The Baily-Borel boundary: Hodge theory and group theory on algebraic tori fibers

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Introduction

This is the write-up of my lectures *The role of algebraic tori in the Baily-Borel compactifications: Hodge and group theoretic aspects* given in the workshop *Perverse Sheaves and Applications*, Dijon, 7–8 April 2015.

The notes follow the informal style of the lectures, but I tried to give rather precise references for the results I discussed. These are based on two main sources: [HaZ] and [Bu-Wi].

1 Hodge theory on modular curves

1.1 The modular curve away from the cusps

The upper half plane $\mathfrak{h} = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ is homogeneous under $\mathrm{SL}(2;\mathbf{R})$:

$$\mathfrak{h} = \mathrm{SL}(2;\mathbf{R})/\mathrm{U}(1).$$

Its points parametrize weight one polarized Hodge structures on the symplectic lattice $V_{\mathbf{Z}} = \left(\mathbf{Z}^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$. Indeed, one has the following correspondence between points of $\mathfrak{h}$ and lines in $\mathbf{C}^2$ which give the first step in the Hodge filtration $0 \subset F^1 \subset F^0 = \mathbf{V}_{\mathbf{C}} = \mathbf{V}_{\mathbf{Z}} \otimes \mathbf{C}$:

$$\mathrm{i} \iff F_{\mathrm{i}} := \mathbf{C}\begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} \subset \mathbf{C}^2, \quad g \cdot \mathrm{i} \iff gF_{\mathrm{i}} \subset \mathbf{C}^2.$$

This assignment defines the tautological variation of Hodge structure on $\mathfrak{h}$ and it descends to the modular curve

$$X_0(N) := \Gamma_N \backslash \mathfrak{h}, \quad \Gamma_N = \{A \in \mathrm{SL}(2;\mathbf{Z}) \mid A \equiv \mathrm{Id}_2 \bmod N\}.$$

The variation one obtains, is called the *standard construction* $\mu(V)$ for $V = V_{\mathbf{Z}} \otimes \mathbf{Q}$ on $X_0(N)$. We'll see below that this construction extends from the standard representation V for the group $\mathrm{GL}^+(2) =$ (invertible 2×2 rational or real matrices with positive determinant) to any representation of this group.

1.2 Compactification

For this section I'll simplify matter by taking $\Gamma = \Gamma(1)$ which does not act freely on $\mathfrak{h}$ but has the advantage of having only one cusp. Of course this is all well known but the point is to introduce the terminology of the Baily-Borel compactification as introduced in [B-B].

The upper half plane has a realization as a bounded domain, namely the unit disk: just apply the holomorphic map $z \mapsto w = (z - \mathrm{i})(z + \mathrm{i})$. Any bounded domain can be embedded in its *compact dual*; in this case the compact dual is $\mathbf{P}^1(\mathbf{C})$ and the rational boundary points of the disk give the cusps in the quotient under Γ. In terms of homogeneous spaces

$$\mathfrak{h} = \mathrm{SL}(2;\mathbf{R})/\mathrm{U}(1) \subset \mathrm{SL}(2;\mathbf{C})/P(\mathbf{C}) = \mathbf{P}^1(\mathbf{C}), \quad P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

and the *rational boundary components* in this example are the points in $\mathbf{P}^1(\mathbf{Q})$. They form one orbit under Γ and I take the point $\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the extended upperhalf plane as a model cusp.

The next issue is the description of the gluing of a neighborhood at infinity to $X_0(1) = \Gamma\backslash\mathfrak{h}$. In the Baily-Borel compactification this is done group theoretically as follows. The parabolic group $P = P(\mathbf{R})$ fixes the point ∞ and its unipotent radical U consists of the real translations $z \mapsto z + b$. The integer translations forming the group $U_{\mathbf{Z}}$ have a remarkable property: for $\operatorname{Im} z > 1$ two points are in the same Γ–orbit precisely if they are in the same $U_{\mathbf{Z}}$–orbit. This means that $U_{\mathbf{Z}}\backslash\mathfrak{h}$ and $X_0(\Gamma)$ near the cusp ∞ look the same. The algebraic torus "fibration" over the boundary component ∞ in this case is just $U_{\mathbf{Z}}\backslash\mathbf{C} \simeq \mathbf{C}^*$ which contains a small pointed disk over which this happens:

$$\begin{array}{ccc} \mathbf{Z}\backslash\mathbf{C} & \xrightarrow[\simeq]{\mathbf{e}} & \mathbf{C}^* \\ \uparrow & & \uparrow \\ \mathbf{Z}\backslash\{\operatorname{Im} z > 1\} & \xrightarrow[\simeq]{\mathbf{e}} & \Delta_r^* = \{w \in \mathbf{C} \mid 0 < |w| < r\} \end{array}$$

where $w = \mathbf{e}(z) = \exp(2\pi\mathrm{i}z)$ and $r = e^{-2\pi}$. To compactify, just glue in the disk Δ_r.

Hodge structure in the limit

Notice that the group $U(\mathbf{C})$ of complex translations acts transitively on the entire complex plane and hence the Hodge bundle extends giving the flag

$$F_z = \mathbf{C}\begin{pmatrix} z \\ 1 \end{pmatrix} \subset \mathbf{C}^2.$$

Of course, if $z \in \mathbf{R}$ this defines no longer a Hodge structure. The constant local system $V_{\mathbf{Z}}$ descends to give $\mathbf{V}_{\mathbf{Z}}$ on the torus $\mathbf{C}^*$. The monodromy of $\mathbf{V}_{\mathbf{Z}}$ is given by the matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The "remaining part" of the group P, namely the one parameter subgroup

$$w_P : t \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

defines a canonical grading on $V_{\mathbf{Q}}$ via the character spaces $\mathbf{Q}e_1$ and $\mathbf{Q}e_2$, and hence a filtration, which by convention is chosen as follows:

$$W^P = \left\{W_0^P = W_1^P = \mathbf{Q}e_2 \subset W_2^P = V_{\mathbf{Q}}\right\}.$$

This filtration is called[1] the *Cayley filtration* and one can easily see that the pair (W^P, F_z) defines a mixed Hodge structure isomorphic to $\mathbf{Q}(0) \oplus \mathbf{Q}(-1)$. Comparing this with Schmid's results [Sch, Theorem 6.16], one sees:

Lemma 1.1. *1. With $N = \log T$, one has*

$$F_z = \exp(zN)F_0,$$

i.e. one has a nilpotent orbit: for $\operatorname{Re} z > 0$ the filtration F_z is a pure Hodge structure;

2. the filtration F_0 is Schmid's limit Hodge filtration and (W^P, F_0) is Schmid's limit mixed Hodge structure.

