Lectures on Torelli Theorems
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Summary

Lecture 1: Andreotti's classical beautiful proof of the Torelli theorem for curves (dating from 1958). The lecture ends with variational techniques due to Carlson, Green, Griffiths and Harris (1980), the curve case serving as a model case.
Lecture 2 and 3 : A proof of the Torelli theorem for projective K3–surfaces modeled on the original proof of Piatečkii-Shapiro and Šafarevič (1971) but using the approach in the Kähler case as given by Burns and Rapoport (1975), with modifications and simplifications by Looijenga and Peters (1981). If time allows for it I shall briefly point out some related developments. I particularly want to say something about derived Torelli and also about Verbitsky's recent proof of Torelli for hyperkähler manifolds.

Contents

Introduction

These Lectures are intended for (prospective) PhD-students with some background in complex algebraic geometry.

The first Lecture should be accessible to many since very little is assumed. It mainly explains Andreotti's classical beautiful proof [An] of the Torelli theorem for curves (dating from 1958). It ends with a section making the bridge to a more recent approach, namely the one using variational techniques. The curve case serves as a model case for these techniques. This approach is due to Carlson, Green, Griffiths and Harris [CaGGH] (1980).

The second Lecture is much more demanding but this is compensated by giving ample references. I explain here a proof of the Torelli theorem for projective K3–surfaces modeled on the original proof of Piatečkii-Shapiro and Šafarevič [Pi-S] (1971) but using the approach in the Kähler case as given by Burns and Rapoport [B-R] (1975), with modifications and simplifications by Looijenga and Peters [L-P] (1981).

The last Lecture is meant to complement the first two by briefly pointing out some recent (and less recent) developments. I particularly want to mention the recent proof of Torelli for hyperkähler manifolds [V] due to Verbitsky.

Acknowledgments I thank Hans Sterk whose remarks led to improvements in the presentation.

1 Torelli for Curves

1.1 What You Should Know

Consult [G-H, Ch 2.7] and [A-C-G-H, Ch. 1]. The original result is [T].

By a **curve** we mean a smooth complex projective curve. Let C be a curve of genus $g \geq 2$ and let

$$\phi_K : C \to \mathbb{P}(H^0(K_C)^*)$$

be the canonical map. Concretely, if $\{\omega_1, \dots, \omega_g\}$ is a basis of $H^0(K_C)$,

$$\phi_K(x) = (\omega_1(x) : \cdots : \omega_g(x)) \in \mathbb{P}^{g-1}.$$

It is known that ϕ_K is biholomorphic onto its image for non-hyperelliptic C while, if C is hyperelliptic, ϕ_K is 2 to 1 onto a rational normal curve in $\mathbb{P}^{g-1}$ of degree $g-1$. In that case ϕ_K is ramified in its $2g+2$ Weierstraß points.

Let $x_1, \dots x_d \in C$ and $D = x_1 + \dots + x_d$ the corresponding divisor. Put

$$\langle \phi_K D \rangle = \text{proj. subspace spanned by } \phi_K(x_1), \dots, \phi_K(x_d).$$

Since

$$h^1(D) = h^0(K_C - D) = \operatorname{codim}\langle \phi_K D \rangle,$$

Riemann-Roch implies

$$\dim\langle \phi_K D \rangle = \deg(D) - h^0(D) \textbf{ (geometric form of Riemann-Roch)}.$$

In particular, for generic points x_j one has $h^0(D) = 1$ and so the images $\phi_K(x_j)$ span a projective subspace of maximal dimension $d - 1$.

Integrating along a 1-cycle γ gives a function on $H^{1,0}(C) = H^0(K_C)$ which only depends on the homology class $[\gamma] \in H_1(X, \mathbb{Z})$. This gives an injective homomorphism $\iota : H_1(C, \mathbb{Z}) \hookrightarrow H^0(K_C)^*$; by definition the quotient

$$J(C) = H^0(K_C)^* / \iota(H_1(C, \mathbb{Z}))$$

is the **Jacobian** $J(C)$. It is a complex torus of dimension g since $\operatorname{Im}(\iota)$ can be shown to be a lattice in $H^0(K_C)$. This torus admits a projective embedding as we are now going to explain.

To start with, there is a canonical identification $H^1(J(C), \mathbb{Z}) = H^1(C, \mathbb{Z})$ and so $H^2(J(C), \mathbb{Z}) = \Lambda^2 H^1(J(C), \mathbb{Z}) = \Lambda^2 H^1(C, \mathbb{Z})$ which implies that the cup product form Q on $H^1(C, \mathbb{Z})$ can be viewed as an element $q \in H^2(J(C), \mathbb{Z})$. The cup product form obeys the (**Riemann bilinear relations**):

$$Q(\omega, \omega') = \int_X \omega \wedge \omega' = 0, \quad \mathrm{i} \cdot Q(\omega, \bar{\omega}) > 0 \text{ for } \omega \neq 0. \tag{1}$$

These can be reinterpreted in terms of the **Hodge decomposition**

$$H^1(C, \mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$$

by saying that $H^{1,0}(C)$ is Q-isotropic and that the **Hodge metric** $h(x, y) := \mathrm{i}\omega(x, \bar{y})$ is a positive definite metric. The Hodge decomposition on $H^2(J(C), \mathbb{C})$ reads

$$\begin{aligned} H^2(J(C), \mathbb{C}) &= H^{2,0}(J(C)) \oplus H^{1,1}(J(C)) \oplus H^{0,2}(J(C)) \\ &= \Lambda^2 H^{1,0}(C) \oplus H^{1,0}(C) \otimes H^{0,1}(C) \oplus \Lambda^2 H^{0,1}(C) \end{aligned}$$

and then $q \in H^{1,1}(J(C)) \cap H^2(X, \mathbb{Z})$ and $q > 0$. Hence (by Lefschetz' theorem on $(1,1)$-classes and Kodaira's ampleness criterion), $q = c_1(\Theta)$, where $\Theta_C \subset J(C)$ is an ample divisor, the **theta divisor**. Such a class is called a polarization. Since q is unimodular, this polarization is special: it is a **principal polarization**.

