VASSILIEV INVARIANTS OF ORDER TWO FOR STRING LINKS

JEAN-BAPTISTE MEILHAN*

Abstract. In this talk, we show that the Casson knot invariant, linking number and Milnor's triple linking number, together with a certain 2-string link invariant V_2, are necessary and sufficient to express any string link Vassiliev invariant of order two. Some applications are given.

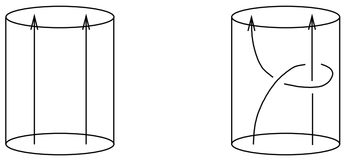
1. Preliminaries

1.1. **String links and Vassiliev invariants.** Let D^2 be the standard two-dimensional disk, and $x_1, ..., x_n$ be n marked points in the interior of D^2.

Definition 1.1. *An n-*component string link *(or n-*string link*) is a proper, smooth embedding*

$$\sigma : \bigsqcup_{i=1}^{n} I_i \longrightarrow D^2 \times I$$

of n disjoint copies I_i of the unit interval such that, for each i, the image σ_i of I_i runs from $(x_i, 0)$ to $(x_i, 1)$. σ_i is called the i^{th} string of σ*. Two examples are depicted below.*



Note that each string of an n-string link is equipped with an (upward) orientation induced by the natural orientation of I.

The set $\mathcal{SL}(n)$ of all n-string links (up to isotopy with respect to the boundary) has a monoid structure, with composition given by the *stacking product* and with the trivial n-string link 1_n as unit element.

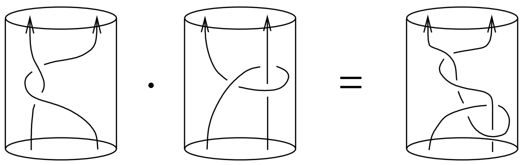
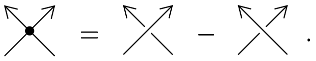


Figure 1.1. Multiplying two n-string links.

For the study of Vassiliev invariants for string links, we consider *singular n-string links*, which are immersions $\bigsqcup_i I_i \longrightarrow D^2 \times I$ whose singularities are (a finite number of) transversal double points. A singular n-string link σ with k double

* The author is supported by a JSPS postdoctoral fellowship.

points can be thought of as an element of $\mathbf{Z}\mathcal{SL}(n)$ by the following well-known skein formula

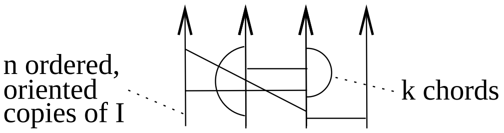


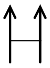
Definition 1.2. *Let A be an Abelian group. An n-string link invariant $f : \mathcal{SL}(n) \to A$ is a* Vassiliev invariant of order k *if its natural extension to $\mathbf{Z}\mathcal{SL}(n)$ vanishes on every n-string-link with (at least) $k+1$ double points.*

Example 1.3. The z^n coefficient in the Alexander-Conway polynomial is a Vassiliev invariant of order n:
· The linking number is of order 1 (for links *and* string links),
· The Casson knot invariant φ is of order 2...

For more about Vassiliev invariants, the reader is refered to [BN].

1.2. **Chord diagrams.** A *chord diagram of order k* is a disjoint union $\bigsqcup_{i=1}^n I_i$ of n oriented and ordered copies of the unit interval, with k chords on it. We denote by $\mathcal{D}_k^n$ the set of chord diagrams of order k (up to order and orientation preserving diffeomorphisms of the copies of I).

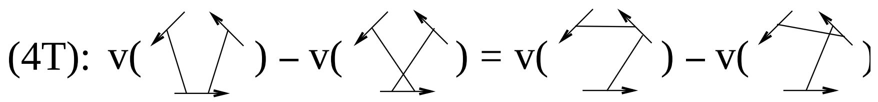


Let σ be a singular n-string link with k double points. The preimages of the double points of σ form a subset of $\bigsqcup_{i=1}^n I_i$ which consists in k pairs of points. The chord diagram D of order k *associated to* σ is the element of $\mathcal{D}_k^n$ obtained by joining each of these k pairs by a chord (σ is said to *respect* the chord diagram D).
For example, the singular 2-string link  respects the chord diagram  .