Glueing mixed Hodge structures at the boundary

Consider the diagram

$$X^0(\Gamma) \underset{j}{\hookrightarrow} X(\Gamma) \underset{i}{\hookleftarrow} \{\infty\}$$

This is an example of a normal crossing compactification

$$X^0 \underset{j}{\hookrightarrow} X \underset{i}{\hookleftarrow} D = X - X^0$$

In this situation, given a local system $\mathbf{V}$ on X^0, the complex $i^*Rj_*\mathbf{V}$ is called the *deleted neighborhood complex* for $\mathbf{V}$ along the boundary $\partial X = D$. The reason for this terminology is that the cohomology of this complex is the cohomology of $\mathbf{V}$ in a small enough tubular neighborhood T_ϵ of D from which D has been deleted:

$$H^\bullet_{\mathrm{dnbh}}(D, \mathbf{V}) := H^\bullet(D, i^*Rj_*\mathbf{V}) = H^\bullet(T_\epsilon - D, \mathbf{V}).$$

This shows that for the deleted neighborhood complex only the local monodromy around D plays a role. In our example we have $T_\epsilon = \Delta_r^*$, $D = \{\infty\}$ with monodromy generated by T. Moreover, we see that Δ_r^* is a deformation retract of the full torus $\mathbf{C}^*$ and so as far as cohomology is concerned, on may replace T_ϵ by a full torus bundle. This is one of the crucial features that is going to play a role in the general Baily-Borel compactification.

Continuing with the deleted neighborhood complex, I want to put the limit Hodge filtration into play by replacing it by a quasi-isomorphic complex of sheaves supported at the boundary point ∞:

$$K(N, F_0) = \left\{0 \to K^0 = (V, F_0) \xrightarrow{N} K^1 = (V, F_0)(-1) \to 0.\right\} \tag{1}$$

I leave it as an exercise to show that it is indeed quasi-isomorphic to $i^*Rj_*\mathbf{V}$ and that

$$H^0(K(N, F_0)) = V^T \simeq \mathbf{Q}(0), \quad H^1(K(N, F_0)) = V/V^T \simeq \mathbf{Q}(-1),$$

[1] Compare [Del73].

where V^T stands for the T–invariant subspace of V. Moreover, the Cayley-filtration W^P being invariant under monodromy, can be put on $K(N, F_0)$ as a second filtration and descends to cohomology in a compatible way. All this is completely obvious here, but I made it explicit because it is a perfect illustration for the much more complicated general situation.

Group theory aspects

Recall the group theoretic set up: $G = \mathrm{SL}(2; \mathbf{R}) \supset P \supset U$. The standard G–representation V can be viewed as a P–representation. This defines a functor Res_P^G on group representations. The complex (1) is an instance of the functor

$$\mathrm{inv}^U : D^b(P\text{-}\mathrm{Reps}_{\mathbf{Q}}) \to D^b(P/U\text{-}\mathrm{Reps}_{\mathbf{Q}}),$$

which to a bounded complex of P–representations associates a bounded complex of P/U–representations. The cohomology of this complex computes the total group cohomology for the group U. For background on this, see [Bro, Ch. III.1] and [Bu-Wi, Section 3].

Let me recapture what I did: I started with the standard construction $\mu_G V$, a variation of Hodge structures on $X_0(\Gamma)$ and applied the deleted neighborhood construction to it and I ended up with the total cohomology $H^\bullet(U, V)$ considered at as a vector space placed at the cusp at infinity. This vector space is a direct sum of (in this case two) irreducible $P/U \simeq \mathbf{G}_m$–representations which each have a natural pure Hodge structure (in this case $\mathbf{Q}(0)$, respectively $\mathbf{Q}(-1)$). This last phenomenon is an instance of the functor $\mu_{P/U}$ applied to the (reducible) P/U–representation $H(U, V)$. In other words, I have shown

$$\boxed{i^* Rj_*(\mu_G V) = \mu_{P/U} \mathrm{inv}^U V.}$$

It is this equality of variations of Hodge structures that I want to discuss in general.

2 Group theoretic interpretation of Hodge structures

Hodge structures revisited

Hodge structures can be viewed as (algebraic) representations of the Deligne torus

$$\mathbf{S} = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in \mathrm{GL}^+(2; \mathbf{R}) \right\} \iff \left\{ x + \mathrm{i}y \in \mathbf{C}^\times \right\} \text{ (as real groups).}$$

Indeed, if $\rho : \mathbf{S} \to \mathrm{GL}(V)$ is such a representation, with V a $\mathbf{Q}$–vector space, the character space decomposition of $V_{\mathbf{C}}$ gives a Hodge decomposition:

$$V^{p,q} = \{ v \in \mathbf{V}_{\mathbf{C}} \mid \rho(z)v = z^p \bar{z}^q \cdot v. \}$$

Since V is real, $\overline{V^{p,q}} = V^{q,p}$ and assembling the $V^{p,q}$ with $p + q = w$ fixed, one gets a weight w Hodge structure. In total, V is a direct sum of Hodge structures of possibly different weights. To determine these weight spaces one should look at the action of the weight co-character:

$$w : \mathbf{R}^\times \hookrightarrow \mathbf{S}, \quad t \mapsto \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} :$$

on weight w Hodge structures it acts as multiplication by t^k.