Fix a point $o \in C$. Recall the **Abel-Jacobi map**

$$\begin{aligned} u : C &\to J(C) \\ x &\mapsto u(x) = \text{integration functional } \omega \mapsto \int_o^x \omega. \end{aligned}$$

It induces the morphisms

$$\begin{aligned} u^{(d)} : C^d &\to J(C) \\ (x_1, \dots, x_d) &= u(x_1) + \dots + u(x_d). \end{aligned}$$

In what follows I need the Jacobi matrix of $u^{(d)}$ (with respect to a basis $\{\omega_1, \dots, \omega_g\}$ of $H^0(K_C)$) at the point $\mathbf{x} = (x_1, \cdots, x_d) \in C^d$:

$$\mathcal{J}(u^{(d)})_{\mathbf{x}} := \begin{pmatrix} \omega_1(x_1) & \cdots & \omega_g(x_1) \\ \ddots & & \ddots \\ \omega_1(x_d) & \cdots & \omega_g(x_d) \end{pmatrix}. \tag{2}$$

The image of the Abel-Jacobi map is denoted classically as

$$W_d := \mathrm{Im}(u^{(d)}).$$

To simplify notation, I do not make a distinction between the point $\mathbf{x}$ and the degree d divisor $D = x_1 + \cdots x_d$ and then $u(D)$ means $u^d(\mathbf{x})$ From (2) one sees:

$$T_{u(D)}(W_d) = \{\text{subspace of } (H^0(K_C)^*) \text{ spanned by the rows of } \mathcal{J}(u^{(d)})_{\mathbf{x}}\}.$$

By the geometric form of the Riemann-Roch theorem one deduces:

Lemma 1.1. *Let W_d be the image of $u^{(d)}$. Then W_d is smooth at the point $u(D)$ if $h^0(D) = 1$, and if this is the case, one has:*

$$\mathbb{P}T_{u(D)}W_d = \langle \phi_K D \rangle.$$

Finally, recall some classical theorems involving the Abel-Jacobi map:

Theorem 1.2. (1) *$u(D) = u(D')$ if and only if D are D' linearly equivalent;*

(2) *W_d is smooth at $u(D)$ if and only if $h^0(D) = 1$;*

(3) *W_{g-1}, the image of $u^{(g-1)}$, is a translate of the theta-divisor.*

1.2 Statements

Theorem 1.3 (Torelli)**.** *Let C, C' be two genus g curves such that $(J(C), \Theta_C) \simeq (J(C'), \Theta_{C'})$, then C and C' are isomorphic.*

Let us explain how this can be rephrased purely in terms of Hodge theory. Giving the Jacobian $J(C)$ is the same as giving $H^{1,0}(C)$ together with the integral lattice $\iota H_1(C, \mathbb{Z})$. The differentiable torus underlying $J(C)$ is just $H^1(C, \mathbb{R})/H^1(C, \mathbb{Z})$. The Hodge structure on $H^1(C)$ defines the Weil-operator $W : H^1(C, \mathbb{C}) \to H^1(C, \mathbb{C})$ given by $W|H^{1,0}$ = multiplication by i while $W|H^{0,1}(C)$ is multiplication by $-$i. This operator is real; hence an operator $W : H^1(C, \mathbb{R}) \to H^1(C, \mathbb{R})$. Since $W^2 = -1$, this is a complex structure. Hence $J(C)$ is a complex torus. The cup product form Q on $H^1(C, \mathbb{Z})$ satisfying the Riemann bilinear relations (1) is exactly what is called a **polarization for the Hodge structure** on $H^1(C, \mathbb{Z})$. Summarizing:

Corollary 1.4. *A genus g curve is determined up to isomorphism by the polarized Hodge structure $(H^1(C, \mathbb{Z}), Q)$.*

Torelli can also be reformulated in terms of the period map. Let M_g be the coarse moduli space of genus g curves. It is a quasi-projective variety of dimension $3g - 3$ (provided $g \geq 2$). A point of M_g represents a genus g curve, or, rather, a class $[C]$ of a curve up to isomorphism. The coarse moduli space A_g of principally polarized abelian varieties of dimension g is a quasi-projective variety of dimension $\frac{1}{2}g(g+1)$. The map $[C] \mapsto [J(C)]$ induces a morphism of algebraic varieties $p : M_g \to A_g$, **the period map**, and Torelli is the statement:

Corollary 1.5. *The period map $p : M_g \to A_g$ is an injective morphism.*

Remark. It does not follow that p is an immersion since M_g and A_g have singularities. Nevertheless, this is true [O-S] and goes by the name **local Torelli theorem**.

1.3 Andreotti's Proof

Consult [G-H, p. 358–362], [A-C-G-H, Ch VI,§3]. The proof is in [An].

