre, where C_i is irreducible. Let $D = \sum_i r_i C_i$ be a $\mathbb{Q}$-divisor. Then $(D, D) \le 0$ and equality holds if and only if $D = rF$ for some $r \in \mathbb{Q}$.*

Proof: Let $F_i = m_i C_i$ so that $F = \sum_i F_i$. Also, set $s_i = r_i / m_i$ so that $D = \sum_i s_i F_i$. One has

$$\begin{aligned}(D, D) =& \sum_i s_i^2 (F_i, F_i) + 2 \sum_{i<j} s_i s_j (F_i, F_j) \\ =& \sum s_i^2 (F_i, F) - \sum_{i<j} (s_i^2 + s_j^2 - 2 s_i s_j)(F_i, F_j) \\ =& 0 - \sum_{i<j} (s_i - s_j)^2 (F_i, F_j) \le 0\end{aligned}$$

with equality if and only if $s_i = s_j = r$ for all i and j, which means $D = r \sum_i F_i = rF$. ■

By Zariski's Lemma $D = a/b \cdot F$ with a, b positive integers. Then $bD = f^*(a[c])$ and hence $h^0(ndD)$ and $h^0(ndK_S)$ grow indefinitely when n tends to infinity. This contradicts $\kappa(S) = 0$. The possibility that K_S is trivial is left. In this case, simply take an unramified cover $C' \to C$ of degree ≥ 2 and pull back your fibration. You get an unramified cover $S' \to S$ of degree ≥ 2 and $K_{S'}$ is still trivial, $\chi(\mathcal{O}_{S'}) = 0$ and hence $q(S') = 2$ by what we have seen. But $q(S') \ge q(C') \ge 3$, a contradiction.

There remains the case that $\operatorname{Alb} S$ is a 2-torus and that the Albanese maps surjectively onto it. It is an elementary fact that in this case $\alpha^* : H^2(\operatorname{Alb} S) \to H^2(S)$ is injective (dually: every 2-cycle on $\operatorname{Alb} X$ is homologous to a cycle which lifts to a 2-cycle on X). Since $b_2(S) = 6$ this then is an isomorphism and so no fundamental cohomology-class of a curve maps to zero. In particular, the Albanese map must be a finite morphism. So, if D is a connected component of the canonical divisor it cannot map to a point and hence it must be an elliptic curve E which maps to an elliptic curve $E' \subset \operatorname{Alb} S$. Now form the quotient elliptic curve $E'' = \operatorname{Alb} S / E'$ and consider the surjective morphism $S \to E''$. The Stein-factorisation then yields an elliptic fibration and D is contained in a fibre. By the Zariski lemma 14.6 it follows that D is a rational multiple of a fibre and as before one concludes that $\kappa(S) = 1$. It follows that the only possibility is that K_S is trivial, but then, by the formula for the canonical divisor of coverings (Problem 14.2) I conclude that the Albanese map is a finite unramified covering and therefore S itself is a torus. ■

15. The canonical bundle formula for elliptic fibrations

The formula refered to in the section heading will be used in the final step of the classification theorem but is also of independent interest. The proof given here does not use relative duality in contrast with the proof of [B-P-V].

Multiple fibres of a fibration $f : S \to C$ of a surface S onto a curve C are defined in the following way. If $F = \sum_j m_j F_j$ is a singular fibre with F_j irreducible, the *multiplicity* of F is the greatest common divisor m of the numbers m_i and F is called *multiple* if this multiplicity is > 1. If $F = f^{-1}c = mF'$ is such a multiple fibre and $\{U, z\} \subset C$ a coordinate disk about c, the function $z \circ f$ vanishes of multiplicity m along F'. Since all fibres are connected, all $g \in \mathcal{O}(f^{-1}U)$ are of the form $g = h \circ f$ with $h \in \mathcal{O}(U)$ and so, if g vanishes along F' it must vanish with order $\geq m$.

Now $\mathcal{O}_{f^{-1}U}(mF') = \mathcal{O}_{f^{-1}U}(F)$ is trivial and so $\mathcal{O}_{f^{-1}U}(F')$ has order $\leq m$. Smaller order is impossible, since it would give a function vanishing to order $< m$ along F'. It follows that $\mathcal{O}(F')$ has order precisely m in $\mathrm{Pic}(f^{-1}U)$. I claim that this also holds for the restriction of the bundle to F'.

Lemma 1. *The bundle $\mathcal{O}_{F'}(F')$ has order m in* $\mathrm{Pic}\, F'$.

Proof: The proof depends on the existence of an exponential sequence on F'.

Sublemma 2. *Let $C = \sum_i n_i C_i$ be any effective divisor on a surface S. Let $\mathcal{O}_C^*$ be the sheaf of functions on C which are restrictions to C of functions which are nonzero in a neighbourhood in S of a point of C. There is a commutative diagram*

$$\begin{array}{ccccc} \mathcal{O}_S & \xrightarrow{\text{restriction}} & \mathcal{O}_C & \to 0 \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & \\ \mathcal{O}_S^* & \xrightarrow{\text{restriction}} & \mathcal{O}_C^* & \to 1 \end{array}$$

and the map exp *fits into an exact sequence*

$$0 \to \mathbb{Z}_C \to \mathcal{O}_C \xrightarrow{\exp} \mathcal{O}_C^* \to 1.$$

Proof of sublemma: For simplicity assume that C has connected support. If $g \in \mathcal{J}_C$, from $\exp(g) - 1 = \sum_{m \geq 1} (2\pi i g)^m / m!$ it follows that $e^{2\pi i g} - 1 \in \mathcal{J}_C$. Here you use 'closedness of submodules' [Gr-Re, Chapt. 2 §2.3]. This remark implies that the diagram is commutative. the only non-trivial part in the exactness statement for the exponential sequence is the fact that $\exp(f) = 1$ for $f \in \mathcal{O}_C$ implies that $f|C = n \in \mathbb{Z}$. To see this, let $g \in \mathcal{O}_S$ restrict to f and consider $e^{2\pi i g} - 1 \in \mathcal{J}_C$. Look at $g|C_i$. This function must be equal to some fixed integer n by the usual exponential sequence and connectedness of the support of C. So $g - n$ vanishes along every C_i, say precisely to order m_i. Then also $\exp(g) - 1 = \sum_i (2\pi i g)^m / m!$ vanishes precisely to order m_i along C_i and hence $m_i = n_i$, i.e. $g - n \in \mathcal{J}_C$ and so $f = n$ on C. ■

Proof of the lemma:

Let me triangulate S such that F' supports a subcomplex. See Appendix A2, Example A2.1 (this treats the case when F' is smooth; in the case at hand, there are finitely many singular points and the same methods apply to yield the desired triangulation; anyway, I advise you to accept these topological fineries upon first reading). In particular it follows that F' is a deformation retract of a neighbourhood, which one may assume to be of the form $f^{-1}U$ with U an open neighbourhood of $c \in C$. Then the restriction maps $H^p(f^{-1}U, \mathbb{Z}) \to H^p(F', \mathbb{Z})$ are isomorphisms. Now consider the exponential sequence on $f^{-1}U$ and its restriction to F'. I find a commutative diagram

$$\begin{array}{ccccccc}
H^1(f^{-1}U, \mathbb{Z}) & \to & H^1(\mathcal{O}_{f^{-1}U}) & \to & H^1(\mathcal{O}^*_{f^{-1}U}) & \to & H^2(f^{-1}U, \mathbb{Z}) \\
\| & & \downarrow & & \downarrow & & \| \\
H^1(F', \mathbb{Z}) & \to & H^1(\mathcal{O}_{F'}) & \to & H^1(\mathcal{O}^*_{F'}) & \to & H^2(F', \mathbb{Z}).
\end{array}$$

Chasing through this diagram now gives the result. ■

Next, an explicit description of the possible multiple singular fibres in an elliptic fibration is needed.

Lemma 3. *If $F = mF'$ is a multiple fibre of an elliptic fibration, F' is either a smooth elliptic curve, a rational curve with one ordinary double point or a cycle of non-singular rational curves.*

Proof: For a smooth fibre $(K, F) = 0$ by the adjunction formula and hence $(K, F') = 0$ which implies that $\chi(O_{F'}) = 0$ by Lemma 5.7. So, if F' is connected, it is either an elliptic curve or a rational curve with an ordinary node by remark 5.8. If F' is reducible and E_i is a component of the fibre F', Zariski's lemma 14.6 implies that $(E_i, E_i) < 0$ and hence, by minimality and the adjunction formula, E_i must be a smooth rational curve with $(E_i, E_i) = -2$. Again by Zariski's lemma, we see that the intersection number of two distinct components components E_i and E_j is 1 or 2. Now I claim that the topological space F' cannot be simply connected. Indeed, by the previous lemma, there is a non-trivial torsion bundle on F' and so $H^1(F', \mathbb{Z}) \neq 0$. If some intersection number (E_i, E_j) is equal to 2 there must be exactly two components forming a cycle as can be seen as follows. If $F' = \sum_i n_i E_i$ one has

$$0 = (E_i, F') = -2n_i + \sum_{j \neq i} n_j (E_i, E_j).$$

If $n_i \leq n_j$ it follows from $2n_i = 2n_j + \sum_{k \neq i,j} n_k (E_k, E_i)$ that all coefficients n_i must be one and that there are exactly two E_i. If $(E_i, E_j) \leq 1$ and three components meet in a single point one similarly finds that there are no more components, contradicting non-simply connectedness of F' in this case. So there must be a cycle contained in F' and it is easily seen that then there can be no more components. ■

Corollary 4. *The dualising sheaf $\omega_{F'} = \mathcal{O}_{F'}(K_S \otimes \mathcal{O}(F'))$ is trivial.*

Proof: To compute the dualising sheaf recall Proposition 5.9. If F' is elliptic, of course $\omega_{F'} = K_{F'} = \mathcal{O}_{K'}$. Otherwise, there is a global meromorphic one-form α on each component $E_i \cong \mathbb{P}^1$ with poles in two points with opposite residues (the form dz/z has this property with respect to 0 and ∞). If one scales such a form on each of the components suitably, the residues in points belonging to two components cancel. This defines a trivialising section of the sheaf $\omega_{F'}$. ■

Let me come now to the main result.

Theorem 5. *Let S be surface with K_S nef and let $f : S \to C$ be an elliptic fibration. Let F_i, $i = 1, \ldots, m$ be the multiple fibres and let m_i be the multiplicity of $F_i = m_i F_i'$. One has*

$$K_S = f^*L + \sum_{i=1}^{m}(m_i - 1)F_i'$$

with L a divisor on C of degree $\chi(\mathcal{O}_S) - \chi(\mathcal{O}_C)$.