This interpretation can be used define a variation of Hodge structure on $X_0(\Gamma)$ associated to any representation $\rho : \mathrm{GL}^+(2;\mathbf{R}) \to \mathrm{GL}(V)$. Indeed $\mathbf{S}$ sits naturally in the group $\mathrm{GL}^+(2;\mathbf{R})$ and since $\mathfrak{h} = \mathrm{SL}(2;\mathbf{R})/\mathrm{U}(1) = \mathrm{GL}^+(2;\mathbf{R})/K$, K the group generated by rotations and dilatations, the composition $\mathbf{S} \to \mathrm{GL}^+(2;\mathbf{R}) \xrightarrow{\rho} \mathrm{GL}(V)$ defines a Hodge structure $F = F_{\mathrm{i}}$ on V which is viewed as sitting over the point $\mathrm{i} \in \mathfrak{h}$. The group $\mathrm{GL}^+(2;\mathbf{R})$ transports this to $\rho(g)F_{\mathrm{i}}$ over $g \cdot \mathrm{i}$. This defines the standard construction, a functor

$$\mu_G : G\text{-}\mathrm{Reps}_{\mathbf{Q}} \to \mathrm{VHS}(X_0(\Gamma)) \text{ for } G = \mathrm{GL}^+(2;\mathbf{R}),$$

from representations of G to (polarizable) variations of Hodge structure on $X_0(\Gamma)$.

Bounded symmetric domains

How does this generalize to bounded symmetric spaces? Recall:

Definition 2.1. A complex manifold D equipped with a hermitian metric is called a *hermitian symmetric space*, if for every $x \in D$ there exists a holomorphic isometric involution s_x with x as isolated fixed point.

These have been classified by E. Cartan. See e.g. [He]: each D is a product of irreducible such symmetric hermitian manifolds. If D is an irreducible hermitian space it is either of compact, non-compact or of euclidean type. We are only considering the non-compact ones which are known to have a realization as a bounded domain in some $\mathbf{C}^N$ and thus are called *bounded symmetric (hermitian) domain* One has in fact:

Theorem ([He, Ch. VIII, Prop. 4.4.])**.** *Let D be an irreducible bounded symmetric domain and G the connected component of its group of holomorphic automorphisms. The group G acts transitively on D; the isotropy group K of $o \in D$ is a maximal compact subgroup of G with connected 1-dimensional center: $Z(K) \simeq S^1$ and there is an algebraic homomorphism*

$$\nu : \mathbf{U}(\mathbf{R}) \to G, \quad \mathrm{Im}(\nu) = Z(K). \tag{2}$$

The circle S^1 sits naturally in $\mathbf{S}$ as the subgroup of $z \in \mathbf{C}^*$ for which $|z|^2 = 1$. We would like to have a morphism $h : \mathbf{S} \to G$ extending the map ν but this does not exist in general; as for $\mathfrak{h}$ we have to enlarge G to include dilatations. This gives G^+ and $D = G^+/K^+$, with $K^+ = (\mathbf{R}_{>0} \cdot K)$. The point o corresponds to an enlarged homomorphism, a so called *Hodge homomorphism* $h : \mathbf{S} \to G^+$ and by conjugating under the G^+–action one gets new Hodge homomorphisms corresponding to G^+–translates of o:

$$h^{(g)} = ghg^{-1} \iff g \cdot o \in D.$$

Indeed $\mathrm{Im}(\nu^g) = Z(K^{(g)})$ and $K^{(g)}$ is the isotropy group of $g \cdot o$.

So we see that D is realized as a single conjugacy class of a Hodge homomorphism with target G^+. This is the description of D as a *Shimura domain*. For this see [De77, Mi04].

If we use this terminology, it is always assumed that G is an algebraic group defined over $\mathbf{Q}$. That this is no restriction follows from [Bo91, 7.9], since $G(\mathbf{R})$ is connected and of adjoint type.

The quotients to be considered, the so called *Shimura varieties* are quotients under arithmetic subgroups Γ of $G(\mathbf{Q})$. Let me recall what these are. We choose a fixed representation of G as a matrix subgroup of $\mathrm{GL}(n)$.

Definition 2.2. 1. A subgroup Γ of $G(\mathbf{Q})$ is called *arithmetic* if it is commensurable with $G_{\mathbf{Z}} = G(\mathbf{Q}) \cap \mathrm{GL}(n; \mathbf{Z})$ (i.e. $\Gamma \cap G_{\mathbf{Z}}$ has finite index in both Γ and G_Z);
2. A *congruence subgroup* of $G(\mathbf{Q})$ is a subgroup of $G(\mathbf{Q})$ containing

$$G(\mathbf{Q}) \cap \{g \in \mathrm{GL}(n; \mathbf{Z}) \mid g \equiv \mathrm{id} \bmod N\}$$

as a subgroup of finite index. A congruence subgroup is arithmetic;
3. A subgroup of $G(\mathbf{Q}) \subset \mathrm{GL}(n; \mathbf{Q})$ is *neat* if for any given element its eigenvalues generate a torsion free subgroup of $\mathbf{C}^\times$ (in particular it cannot have finite order).

It is a fact [Bo69, 17.4] that arithmetic subgroups contain neat congruence subgroups of finite index.

Example 2.3. The group $\mathrm{Sp}(g, \mathbf{Z})$ is *not* neat: it contains torsion elements. For instance, $\mathbf{J}_g$ has order 4. As is well known, the congruence subgroup of matrices $U \equiv \mathrm{id} \bmod N$ is neat whenever $N \geq 3$.

Siegel's upper half space

Let me treat now the model example, the Siegel upper half space

$$\mathfrak{h}_g = \left\{ Z \in \mathbf{C}^{g \times g} \mid Z = {}^{\mathsf{T}}Z, \, \mathrm{Im}\, Z > 0 \right\}.$$

As a homogeneous space equals $\mathrm{Sp}(g, \mathbf{R})/\mathrm{U}(g$. Indeed, writing a symplectic matrix as usual as a 2×2 matrix of matrices of size g, the action can be written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}.$$