Let v be a Vassiliev invariant of order k for n-string links. If two singular string links σ and σ' with k double points respect the same chord diagram D, then

$$v(\sigma) = v(\sigma').$$

So, for any chord diagram D of order k, $v(D)$ denotes the value of v on *any* singular string link respecting D.
It is known [B] that a Vassiliev invariant satisfies the $(1T)$ and $(4T)$ relations below



Definition 1.4. *The* initial data of v *consists in the following :*

$$\begin{cases} v(D) & \textit{for every } D \in \mathcal{D}_k^n. \\ v(\sigma) & \textit{for every chord diagram in } \mathcal{D}_l^n,\ l < k, \\ & \textit{where } \sigma \textit{ is a fixed singular } n\textit{-string link respecting it.} \end{cases}$$

As outlined in [M], two vassiliev invariants of order k with the same initial data coincide.

2. Vassiliev invariants of order two for n-string links

Let us denote by φ_i the Casson knot invariant of the i^{th} component of a link, and by μ_{ij} the linking number of its i^{th} and j^{th} components. In the case of links, we have the following:

Theorem 2.1 (H. Murakami). *Any Vassiliev invariant v_2 of order two for links is of the form*

$$v_2 = a + \sum_{i=1}^{n} b_i.\varphi_i + \sum_{i<j} \left(c_{ij}.\mu_{ij} + d_{ij}.\mu_{ij}^2\right)$$
$$+ \sum_{i\ ;\ j<k} e_{ijk}.\mu_{ij}\mu_{ik} + \sum_{i<j\ ;\ k<l\ ;\ i<k} f_{ijkl}.\mu_{ij}\mu_{kl},$$

where the constants a, b_i, c_{ij}, d_{ij}, e_{ijk} and f_{ijkl} are explicitly given by initial data of v_2.

We aim to state a similar result for string links. Note that in the link case, a similar result is known Vassiliev invariants of order 3 [KMT].

2.1. Invariants of order two for 2-string link.

Let $V_2 : \mathcal{SL}(2) \longrightarrow \mathbf{Z}$ be the map defined by

$$V_2(\sigma) := \varphi\left(p(\sigma)\right) - \varphi(\sigma_1) - \varphi(\sigma_2), \tag{2.1}$$

where $p(\sigma)$ denotes the plat-closure of σ : it is the knot obtained by identifying the origin (resp. the end) of σ_1 with the origin (resp. the end) of σ_2, with orientation induced by the orientation of σ_1. An example is given in Fig. 2.1.

As an immediate consequence of the definition of V_2, we have :

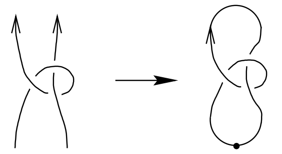
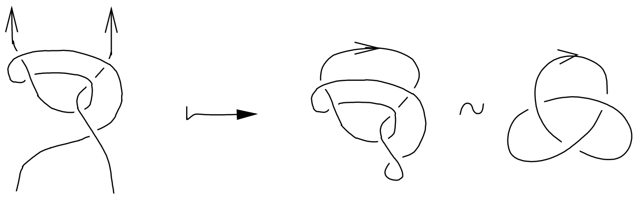


FIGURE 2.1. The plat-closure of a 2-string link.

Proposition 2.2. *V_2 is a $\mathbf{Z}$-valued Vassiliev invariant of order two for 2-string links.*

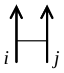
Let us compute this invariant for the string link version W of the Whitehead link, depicted below.