Let $X \subset J(C)$ be a dimension d subvariety and let $x \in X$ be a smooth point of X. Let $L_x : J(C) \to J(C)$ be the addition map $a \mapsto a + x$. Then $(L_{-x})_* T_x X$ is a subspace of $T_o(J(C)) = H^{1,0}(C)^*$ of dimension d. The map $x \mapsto (L_{-x})_* T_x X$ defines the rational **Gauß map**

$$\gamma_X : X \dashrightarrow \mathbb{G}(d, H^{1,0}(C)^*).$$

The normalization $\widetilde{\Gamma}_X$ of the closure of its graph $\Gamma_X \subset X \times \mathbb{G}(d, H^{1,0}(C)^*))$ can be projected onto the second factor which gives a morphism

$$\tilde{\gamma}_X : \widetilde{\Gamma}_X \to \mathbb{G}(d, V), \quad V := H^{1,0}(C)^*,$$

the **Gauß morphism**. Note that in case X is a hypersurface, $d = g - 1$ and then

$$\tilde{\gamma}_X : \widetilde{\Gamma}_X \to \mathbb{P}(V^*).$$

The case which is of interest to us is when this morphism is surjective and generically finite of degree $k \geq 2$. If this is the case, the **ramification locus** $R(\tilde{\gamma}_X) \subset \mathbb{P}(V^*)$ is the complement of the set where the Gauss morphism is unramified.

Andreotti's proof uses the geometry of the ramification locus of the Gauß morphism for $\Theta_C \subset J(C)$ and starts with the following observations:

Lemma 1.6. *Let C be a non-hyperelliptic curve. Then*

(1) $\widetilde{\gamma}_{\Theta_C} : \widetilde{\Gamma}_{\Theta_C} \to \mathbb{P}(V^*))$ *is finite;*

(2) *Let $C^* \subset \mathbb{P}(V^*)$ be the dual variety of $\phi_K C$, i.e., the variety whose points correspond to hyperplanes $H \subset \mathbb{P}(V)$ that are tangent to $\phi_K C$. Then $C^* = R(\tilde{\gamma}_{\Theta_C})$, the ramification locus of the Gauß morphism for Θ_C.*

Proof: (1) A hyperplane $H \subset \mathbb{P}(V)$ corresponds to a point $[H] \in \mathbb{P}(V^*)$. Suppose $[H] = \widetilde{\gamma}_{\Theta_C}(u(D))$ with $D = x_1 + \cdots + x_{g-1}$. For generic points x_j by Lemma 1.1 one has $\mathbb{P}T_{u(D)}\Theta = \langle \phi_K(x_1), \cdots \phi_K(x_{g-1}) \rangle$. By the definition of the Gauß map, we have $\mathbb{P}T_{u(D)}\Theta = H$. A limit argument shows that this remains true for all $D \in W_d$. The hyperplane H meets the canonical curve $\phi_K(C)$ in $2g - 2$ points (counted with multiplicity) and so the fiber of the Gauß morphism consists of

$$\delta_g := \binom{2g-2}{g-1}$$

points.
(2) Let $E \subset \mathbb{P}(V^*)$ be the set of hyperplanes $H \subset \mathbb{P}(V)$ such that $H \cap \phi_K(C)$ is a collection of points which are not in general position. If $H \in \mathbb{P}(V) - E$ is tangent to $\phi_K(C)$, the fiber of the Gauß morphism over H consists of less than δ_g points and hence $[H] \in R := R(\tilde{\gamma}_X)$. It is not hard to see that, conversely, if H is not tangent to the canonical curve $\phi_K(C)$, the corresponding point $[H]$ cannot be a ramification point of the Gauß morphism. It follows that $R \subset C^*$ and that $R - (E \cap C^*) = C^* - (E \cap C^*)$.

Then one must have equality, $R = C^*$, since C^* is an irreducible variety: it is the image under projection of the incidence correspondence:

$$\{(x, H) \in C \times \mathbb{P}(V) \mid T_x C \subset H\} \to \mathbb{P}(V). \quad \square$$

Remark 1.7. If C is hyperelliptic, $\phi_K(C)$ is $2:1$ onto a rational normal curve in $\mathbb{P}(V)$ and there are $2g+2$ ramification points $x_1, \dots, x_{2g+2}$. In this situation the ramification locus R of the Gauß morphism is reducible:

$$R = C^* \cup x_1^* \cup \dots \cup x_{2g+2}^*$$

where the $x_j^* \subset \mathbb{P}(V^*)$ are the hyperplanes in $\mathbb{P}(V)$ that pass through $x_j \in \mathbb{P}(V)$ [1].

Concluding Step of the Proof. One uses bi-duality

$$(C^*)^* = \phi_K(C).$$

This suffices to reconstruct C, if C is not hyperelliptic. Otherwise, the canonical image of C can be reconstructed as $(C^*)^*$ and the other components of R give $x_1, \cdots, x_{2g+2}$; together these give back the hyperelliptic curve C. $\square$

1.4 A Variational Version

For this subsection consult [C-MS-P, Ch. 4 and 5], [A-C-G-H, Ch. III §3].