Proof:
Step 1.: $K_S = f^*L + \sum_i (n_i - 1)F_i'$ for some divisor L on C.
Take N smooth fibres $G_1, \ldots, G_N$ and tensor the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}(\sum_j G_j) \to \oplus_j \mathcal{O}_{G_j}(G_j) \to 0$$

with K_S. Since $\mathcal{O}_{G_j}(K_S + G_j) \cong \mathcal{O}_{G_j}$ by the Adjunction Formula and the fact that G_j is a smooth elliptic curve, one finds

$$0 \to \mathcal{O}(K_S) \to \mathcal{O}(K_S + \sum_j G_j) \to \oplus_j \mathcal{O}_{F_j} \to 0.$$

The long exact sequence in cohomology gives $h^0(K_S + \sum_j G_j) \geq p_g - q + (N - 1)$ and so for N large enough, there exists a divisor $D \in |K_S + \sum_j G_j|$. Now $(F, G_j) = 0 = (K_S, F)$ and so $(D, F) = 0$ implies that D consists of linear combinations of fibral components D_i, $i = 1, \ldots, M$. Since D is effective, and K_S nef, one has $(D, D_i) = (K_S, D_i) \geq 0$. On the other hand, from Proposition 12.4 one sees that $(K_S, K_S) = 0$ and hence

$$0 = (K_S, K_S) = (D, K_S) = (\sum d_i D_i, K_S).$$

It follows that $(D, D_i) = (K_S, D_i) = 0$ and so $(D, D) = 0$. If $D = \sum_i D_i'$ with D_i' the part supported in exactly one fibre, it follows that $(D_i', D_i') = 0$ and from Zariski's lemma one concludes that D_i' is a rational multiple of a fibre. Any part supported on a multiple fibre F_i can be written as $n_i F_i' + r_i F_i$ with $0 < n_i < m_i$ and $r_i \in \mathbb{Z}$. The parts supported on a non-multiple fibre G_j must be of the form $r_j G_j$ with $r_j \in \mathbb{Z}$ and so $K_S = f^*(L) + \sum_i n_i F_i'$ and since $\mathcal{O}_{F_i'}(K_S + F_i') = \mathcal{O}_{F_i'} = \mathcal{O}_{F_i'}((n_i + 1)F_i')$, by the previous Corollary, one finds that $n_i + 1 = m_i$.
Step 2. deg $L = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_C)$.
Compute $h^0(K_S + \sum_j G_j)$ in two ways. First, note that $\sum_i (m_i - 1)F_i'$ is a fixed component

of $|K_S + \sum_j G_j|$ and so $h^0(K_S + \sum_j G_j)) = h^0(\mathcal{O}_S(f^*L + \sum G_j) = h^0(\mathcal{O}_C(L + \sum_j c_j))$, where $c_j = f(G_j) \in C$. Riemann-Roch then shows that

$$h^0(K_S + \sum_j G_j) = \deg L + N + 1 - g(C)$$

provided N is large enough. On the other hand, the computation from Step 1 shows that

$$h^0(K_S + \sum_j G_j) = \chi(\mathcal{O}_S) + N - 1 + \dim\Big(\mathrm{im}\ H^1(\mathcal{O}_S(K_S)) \to H^1(\mathcal{O}_S(K_S + \sum_j G_j))\Big).$$

The dimension of the image in the preceding formula, by Serre-duality translates into the dimension of the image of $H^1(\mathcal{O}_S(-\sum_j G_j)) \to H^1(\mathcal{O}_S)$, which, by exactness of the usual cohomology sequence, is the dimension of the kernel of $H^1(\mathcal{O}_S) \to \oplus_j H^1(\mathcal{O}_{G_j})$. Now the Hodge decomposition for H^1 tells us that $H^1(\mathcal{O}) = \overline{H^0(\Omega^1)}$ and so, the dimension I am after is equal to the dimension of the kernel of $H^0(\Omega^1_S) \to \oplus_j H^0(\Omega^1_{G_j})$. But this kernel consists precisely of the holomorphic one forms which are pull backs of 1-forms on C and these form a space of dimension $g(C)$. See Problem 17.3.

Combining everything, you get

$$h^0(K_S + \sum_j G_j) = \deg L + N + 1 - g(C) = \chi(\mathcal{O}_S) + N - 1 + g(C)$$

and so $\deg L = \chi(\mathcal{O}_S) - 2 + 2g(C) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_C)$.

■

16. Two technical results and the final step

For the final step of the proof of the classification theorem, an easy topological result about fibrations of surfaces over curves is needed but also an involved result about isotrivial fibrations. Then the proof of the classification theorem can be completed by proving Proposition 14.1.

First I state and prove the fact about the topology of fibrations.

Proposition 1. *Let S be a surface with K_S nef and let $f : S \to C$ be a fibration onto a curve. Let $\delta(f) \subset C$ be the (finite) set of critical values of f, i.e. $c \in \delta(f)$ if and only if at some $s \in f^{-1}(c)$ the map $df(s)$ vanishes. Let F be a smooth fibre and let $F_c = f^{-1}(c)$. One has*

$$e(S) = e(C)e(F) + \sum_{b \in \delta(f)} (e(F_b) - e(F)).$$

Furthermore, $e(F_b) - e(F) \geq 0$ with equality if and only if F_b supports a smooth elliptic curve.

Proof: You triangulate C in such a way that $\delta(f)$ becomes a subcomplex L' of the resulting complex K'. Likewise you triangulate $f^{-1}L$ and extend the triangulation to S Let (K, L) be the resulting pair of complexes. Then $e(K) = e(K \setminus L) + e(L) = e(K' \setminus L')e(F) + \sum_{b\in\delta(f)} e(F_b)$. The last equality follows, since f is topologically a locally trivial fibration over the set of non-critical values of f. See Problem 17.1. So $e(S) = e(C)e(F) - \sum_{b\in\delta(f)} e(F) + \sum_{b\in\delta(f)} e(F_b)$, which proves the first statement. For the second statement I need

Lemma 2. *Let $C = \sum_i C_i$ be any curve on a surface, where C_i are the irreducible components. Then $e(C) \geq 2\chi(\mathcal{O}_C)$ with equality if and only if C is smooth.*

Proof: (of the Lemma) Let $\nu : \tilde{C} \to C$ be the normalisation of the curve C. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{C}_C & \to & \nu_*\mathbb{C}_{\tilde{C}} & \to & \delta & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j} & & \\ 0 & \to & \mathcal{O}_C & \to & \nu_*\mathcal{O}_{\tilde{C}} & \to & \Delta & \to & 0. \end{array}$$

From the diagram one finds that

$$e(\tilde{C}) = e(C) + h^0(\delta)$$
$$\chi(\mathcal{O}_{\tilde{C}}) = \chi(\mathcal{O}_C) + h^0(\Delta)$$

and since $e(\tilde{C}) = 2\chi(\mathcal{O}_{\tilde{C}})$, one finds

$$e(C) = 2\chi(\mathcal{O}_C) + 2h^0(\Delta) - h^0(\delta).$$

One checks that j is injective and hence $h^0(\delta) \leq h^0(\Delta)$ and so $e(C) \geq 2\chi(\mathcal{O}_C)$ with equality if and only if $h^0(\Delta) = 0$. ■

Now I can finish the proof of the Proposition. For a singular fibre $F_b = \sum_i m_i C_i$, put $F' = \sum C_i$. The lemma says that $e(F') \geq 2\chi(\mathcal{O}_{F'})$. On the other hand $2\chi(\mathcal{O}_{F'}) = -(F', F') - (F', K_S)$ by the Adjunction Formula. The first term is ≥ 0 by Zariski's Lemma. For the second term write $-\sum_i (C_i, K_S) \geq \sum_i -m_i(C_i, K_S)$ (since K_S is nef)$=$ $-(F_b, K_S) = -(F, K_S) = e(F)$). It follows that $e(F_b) \geq e(F)$ and equality implies that F' is smooth, and hence $F_b = nF'$. Closer inspection of the preceding computation then reveals that $e(F') = 1/n \cdot e(F)$. Now f cannot be a fibration with rational fibres, since K_S is nef. So $e(F) \leq 0$ and hence $e(F') = e(F) = 0$, i.e. F' is a smooth elliptic fibre. ■

Secondly, I need a result about families of curves over a base curve of low genus. To this end I introduce the notion of isotrivial fibration.

Definition 3. *A fibration $f : X \to Y$ between projective manifolds is called isotrivial if there exists a finite unramified covering $g : Y' \to Y$ such that the pull back $f' : X' = X \times_Y Y' \to Y'$ of f is isomorphic to a product-fibration $X' \cong Y' \times F$, for some projective manifold F.*

Example 4. Let G be a finite group which is the quotient of $\pi_1(Y)$ and which acts on a manifold F. Let $g : Y' \to Y$ be the covering defined by G and consider the product action of G on $Y' \times F$. The quotient manifold $(Y' \times F)/G$ admits an isotrivial fibration onto Y. Conversely, any fibre bundle $X \to Y$ such that the fibre F has a finite group of automorphisms arises in this way. See Problem 4.

The main result in this section is:

Proposition 5. *Suppose that $f : S \to C$ is a fibration of a surface onto a curve of genus* 0 *or* 1 *and suppose that f has everywhere maximal rank. Then f is isotrivial.*

Before I give a proof of this proposition I make a few comments on the situation arising in this proposition. Any proper surjective morphism $f : X \to Y$ between complex manifolds (not necessarily compact) which is everywhere of maximal rank is called a *family of complex manifolds*. Note that all fibres in a family are compact manifolds. By Ehresmann's theorem (Problem 1), the family is differentiably locally trivial. I can therefore assume that the sheaf $\cup_{y\in Y} H_1(X_y, \mathbb{Z})$ is locally constant. In case the fibres are curves, this is a locally constant sheaf of $\mathbb{Z}$-modules of rank $2g$, where g is the genus of the fibre. Now consider f_*K_Y. This is a sheaf whose fibres are $H^0(K_{X_y})$ and hence have constant dimension. It is a non-trivial fact that f_*K_Y is locally trivial on Y and has rank g. This follows for example from deep results of Grauert. See [Gr-Re, Chapter 10, §5] or [Ha, p. 288] for a proof in the slightly easier algebraic setting. In the case at hand this implies that you can choose a basis $\omega_1(y), \ldots, \omega_g(y)$ for $H^0(K_{X_y})$ depending holomorphically on y. Now you take any small open subset $U \subset Y$ over which the family is differentiably trivial and so there is a constant basis $\gamma_1, \ldots, \gamma_{2g}$ for $H_1(X_y, \mathbb{Z})$ over U such that the periods matrices

$$\begin{pmatrix} \int_{\gamma_1} \omega_1(y) & \cdots & \int_{\gamma_{2g}} \omega_1(y) \\ \vdots & \ddots & \vdots \\ \int_{\gamma_1} \omega_g(y) & \cdots & \int_{\gamma_{2g}} \omega_g(y) \end{pmatrix}$$

are all normalised (as defined in Example 10.5).

For the rest of the proof I assume that the reader is acqainted with some more advanced topics from the theory of curves, for instance with the material presented in [G-H, Chapter 2, §6-7].