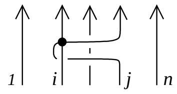


The plat-closure of W being the trefoil, and each component being trivial, we have

$$V_2(W) = 1.$$

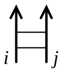
It follows that V_2 can not be expressed, as in theorem 2.1, as a linear combination of the linking number and the Casson invariants of the components.

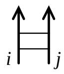
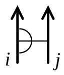
2.2. **Main result.** Let us denote by  the singular n-string link respecting  depicted below.

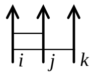
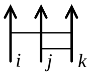


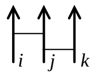
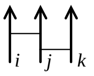
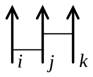
Theorem 2.3. *Let v be a Vassiliev invariant of order two for n-string links. For any $\sigma \in \mathcal{SL}(n)$, $v(\sigma)$ is given by the following formula*

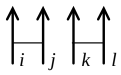
$v(\sigma) = v(1_n) + \sum_{i=1}^{n} v($  $)\varphi_i(\sigma)$

$+ \sum_{i<j} \Big(v($  $) - \frac{1}{2} v($  $) \Big) \mu_{ij}(\sigma) +$

$\frac{1}{2} v($  $).\mu_{ij}^2(\sigma) - v($  $).(V_2)_{ij}(\sigma)$

$+ \sum_{i<j<k} \quad v($  $).\mu_{ij}\mu_{ik}(\sigma) + v($  $).\mu_{ik}\mu_{jk}(\sigma) +$

$v($  $).\mu_{ij}\mu_{jk}(\sigma) + \Big(v($  $) - v($  $) \Big) .\mu_{ijk}(\sigma)$

$+ \sum_{\substack{i<j \\ k<l\ ;\ i<k}} \quad v($  $).\mu_{ij}\mu_{kl}(\sigma),$

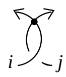
where $\mu_{ijk}(\sigma)$ denotes Milnor's triple linking number, and $(V_2)_{ij}(\sigma)$ denotes the invariant V_2 of the 2-string link $\sigma_i \cup \sigma_j$.

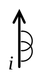
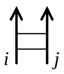
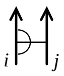
Sketch of the Proof.
For arbitrary constants A, B_i, C_{ij}, D_{ij}, E_{ij}, F_{ijk}, G_{ijk}, H_{ijk}, P_{ijk}, Q_{ijkl}, R_{ijkl} and S_{ijkl},

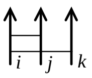
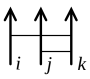
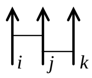
$$\begin{aligned} w = A \quad + \quad & \textstyle\sum_{i=1}^{n} B_i.\varphi_i \quad + \quad \sum_{i<j} \left(C_{ij}.\mu_{ij} + D_{ij}.\mu_{ij}^2 + E_{ij}.(V_2)_{ij} \right) \\ & + \textstyle\sum_{i<j<k} \left(F_{ijk}.\mu_{ij}\mu_{ik} + G_{ijk}.\mu_{ik}\mu_{jk} + H_{ijk}.\mu_{ij}\mu_{jk} + P_{ijk}.\mu_{ijk} \right) \\ & \quad + \textstyle\sum_{i<j<k<l} \left(Q_{ijkl}.\mu_{ij}\mu_{kl} + R_{ijkl}.\mu_{ik}\mu_{jl} + S_{ijkl}.\mu_{il}\mu_{jk} \right) \end{aligned}$$

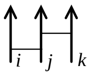
is a Vassiliev invariant of order two. Indeed, recall from [BN2, Lin] that the Milnor μ invariants of lenght k are Vassiliev invariants of order $k-1$. Following the proof of [M, Thm. 3.3], we now choose these constants so that v and w coincide.