The stabilizer of $\mathrm{i}\mathbf{1}_g \in \mathfrak{h}_g$ is

$$\mathrm{U}(g) = \left\{ A + \mathrm{i}B \mid \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{Sp}(g; \mathbf{R}) \right\}$$

with center

$$Z(\mathrm{U}(g)) = \{x\mathbf{1}_g + \mathrm{i}y\mathbf{1}_g \mid x + \mathrm{i}y \in \mathbf{S} \cap \mathrm{Sp}(g; \mathbf{R})\}$$

which gives the homomorphism $\nu : S^1 \to \mathrm{Sp}(g, \mathbf{R})$ and hence the Hodge homomorphism

$$h : \mathbf{S} \to \mathbf{R}^+ \cdot \mathrm{Sp}(g, \mathbf{R})$$

corresponding to the point i. Finally, one has an explicit formula which gives the transitivity:

$$Z = \begin{pmatrix} \mathbf{1}_g & \operatorname{Re} Z \\ 0_g & \mathbf{1}_g \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & {}^{\mathsf{T}}U^{-1} \end{pmatrix} \mathrm{i}\mathbf{1}_g, \quad \operatorname{Im} Z = U^{\mathsf{T}}U.$$

The last equality follows since $Y = \operatorname{Im} Z$ defines a euclidean metric on $\mathbf{R}^g$ and as such it can be diagonalized with respect to an orthonormal basis; hence $Y = {}^{\mathsf{T}}PDP$ with D a diagonal matrix with diagonal entries the positive eigenvalues of Y. So one may form $D^{\frac{1}{2}}$, the diagonal matrix with the corresponding positive roots on the diagonal. Now set $U := {}^{\mathsf{T}}PD^{\frac{1}{2}}P$. This also shows that under the action on metrics Y, given by $P(Y) = {}^{\mathsf{T}}P{\circ}Y{\circ}P$ one has:

$$\begin{aligned} \mathrm{GL}(g, \mathbf{R})/\,\mathrm{O}(g) &= \{\text{euclidean metrics on } \mathbf{R}^g\} \\ [P] &\mapsto {}^{\mathsf{T}}P{\circ}P. \end{aligned} \tag{3}$$

The group of bihomorphic automorphisms of $\mathfrak{h}_g$ is the group $\mathrm{Sp}(g, \mathbf{R})/\pm\mathbf{1}_g$. The Cayley transformation $Z \mapsto \tau(Z) := (Z - \mathrm{i}\mathbf{1}_g)(Z + \mathrm{i}\mathbf{1}_g)^{-1}$ maps $\mathfrak{h}_g$ biholomorphically to the bounded domain

$$D_g = \left\{\tau \in \mathbf{C}^{g\times g} \mid \tau = {}^{\mathsf{T}}\tau,\ \tau\, {}^{\mathsf{T}}\bar{\tau} < \mathbf{1}_g\right\}.$$

It sends $\mathrm{i}\mathbf{1}_g$ to the origin and the symmetry $Z \mapsto -Z^{-1}$ becomes $\tau \mapsto -\tau$ with fixed point 0.

The involution at $x = g(0)$ thus equals $s_x = g{\circ}s_0{\circ}g^{-1}$. The domain D_g is a hermitian symmetric space: it carries a metric which is invariant under the action of the symplectic group [He, Ch. VIII §3] and s_0 is an isometry since it is induced by the action of standard symplectic matrix

$$\mathbf{J}_g := \begin{pmatrix} 0_g & \mathbf{1}_g \\ -\mathbf{1}_g & 0_g \end{pmatrix}.$$

3 A double fibration associated to boundary strata

In this section I follow closely the book [AMRT] and, for the special case of the Siegel upper half space, [Na]. The double fibration, which plays a central role, is also explained from a Hodge theoretic point of view in [Mi90].

The Baily-Borel compactification.

Let $D = G/K$ be a bounded hermitian symmetric domain with G an algebraic group defined over $\mathbf{Q}$ and let $\Gamma \subset G(\mathbf{Q})$ be an arithmetic subgroup. To construct the Baily-Borel compactification one first embeds $D = G/K$ in its compact dual $\check{D} = G(\mathbf{C})/\tilde{P}$ and one divides $\partial D \subset \check{D}$ into connected components each of which is left stable by some (maximal) parabolic subgroup P of G and if P is defined over $\mathbf{Q}$ one calls the corresponding component a *rational boundary component*. It is completely characterized by P, and hence denoted D_P.

Heuristically, the Baily-Borel compactification $D(\Gamma)^*$ is the union of these rational boundary components modulo the action of Γ. For our example of the Siegel upper half space $\mathfrak{h}_g$ and $\Gamma = \mathrm{Sp}(g; \mathbf{Z})$ this becomes $\mathfrak{h}_g^* = \coprod_{h=0}^{g} \mathfrak{h}_h$.

Structure of the maximal parabolic subgroups

I sketch the general picture and give details only[2] in the example of $\mathfrak{h}_2$. In contrast with what happens for $\mathfrak{h}$, in general U_P, the unipotent radical needs no longer be abelian; one has a 2-step filtration

$$1 \subset Z(U_P) \subset U_P \subset P,$$

with abelian quotients; the underlying real vector spaces will be denoted as follows:

$$E_P = Z(U_P), \quad E'_P = P/U_P.$$

There is also a one-parameter subgroup w_P of P which generalizes what we had for $\mathfrak{h}$ and which is going to induce the Cayley filtration on G–representation spaces

Furthermore, the semi-simple group P/U_P contains a subgroup G'_P which acts as the identity on D_P and there is an almost direct product decomposition

$$P/U_P = G_P \cdot G'_P \quad \text{such that } D_P = G_P/K_P.$$

It is also useful to consider the following group

$$P_1 = \mathrm{Ker}(P \to P/U \to G'_P)$$

which maps onto G_P with finite kernel.

The factor G'_P contains an abelian factor A_P and G'_P acts on E_P by conjugation; the A_P–factor acts by homotheties. There is a preferred point $o_P \in E_P$ whose orbit, $C(P)$, is a cone:

$$C(P) \subset E_P, \quad C(P) = \left\{ g \cdot o_P \cdot g^{-1} \mid g \in G'_P \right\}. \tag{4}$$

While we don't prove these assertions here, let me illustrate them in an example.