Any normalised period matrix Ω is a symmetric matrix whose imaginary part is positive definite and hence defines a point in the Siegel upper half space

$$\mathfrak{h}_g = \{\Omega \in \mathbb{C}^{g,g} \ ; \ \Omega = \Omega^T, \operatorname{Im}(\Omega) > 0\}.$$

Note that different choices of symplectic bases for H_1 give a different normalised period matrix Ω and one can compute that these are in the same Γ_g-orbit, where Γ_g is the symplectic group modulo its center $\pm\mathbb{1}_g$ acting by means of fractional linear transformations: if

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

the action is given by $\gamma(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}$.

Fixing one symplectic basis for the first homology of curves over U, you get a holomorphic map

$$U \to \mathfrak{h}_g,$$

the *local period map* associated to the family over U. In general, one cannot extend this map to a uni-valent map $Y \to \mathfrak{h}_g$, due to the monodromy action on $H_1(X_y, \mathbb{Z})$. This action can be trivialised by passing to the universal cover $\tilde{Y}$ of Y. The holomorphic map $p : \tilde{Y} \to \mathfrak{h}_g$ thus obtained is the *period map.*

Let me now sketch the proof of the Proposition.

Proof: (Sketch) The universal cover of C is $\mathbb{P}^1$ or $\mathbb{C}$. In both cases the period map then has to be constant since $\mathfrak{h}_g$ is isomorphic to a bounded domain (see Problem 5). Now apply Torelli's theorem:

Theorem (Torelli's theorem [G-H,p. 359]) *Let C and C' be two smooth curves of genus g such that their Jacobians together with their natural polarisations are isomorphic, or, equivalently such that their normalised period matrices are in the same Γ_g-orbit. Then C and C' are isomorphic.*

It follows that all fibres of f are isomorphic. Then you can for instance apply the local-triviality theorem of Grauert-Fischer [F-G] which says that f must be a locally trivial fibre bundle in this case. If the genus of the fibre is ≥ 2, the automorphism group of the fibre is always finite and hence f is isotrivial by Example 16.4. If the fibre is elliptic, this is also the case, but slightly more involved. Consider the action of the fundamental group of C on the group of n-torsion points of a fibre F. It can be trivialised by passing to a finite unramified covering. In particular you have the zero-section globally. Since the group of automorphisms of an elliptic curve preserving the origin is finite, it follows that after taking some unramified covering, the family becomes trivial. ■

Now, finally, the proof of Proposition 14.1 can be given. Let me recall it before giving the proof.

Proposition 6. *Suppose S is a surface with K_S nef and $(K_S, K_S) = 0$, $q = 1$ and $p_g = 0$. Then $\kappa(S) = 0$ or 1 and $\kappa(S) = 0$ if and only if S is bielliptic.*

Proof: Since $q(S) = 1$, the Albanese of S is an elliptic curve C and by Lemma 10.6 the Albanese mapping $\alpha : S \to C = \operatorname{Alb} S$ has connected fibres. Recall (Proposition 12.4) that $b_2(S) = 2$ and hence $e(S) = 2 - 2b_1(S) + b_2(S) = 0$. Now apply the topological lemma 16.1 to conclude that $\alpha : S \to C = \operatorname{Alb} S$ is either a genus g fibration with $g \geq 2$ and α everywhere of maximal rank, or an elliptic fibration with only smooth fibres, some of which are possibly multiple. In the first case, apply Proposition 16.5 to conclude that $\alpha : S \to C = \operatorname{Alb} S$ is isotrivial, so that there exists a finite unramified covering $\hat{S} \to S$ which is a product. By Proposition 9.9 the Kodaira-dimension does not change under finite unramified covers and so $\kappa(S) = 1$ in this case. So S is not bielliptic. If $\alpha : S \to C = \operatorname{Alb} S$ is elliptic and has multiple fibres, an application of the elliptic bundle formula 15.5 shows that $\kappa(S) = 1$ in this case too. There remains the possibility that $\alpha : S \to C = \operatorname{Alb} S$ is an elliptic fibration which is everywhere of maximal rank and then, again by Proposition 16.5 one has an isotrivial fibration. There exists therefore an elliptic curve F, an elliptic curve

E, a group of automorphisms G of F acting as translations on E such that $S = E \times F/G$. Since $p_g(S) = 0$, by definition S is bi-elliptic. So only in the case $\kappa(S) = 0$ you get a bi-elliptic surface, in all other cases $\kappa(S) = 1$.

■

Problems.

16.1. Let $f : X \to Y$ be a surjective differentiable map between differentiable manifolds which is everywhere of maximal rank and which is proper. Show that f is locally a differentiably trivial fibre bundle. This is sometimes called *Ehresmann's Theorem*. Hint: see e.g. [We, Chapter V, Proposition 6.4].

16.2. Let $f : X \to Y$ be a locally trivial fibre bundle with compact fibres $X_y = f^{-1}y$. The fundamental group $\pi_1(Y, y_0)$ acts on $H^k(X_y, \mathbb{Q})$ as follows (*monodromy representation*). Choose a loop $\gamma : I \to Y$ based at y and choose a differentiable trivialisation of the pull back of f to γ. So there are diffeomorphisms $g_t : X_{\gamma(0)} \to X_{\gamma(t)}$ and hence an induced isomorphism g_1^* on $H^k(X_y, \mathbb{Q})$.
Show that it is independent of the choosen trivialisation and that g_1^* only depends on the class of γ in the fundamental group.
The groups $H^k(X_y, \mathbb{Q}), y \in Y$ form a locally constant sheaf on Y which is nothing but $R^k f_* \mathbb{Q}$ and any class a_y invariant under monodromy yields a global section of this sheaf.
Let $a \in H^k(X, \mathbb{Q})$ be given and consider the restrictions $a_y \in H^k(X_y, \mathbb{Q})$ of a to the fibres. Show that this yields a global section of $R^k f_* \mathbb{Q}$.

16.3. Let $f : S \to C$ be an ellipic fibration and let ω be a holomorphic 1-form on S which restricts to zero on a smooth fibre F. Show that $\omega = f^*\omega'$ where ω' is a holomorphic 1-form on C. Hint: Use the previous exercise to see that ω restricts to zero on all smooth fibres. Now fix a regular value $x_0 \in S$ for f and define $g(x) = \int_{x_0}^{x} \omega$. This yields a well-defined function on $f^{-1}U$ where U is a suitable neighbourhood of x_0. It is constant on any fibre and so is of the form $g = f^*h$ with $h \in \mathcal{O}_C(U)$ and $\omega = f^*(dh)$ on $f^*(U)$. The local forms dh define a global meromorphic form ω' with $\omega = f^*\omega'$. Show that ω' must be holomorphic.

16.4. Show that any fibre bundle is an isotrivial fibration if the fibre has a finite group of automorphisms. Hint: any fibre bundle determines a homomorphism of the fundamental group of the base manifold to the group of automorphisms of the fibre ('monodromy') and the bundle is trivial if and only if this homomorphism is trivial.

16.5. Prove that $\mathfrak{h}_g$ is isomorphic to the bounded domain

$$\{U \in \mathbb{C}^{g,g} \; ; \; \mathbb{1}_g - U\bar{U}^T \geq 0\}.$$

Hint: consider the map $\mathfrak{h}_g \ni Z \mapsto (i\mathbb{1}_g + Z)(i\mathbb{1}_g - Z)^{-1}$.

A 1. Appendix: Some algebra

A1.1 Direct limits of modules

In this section R is any commutative ring with unit.

One starts with a partially ordered set I with partial ordering $<$. Assume that the partial ordering is directed. This means that every two elements $i, j \in I$ have a common upperbound k, i.e. $i < k$ and $j < k$. Assume furthermore that for every $i \in I$ there is some R-module M_i and for any pair $i, j \in I$ with $i < j$ there are homomorphisms $h^i_j : M_i \to M_j$ which satisfy a cocycle relation $h^j_k \circ h^i_j = h^i_k$.

A direct limit of this system of modules is a module M together with homomorphisms $h^i : M_i \to M$ with $h^j \circ h^i_j = h^i$ whenever $i < j$ and such that the usual universality property holds: Given any module N with homomorphisms $k^i : M_i \to N$ which also satisfy $k^j \circ h^i_j = k^i$ whenever $i < j$ there is a unique homomorphism $k : M \to N$ such that $k^i = k \circ h^i$ for all $i \in I$.

It follows that any two direct limits are isomorphic by a unique isomorphism and it makes sense to speak of the direct limit denoted by

$$\operatorname*{dirlim}_I M_i.$$

There is the following standard construction of the direct limit. One takes the direct product $\prod_{i \in I} M_i$ and identifies $m \in M_i$ (viewed as submodule of the product) with $m' \in M_j$ whenever there is some k with $i < k$ and $j < k$ such that $h^i_k(m) = h^j_k(m')$. The quotient module M and the natural maps $h^i : M_i \to M$ then satisfy the properties needed for direct limit as one may readily verify.

A useful remark is that in forming the direct limit one need not take the entire set I. Any subset $J \subset I$ which itself directed under $<$ and which is co-final in it will do. This means that for any $i \in I$ there is some $j \in J$ with $i < j$. So, the remark is that the homomorphism resulting from the universal property of direct limits

$$\operatorname*{dirlim}_J M_j \to \operatorname*{dirlim}_I M_i$$

is indeed an isomorphism. It is straightforward to see that this map is surjective because of the fact that J is cofinal. That it is injective is slightly more involved, and is left to the reader.

A1.2 Some basic commutative algebra

The following concepts and theorems are used freely. For background and proofs see [Reid, p.48-49].

A commutative ring R with unit is *Noetherian* if every ideal in it is finitely generated. Equivalently, every ascending chain of ideals in R becomes stationary. A basic fact is:

Theorem *(Hilbert's basis theorem) If R is Noetherian, then so is $R[X]$.*

The concept of *localisation* of a ring R is used throughout. One starts with a multiplicative set $S \subset R$, i.e. $1 \in S$ and if $f, g \in S$ then $fg \in S$. Then one considers the equivalence relation on $R \times S$ given by $(r, s) \equiv (r', s')$ if and only $s''(rs' - r's) = 0$ for some $s'' \in S$. The equivalence class of (r, s) is denoted r/s. The equivalence classes form a ring R_S, the localisation of R in S. The map which sends $r \in R$ to $r/1$ is a homomorphism $R \to R_S$. Important special cases are when $S = \{\text{non-zero divisors in } R\}$ or when $S = \{f^n \; ; \; n \in \mathbb{Z}_{\geq 0}\}$ with f a non-zero divisor. In the first case you get the ring of fractions $Q(R)$ of R and in the second case you get a ring denoted by R_f. If R is an integral domain (i.e. there are no zero-divisors except 0), the ring $Q(R)$ is a field, the field of fractions and R embeds in it and if S does not contain 0 the localisation R_S also embeds naturally in the field of fractions.

If R is Noetherian, any localisation is.

The concepts of Noetherian goes over to R-modules M by replacing 'ideal' with 'submodule' of M if appropriate in the above. Also, by replacing R with M one can define the localisation M_S of M in S. It is in a natural way an R_S-module and homomorphisms between R-modules induce homomorphism between their localisations.