We have seen that the Torelli theorem can be rephrased as the injectivity of the period map. Let us now look at the period map for an arbitrary family $f : \mathcal{C} \to S$ of genus g curves $\mathcal{C}_s = f^{-1}s, s \in S$ over a smooth complex manifold S. We have also seen that the data of the polarized Jacobian is the same as the datum of the polarized Hodge structure on H^1. So one can equally consider the local system $R^1 f_* \mathbb{Z}$ on S whose stalk at s is $H^1(\mathcal{C}_s, \mathbb{Z})$. Let us for simplicity assume that this is a trivial local system, so that we can identify $H^1(\mathcal{C}_s, \mathbb{Z})$ with a fixed free lattice H. The Hodge decomposition singles out a holomorphic subbundle of $H \otimes_{\mathbb{Z}} \mathcal{O}_S$, namely the bundle F whose fibre at s is $H^{1,0}(C_s)$. The two bilinear relations for the cup product form Q translate into $Q|F_s$ being totally isotropic plus the positivity condition $\mathrm{i}Q(z, \bar{z}) > 0$ for $z \neq 0$. These positive isotropic g-dimensional subspaces of $H \otimes \mathbb{C}$ are parametrized by Siegel's upper half space $\mathfrak{h}_g$ so that we get a period map

$$p : S \to \mathfrak{h}_g, \quad s \mapsto F_s \subset H_{\mathbb{C}}.$$

The tangent space of $\mathfrak{h}_g$ at a point F turns out to be the set of Q–endomorphisms $\alpha : F = H^{1,0} \to H_{\mathbb{C}}/F^{1,0} = H^{0,1}$, i.e., for which $Q(\alpha x, y) + Q(x, \alpha y) = 0$ for all $x, y \in H_{\mathbb{C}}$. Identifying $H^{0,1}$ with F^* via the polarization, we then see that

$$T_F \mathfrak{h}_g = \mathrm{Hom}^Q_{\mathbb{C}}(F, F^*) := \{\alpha \in \mathrm{Hom}_{\mathbb{C}}(F, F^*) \mid \alpha \text{ is a } Q\text{–endomorphism.}\}$$

[1] x_j^* is indeed a hyperplane in $\mathbb{P}(V^*)$

Fix a reference point $o \in S$ and let $C = \mathcal{C}_0$. Recall that we have a **Kuranishi map** $\kappa : T_{S,o} \to H^1(C, T_C)$ and that there is a natural cup product morphism

$$H^1(C, T_C) \otimes \underbrace{H^0(\Omega^1_C)}_{F} \to \underbrace{H^1(\mathcal{O}_C)}_{F^*} \tag{3}$$

which intervenes when computing the tangent map.

Lemma 1.8. *The tangent map* $(dp)_o : T_{S,o} \to \mathrm{Hom}^Q_{\mathbb{C}}(F, F^*)$ *sends a tangent vector* $\xi \in T_{S,o}$ *to the homomorphism*

$$\begin{aligned} \alpha(\xi) : \quad F &\to \quad F^* \\ z &\mapsto \kappa(\xi) \cup z \end{aligned}$$

induced by (3).

The tangent map $(dp)_o$ is also called **infinitesimal variation of Hodge structure** associated to the variation. We shall give two applications of Lemma 1.8.

Corollary 1.9. *Suppose that the Kuranishi map* κ *is injective. The period map is an immersion at* o *if and only if the homomorphism* $\xi \mapsto \alpha(\xi)$ *from Lemma 1.8 is injective on the image of* κ*.*

This can be applied to any **Kuranishi family** $f : \mathcal{C} \to B$ (i.e., a locally universal family) since for such a family κ is an isomorphism. One constructs such a family as follows. Let U be a germ at o of the moduli space M_g and let C be a curve corresponding to o. Then, as is well known, if $G = \mathrm{Aut}(C)$, we can write $U = B/G$ where B is smooth; then the germ (B, o) is the base of a locally universal family. In particular $T_{B,o} = H^1(C, T_C)$. By Serre duality $H^1(C, T_C) = H^0((\Omega^1_C)^{\otimes 2})$ and so the map α from Lemma 1.8 dualizes to the cup product

$$H^0(\Omega^1_C) \otimes H^0(\Omega^1_C) \to H^0((\Omega^1_C)^{\otimes 2}). \tag{4}$$

It follows that the period map is an immersion if and only if this product is surjective. This is known to hold if $g = 2$ and for higher genus if and only C is not hyperelliptic. So we see that although Torelli is true for all curves, this fails infinitesimally. As remarked before, it is still true that the period map on the level of period spaces is injective. The reason is that the hyperelliptic locus is singular which compensates for the failure of infinitesimal Torelli.

The second application is a **variational Torelli theorem**: the infinitesimal variation for a Kuranishi family for C determines C up to isomorphism. The crucial observation is that (4) can be viewed as part of the structure of the so called **canonical ring**

$$R_C := \bigoplus_{k \geq 0} H^0(C, \omega_C^{\otimes k}), \quad \omega_C = \Omega^1_C.$$

To explain this, recall the canonical map $\phi_K : C \to \mathbb{P}(V)$. The coordinate ring of $\mathbb{P}(V)$ equals

$$R := \bigoplus_{k\geq 0} (H^0(C, \omega_C))^{\otimes k}$$

since $H^0(\mathbb{P}(V), \mathcal{O}(1)) = H^0(C, \omega_C)$. So, there is an exact sequence

$$0 \to I_C \to R \to R_C \to 0$$

where I_C is the ideal of the canonical image. If C is not hyperelliptic, the ideal I_C permits to reconstruct C. What the infinitesimal variation gives back is I_2, the quadrics passing through the canonical curve. Indeed, this is the kernel of the cup product map (4).