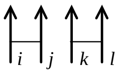
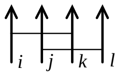
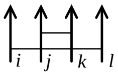
The following computations give us the initial data of w : for all $1 \leq i < j < k < l \leq n$,

$w(1_n) = A, \quad w($  $) = C_{ij} + D_{ij},$

$w($  $) = B_i, \quad w($  $) = 2D_{ij}, \quad w($  $) = -E_{ij},$

$w($  $) = F_{ijk}, \quad w($  $) = G_{ijk}, \quad w($  $) = H_{ijk},$

$w($  $) = H_{ijk} - P_{ijk},$

$w($  $) = Q_{ijkl},$ $w($  $) = R_{ijkl},$ $w($  $) = S_{ijkl}.$

The result follows (there is just an elementary linear system to solve). □

3. Applications

In this last section, we show several consequences of Theorem 2.3.

3.1. **Milnor's triple linking numbers.** We start this section by a brief recall of the definition of Milnor's triple linking numbers.

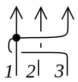
Let $\sigma \in \mathcal{SL}(n)$. $E(\sigma) := D^2 \times I \setminus \sigma$ denotes the exterior of σ and $D^2_n := D^2 \setminus \{x_1, ..., x_n\}$. Let $i_1 : D^2_n \longrightarrow D^2_n \times \{1\} \subset E(\sigma)$ be the inclusion.

Consider a parallel copy of the k^{th} string of σ: it defines an element of $\pi_1(E(\sigma))$, which correspond under $(i_1)^{-1}_*$ to an element $\lambda_k \in F$, where F is the free group on n generators $x_1, ..., x_n$.
Let P be the ring of formal power series in non-commuting variables $X_1, ..., X_n$.
The *Magnus expansion* is the group homomorphism $\mu : F \longrightarrow P$ defined by $\mu(x_i) = 1 + X_i$.

The *Milnor triple linking number* μ_{ijk} of σ (or Milnor invariant of lenght 3) is the coefficient of $X_i X_j$ of the Magnus expansion of λ_k.

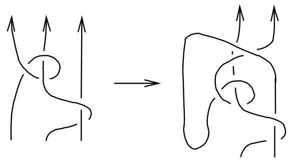
In the case of links, the triple linking number μ_{ijk} is only defined modulo the linking numbers of the components. We set $\overline{\mu}_{ijk} \equiv \mu_{ijk} \quad mod \ \gcd(\mu_{ij}, \mu_{ik}, \mu_{jk})$.
$\overline{\mu}_{ijk}$ is known to be invariant under cyclic permutations of the indices, and to change its sign under exchange of two indices.
As a consequence of Theorem 2.3, we have the following formulas for such operations on μ_{ijk}, with $i < j < k$:

Indices Exchange	μ_{ijk}	$=$	$-\mu_{jik} + \mu_{ik}\mu_{jk}$
		$=$	$-\mu_{ikj} + \mu_{ij}\mu_{ik} - \mu_{ik}$
		$=$	$-\mu_{kji} + \mu_{ik}\mu_{jk} + \mu_{ij}\mu_{ik} - \mu_{ij}\mu_{jk}$
Cyclic Permutation	μ_{ijk}	$=$	$\mu_{jki} - \mu_{ij}\mu_{jk} + \mu_{ik}\mu_{jk}$
		$=$	$\mu_{kij} - \mu_{ij}\mu_{jk} + \mu_{ij}\mu_{ik} - \mu_{ik}$

Indeed, the singular string link $\sigma_s =$  satisfies $\mu_{312}(\sigma_s) = 1$, $\mu_{132}(\sigma_s) = -1$ and $\mu_{123}(\sigma_s) = \mu_{213}(\sigma_s) = \mu_{321}(\sigma_s) = \mu_{231}(\sigma_s) = 0$.

Moreover, we can give a formula for the triple linking numbers in terms of the invariant V_2.