A1.3 Normalisation of rings

Let there be given a commutative ring R with unit 1 and let S be a subring. An element $r \in R$ is called integral over S if it satisfies an equation of the form

$$x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0 = 0, \quad a_j \in S.$$

The integral elements form a subring of R containing S, the integral closure of S in R. This is not entirely trivial. See [Ma §9]. If it coincides with S, the ring S is said to be integrally closed in R. A ring which is integrally closed in its quotient field is called *normal*.

It is easy to see that if an integral domain is normal, all of its localisations are normal. The converse is also quite easy and in fact already follows as soon as the localisations in all maximal ideals are normal. Indeed, any element in the field of fractions of a given integral domain R which is integral over the locatisation in a maximal ideal then must belong to this localisation and the intersection of these localisations is precisely R (this is a nice exercise).

The following fact is less elementary. The proof uses Galois theory. See for instance [Ii, §2.2].

Theorem *Let R be a normal Noetherian ring with K its field of fractions. Let L be a finite separable field extension. Then the integral closure of R in L is a finite R-module.*

A 2. Appendix: Algebraic Topology

A2.1 Chain complexes

Let me start out with some commutative ring R with a unit and a collection of R-modules K_i. A *chain complex* is a sequence of R-module homomorphisms

$$K_\bullet = \{\cdots \to K_{i-1} \xrightarrow{d_{i-1}} K_i \xrightarrow{d_i} K_{i+1} \to \cdots\}$$

with the property that $d_i \circ d_{i-1} = 0$. The homology groups are defined by

$$H_p(K_\bullet) = \frac{\ker\left(K_i \xrightarrow{d_i} K_{i+1}\right)}{\operatorname{im}\left(K_{i-1} \xrightarrow{d_{i-1}} K_i\right)}.$$

If preceding maps increase the index-degree you have a *cochain complex*. Usualy one uses upper-indices in this case. It should be clear what is meant by a homomorphism $f : K^\bullet \to L^\bullet$ of cochain complexes. These induce maps $H^p(f) : H^p(K^\bullet) \to H^p(L^\bullet)$ in cohomology. For a short exact sequence

$$0 \to K'^\bullet \xrightarrow{f} K^\bullet \xrightarrow{g} K''^\bullet \to 0$$

of cochain complexes one can define coboundary maps $\delta : H^p(K''^\bullet) \to H^{p+1}(K'^\bullet)$ such that the resulting cohomology sequence

$$\ldots \to H^p(K'^\bullet) \xrightarrow{H^p(f)} H^p(K^\bullet) \xrightarrow{H^p(g)} H^p(K''^\bullet) \xrightarrow{\delta} H^{p+1}(K'^\bullet) \to \ldots$$

is exact.

A2.2 Polyhedra, cell-complexes

Classically, (co)-homology groups were first defined for polyhedra. These are spaces built up from linear simplices which make computation of (co)-homology an almost mechanical task.

The basic building blocks are the (linear) p-simplices, i.e. the convex hulls of $p+1$ independent points in some $\mathbb{R}^n$. Each p-simplex has a boundary consisting of $p-1$-simplices and there are $p+1$ of them. These form the ($p-1$-*faces*. By induction one defines the q-*faces* of a p-simplex for $q < p$. The 0-dimensional faces are also called *vertices*. A (compact) *polyhedron* is a topological space X which admits a triangulation, i.e. a homeomorphism $t : K \to X$, where K is a *simplicial complex*, i.e. a closed subset of $\mathbb{R}^n$ which is the finite union of simplices such that two simplices have at most an entire face in common. A *simplicial map* between simplicial complexes K and K' is a homeomorphism $K \to K'$ which maps every simplex of K in an affine-linear way to a simplex of K'. It should be clear what is meant by a subcomplex of a given simplicial complex and a compact polyhedral pair (X, A) of topological spaces.

Example A2 1. Any compact differentiable manifold X with a compact submanifold A is a polyhedral pair. In fact one may choose a differentiable triangulation for the pair (X, A). See [Mun, Problem 10.8].

Order the vertices occurring in a given simplicial complex K once and for all. Now given any simplex any ordering of its set of vertices $\{P_0, \ldots, P_q\}$ can be compared with the fixed order and hence determines a sign. In this way one can unambiguously speak of *ordered simplices* in a given complex and one can define a q-chain as a finite formal linear combination of ordered q-simplices. These form an abelian group $C_q(K)$. The 'dual' group $\mathrm{Hom}_{\mathbb{Z}}(C_q(K), \mathbb{Z})$ is called the group of q-cochains and denoted $C^q(K)$. There is the boundary homomorphism $\delta_p : C_p(K) \to C_{p-1}(K)$ defined for an ordered p-simplex $[P_0, \ldots, P_p]$ by $\delta([P_0, \ldots, P_p]) = \sum_{q=0}^{p}(-1)^q[P_0, \ldots, \widehat{P_q}, \ldots, P_p]$ and then extended by linearity. The coboundary $\partial^{p-1} : C^{p-1}(K) \to C^p(K)$ is its transpose. One verifies that $\delta_{p-1} \circ \delta_p = 0$ and so this gives a complex with homology group

$$H_p(K) = H_p(C_\bullet(K)) = \frac{\ker\left(C_p(K) \to C_{p-1}K\right)}{\operatorname{im}\left(C_{p+1}(K) \to C_p(K)\right)}.$$

Similarly one has the cohomology groups

$$H^p(K) = H^p(C^\bullet) = \frac{\ker\left(C^p(K) \to C^{p+1}(K)\right)}{\operatorname{im}\left(C^{p-1}(K) \to C^p(K)\right)}.$$

Clearly, simplicial maps $f : K \to K'$ induce homomomorphisms $C_q(f)$ between the groups of p-chains compatible with the boundaries and likewise for the p-cochains. So there are induced maps $H_q(f) : H_q(K) \to H_q(K')$ in homology and $H^q(f) : H^q(K') \to H^q(K)$ in cohomology with the obvious functoriality properties ($H^q(\mathrm{Id}) = \mathrm{Id}$ and $H^q(f \circ g) = H^q(g) \circ H^q(f)$).

If L is a subcomplex of K with inclusion $i : L \to K$, define

$$C^p(K, L) = \frac{C^p(K)}{C^p(i)\big(C^p(L)\big)}$$

and let

$$j : C^p(K) \to C^p(K, L)$$

be the natural projection. From Appendix A2.1 one concludes that there are coboundary homomorphisms $H^q(L) \xrightarrow{\delta^q} H^{q+1}(K, L)$ fitting into a long exact sequence

$$\cdots \xrightarrow{\delta^{q-1}} H^q(K, L) \xrightarrow{H^q(j)} H^q(K) \xrightarrow{H^q(i)} H^q(L) \xrightarrow{\delta^q} H^{q+1}(K, L) \to \cdots$$

Similar assertions hold for homology.

If one considers a polyhedron X with triangulation $t : K \to X$, it is by no means clear that the groups $H_q(K)$ are intrinsically attached to X. This however is true, and I'll come back to this in section A2.3 where singular (co)-homology is introduced. For the moment, assuming this fact, note that the homology groups of many topological spaces now can be computed 'by hand' by choosing an appropriate triangulation.

Examples A2 2. 1. The sphere S^n. One easily finds that $H_q(S^n) = H^q(S^n) = 0$ unless $q = 0$ or $q = n$ in which case these groups are infinite cyclic.

2. A compact Riemann surface of genus g. One finds that $H_0 \cong H^0 \cong H_2 \cong H^2 \cong \mathbb{Z}$ and $H_1 \cong H^1 \cong \mathbb{Z}^{2g}$, where g is the genus of the Riemann-surface.

Disappointingly, even for relatively simple spaces such as a torus, one needs a lot of simplices to triangulate. For this reason one needs larger building blocks, so called *cells.* A singular q-cell inside X is the continuous image in X of the closed q-ball by means of a continuous map, the *characteristic map* which restricts to a homeomorphism from the open q-ball onto its image. A (finite) *cell complex, or CW-complex* is a compact Hausdorff space which is the union of a finite number of (singular) cells such that the boundary of a cell is a union of cells of strictly lower dimension and two cells have no interior points in common. The union of the q-cells is called the *q-skeleton* and for a $q+1$-cell, the characteristic map restricted to the boundary sphere is a continuous map of the q-sphere onto the q-skeleton and is called the *attaching map* of the q-cell.

As in the case of a triangulation one could define homology-groups for cellular complexes, but the definition is a bit more involved. See [M-S, p. 260–263] for details. Instead of carrying this out, in Appendix A2.3 it is indicated how one can use these building blocks to compute homology using the Mayer-Vietoris sequence.

Examples A2 3.

1. The n-sphere is the standard example consisting of one n-ball and one 0-ball.

2. The complex projective space $\mathbb{P}^n$ is the union of $\mathbb{C}^n$ and $\mathbb{P}^{n-1}$. One can take the standard closed $2n$-ball $B_{2n} \subset \mathbb{C}^{2n}$ and define a surjective continuous map $f_n : B_{2n} \to \mathbb{P}^n$ by setting $f_n(z_1, \ldots, z_n) = (\sqrt{1 - |z_1|^2 - \ldots - |z_n|^2}, z_1, \ldots, z_n)$. This map restricts to the boundary sphere as the restriction to S^{2n-1} of the defining projection $\mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$. This is the *Hopf fibration* $S^{2n-1} \to \mathbb{P}^{n-1}$. Then $\mathbb{P}^n$ is also the union of the open cell $f_n(\overset{o}{B}_{2n})$ and $\mathbb{P}^{n-1}$.

Inductively one can define a cell-complex by letting $\mathbb{P}^0$ be the complex consisting of one point and use f_i, $i = 1, 2, \ldots, n$ to attach successively cells of dimensions $2i$. The resulting complex has only even-dimensional cells, one in each even dimension.

3. The direct product of two cell-complexes is again a cell-complex.

A2.3 The axiomatic approach.

Let R be a commutative ring with 1 (mostly $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{C}$). Consider a collection of pairs of topological spaces (X, A) (this means $A \subset X$) and certain continuous maps $f : (X, A) \to (Y, B)$ between them (this means a continuous $f : X \to Y$ with $f(A) \subset B$. A cohomology theory with coefficients R assigns to each such pair (X, A) R-modules $H^q(X, A)$, $q \in \mathbb{Z}$ (the cohomology-groups) and to each of the allowed continuous $f : (X, A) \to (Y, B)$ R-module homomorphisms $H^q(f) : H^q(Y, B) \to H^q(X, A)$ (the induced maps in cohomology) such that the usual functorial properties hold:

1. $H^q(f \circ g) = H^q(g) \circ H^q(f)$,
2. $H^q(\mathrm{Id}) = \mathrm{Id}_{H^q}$.

Furthermore the following axioms are to hold:

3. (Homotopy Axiom) Homotopic continuous maps induce the same map in cohomology.