In many cases this suffices to reconstruct the canonical curve. Indeed, an old result of Enriques, Babbage and Petri (for exact references, see the bibliographical notes to [A-C-G-H, Ch.III]) shows that the intersection of all quadrics in I_C is exactly the canonical image provided C is not hyperelliptic, a plane quintic, or a trigonal and hence:

Proposition 1.10. *For a non-trigonal curve of genus $g = 3, 4, 5$, $g > 6$, variational Torelli holds.*

Let me finish this section by discussing the meaning of variational Torelli. After passing to a finite cover of M_g we may assume that we have a universal family over a smooth base B and that the period map $p : B \to A_g$ is an immersion. In particular, the infinitesimal variations at s, s' are equal (i.e. $dp_s = dp_{s'}$) if and only if $p(s) = p'(s)$. Variational Torelli for states that the infinitesimal variations at s, s' are equal (i.e. $dp_s = dp_{s'}$) if and only if $C_s \simeq C_{s'}$. Of course, this is a consequence of the Torelli theorem. The above variational method allows to conclude this only for *non-trigonal* curves of genus $g \neq 2, 6$, i.e., in this range of the genus, variational Torelli says that the period map is generically injective.

The crucial remark here is that variational Torelli only uses part of the geometry of the canonical image, while Torelli uses a much richer geometry: Gauß maps, etc. The variational approach is therefore in principal applicable in a wider range of settings as we shall see later in § 3.2.

2 Torelli for K3 Surfaces

Consult [B-P-H-V, Ch. VIII] and the references given at the end of this chapter.

2.1 Topology

By definition a K3–surface S is a compact complex surface which is simply connected and which has a nowhere vanishing holomorphic 2-form ω_S. For simplicity I shall restrict the entire discussion to *projective* K3-surfaces.

Examples 2.1. (1) A smooth hypersurface of degree 4 in $\mathbb{P}^3$ is a K3–surface; (2) Let A be a complex 2–torus and let $\iota : A \to A$ be the standard involution, i.e., $\iota(x) = -x$. The quotient surface has 16 ordinary double points which are the images of the 16 order 2 points on A. The minimal resolution $\mathrm{Km}(A)$ is called a **Kummer surface**. If A is projective, also $\mathrm{Km}(A)$ is projective; (3) A surface S admitting a double cover $\pi : S \to \mathbb{P}^2$ branched along a smooth sextic curve is a K3–surface.

The invariants of a K3–surface are as follows:

Lemma 2.2. (1) *The Betti numbers of a K3–surface S are as follows: $b_1(S) = 0$, $b_2(S) = 22$;*

(2) *The Hodge numbers are $h^{2,0}(S) = h^{0,2}(S) = 1$, $h^{1,1}(S) = 20$;*

(3) *The cup product form on $H^2(S, \mathbb{Z})$ is even, unimodular of signature $(3, 19)$.*

Proof: Since S is simply connected $b_1(S) = b_3(S) = 0$, and hence also the irregularity $q(S) = H^{1,0}(S) = 0$. The existence of ω_S implies that $p_g(S) = h^{2,0}(S) = 1$. Riemann-Roch reads

$$p_g(S) - q(S) + 1 = \frac{c_1^2(S) + c_2(S)}{12}$$

and hence $24 = c_2(S) = 2 - 2b_1(S) + b_2(S)$ (since $c_1(S) = -K_S = 0$) and so $b_2(S) = 22$. This proves (1). Then (2) is immediate. Item (3) can be seen as follows: Poincaré-duality implies the unimodularity of the lattice $H^2(S, \mathbb{Z})$. That the pairing is even, is slightly more involved. See e.g. [B-P-H-V, Lemma VIII 3.1]. The signature theorem directly implies the assertion about the signature. See loc. cit. □

Since, up to isometry there is only one even unimodular lattice of signature $(3, 19)$ [Se, § 2.2–2.3] we fix one and call it the **K3–lattice** L.

The Hodge structure on $H^2(S, \mathbb{Z})$ gets polarized by the cup product form Q which means that, as in the curve case, the two Riemann bilinear relations hold. In the present setting these can be rephrased by saying that $H^{2,0}(S) \subset L_{\mathbb{C}} := L \otimes_{\mathbb{Z}} \mathbb{C}$ is a Q–isotropic line which is positive in the sense that $h(z) := Q(z, \bar{z}) > 0$. The collection of such lines forms the open manifold

$$D := \{[z] \in \mathbb{P}(L \otimes \mathbb{C}) \mid Q(z, z) = 0;\ h(z) > 0\} \subset \mathbb{P}(L \otimes \mathbb{C}).$$

2.2 The Ample Cone

Consider the polarized Hodge structure $L(S) = (H^2(S, \mathbb{Z}), Q)$. Simplifying notation, I write $c \cdot d = Q(c, d)$ and $c^2 = c \cdot c$. The Hodge structure permits to locate divisor classes as the $(1,1)$ classes in $L(S)$. They span the **Néron-Severi sublattice** $N(S) \subset L(S)$. The orthogonal complement $T(S) = N(S)^\perp$ is called the **transcendental lattice**. The cone $\{x \in N_\mathbb{R}(S) \mid x^2 > 0\}$ is a disjoint union of two half-cones. One of these contains the effective divisors and is called the **positive cone** $N^+_\mathbb{R}(S)$. The ample classes span a subcone $C(S) \subset N^+_\mathbb{R}(S)$, the **ample cone**.