Example 3.1. Let us look at $\mathfrak{h}_2$ with boundary component given by

$$P = \left\{ \begin{pmatrix} a & 0 & b & * \\ * & g & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & g^{-1} \end{pmatrix} \in \mathrm{Sp}(2, \mathbf{R}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R}), \quad g \in \mathbf{R}^\times. \right\}$$

In this case

$$w_P : t \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}.$$

$$U_P = \left\{ \begin{pmatrix} 1 & 0 & 0 & n \\ m & 1 & n & b \\ 0 & 0 & 1 & -m \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid b, m, n \in \mathbf{R} \right\}$$

[2]It is only notationally more complicated to deal with $\mathfrak{h}_g$.

with center and quotient by its center given by

$$
\begin{aligned}
Z(U_P) &= \left\{ [b] \mid b \in \mathbf{R},\ \text{where } [b] := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \\
U_P/Z(U_P) &= \{[m:n] \mid n, m \in \mathbf{R}\},\ \text{where} \\
& \quad [m:n] := \begin{pmatrix} 1 & 0 & 0 & n \\ m & 1 & n & 0 \\ 0 & 0 & 1 & -m \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

In the case of a zero-dimensional boundary component, one has

$$
P = \left\{ \begin{pmatrix} g & * \\ 0 & {}^{\mathsf{T}}g^{-1} \end{pmatrix} \in \mathrm{Sp}(2, \mathbf{R}) \mid g \in \mathrm{GL}(2, \mathbf{R}). \right\}
$$

and

$$
\begin{aligned}
w_P &= \begin{pmatrix} t\mathbf{1}_2 & 0 \\ 0 & t^{-1}\mathbf{1}_2 \end{pmatrix}, \\
U_P = Z(U_P) &= \left\{ \begin{pmatrix} \mathbf{1}_2 & B \\ 0 & \mathbf{1}_2 \end{pmatrix} \mid B = {}^{\mathsf{T}}B \right\}.
\end{aligned}
$$

Here $P/U_P = G'_P = \mathrm{GL}(2, \mathbf{R})$ with $A_P = \mathbf{R}^\times$. The action on E_P, the vector space of 2×2 symmetric matrices is as follows

$$
\begin{pmatrix} g & 0 \\ 0 & {}^{\mathsf{T}}g^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_2 & B \\ 0 & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & {}^{\mathsf{T}}g \end{pmatrix} = \begin{pmatrix} 1 & gB\,{}^{\mathsf{T}}g \\ 0 & 1 \end{pmatrix}
$$

and we see that the orbit of the matrix $o_P = \begin{pmatrix} \mathbf{1}_2 & \mathbf{1}_2 \\ 0 & \mathbf{1}_2 \end{pmatrix}$ corresponds to the cone of positive definite symmetric matrices.

The double fibration

As in the case of the upper half plane, one has $\check{D} \supset D(P) = P(\mathbf{C}) \cdot D$ with $D(P)$ fibered over $D_P = G_P/K_P$ in a natural way:

$$
D(P) \xrightarrow{\pi_1} Z(U_P)(\mathbf{C}) \backslash D(P) \xrightarrow{\pi_2} D_P,
$$

where the first fibration is a trivial holomorphic fibration with fibers $E_P(\mathbf{C}) = Z(U_P)(\mathbf{C})$, while the second fibration has fibers isomorphic to E'_P which apriori only is a real vector space. It turns out that $\pi_2^{-1}(z)$ has a complex structure depending on z which makes π_2 a holomorphic fibration as well.

Next, divide out by the action of $E_P(\mathbf{Z}) = \Gamma \cap E_P$. The first fibration becomes a fibration in algebraic tori

$$
T(P) = Z(U_P)(\mathbf{C})/E_P(\mathbf{Z})
$$

while the second does not change. Dividing out instead by the larger discrete group $\Gamma \cap U_P$ has the same effect on the fibration π_1 but it replaces the fibers of π_2 by compact tori $E'_P/E'_P(\mathbf{Z})$ where $E'_P(\mathbf{Z}) = \Gamma_{U_P}/Z(U_P) \cap \Gamma$. These tori have a complex structure; they are also algebraic and hence they form abelian varieties. The result is the double fibration

$$\pi_P : \Gamma_{U_P} \backslash D(P) \xrightarrow{\pi_{\text{torus}}} (Z(U_P)_{\mathbf{C}} \cdot E'_P(\mathbf{Z}))\backslash D(P) \xrightarrow{\pi_{\text{Ab}}} D_P,$$

where π_{torus} signifies "$T(P)$-torsor" and π_{Ab} "fibration in abelian varieties". For the last step we divide out by the remaining non-trivially acting discrete subgroup $\Gamma \cap P_1/\Gamma \cap U_P$. This acts freely on the double fibration and has the effect of identifying fibers over the points of the same Γ_{G_P}–orbit in D_P, where Γ_{G_P} is the trace of Γ in G_P. The end result is the following commutative diagram where $\Gamma_{P_1} = P_1 \cap \Gamma$:

$$\begin{array}{ccccc} & & \overset{\pi_{P_1}}{\curvearrowright} & & \\ M(P_1) & \xrightarrow[\pi_{P,\text{tor}}]{} & \mathcal{A}(P) & \xrightarrow[\pi_{P,\text{Ab}}]{} & D_P(\Gamma_{G_P}) \\ \text{def}\Big\| & & \text{def}\Big\| & & \Big\| \\ \Gamma_{P_1}\backslash D(P) & \longrightarrow & (\Gamma_{G_P} \cdot Z(U_P)_{\mathbf{C}} \cdot E'_P(\mathbf{Z}))\backslash D(P) & \longrightarrow & D_P(\Gamma_{G_P}). \end{array} \tag{5}$$

Example 3.2. Again look at $\mathfrak{h}_2$. Over the zero-dimensional boundary component $\mathcal{A}(P) = D_P =$ point and the torus fibration is just $T(P) = (\mathbf{C}^*)^3 \to \text{pt}$. It comes from the vector space of 2×2 symmetric complex matrices. Notice we may partially compactify this to obtain $\mathbf{C}^3$. Then the divisor $Z(P)$ at infinity is a normal crossing divisor corresponding to the union of the 3 coordinate planes. Over the one dimensional boundary $T(P) = \mathbf{C}^*$, but

$$\mathcal{A}(P) \xrightarrow{\pi_{\text{Ab}}} X_0(\Gamma)$$

is the universal fibration in elliptic curves (now Γ has to be a congruence subgroup). Hence, the zero-section $Z(P)$ for the total fibration π_{P_1} is also a divisor.