4. (Exactness Axiom) For every pair (X, A) and every $q \in \mathbb{Z}$ there are R-module homomorphisms $\delta^q : H^q(A) \to H^{q+1}(X, A)$ (the *coboundary-operators*) such that the following sequence is exact ($i : A \to X$ and $j : X \to (X, A)$ are the obvious inclusions)

$$\xrightarrow{\delta^{q-1}} H^q(X,A) \xrightarrow{H^q(j)} H^q(X) \xrightarrow{H^q(i)} H^q(A) \xrightarrow{\delta^q} H^{q+1}(X,A) \to \dots$$

5. (Naturality of the Coboundary) If $f : (X, A) \to (Y, B)$ a continuous map there are commutative diagrams

$$\begin{array}{ccc} H^q(A) & \xrightarrow{\delta^q} & H^{q+1}(X,A) \\ \uparrow & & \uparrow \\ H^q(B) & \xrightarrow{\delta^q} & H^{q+1}(Y,B), \end{array}$$

where the vertical arrows are induced by f.

6. (Excision Axiom) Let (X, A) be a pair and $U \subset X$ open with closure contained in the interior of A, then the excision map $(X - U, A - U) \subset (X, U)$ induces isomorphisms in cohomology.
7. (Dimension Axiom) Let p be a point. Then $H^0(p) = R$ and $H^q(p) = 0$ for $q \neq 0$.

Examples A2 4.

1. Simplicial Cohomology. Restrict to complexes $K, K', \dots$ and and simplicial maps $K \to K', \dots$ between them. The cohomology groups $H^q(K)$ and induced maps give an example of a cohomology theory, which is called *simplicial cohomology.* The verification of the axioms is done for instance in [Sp,Chapter 4] by relating these groups to the singular cohomology groups, which are introduced next.

2. Singular Cohomology. The singular cohomology groups are defined for all pairs (X, A) of topological spaces. To define them a couple of concepts are needed.

The standard p-simplex. This is the convex hull in $\mathbb{R}^{p+1}$ of the $p+1$ standard unit-vectors:

$$\Delta_p = \{(x_1, \dots, x_{p+1}) \ ; \ x_i \geq 0, \sum_i x_i = 1\}.$$

The boundary of Δ_p consists of the $p-1$-simplices $\Delta_p^q = \Delta_p \cap \{x_q = 0\}$, $q = 1, \dots, p+1$. There are natural embeddings $i^q : \Delta_p \to \Delta_p^q$.

A singular p-simplex in a topological space X is a continuous map of the standard p-simplex to X.

A singular p-chain in X is a formal finite linear combination of singular p-simplices with coefficients in R. These form a R-module $S_p(X; R)$. A *singular p-cochain* is a R-module homomorphism $S_p(X; \mathbb{Z}) \to R$. The singular p-cochains form the R-module $S^p(X; R)$.

The boundary homomorphism $\delta_p : S_p(X;R) \to S_{p-1}(X;R)$ is defined first for a singular simplex $\sigma : \Delta_p \to X$ by $\delta(\sigma) = \sum_{q=1}^{p+1}(-1)^{q+1}\sigma \circ i^q$ and then extending it as an R-module-homomorphism. The singular chains and their boundary maps form a chain complex with homology groups $H_p(X;R)$. Explicitly

$$H_p(X;R) = \frac{\ker\left(S_p(X;R) \to S_{p-1}(X;R)\right)}{\operatorname{im}\left(S_{p+1}(S;R) \to S_p(X;R)\right)}.$$

The *coboundary* $\partial^{p-1} : S^{p-1}(X;R) \to S^p(X;R)$ is the R-dual or transpose of the boundary map, yielding a cochain complex $\{S^\bullet, \partial^\bullet\}$. Its p-th cohomology by definition is the p-th singular cohomology with coefficients in R, notation $H^p(X;R)$.

If $f : X \to Y$ is continuous, there are obvious R-module homomorphisms $S_q(f) : S_q(X) \to S_q(Y)$ resp. $S^q(f) : S^q(Y) \to S^q(X)$ compatible with the boundary, resp. coboundary maps and which induce R-module homomorphisms $H_q(f) : H_q(X;R) \to H_q(Y;R)$ and $H^q(f) : H^q(Y;R) \to H^q(X;R)$. For the latter the axioms 1. and 2. are obvious. They imply cohomological invariance.

The singular cohomology is a topological invariant.

Special Case I want to mention that $H_1(X)$ is the fundamental group modulo its commutator subgroup. See [Gr, section 12]. In particular, simply connected spaces have no first homology groups. Also, if T is a g-torus, $H_1(T) \cong \mathbb{Z}^{2g}$.

For a pair (X, A) the inclusion $i : A \to X$ induces $S(i_q)$ and the cokernel is denoted $S_q(X, A; R)$ and its R-dual by $S^q(X, A; R)$. One verifies that the boundaries, resp. the coboundaries give $S_\bullet(X, A; R)$ resp. $S^\bullet(X, A; R)$ the structure of a chain, resp. cochain complexe and by definition $H_q(X, A; R) = H_q(S_\bullet(X, A; R))$ resp. $H^q(X, A; R) = H^q$ $(S^\bullet(X, A; R))$. The usual theory of complexes then shows that the axioms 4. and 5. are valid. Also the dimension axiom is almost trivial. The remaining two axioms however require some work. See [Gr, sections 11,15].

Later I shall compare homology and cohomology with coefficients in a ring R. First note that the tautological pairing $S^q(X, A; R) \times S_q(X, A; R) \to R$ is compatible with boundary and coboundary and hence one gets a pairing (Kronecker pairing)

$$H^q(X, A; R) \times H_q(X, A; R) \to R.$$

This pairing will be denoted by $\langle\ ,\ \rangle$, so that

$$\langle [f], [c] \rangle = f(c) \quad f \text{ a } q\text{-cocycle, } c \text{ a } q\text{-cycle,}$$

and the square brackets denote the corresponding classes in (co)-homology. This pairing induces the Kronecker homomorphism

$$H^q(X, A) \to \operatorname{Hom}_R(H_q(X, A), R).$$

For a principal ideal ring R this map is surjective. In fact [Gr, section 23]:

Proposition A2 5.

1. *If R is a field, the Kronecker homomorphism is an isomorphism.*

2. *For $R = \mathbb{Z}$ the Kronecker homomorphism is surjective. If $H^q(X, A)$ and $H_{q-1}(X, A)$ are finitely generated, the kernel of the Kronecker map (which is precisely the torsion subgroup of $H^q(X, A)$) is isomorphic to the torsion subgroup of $H_{q-1}(X, A)$.*

In the sequel I shall omit R in case $R = \mathbb{Z}$, hence $H^q(X, A)$ denotes the singular relative cohomology group with integral coefficients. The *Universal coefficient theorem* gives a recipe to determine the (co)-homology groups with coefficients in any principal ideal domain R from the groups with values in $\mathbb{Z}$. See [Gr, Section 29]. I only need the result for fields:

Proposition A2 6. *Let R be a field. For any topological space, the natural homomorphism*

$$H_n(M) \otimes R \to H_n(M; R)$$

is an isomorphism.

For a polydron $t : K \to X$ there is an obvious map of complexes $C_\bullet(K) \to C_\bullet(X)$. This assignment extends to polyhedral pairs and maps between them in an obvious way and induces isomorphisms between simplicial (co)-homology and singular (co)-homology [Sp, 4.6, Theorem 8]. There are some useful consequences.

Proposition-Definition A2 7. *The cohomology groups of compact polyhedral pairs are finitely generated abelian groups. The rank of $H_q(X)$ (here X is a polyhedron) is called the Betti-number $b_q(X)$. The alternating sum $e(X) = \sum_j (-1)^q b_q(X)$ is called the Euler number. It is equal to the alternating sum $\sum_q (-1)^q n_q$ of the number of q-simplices n_q.*

Corollary A2 8. *If $f : X \to Y$ is an unramified covering between polyhedra of degree d one has $e(Y) = d \cdot e(X)$.*

There is a way to compute the cohomology of a product of polyhedra, or more generally, cell complexes, from the cohomology groups of the factors. This is expressed by the *Künneth formula.* I only give the result for the ranks of the cohomology groups and refer to [Gr, 29.11] for the full statement.

Proposition A2 9. *Let X and Y be finite cell complexes. Then $X \times Y$ is a finite cell compex and one has*

$$b_n(X \times Y) = \sum_{i=0}^{n} b_i(X) \cdot b_{n-i}(Y).$$

From the axioms one can derive the Mayer-Vietoris sequence (see [Sp, 4.6 and 5.4] which tells one how to compute the (ho)mology of a union of (suitable sets) from its parts. Since many spaces are cell-complexes, this is very useful. Suppose that a topological space X is a union $X = X_1 \cup X_2$ of two parts for which the inclusion maps $(X_1, X_1 \cap X_2) \to (X, X_2)$ and $(X_2, X_1 \cap X_2) \to (X, X_1)$ induce isomorphisms in the cohomology. The most important cases are when X_1 and X_2 are open in X or when X is a simplicial complex and X_1 and X_2 are subcomplexes or when $X_1 = X_{q-1}$ is the $q-1$-skeleton of a cell-complex and X_2 a given q-cell. With $i_k : X_k \to X$ and $j_k : X_1 \cap X_2 \to X_k$ the inclusion maps, the exact Mayer-Vietoris sequence reads as follows

$$\cdots \to H^q(X) \xrightarrow{(H^q(i_1),H^q(i_2))} H^q(X_1) \oplus H^q(X_2) \xrightarrow{j_1 - j_2} H^q(X_1 \cap X_2) \to H^{q+1}(X) \to \cdots$$

Example A2 10. One easily computes the cohomology of $\mathbb{P}^n$ using the previous description of it as a cell-complex. You find that $H^q(\mathbb{P}^n) = 0$ when q is odd or $q < 0$ or $q > 2n$ and the remaining cohomology groups are all infinite cyclic. See [Gr, section 19] for details.

A2.4 Manifolds

If M is any n-dimensional topological manifold one defines an orientation sheaf O_M, as the locally constant sheaf of $\mathbb{Z}$-modules defined by the presheaf $U \to H_n(M, M \setminus U)$. By the Excision theorem these groups are isomorphic to $H_n(U, U \setminus \{x\}) \cong \mathbb{Z}$, where x is any point of a coordinate ball U. So O_M is locally free of rank 1. M is *orientable* if O_M is constant. An orientation is a choice of one of the two generators of $\Gamma(O_M) \cong \mathbb{Z}_M$. One can easily see that M is orientable if and only if one can orient the tangents spaces $T_x(M)$ in a coherent way, i.e. if the line bundle $\det T_M$ is trivial. In the differentiable context this is equivalent to the existence of a nowhere zero differential form of maximal degree n. One can show [Gr, section 22]:

Lemma-Definition A2 11. *A connected compact manifold M of dimension n is orientable if and only if $H_n(M) \cong \mathbb{Z}$. The choice of a generator is equivalent to choosing an orientation. The generator corresponding to a chosen orientation is called the fundamental class of M, denoted $o_M \in H_n(M, \mathbb{Z})$,*

Corollary A2 12. *For any compact complex manifold of dimension n one has $H_{2n} \cong \mathbb{Z}$.*

Next I state the topological version of the Poincaré-duality theorem for compact manifolds, which is more refined than the version for differentiable manifolds in terms of the de Rham cohomology.