Riemann-Roch implies that $\pm d \in N(S)$ is the class of an effective divisor as soon as $d^2 \geq -2$. This holds in particular for **roots** $r \in N(S)$, i.e., divisor classes r with $r^2 = -2$. An effective such class is called a **nodal class**. To any root there is associated a corresponding reflection

$$\begin{aligned} s_r : L(S) &\to L(S) \\ x &\mapsto x + (x \cdot r)r. \end{aligned}$$

Let $W(S) \subset \mathrm{O}(L(S))$ be the group these generate. They preserve the positive cone. The complement of the reflection hyperplanes $H_r := r^\perp \subset N_\mathbb{R}(S)$ is a union of connected half-cones of $N^+_\mathbb{R}(S)$ and one of these is the ample cone. Its closure is a fundamental domain for the action of $W(S)$ on the positive cone. This has the following consequence:

Proposition 2.3 ([B-P-H-V, Prop. VIII, 3.11])**.** *An isometry of $L(S)$ preserves classes of effective divisors if and only if it preserves the ample cone $C(S)$. After composing the isometry with an isometry in $\{\pm 1\} \times W(S)$ these equivalent conditions are satisfied.*

Definition 2.4. An isometry of L_S satisfying one of the above equivalent properties is called an **effective** isometry.

Remark. Suppose that $N(S)$ has rank 1 and hence is spanned by an ample divisor. In this case the positive cone is the same as the ample cone. There are no roots in $N(S)$: every isometry is either effective or its negative is.

2.3 Infinitesimal Torelli

By Serre duality we have

$$\begin{aligned} \dim H^0(S, T_S) &= \dim H^2(\Omega^1_S) = 0, \\ \dim H^1(S, T_S) &= \dim H^1(\Omega^1_S) = h^{1,1}(S) = 20, \\ \dim H^2(S, T_S) &= \dim H^0(\Omega^1_S) = 0. \end{aligned}$$

By general deformation theory it follows from this that $S = \mathcal{S}_0$ has a locally universal deformation $f : \mathcal{S} \to B$ over a base of dimension 20. Assume for a moment that this base is contractible so that $R^2 f_* \mathbb{Z}$ is a constant local system with stalks $H^2(\mathcal{S}_s, \mathbb{Z})$ isomorphic to L. The choice of a marking $L_B \xrightarrow{\simeq} R^2 f_* \mathbb{Z}$ defines the period map $p : B \to D$.

Proposition 2.5 (Infinitesimal Torelli)**.** *If $(\mathcal{S}, S) \to (B, o)$, $S = \mathcal{S}_0$ is a Kuranishi family of K3–surfaces, the period map $p : B \to D$ is an immersion at o.*

Proof: The complex tangent space at a point $[\omega] \in D$ is canonically isomorphic to $\mathrm{Hom}(\mathbb{C} \cdot \omega, \omega_{\mathbb{C}}^{\perp}/(\mathbb{C} \cdot \omega))$. If $[\omega] = p(S)$, this space is just $\mathrm{Hom}(H^{2,0}(S), H^{1,1}(S)$. Hence (after a small computation similar to the curve case (3)) the derivative of the period map at $o \in B$ can be identified as

$$\begin{aligned} T_oB = H^1(S, T_S) &\to \mathrm{Hom}(H^{2,0}(S), H^{1,1}(S)) \\ \xi &\mapsto \{\omega \mapsto \omega \cup \xi\}. \end{aligned}$$

Making appropriate identifications, this map is the isomorphism

$$H^1(\Omega^1_S) \xrightarrow{\simeq} \mathrm{Hom}(H^{2,0}(S), H^{1,1}(S)). \quad \square$$

2.4 Global Torelli: Statement

The Hodge structure on $H^2(S)$ then determines a point in D after choosing an isometry $\varphi : L \xrightarrow{\simeq} H^2(X, \mathbb{Z})$. Such an isometry is called a **marking** and the resulting point in D the **period point** of (S, φ).

In the projective setting S has an ample divisor[2] which under the marking gives a primitive element $\ell \in L$. All K3–surfaces S admitting such a polarization define a period point $p(s)$, $s \in S$ such that ℓ corresponds to a $(1,1)$-class which means $Q(\ell, p(s)) = 0$: the period point belongs to the hyperplane in $\mathbb{P}(L_{\mathbb{C}})$ orthogonal to ℓ. The corresponding marking is called an **ℓ-marking** and gives points in the 19–dimensional manifold

$$D(\ell) = D \cap \ell^{\perp}.$$

This manifold consists of two connected components. The orthogonal group $\mathrm{O}(L)$ acts properly and discontinuously on D. Of course $\ell' = \gamma(\ell)$, $\gamma \in \mathrm{O}(L)$ has the same (even) norm as ℓ, say $\ell^2 = 2k > 0$. If can be shown that conversely, any $\ell' \in L$ with $(\ell')^2 = 2k$ belongs to the $\mathrm{O}(L)$–orbit of ℓ. Divisor classes that correspond to any element in this orbit is called a **polarization of type** k. So, we may write

$$M_k := \mathrm{O}(L) \backslash D(\ell)$$

for the resulting quotient space, the **moduli space of K3-surfaces with a polarization of type** k. Since $\mathrm{O}(L)$ contains elements interchanging the two components, this quotient is connected. It can be shown that in fact M_k is a quasi projective variety. I can now state the main result in this chapter.

[2]For technical reason one allows divisors for which a multiple gives an embedding up to A-D-E singularities; these are the so called almost-ample divisors.

Theorem 2.6 (Torelli Theorem for Projective K3–surfaces)**.** *Two K3-surfaces admitting a polarization of type k are isomorphic if and only if they have the same period point in M_k.*

Remark. One can actually prove that all points of M_k occur as a period point of a K3–surface admitting a polarization of type k. Here it is important to allow almost-ample divisors: any point on a hypersurface in M_k orthogonal to a root r in the lattice $\ell^\perp$ corresponds to a K3 on which the divisor corresponding to ℓ is not ample since $Q(\ell, r) = 0$ and $\pm r$ corresponds to an effective divisor (by Riemann-Roch).