Notice also that for the zero dimensional boundary the monodromy around $Z(P)$ is generated by three independent commuting unipotent operators while for the one-dimensional boundary there is only one such operator. In all cases, if T is one of the monodromy operators, $(T - \mathbf{1})^2 = 0$ and hence

$$N = \log T = T - \mathbf{1}.$$

Remark. The variety $M(P_1)$ is an example of a so called *mixed Shimura variety* as introduced by Pink in his thesis [Pink]. These come naturally with variations of mixed Hodge structures as we have explained. This point of view is exploited systematically in [HaZ] as well as [Bu-Wi].

Neighborhoods at infinity

The topology in the Satake compactification as well as in the toroidal compactifications can best be compared through the toric fibration $\pi_{P,\text{tor}}$. Adding in every fiber $T(P) \simeq (\mathbf{C}^*)^n$ the n axes gives a boundary divisor $Z(P)$. A small deleted tubular neighborhood $T_\epsilon - Z(P)$ in the resulting partial compactification can be identified with a neighborhood at infinity of the component $D_P(\Gamma_{G_P})$. This follows from reduction theory [Bo69]. Notice that $Z(P) \to D_P(\Gamma_{G_P})$ is a fibration with fibers $\simeq (\mathbf{C}^n - (\mathbf{C}^*)^n) \times$(abelian variety) and to obtain a neighborhood of $D_P(\Gamma_{G_P})$ in $D(\Gamma)^*$ one still needs to contract $Z(P)$ inside T_ϵ along the fibers of this fibration.

The gluing in of $Z(P)$ is an instance of a *toroidal compactification*, but in general this does not lead to a smooth projective compactification. This can be achieved by compactifying each fiber $\simeq T(P)$ as a toric variety[3] associated to a polyhedral decomposition of the cone $C(P)$, its vertex corresponding to $T(P)$.

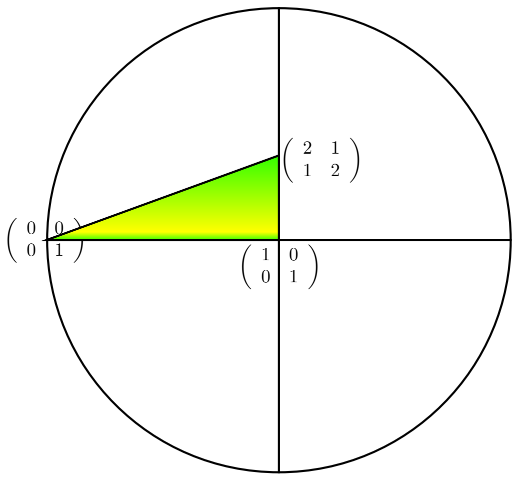
Since this toric variety needs to be equivariant with respect to the "trace" of Γ in G'_P, i.e. the group $\bar{\Gamma}_P := P \cap \Gamma / G_P \cap \Gamma$, the polyhedron usually is infinite and hence such a toric variety need *not* be of finite type. Of course, one needs to divide out by the action of this group to construct the toroidal compactification so that the end result is a nice algebraic scheme.

While again, I don't explain this here in detail, let me look at our example $\mathfrak{h}_2$. Here only the zero dimensional boundary component needs to be blown up by a suitable toroidal compactification. The primitive partial compactification $\simeq \mathbf{C}^3$ corresponds to the polyhedral cone on the three semi-positive define matrices

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and } B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

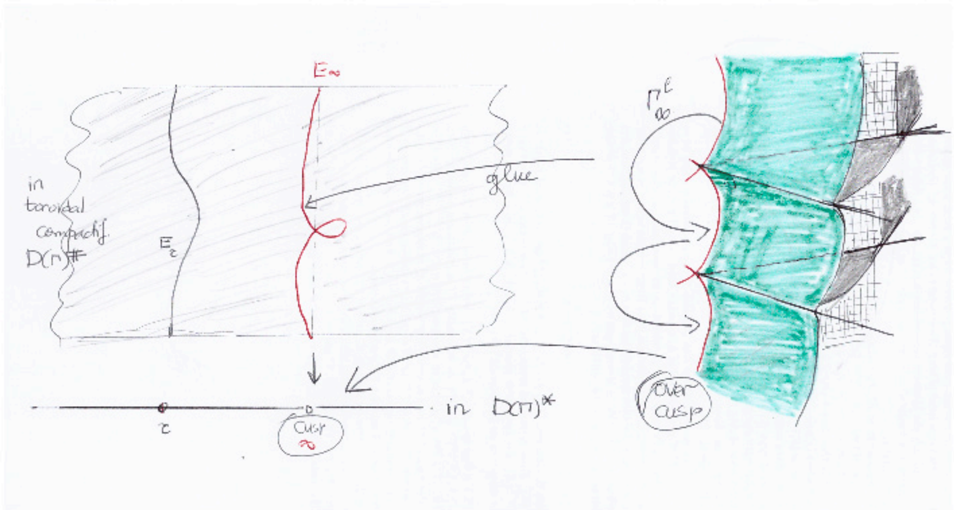
These give the three local monodromy operators around the three coordinate hyperplanes.

The so called *second Voronoi compactification* is more more sophisticated; it is constructed using the following fundamental domain for the action of $\mathrm{GL}(2;\mathbf{Z})$:



The compactification then looks like:

[3]The reader may use [Fu] to get acquainted with toric varieties.



Here the red cuspidal curve on the left is glued to the red curve on the right quotiented out by the action of $\mathrm{GL}(2;\mathbf{Z})$. On the right one sees a tubular neighborhood of a $\mathrm{GL}(2,\mathbf{Z})$ invariant set of triples of irreducible divisors forming a normal crossing divisor. Two components from each triple meet along the red curve and the third is transversal to it.