I first say a few words about the cup products in singular cohomology. Introduce a product on $\oplus_p S^p(X)$ (now X is an arbitrary topological space) as follows. Define $f \cup g$ for $f \in S^p(X)$ and $g \in S^q(X)$ by evaluating it on a singular $p+q$-simplex $\sigma : \Delta \to X$. Define σ_p to be the singular p-simplex obtained by restricting σ on the standard-simplex spanned by the first $p+1$ unit vectors and σ_q by restricting to the 'complementary face' spanned by the last q unit vectors. Then define

$$f \cup g(\sigma) = f(\sigma_p) \cdot g(\sigma_q).$$

One shows that this cup-product induces a (non)-commutative ring structure on the direct sum $H^*(X) = \oplus_q H^q(X)$. This ring has a unit $1 \in H^0(X)$ given by the constant cochain $x \mapsto 1$ for any point $x \in X$. The ring is skew-commutative in that

$$a \cup b = (-1)^{pq} b \cup a, \quad a \in H^p(X),\, b \in H^q(X).$$

If M is a compact connected oriented n-dimensional manifold, one can use the cup product also to define the duality homomorphism. Let me follow [Gr, section 22]. Define the duality homomorphism

$$D_M : H^q(M) \to H_{n-q}(M)$$

by demanding that

$$\langle b, D_M a\rangle = \langle a \cup b, o_M\rangle,$$

where the Kronecker-pairing and the fundamental class o_M are used.

Theorem A2 13. (Poincaré-duality) *The duality homomorphism*

$D_M : H^q(M) \to H_{n-q}(M)$ is an isomorphism for all $q \in \mathbb{Z}$.

Combining this with A2.5 one immediately gets:

Corollary A2 14. *There is only cohomology for $q = 0, \ldots, n = \dim M$. For the Betti-numbers one has $b_q = b_{n-q}$ and the torsion subgroup of H_q is isomorphic to the torsion subgroup of H_{n-q-1}.*

Poincaré-duality also shows that the cup-product pairing between cohomology groups of complentary degrees is perfect.

Corollary A2 15. *The cup-product pairing*

$$\begin{array}{cccc} I^q : H^q(M,\mathbb{Z}) \times H^{n-q}(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ (a,b) & \longmapsto & \langle a\cup b, o_M\rangle = \langle D_M a, b\rangle \end{array}$$

is perfect in the sense that if $I^q(a,b) = 0$ for all $a \in H^q(M,\mathbb{Z})$ then b is torsion and similarly if $I^q(a,b) = 0$ for all $b \in H^{n-q}(M,\mathbb{Z})$ then a is torsion.

If $n = 2m$ is even, there is the cup product pairing on $H^m(M,\mathbb{Z})$ and Poincaré-duality says that it is unimodular. Recall that this means the following. Choose a basis for $H^m(M,\mathbb{Z})$ mod torsion. Then the Gram matrix of the cup product pairing is integral with determinant ± 1. If m is odd, this pairing is skew-symmetric and it is an easy exercise in linear algebra to show that the rank of $H^m(M,\mathbb{Z})$ must be even, say $2g$ and that one can find a basis so that the Gram matrix becomes the standard symplectic form

$$J_g := \begin{pmatrix} 0_g & -\mathbb{1}_g \\ \mathbb{1}_g & 0_g \end{pmatrix}.$$

One says that the form is *isometric* to J_g.

Example A2 16.

1. Compact Riemann-surface. The preceding considerations show that $H^1 \cong H_1 \cong \mathbb{Z}^{2g}$ the cup-product form is isometric to J_g.

2. Compact complex surfaces. Let T be the torsion subgroup of H_1. Then T is isomorphic to the torsion subgroup in H_2, in H^2 and H^3, whereas H^1 and H_3 are free modules. Moreover $b_1 = b_3$ so that $e = 2 - 2b_1 + b_2$.

One can introduce a dual pairing on $H_m(M,\mathbb{Z})$, the intersection pairing, by setting $I(a,b) = I^m(D_M^{-1}b, D_M^{-1}a)$. I shall give a more geometric description of this pairing for special classes, coming from submanifolds. Recall that for any compact oriented p-dimensional manifold P there is a fundamental class $o_P \in H_p(P,\mathbb{Z})$. If now $i : P \hookrightarrow M$ realises P as a submanifold of M, I define $h(P) = i_*(o_P) \in H_p(M,\mathbb{Z})$. This class is called the *fundamental homology class of P.* Its Poincaré-dual class in $H^{n-p}(M,\mathbb{Z})$ is called the *fundamental co-homology class* and denoted by $c(M)$.

Suppose that M is a compact differentiable oriented manifold of dimension $n = 2m$ as before and that P and P' are two submanifolds of dimension m intersecting transversally in a finite number of points. Any choice of an orientation for P and P' makes it possible to define the homology classes $h(P)$ and $h(P')$. Now at an intersection point m of P and P' taking first the induced orientation of T_xP and then of T_xP' yields an orientation of T_xM which may or may not be compatible with the given orientation of M. In the first case set $(P,P')_x = +1$ and in the second case set $(P,P')_x = -1$. Then set $(P,P') = \sum_x (P,P')_x$. This pairing is the *geometric intersection pairing.*

Claim A2 17. *In the preceding set-up $(P,P') = I(h(P),h(P')) = \langle c(P) \cup c(P'), o_M\rangle$.*

There are various ways to prove this. See [G-H, Chapter 0.4] for a proof. In rough outline this goes as follows. Consider a compact differentiable manifold M and fix a smooth triangulation. First observe that one can assign a fundamental class to topological manifolds, in particular to to piecewise smooth submanifolds. Hence also the geometric intersection pairing for submanifolds can be extended to piecewise smooth submanifolds meeting transversally. Now any p-cycle a can be represented by a linear combination of smooth p-simplices and one can then see that there is a piecewise smooth submanifold A such that a is a multiple of $h(A)$. Next, one shows that a given $n-p$-piecewise linear submanifold B' is homologous to a piecewise linear submanifold B meeting A transversally and one extends the geometric intersection pairing by setting $(A,B') = (A,B)$. Now one shows that Poincaré duality can be given by

$$\langle a,b\rangle = (D_M a, b), \quad a \in H^{n-p}(M,\mathbb{Z}),\ b \in H_{n-p}(M,\mathbb{Z}).$$

From this formula one easily sees that $I(h(A),h(B)) = (A,B)$ for any two piecewise linear submanifolds of complementary dimension. Indeed $I(h(A),h(B)) = \langle D_M^{-1}h(B) \cup D_M^{-1}h(A), o_M\rangle$ by definition, while $(A,B) = (D_M \circ D_M^{-1}h(A), h(B)) = \langle D_M^{-1}h(A), h(B)\rangle = \langle D_M^{-1}h(B) \cup D_M^{-1}h(A), o_M\rangle$ by the previous formula and the meaning of the Poincaré duality isomorphism D_M.

It should be remarked that in [G-H] the preceding formula for the Poincaré duality isomorphism is derived using differential forms and that De Rham's theorem is used implicitly. To treat this properly, introduce the groups $S^p_\infty(M)$ of singular smooth p-cochains which are simply functionals on the free group S_p^∞ on the smooth singular p-simplices on M. These form a subcomplex $S^\bullet_\infty(M)$ of $S^\bullet(M)$ and one shows ([Wa, 5.31, 5.32]) that there are canonical isomorphisms

$$H^p(\mathbb{R}_M) \cong H^p(S^\bullet_\infty(M,\mathbb{R})) \cong H^p(M;\mathbb{R}).$$

By the De Rham theorem $H^p_{DR}(M)$ is isomorphic to $H^p(\mathbb{R}_M)$ and so there is an isomorphism from the p-th De Rham group to the p-th (smooth) singular cohomology group with real coefficients. There is a homomorphism

$$\mathcal{E}^p(M) \longrightarrow S^p_\infty(M, \mathbb{R})$$

given by integrating p-forms over smooth p-simplices. Stokes' theorem implies that it induces a homomorphism in cohomology

$$\int^{[p]} : H^p_{DR}(M) \to H^p(M; \mathbb{R}).$$

There is the following refined version of the theorem of De Rham, a proof of which can be found in [Wa, p. 205-214].

Theorem A2 18. (Explicit Form of De Rham) *The integration map*

$$\int^{[p]} : H^p_{DR}(M) \to H^p(M; \mathbb{R}).$$

is an isomorphism. If you endow $\oplus_p H^p_{DR}(M)$ with the ring structure coming from the wedge product of differential forms and put a ring structure on $\oplus_p H^p(M; \mathbb{R})$ by means of the cup product, the isomorphism $\oplus_p \int^{[p]}$ becomes an isomorphism of (graded) rings.

Finally, I can reformulate Poincaré duality in terms of this isomorphism.

Corollary A2 19. *Let M be a compact oriented manifold of dimension n with orientation class o_M. There is a commutative diagram*

$$\begin{array}{ccccc}
H^p(M, \mathbb{R}) & \times & H^{n-p}(M, \mathbb{R}) & \xrightarrow{t} & \mathbb{R} \\
\downarrow & & \downarrow & & \| \\
H^p_{\mathrm{DR}}(M) & \times & H^{n-p}_{\mathrm{DR}}(M) & \xrightarrow{t_{\mathrm{DR}}} & \mathbb{R}.
\end{array}$$

Here $t(a, b) = \langle a \cup b, o_M \rangle$ and $t_{\mathrm{DR}}(\alpha, \beta) = \int_M \alpha \wedge \beta$

A.2.5 Lefschetz theory

Here I review the theory of hyperplane sections and its consequences for hypersurfaces in $\mathbb{P}^n$. An excellent treatment, using Morse theory, can be found in [Mi].