Variants and Complements

There are several variants of the Torelli Theorem.

- There is a more precise theorem (the *refined Torelli theorem*) which states that whenever there is a polarized Hodge isomorphism $\varphi : L(S) \xrightarrow{\simeq} L(S')$ (so it preserves the lattice structure) which is moreover effective (see Definition 2.4), then there is a *unique* isomorphism $S' \xrightarrow{f} S$ with $f^* = \varphi$. Applying this to $S = S'$ makes it possible to describe the group of automorphisms as the group $G(S)$ of effective isometries inside $L = L(S)$. It is the subgroup of $\mathrm{O}(L)$ preserving the ample cone and one has $\mathrm{O}(L) = G(S) \times \{\pm 1\} \ltimes W(S)$. This description is fundamental for many recent studies of the possible automorphism groups of K3–surfaces.

  Note that the statement of the refined Torelli theorem simplifies if $N(S)$ (and hence $N(S')$) has rank 1 since in that case $\pm\varphi$ is effective for any polarized Hodge isometry φ. The assumption for the Néron-Severi group holds generically on the moduli space and so one has: a polarized Hodge isometry $\varphi : L(S) \xrightarrow{\simeq} L(S')$ is up to sign induced by a (unique) isomorphism $S' \to S$ provided S, S' are generic. This form of the Torelli theorem holds also for hyperkähler manifolds (except for the uniqueness statement). See § 3.2 below.

- There is a version for all Kähler K3–surfaces (this is no restriction: all K3–surfaces are Kähler, see below). Of course one has to substitute the proper notion for the ample cone: the so-called Kähler cone. The proof I sketch below in § 2.5 applies in this setting with some more or less obvious modifications. Also a refined Torelli holds in this case.

- **Derived Torelli**. Let X be any complex projective manifold. Then $D^{\mathrm{b}}(X)$ denotes the derived category of bounded complexes of coherent sheaves on X. Two varieties X and X' are called **derived equivalent** if there exists an equivalence of triangulated categories $D^{\mathrm{b}}(X) \simeq D^{\mathrm{b}}(X')$.

Orlov shows that derived equivalence for K3-surfaces can be seen from the transcendental part of the K3-lattice $L(S)$,

$$T(S) := N(S)^\perp \subset L(S),$$

namely, one has:

Theorem ([O]). *Two K3–surfaces S and S' are derived equivalent if and only if there exists a Hodge isometry $T(S) \xrightarrow{\simeq} T(S')$.*

There are several variants of this result, for instance the variant which uses the **Mukai lattice** instead of the transcendental lattice:

$$\begin{aligned} M(S) &= H^0(S) \oplus H^2(S) \oplus H^4(S), \\ q_{M(S)}(x,y) &= -x_2 \cdot y_2 + x_0 \cdot y_4 + x_4 \cdot y_0, \\ &= -L(S) \oplus U, \end{aligned}$$

where $x = (x_0, x_2, x_4)$, $y = (y_0, y_2, y_4)$ and where U is the standard hyperbolic lattice. In this setting, Hodge isometry only means that under the isometry the class of $[\omega_S]$ is mapped to the class of $[\omega_{S'}]$. See [H1] for details.

Remark. (1) The proof that **all K3–surfaces are Kähler** is relatively recent (1982/1983 Siu [Si], Todorov [To]). This proof at the same time shows surjectivity of the period map: all points in D occur as period points. See e.g. [Be, XII] and the references given there. This surjectivity proof has been generalized by Verbitsky [V] in order to prove Torelli for hyperkähler manifolds. See § 3.3.

Sixteen years later a completely different proof for the Kähler property (by Buchdahl [Bu] and Lamari [La] both in 1999) has been found which applies to all surfaces with even b_1. See [B-P-H-V, Ch. IV.3].

(2) Extending the notion of a K3–surface S of polarization type $\ell \in L$, fixing $M \subset L$ of signature $(1, t)$, one says that S is of **polarization type** M if there exists a marking $\varphi : L(S) \xrightarrow{\simeq} L$ such that $\varphi^{-1} M$ belongs to the Néron-Severi lattice $N(S)$. The relevant period domain is

$$D_M = D \cap M^\perp,$$

a domain of dimension $19 - t$ having two connected components. Such domains come up when studying special families of K3–surfaces. See for example [B-P-H-V, Ch.VIII.22] where the relation with mirror symmetry is explained, or [L-P-S] where a link with special Abelian varieties is pursued.

2.5 Global Torelli: Sketch of Proof

One has to compare the Hodge structures of two different K3–surfaces S and S' with the same period point. In particular, there is a Hodge isometry

$\varphi : H^2(S,\mathbb{Z}) \xrightarrow{\sim} H^2(S',\mathbb{Z})$. Applying Prop. 2.3 I may assume that φ is effective: it sends $C(S)$ to $C(S')$ and it sends effective divisor classes on S to similar such classes on S'. The proof proceeds in several steps.
Step 1. Torelli for Kummer surfaces, i.e. one shows that if S is a projective Kummer surface, S' any K3–surface such that their period points are the same, then S and S' are isomorphic.