In what follows we shall use the convention that $Z(P)$ is the "essential part" of the divisor at infinity in a (suitable) toric compactification, meaning that it is the smallest connected divisor with finitely many components such that its G'_P–orbit give the entire divisor at infinity. It fibers over $D_P(\Gamma_{G_P})$ by a morphism extending π_{P_1} and denoted

$$\boxed{\pi_P : Z(P) \to D_P(\Gamma_{G_P}).} \tag{6}$$

4 Hodge theory along toric fibers of the double fibration

This is analogous to what we did in the modular curve case. The parabolic subgroup P has the one-parameter subgroup w_P which we made explicit for $\mathfrak{h}_2$. It determines the Cayley filtration W_P: with V^ℓ the subspace of the representation (V,ρ) on which $\rho\circ w_P$ acts as t^ℓ, one has

$$W_k^P V = \bigoplus_{\ell \geq -k} V^{(\ell)}.$$

This filtration relates to the monodromy weight filtration around the divisor $Z(P)$ as follows.

First, consider any $T \in C(P)$. Via the representation (V,ρ) of weight k this induces a unipotent operator on V, also denoted T, with $(T-\mathbf{1})^{k+1} = 1$. Introduce

$$N = \log(T) = \sum_{\ell \geq 0} (-1)^\ell (T-\mathbf{1})^{\ell+1}.$$

By [Sch, Lemma 6.3], any nilpotent endomorphism N on a vector space V defines a canonical *N–weight filtration (centered at 0)* characterized by the following properties:

1. $NW_p(N) \subset W_{p-2}(N)$ for all p;
2. there are isomorphisms $N^\ell : \mathrm{Gr}_\ell^{W(N)} \xrightarrow{\simeq} \mathrm{Gr}_{-\ell}^{W(N)}$ for all $\ell \geq 0$.

One can show [HaZ, Prop. 4.4.10]:

$$W^P V = W(N), \quad N = \log(T), \text{ where } T \in C(P). \tag{7}$$

As before, the Hodge F–filtration on $V_{\mathbf{C}}$ coming from the standard construction extends to $D(P)$, but not necessarily as a Hodge filltration. This descends to the variety $M(P_1)$ introduced in (5). Along the boundary of the toric fiber $T(P) \simeq (\mathbf{C}^*)^n \subset \mathbf{C}^n$, the resulting fibration has a limit, e.g. $F_0 \subset V_{\mathbf{C}}$ at the "missing" zero of the torus. We then have:

Proposition 4.1. *1. On $M(P_1)$ the Cayley filtration together with the F–filtration defines a variation $(\mathbf{V}, W^P, F)$ of mixed Hodge structures, the Cayley variation;*

2. The limit mixed Hodge structure at 0 is given by (W^P, F_0);
In particular, the Cayley variation is constant along each fiber of $\pi_{\text{torus}} : M(P_1) \to \mathcal{A}(P)$.

For a proof see [HaZ, Prop. 4.3.3]. Crucial is the role of the Cayley filtration: note that *all* operators in the cone $C(P)$ lead to the same weight filtration. This holds in particular for any point in the interior of a polyhedral cone in the closure of $C(P)$ with at least one ray in the interior of this cone. Such cones come up when constructing smooth toroidal compactifications of $D(\Gamma)$: each of the k rays defining such a polyhedral cone corresponds to a local monodromy operator around one of the k components making up a normal crossing divisor at a given boundary point x. Since the limit mixed Hodge structure at such a point has weight filtration of the form $W(N)$ with N in the interior of such a cone this implies that this weight filtration is always the Cayley filtration! This has an important consequence:

Corollary 4.2. *The Cayley variation is an admissable polarizable variation of mixed Hodge structures.*

I don't dwell on definitions here, but rather refer to [PeSt, § 14.4.1]. The importance of this will become apparent in next section: an admissable polarizable variation of mixed Hodge structures turns out to be a crucial example of a mixed Hodge module (Example 5.1).

We also need the connection with group representations. Recall that

$$\pi_1(T(P)) = \Gamma \cap Z(U_P)$$

is an abelian group generated by k loops around the compactifying divisors which meet at 0, the "missing zero" of the torus $T(P)$. Let N_j, $j = 1, \ldots, k$ be the corresponding logarithms. These generate the Lie algebra $\mathfrak{z}(\mathfrak{u}) = \mathrm{Lie}(Z(U_P))$. On the other hand, we have the G–representation space V. Let us equip it with the two filtrations F_0 and W^P and consider the complex

$$K^\bullet(N_\bullet, V, W^P, F_0) = \{V(0) \to \mathfrak{z}(\mathfrak{u}) \otimes V(-1) \to \cdots \wedge^p \mathfrak{z}(\mathfrak{u}) \otimes V(-p) \to \cdots V(-k)\}$$

with derivative

$$u_{i_1} \wedge \cdots u_{i_p} \otimes v \mapsto \sum_j (-1)^j u_{i_1} \wedge \cdots \wedge \widehat{u_{i_j}} \wedge \cdots u_{i_p} \otimes N_{i_j} v.$$

Its cohomology gets in this way a natural mixed Hodge structure. The complex gives the stalk at 0 of the deleted neighborhood complex of $\mathbf{V}$ along the boundary divisor $Z(P)$. But by [Ho, No] its cohomology is also the Lie algebra cohomology $H^\bullet(\mathfrak{z}(\mathfrak{u}), V) \simeq H^\bullet(Z(U_P), V)$. So:

Lemma 4.3. *For $a \in \mathcal{A}(P)$, one has*

$$H^k(\pi_{\mathrm{torus}}^{-1} a, \mathbf{V}) \simeq H^k_{\mathrm{dnbh}}(\{0\}, \mathbf{V}) \simeq H^k(\mathfrak{z}(\mathfrak{u}), V) \simeq H^k(Z(U_P), V).$$

These have a canonical mixed Hodge structure induced by (W^P, F).

In order to study the Baily-Borel compactification we should not consider the toric fibers, but go all the way down, i.e., we should study the fibers of the map $\pi_{P_1} : M(P_1) \to D_P(\Gamma_{G_P})$ from (5). Clearly such a fiber is a $T(P)$–fibration over an abelian variety (with $\pi_1 \simeq \Gamma \cap U_P / \Gamma \cap Z(U_P)$) and hence a nilmanifold with fundamental group $\Gamma_{U_P} = \Gamma \cap U_P$. Using this together with the previous calculation, one can then show[4] for these fibers that an analog of Lemma 4.3 holds:

Proposition 4.4. *For $z \in D_P(\Gamma_{G_P})$, there is a canonical isomorphism*

$$H^k(\pi_{P_1}^{-1} z, \mathbf{V}) \simeq H^\bullet(U_P, V)$$

of mixed Hodge structures, where the mixed Hodge structures are induced by (W^P, F). These isomorphisms are G_P–equivariant.