Theorem A2 20. (Lefschetz theorem on hyperplane sections) *Let $X \subset \mathbb{P}^N$ be an $(n+1)$-dimensional projective variety and let H be a hyperplane which contains the singular points of X. The inclusion $X \cap H \to X$ induces isomorphisms for the integral homology groups of degree $\leq n - 1$ and a surjection in degree n. A similar result holds for the homotopy groups. In particular, if X is connected and $n \geq 1$, $X \cap H$ is connected. If X is simply connected and $n \geq 2$, also $X \cap H$ is simply connected.*

This can be applied in the following way. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and consider the d-uple Veronese embedding $\mathbb{P}^{n+1} \hookrightarrow \mathbb{P}^N$ using the polynomials of degree d. The hyperplanes in $\mathbb{P}^N$ correspond to the hypersurfaces of degree d in $\mathbb{P}^n$ and the preceding theorem can be applied. More generally one has

Corollary A2 21. *Let $X \subset \mathbb{P}^{n+k}$ be a complete intersection manifold of dimension n. If $n \geq 2$ the manifold X is connected and simply connected. One has $H_m(X, \mathbb{Z}) = 0$ for m odd and $m \leq n-1$, $H_m(X, \mathbb{Z}) \cong \mathbb{Z}$ for m even and $m \leq n-1$.*

A 3. Appendix: Hodge Theory and Kähler manifolds

A3.1 Hodge theory and consequences

Let M be a compact complex n-dimensional manifold with a Hermitian metric h. By definition, this is a smooth section in the bundle $T(M) \otimes \overline{T}(M)$. Taking the anti-symmetric part gives a real $(1,1)$-form ω_h, the metric form associated to h. To fix the normalisation, if in local coordinates h is given by

$$h = \sum_{i,j} h_{i\bar{j}} dz_i \otimes d\bar{z}_j,$$

the form is given by

$$\omega_h = \frac{1}{2}\sqrt{-1} \sum_{i,j} h_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

Let me outline how Hodge theory works. The metric h, which is a metric on the tangent bundle, induces metrics on all the associated tensor bundles. For instance, one obtains pointwise metrics $(\ ,\ )_m$, $m \in M$ on the bundle $\mathcal{E}^{p,q}(M)$, the bundle of complex-valued (p,q)-forms. Now using the volume-form

$$vol_h := \underbrace{\omega_h \wedge \ldots \wedge \omega_h}_{n}$$

global inner products, the *Hodge inner products* can be defined

$$(\ ,\ ) := \int_M (\ ,\ )_m vol_h.$$

With this inner-product, $\mathcal{E}^m_{\mathbb{C}}(M) = \oplus_{p+q=m} \mathcal{E}^{p,q}(M)$ is an an orthogonal splitting ([We, Chapt V, Prop. 2.2]).

The Hodge $*$-operators

$$* : \wedge^m T^\vee_m M \to \wedge^{2n-m} T^\vee_m M$$

are defined by the formula $\alpha \wedge *\beta = (\alpha, \beta) vol_h(m)$. They induce linear operators on $\mathcal{E}^m(M)$ and $\mathcal{E}^m_{\mathbb{C}}(M)$. The corresponding conjugate linear operator

$$\bar{*} : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{n-q,n-p}(M)$$

is defined by $\bar{*}(\alpha) = *\bar{\alpha}$.

The d, ∂ and $\overline{\partial}$-operators have formal adjoints d^*, ∂^* and $\overline{\partial}^*$ with respect to these inner products and one can form the associated Laplacians:

$$\begin{aligned}\triangle_d &= dd^* + d^*d,\\ \triangle_\partial &= \partial\partial^* + \partial^*\partial,\\ \triangle_{\overline{\partial}} &= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}.\end{aligned}$$

The m-forms that satisfy the Laplace equation $\triangle_d = 0$ are called *d-harmonic* and denoted $\mathrm{Harm}^m(M)$. Likewise for the $\overline{\partial}$-harmonic (p,q)-forms. These constitute $\mathrm{Harm}^{p,q}(M)$.

To motivate the Hodge theorems, let me assume that the spaces $\mathcal{E}^p(M)$ are finite dimensional. Look at the short complex

$$\mathcal{E}^{p-1}(M) \xrightarrow{d} \mathcal{E}^p(M) \xrightarrow{d} \mathcal{E}^{p+1}(M).$$

Standard linear algebra yields orthogonal sum decompositions $\mathcal{E}^p(M) = \ker d \oplus \mathrm{im}\ d^* = \mathrm{im}\ d \oplus \ker d^*$ and since $\mathrm{im}\ d \subset \ker d$ and $\mathrm{im}\ d^* \subset \ker d^*$, the direct sum decomposition

$$\mathcal{E}^p(M) = \ker d \cap \ker d^* \oplus \mathrm{im}\ d \oplus \mathrm{im}\ d^*$$

follows. Now the first summand consists precisely of the d-harmonic forms and the d-Laplacian is an isomorphism on the other two summands. In particular one sees that the p-th De Rham group can be canonically identified with the space of harmonic forms. Now in general the spaces $\mathcal{E}^p(M)$ are infinite dimensional, but the results still hold. This is the content of the Hodge theorem.

Theorem A3 1. (Hodge Theorem) *Let M be a compact differentiable manifold equipped with a hermitian metric. Then*

1. $\dim \mathrm{Harm}^m(M) < \infty$.
2. *Let*
$$H : \mathcal{E}^m(M) \to \mathrm{Harm}^m(M)$$
be orthogonal projection onto the harmonic forms. There is a unique operator
$$G : \mathcal{E}^m(M) \to \mathcal{E}^m(M)$$
with kernel containing the harmonic forms and which satisfies
$$\mathrm{Id} = H + \triangle_{\overline{\partial}} \cdot G.$$
In particular, one has a direct sum decomposition
$$\mathcal{E}^m(M) = \mathrm{Harm}^m(M) \oplus dd^*G\mathcal{E}^m(M) \oplus d^*dG\mathcal{E}^m(M)$$
and H induces an isomorphism
$$H^m_{\mathrm{DR}}(M) \xrightarrow{\cong} \mathrm{Harm}^m(M).$$

There is a similar version for complex manifolds. In fact the theorem is valid in a more general context, that of elliptic complexes. See [We Chapt. IV, Theorem 5.2]. In particular, one can apply it to (p,q)-forms with values in a vector bundle E with a hermitian metric h_E, replacing $\mathcal{E}^{p,q}$ by $\mathcal{E}^{p,q}(E) := \mathcal{E}^{p,q} \otimes E$. To introduce a Hodge metric on $\mathcal{E}^m_{\mathbb{C}} \otimes E$, first choose a conjugate linear isomorphism $\tau : E \to E^\vee$ and define

$$\bar{*}_E : \mathcal{E}^{p,q}(E) \to \mathcal{E}^{n-q,n-p}(E^\vee)$$

by $\bar{*}_E(\alpha \otimes e) = \bar{*}\alpha \otimes \tau(e)$. Then one defines the Hodge metric on $\mathcal{E}^{p,q}(E)$ by

$$(-,-') = \int_M - \wedge \bar{*}_E -' \,.$$

Let us now state the version needed in the text.

Theorem A3 2. (Hodge Theorem - Second Version) *Let M be a compact complex manifold with hermitian metric and let E be a vector bundle equipped with an hermitian metric.*

1. $\dim \operatorname{Harm}^{p,q}(E) < \infty$.
2. *Let*

$$H : \mathcal{E}^{p,q}(E) \to \operatorname{Harm}^{p,q}(E)$$

be orthogonal projection onto the harmonic forms. There is a unique operator

$$G : \mathcal{E}^{p,q}(E) \to \mathcal{E}^{p,q}(E)$$

with kernel containing the harmonic forms and which satisfies

$$\mathrm{Id} = H + \triangle_{\overline{\partial}} \cdot G.$$

In particular, one has a direct sum decomposition

$$\mathcal{E}^{p,q}(E) = \operatorname{Harm}^{p,q}(E) \oplus dd^* G\mathcal{E}^{p,q}(E) \oplus d^* dG\mathcal{E}^{p,q}(E)$$

and H induces an isomorphism

$$H^{p,q}_{\overline{\partial}}(E) \xrightarrow{\cong} \operatorname{Harm}^{p,q}(E)$$

where

$$H^{p,q}_{\overline{\partial}}(E) := \frac{\overline{\partial}\text{-closed } (p,q)\text{-forms with values in } E}{\overline{\partial}\mathcal{E}^{p,q-1}(E)} \xrightarrow{\cong} \operatorname{Harm}^{p,q}(E).$$

Combining the last part of this theorem with the Dolbeault-isomorphism 3.3 one finds that the groups $H^q(\Omega^q(E))$ are finite dimensional.

Next, note that the operator $\bar{*}_E$ commutes with the Laplacian $\triangle_{\overline{\partial}}$ as acting on $\mathcal{E}^{p,q}(E)$ and hence harmonic (p,q)-forms with values in E go to harmonic $(n-p, n-q)$-forms with values in $E^\vee$. In particular $\mathrm{Harm}^{p,q}(E)$ and $\mathrm{Harm}^{n-q,n-p}(E^\vee)$ are conjugate-linearly isomorphic. The following classical consequence then follows.

Corollary A3 3. (Serre Duality) *The operator $*_E$ defines an isomorphism*

$$H^q(M, \Omega^p(E)) \xrightarrow{\cong} H^{n-p}(M, \Omega^{n-p}(E^\vee))^\vee.$$

A3.2 Kähler metrics and the Hodge decomposition theorem

A metric h is called *Kähler* if the associated form is closed. Such a form is called a *Kähler form.* Any manifold admitting a Kähler metric is called Kähler manifold.

Examples A3 4.

1. Any hermitian metric on a Riemann surface is Kähler.
2. The Fubini-Study metric on $\mathbb{P}^n$ is Kähler. It is defined by the $\mathbb{C}^*$-invariant form $\dfrac{1}{2\pi}\partial\overline{\partial}\log ||Z||^2$ on $\mathbb{C}^{n+1} \setminus \{0\}$.
3. Any submanifold of a Kähler manifold is Kähler. Indeed, the restriction of the Kähler form restricted to the submanifold is a Kähler form on this submanifold. An important special case are the projective manifolds.

Let now h be a Kähler metric, and ω_h its associated $(1,1)$-form. Let L denote the operator defined by multiplication against the Kähler form: $L(\alpha) = \omega_h \wedge \alpha$. Let Λ denote its formal adjoint. These operators are of types $(1,1)$ and $(-1,-1)$, respectively. I use the square brackets to denote commutators of operators: $[A, B] = AB - BA$. This said, there are the fundamental *Kähler identities* [We, Chapt V, Coroll. 4.10]

$$\begin{aligned} \partial^* &= \sqrt{-1}[\Lambda, \bar{\partial}] \\ \bar{\partial}^* &= -\sqrt{-1}[\Lambda, \partial]. \end{aligned}$$

If you introduce the real operator

$$d_c := -\sqrt{-1}(\partial - \bar{\partial})$$

with formal adjoint d_c^* these can be rewritten as

$$[\Lambda, d] = -d_c^*.$$

The adjoint relation is also useful and reads as follows.

$$[L, d^*] = d_c.$$

In [We] these relations are derived more or less together with the following Claim, using representation theory of $SL(2, \mathbb{C})$. I have separated the latter from the Kähler identities for clarity. Also, the proof given here is somewhat shorter than the proof in [We].