Let me give a sketch of the proof. Let A be a projective 2–torus and $V \subset A$ the collection of 2–torsion points. This set has a natural structure of an affine space of dimension 4 over the field $\mathbb{F}_2$. For every $v \in V$ there is a nodal class e_v on $S = \mathrm{Km}(A)$. It is known (see [B-P-H-V, Ch. VIII.6] for references) that $\sum_{v\in V} e_v = 2f \in L(S)$ and hence this also holds in $L(S')$ as an equality between effective divisors (since φ is effective). But then (loc. cit.) the surface S' also is Kummer, say $S' = \mathrm{Km}(A')$ and φ sends the affine space V to the corresponding affine space V' associated to the collection of 2–torsion points on A'. A nice argument [B-P-H-V, Ch. VIII.5] using detailed affine and projective geometry over the field $\mathbb{F}_2$ then shows that A and A' are isomorphic. A fortiori, $S \simeq S'$.
Step 2. The period points of projective Kummer surfaces forms a dense set in D. This argument is lattice-theoretic and quite elementary. It is spelled out in detail in [B-P-H-V, Ch. VIII. 8]. This step implies also that any two K3–surfaces are diffeomorphic.
Step 3. Let me first change notation. Instead of S, S' the surfaces are now called $\mathcal{S}_o$, $\mathcal{S}'_o$ since I want to put both of them in Kuranishi families $f : \mathcal{S} \to B$, $f' : \mathcal{S}' \to B'$, respectively. Moreover, by the Infinitesimal Torelli theorem I may replace B and B' by their images in the period domain D. We may further replace these images by a sufficiently small polydisk Δ about 0 such that the isometry at 0 has been extended to an isometry $\Phi : R^2 f_*\mathbb{Z} \xrightarrow{\simeq} R^2 f'_*\mathbb{Z}$.

By Steps 1 and 2 there is a sequence $z_n \to 0 \in \Delta$ and there are isomorphisms $f_n : \mathcal{S}_{z_n} \xrightarrow{\simeq} \mathcal{S}'_{z_n}$ between the fibers over z_n of these families inducing Φ_{z_n}. A fairly general principle which is explained in [B-P-H-V, Ch. VIII. 10] then implies that, after passing to a subsequence, the isomorphisms f_n converge (in a suitable topology) to an isomorphism f_0.

3 Other Torelli Theorems

3.1 Infinitesimal Torelli

For this section see [C-MS-P, Ch. 8].

Recall that infinitesimal Torelli (w.r.t. cohomology of rank w) means that the period map for a Kuranishi family is an immersion. Griffiths found a cohomological criterion for this which generalizes what happens for curves and K3-surfaces:

Proposition 3.1. *Infinitesimal Torelli (w.r. to cohomology of rank w) holds for a projective manifold X if the cup product map*

$$H^1(T(X)) \to \bigoplus_{p+q=w} \mathrm{Hom}(H^q(\Omega^p_X), H^{q+1}(\Omega^{p-1}_X))$$

is injective.

Apart from curves and K3–surfaces, this criterion has been applied successfully to hypersurfaces in projective space, to complete intersections and several other cases. See [C-MS-P, Ch. 8.1].

3.2 Variational Torelli

The most widely applicable approach to Torelli theorems has been through the use of the multilinear algebra information provided by the tangent of the period map at any given point. The prototype of argument has been given in Ch.1.4 and, as shown there, leads to generic Torelli theorems. The most explicit results in this direction are for hypersurfaces in projective spaces: except for some special degrees generic Torelli holds for such hypersurfaces. See [C-MS-P, Ch. 8.3]. There are other, less explicit results stating that for all sufficiently ample hypersurface sections of a given projective manifold generic Torelli holds. See [G].

3.3 Torelli for Hyperkähler Manifolds

For this section where Verbitsky's results are discussed, consult [V] and the Bourbaki talk [H2].

Definition 3.2. A simply connected compact Kähler manifold X is said to be **hyperkähler** if there exists an everywhere non-degenerate holomorphic 2–form σ such that $\mathbb{C} \cdot \sigma = H^0(X, \Omega^2_X)$.

Examples 3.3. (1) Any K3–surface;
(2) The Hilbert scheme of finite fixed length zero-dimensional subschemes of a fixed K3–surface.

As for K3–surfaces it turns out that for X hyperkähler, there is an integral (in general not unimodular) form on $H^2(X, \mathbb{Z})$, the Beauville-Bogomolov form q_X. It has signature $(3, b_2(X) - 3)$. It polarizes the Hodge structure on $H^2(X)$. One sets

$$\Lambda(X) = (H^2(X, \mathbb{Z}), q_X).$$

As for K3–surfaces, for a model such lattice, say Λ with form q of signature $(3, b-3)$, one introduces the period domain

$$D_\Lambda = \{x \in \mathbb{P}(\Lambda_\mathbb{C}) \mid q(x) = 0,\, q(x, \bar{x}) > 0\}$$

which is a connected manifold of dimension $b-2$. A marking is just an isometry $\Lambda(X) \xrightarrow{\simeq} \Lambda$ and one lets M_Λ be the moduli space of Λ–marked hyperkähler manifolds. In general it may have many components. Fix one and continue to denote it M_Λ. Clearly, there is a holomorphic period map

$$p_\Lambda : M_\Lambda \to D_\Lambda.$$

Theorem 3.4 ([V]). *The period map p_Λ is a local biholomorphic map which is surjective and generically injective.*

Remark. (1) It is known that p_Λ is in general *not* injective although it is in the bimeromorphic sense. However, Namikawa [N] has found two hyperkähler non-bimeromorphic projective 4-folds X, X' admitting an isometry $H^2(X) \simeq H^2(X')$ of polarized Hodge structures. These must belong to two different components of the moduli space.
(2) As a side remark, the proof of Verbitsky's theorem gives an alternative approach to Torelli for K3–surfaces which is much more direct in that it avoids using the geometry of special K3–surfaces like Kummer surfaces which are dense in moduli.

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