Now recall (6). This morphism can be fitted in a commutative diagram

$$\begin{array}{ccccc} Z(P) & \xhookrightarrow[i_\#]{} & D(\Gamma)^\# & \xhookleftarrow[j_\#]{} & D(\Gamma) \\ \Big\downarrow \pi_P & & \Big\downarrow p_\# & & \Big\| \\ D_P(\Gamma_{G_P}) & \xhookrightarrow[i_P]{} & D(\Gamma)^* & \xhookleftarrow[j]{} & D(\Gamma), \end{array} \qquad (8)$$

where where $D(\Gamma)^\#$ is a suitable smooth toric compactification of $D(\Gamma)$. The fact that the isomorphisms of Proposition 4.4 are equivariant with respect to G_P, implies:

Corollary 4.5. *One has an isomorphism of variations of mixed Hodge structures on $D_P(\Gamma_{G_P})$*

$$R^k \pi_{P_1,*} \underbrace{(\, i_\#^* j_{\#*} \mathbf{V}\,)}_{\substack{\text{del. nbd cmplx} \\ \text{along } Z(P)}} = \mu_{G_P}(\mathrm{inv}^{U_P} \operatorname{Res}_P^G V).$$

[4]See [HaZ, §5] and in particular [HaZ, Appendix to 5.4]

5 The role of mixed Hodge modules

The goal is to compare the result of Corollary 4.5 with the original variation $\mu_G(V)$. Here mixed Hodge modules appear. These have been discussed in the lecture of Claude Sabbah. See for example [Sa88, Sa89, Sa90], [PeSt, Chap. 14].

Example 5.1. An admissible graded polarizable variation of mixed Hodge structures is an example of a mixed Hodge module. In particular, a variation of polarizable Hodge structures is a (pure) Hodge module.

For a complex algebraic manifold X, we let $\mathsf{MHM}_{\mathbf{Q}}X$ stand for the (algebraic) $\mathbf{Q}$–rational mixed Hodge modules on X. With this convention, the main result is as follows:

Theorem 5.2. *Consider*

$$D(\Gamma) \overset{}{\underset{j}{\hookrightarrow}} D(\Gamma)^* \underset{i}{\hookleftarrow} D_P(\Gamma_{G_P}).$$

Let c be the codimension of $D_P(\Gamma_{G_P})$. There is a commutative diagram

$$\begin{array}{ccc} G\text{-Reps}_{\mathbf{Q}}) & \xrightarrow[\mu_G]{} & D^b\mathsf{MHM}_{\mathbf{Q}}D(\Gamma) \\ {\scriptstyle H^\bullet \mathrm{inv}^{U_P}\,\mathrm{Res}^G_P}\downarrow & & \downarrow{\scriptstyle i_P^* j_*[-c]} \\ G_P\text{-Reps}_{\mathbf{Q}}) & \xrightarrow[\mu_{G_P}]{} & D^b\mathsf{MHM}_{\mathbf{Q}}D_P(\Gamma_{G_P}). \end{array}$$

In the remainder of the lecture I want to indicate how the good functorial properties of mixed Hodge modules can be invoked to deduce this theorem from Proposition 4.4.

Recall the following (highly non-trivial!) properties for mixed Hodge modules:

- For any morphism $f : X \to Y$ there are functorial morphisms in the *derived category* of mixed Hodge modules $f_* : D^b\mathsf{MHM}_{\mathbf{Q}}X \to D^b\mathsf{MHM}_{\mathbf{Q}}Y$. In particular, for $\mathcal{M} \in \mathsf{MHM}_{\mathbf{Q}}X$, if $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)_*\mathcal{M} \simeq g_*(f_*\mathcal{M})$.
- One has proper base change: given a cartesian diagram

$$\begin{array}{ccc} X' = X \times_Y Y' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

  with f proper and $\mathcal{M}$ e a mixed Hodge module on X. Then the natural morphism $g^*f_*\mathcal{M} \to {f'}_*{g'}^*\mathcal{M}$ is an isomorphism in $D^b(\mathsf{MHM}_{\mathbf{Q}}Y')$.

We apply this in the set-up of diagram (8). We provisionally set

$$\mathbf{V}^G := \mu_G V \in \mathsf{MHM}_{\mathbf{Q}}D(\Gamma).$$

Recall that

$$\begin{aligned} i_{\#}^{*} j_{\#,*} \mathbf{V}^G =& \text{del. nbh. cmplx for } \mathbf{V}^G \text{ around } Z(P) \\ & \text{with Cayley filtr. } W^P \text{ and limit MHS} \\ =:& \mathbf{V}^P \end{aligned}$$

and that by Corollary 4.5, we have

$$R\pi_{P*}\mathbf{V}^p = \mu_{G_P}(\mathrm{inv}^{U_P} \mathrm{Res}_P^G V).$$

Now apply the functorial properties for mixed Hodge modules:

$$\begin{aligned} i^* j_* \mathbf{V}^G &= i^*(p_{\#*} \circ j_{\#*} \mathbf{V}^G) \\ &= \pi_{P*}(i_{\#}^* j_{\#*} \mathbf{V}^G) \\ &= \pi_{P*} \mathbf{V}^P \\ &= \mu_{G_P}(\mathrm{inv}^{U_P} \mathrm{Res}_P^G V), \end{aligned}$$

which shows (modulo a shift) the asserted commutativity from Theorem 5.2. The shift (by the codimension of the stratum $D(\Gamma_{G_P})$ in $D(\Gamma)^*$) is a result of the conventions one uses for Hodge modules: the underlying topological object is not a local system but a perverse sheaf. Local systems on smooth varieties of dimension d become perverse sheaves when considered as complexes placed in degree $-d$. □

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