From these identities one derives:

Claim *One has $\triangle_d = 2\triangle_{\overline{\partial}}$ and hence in particular*

1. *The Laplacian is real,*
2. *The Laplacian preserves types*
3. *The L-operator preserves harmonic forms.*

Proof: One has $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$ since

$$\begin{aligned}\sqrt{-1}(\partial\overline{\partial}^* + \overline{\partial}^*\partial) =& \partial[\Lambda,\partial] + [\Lambda,\partial]\partial\\ =& \partial\Lambda\partial - \partial\Lambda\partial = 0.\end{aligned}$$

Then

$$\begin{aligned}\triangle_d =& (\partial+\overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial+\overline{\partial})\\ =& \ \triangle_\partial + \triangle_{\overline{\partial}}.\end{aligned}$$

Next observe that

$$\begin{aligned}-\sqrt{-1}\triangle_\partial =& -\partial[\Lambda,\overline{\partial}] - [\Lambda,\overline{\partial}\ ]\partial\\ =& -\partial\Lambda\overline{\partial} + \partial\overline{\partial}\Lambda - \Lambda\overline{\partial}\partial + \overline{\partial}\Lambda\partial\\ =& \ \ \overline{\partial}\Lambda\partial + \partial\overline{\partial}\Lambda - \Lambda\overline{\partial}\partial - \partial\Lambda\overline{\partial}\\ =& \ \ \overline{\partial}[\Lambda,\partial] + [\Lambda,\partial]\overline{\partial}\\ =& -\sqrt{-1}\triangle_{\overline{\partial}}.\end{aligned}$$

Finally, if α is harmonic, one has $d\alpha = d^*\alpha = 0$. So $dL\alpha = d(\omega\wedge\alpha) = \omega\wedge d\alpha = 0$ and if one assumes that α has pure type (this is allowed because of part (2) of the Consequences) one also has $d^*L\alpha = -[L,d^*]\alpha = d_c^*\alpha = 0$. ∎

From the preceding consequences one derives immediately:

Theorem A3 5. (Hodge Decomposition) *Let M be a compact Kähler manifold. There is a direct sum decomposition*

$$H^m_{\mathrm{DR}}(M)\otimes\mathbb{C} = \oplus_{p+q=m} H^{p,q}_{\overline{\partial}}.$$

Moreover $H^{p,q}_{\overline{\partial}} = \overline{H^{q,p}_{\overline{\partial}}}$.

Proof: Since the Laplacian preserves types, there is a homomorphism

$$\mathrm{Harm}^m(M) \to \oplus^{p,q}\,\mathrm{Harm}^{p,q}(M).$$

which is clearly injective and surjective. Since the Laplacian is real, the last statement follows also. ∎

This theorem allows us to see that the Hopf manifolds are not Kähler and hence a fortiori not projective.

Example A3 6. The Hopf manifolds are not Kähler . To see this, recall that a Hopf manifold is homeomorphic to $S^1\times S^{2n-1}$ and so $b_1(S^1\times S^{2n-1}) = b_1(S^1) + b_1(S^{2n-1}) = 1$ since $n > 1$. Here the Künneth formulas are used, see Proposition A2.9. On the other hand, b_1 must be even for any Kähler manifold.

A3.3 Implications for Riemann surfaces

In this section I prove that any compact Riemann surface can be embedded in some projective space and hence, by Chow's Theorem, is projective.

From Hodge theory you know that for any line bundle L on a compact Riemann surface M the space of sections $H^0(\mathcal{O}_M(L))$ as well as the space $H^1(\mathcal{O}_M(L))$ is finite dimensional. It is an elementary observation that L can have no holomorphic sections if deg $L < 0$. Indeed, any holomorphic section of L would vanish in a divisor D which either is zero or effective and hence deg $L =$ deg $D \geq 0$. Now Serre-duality implies that dually $H^1(\mathcal{O}_M(L)) = 0$ if deg $L > 2g(M) - 2$.

Let me start with a divisor D of degree $> 2g(M)$ and let $L = \mathcal{O}_M(D)$ be the corresponding line bundle. I claim that the corresponding meromorphic map $\varphi_L : C \dashrightarrow \mathbb{P}^N$ gives an embedding. By 4.20 it is sufficient to show that $H^1(M, \mathfrak{m}_x \cdot L) = 0$ for all $x \in M$ and that $H^1(M, \mathfrak{m}_x \cdot \mathfrak{m}_y \cdot L) = 0$ for all pairs of points $x, y \in M$. Now $\mathfrak{m}_x = \mathcal{O}_M(-x)$ since you are on a Riemann surface and the bundles involved all have degree $> 2g(M) - 2$ and so, by the previous remark, the desired groups vanish and map φ_L is an embedding.

A3.4 First Chern class

Let M be a compact manifold and let L be a holomorphic line bundle on M with a hermitian metric h. The form

$$c(L, h) := -\frac{i}{2\pi}\partial\overline{\partial}\log h$$

is is a closed $(1,1)$-form and is called the *Chern*-form of the metric h. Any other metric h' on the line bundle is related to h by a relation $h' = e^{\varphi}h$ with φ some C^∞ function on the manifold. It follows that $c(L, h') = c(L, h) + d(\overline{\partial}\varphi)$ and hence the class of $c_1(L)$ in $H^2_{\mathrm{DR}}(M)$ is independent of the chosen metric. It is called the *first Chern class* of the line bundle L and denoted $c_1(L)$. For the proof of the following proposition see [We, Chapt. III, Theorem 4.5].

Proposition A3 7. *Consider the exponential sequence*

$$0 \to \mathbb{Z}_M \to \mathcal{O}_M \xrightarrow{exp} \mathcal{O}_M^* \to 0,$$

where "exp" means the map $f \mapsto \exp(2\pi i f)$. Let $\delta : H^1(\mathcal{O}_M^*) \to H^2(\mathbb{Z}_M)$ be the coboundary map and let $i : H^2(M, \mathbb{Z}) \to H^2_{\mathrm{DR}}(X)$ be the natural map. Then $i \circ \delta(\mathcal{O}(L)) = c_1(L)$.

For a divisor D the class $\delta(\mathcal{O}(D))$ is the fundamental cohomology class as defined in Appendix 2:

Proposition A3 8. *Let M be a compact complex manifold and let D be a smooth hypersurface in M. The fundamental cohomology class of D in $H^2(M, \mathbb{Z})$ coincides with $\delta(\mathcal{O}(D))$, where δ is the coboundary map $H^1(\mathcal{O}_M^*) \to H^2(M, \mathbb{Z})$ from the exponential sequence.*

Proof: (Sketch). With help of the explicit version of the Poincaré-duality isomorphism for De Rham cohomology, the fundamental cohomology class of D can be interpretated as the class $[c(D)]$ for which

$$\int_D i^*\alpha = \int_M c(D) \cup \alpha,$$

where $i : D \hookrightarrow M$ is the embedding. To prove the preceding proposition, it suffices to prove the formula

$$\int_D i^*\alpha = -\frac{i}{2\pi} \int_M \partial\overline{\partial} \log h(s,s) \wedge \alpha,$$

where h is a hermitian metric on the line bundle $\mathcal{O}(D)$ and s is the section of $\mathcal{O}(D)$ defining D. If one takes the example of a point in $\mathbb{P}^1$ this reduces to the residue theorem and the general case is similar. ■

See [G-H, Chapter 1, p. 141] for the details.

A3.5 Kodaira-Vanishing

For details of the following discussion see [We, Chapt. VI, §2].

A line bundle is called *positive* if for some metric h the Chern-form is positive.

Examples A3 9.

1. A line bundle L on a Riemann surface M is positive if and only if its degree is positive. Indeed, the generator of $H^2(M, \mathbb{Z})$ is represented by a positive form, which is a positive multiple of the volume form of any hermitian metric.
2. The bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ with the metric coming from the Fubini-Study metric. It follows that any ample line bundle is positive.

Theorem A3 10. (Kodaira-Nakano Vanishing) *Let L be a positive line bundle on a compact complex manifold M of dimension n. Then*

$$H^q(M, \Omega^p_M \otimes \mathcal{O}_M(L)) = 0, \qquad \textit{if } p + q > n.$$

The version for $p = n$ yields:

Corollary A3 11. (Kodaira Vanishing) *Let L be an ample line bundle on a projective manifold M then*

$$H^q(M, \mathcal{O}(K_M + L)) = 0 \qquad \textit{if } q > 0.$$

A 4. The GAGA Theorems

In this appendix I gather the various GAGA-type theorems from [Se].

Let me start with a projective variety $X \subset \mathbb{P}^n$. It can naturally be regarded as a complex subvariety X_h of $\mathbb{P}^n$. Any morphism $f : X \to Y$ between projective varieties X and Y can be regarded as a holomorphic map $f_h : X_h \to Y_h$ between the associated complex varieties.

If $\mathcal{F}$ is a coherent sheaf on X, one defines a coherent analytic sheaf $\mathcal{F}_h$ in the following manner. Locally (for the Zariski topology) the sheaf $\mathcal{F}$ is a cokernel

$$\mathcal{O}_U^n \xrightarrow{\varphi} \mathcal{O}_U^m \to \mathcal{F}|U \to 0.$$

Now φ is given by a matrix of regular functions on U and these are holomorphic functions on U, which is open in the ordinary topology. So one can define $\mathcal{F}_h$ on U by the cokernel

$$\mathcal{O}_{U_h}^n \xrightarrow{\varphi} \mathcal{O}_{U_h}^m \to \mathcal{F}_h|U \to 0,$$

where $\mathcal{O}_{U_h}$ is the sheaf of germs of holomorphic functions on U.

One can furthermore compare the cohomology groups $H^p(X, \mathcal{F})$ and $H^p(X_h, \mathcal{F}_h)$. The identity map $f : X_h \to X$ is continuous and it induces a natural map $f^{-1}\mathcal{O}_X \to \mathcal{O}_{X_h}$ (you simply regard a regular function on a Zariski-open set as a holomorphic function). Clearly $\mathcal{F}_h \cong f^*\mathcal{F}$ and so there are natural maps of cohomology groups

$$\alpha_p : H^p(X, \mathcal{F}) \to H^p(X_h, \mathcal{F}_h).$$

One now has

Theorem (Serre) *Let X be a projective variety. Then for any coherent analytic sheaf $\mathcal{F}$ on X_h there exists a coherent sheaf $\mathcal{F}^a$ on X such that ${\mathcal{F}^a}_h \cong \mathcal{F}$.*
Furthermore, any homomomorphism $\varphi' : \mathcal{F}' \to \mathcal{G}'$ between coherent analytic sheaves on X_h is induced by a unique homomorphism $\varphi^a : \mathcal{F}^a \to \mathcal{G}^a$ between the corresponding associated sheaves.
The assigment $\varphi \mapsto \varphi^a$ is functorial, i.e. $\mathrm{Id}^a = \mathrm{Id}$ and $(\varphi \circ \psi)^a = \varphi^a \circ \psi^a$.
Finally, the natural maps

$$\alpha_p : H^p(X, \mathcal{F}) \to H^p(X_h, \mathcal{F}_h).$$

are isomorphisms.

The following corollaries should be obvious:

Corollary *If $\mathcal{F}$ and $\mathcal{G}$ are coherent on X and the sheaves $\mathcal{F}_h$ and $\mathcal{G}_h$ are isomorphic on X_h, then $\mathcal{F} \cong \mathcal{G}$.*

Corollary *Let X be projective. The group of holomorphic line bundles up to isomorphism is isomorphic to the group of algebraic line bundles up to isomorphism.*

With little work, one can derive

Corollary (Chow's Theorem) *Any compact subvariety X of $\mathbb{P}^n$ has the structure of a projective variety, i.e. there exists a projective subvariety $X^a \subset \mathbb{P}^n$ such that $X_h^a = X$.*

as well as

Corollary *For any holomorphic map $f : X \to Y$ between projective manifolds, there is a unique morphism $X^a \to Y^a$ inducing f.*

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