Definition 3.1. *Let $\sigma \in \mathcal{SL}(3)$ be a 3-string link. By stacking above σ a negative crossing between the first and third strings (such that both strings pass over the second one), and then identifying the endpoints $x_1 \times \{0\}$ and $x_1 \times \{1\}$, we obtain a 2-string link denoted by $\tilde{\sigma}$ (an example is given below).*



We denote by V_- the Vassiliev invariant of order two $V_- : \mathcal{SL}(3) \longrightarrow \mathbf{Z}$ defined by

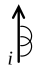
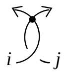
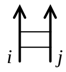
$$V_-(\sigma) := V_2(\tilde{\sigma}).$$

Proposition 3.2. *The following formula holds for Milnor's triple linking number of $\sigma \in \mathcal{SL}(3)$:*

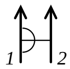
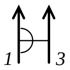
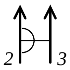
$$\mu_{123}(\sigma) = -V_-(\sigma) + V_2(\sigma_{12}) + V_2(\sigma_{23}) - V_2(\sigma_{13}), \tag{3.1}$$

where σ_{ij} denotes the 2-string link $\sigma_i \cup \sigma_j$ $(1 \leq i < j \leq 3)$.

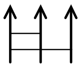
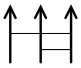
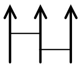
Proof. Using Theorem 2.3, it suffices to compute the initial data of the invariant V_-. For all $1 \leq i < j \leq 3$,

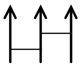
$V_-(1_3) = 0$, $V_-($  $) = 0$, $V_-($  $) = 0$, $V_-($  $) = 0$.

Moreover,

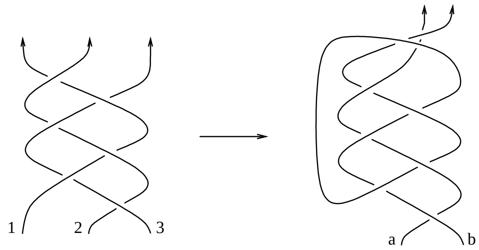
$V_-($  $) = -1$, $V_-($  $) = -1$, $V_-($  $) = +1$,

and

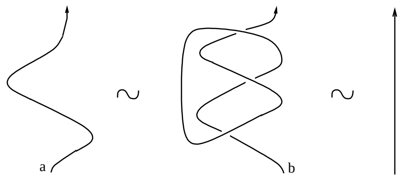
$V_-($  $) = 0$, $V_-($  $) = 0$, $V_-($  $) = 0$,

$V_-($  $) = 1$. □

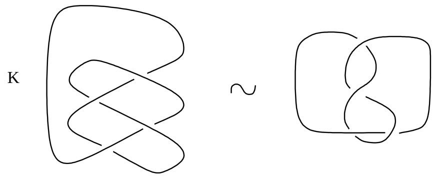
Example 3.3. Consider the following string link version σ of the Borromean rings:



Let $\tilde{\sigma} = \sigma_a \cup \sigma_b$. We have $\varphi(\sigma_a) = \varphi(\sigma_b) = 0$, as both components are trivial:



Moreover, $K = p(\tilde{\sigma})$ is the Figure 8 knot.



So

$$\mu_{123}(\sigma) = -\varphi(K) = +1.$$

3.2. **Clasp-pass equivalence for string links.** The *clasp-pass equivalence* is the equivalence relation on links generated by isotopies and *clasp-pass moves*, which are local moves as illustrated below:

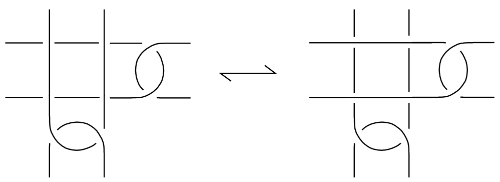


FIGURE 3.1. A clasp-pass move.

In [TY], the authors give a characterisation of this equivalence relation for links.

Theorem 3.4 (K. Taniyama ; A. Yasuhara). *Let L and L' be two n-component algebraically split links in S^3 (that is, links with all linking numbers zero). Then, the following assertions are equivalent:*
1. L and L' are clasp-pass equivalent;
2. L and L' are not distinguished by Milnor's triple linking numbers, nor the mod 2 reduction of the Sato-Levine invariant and the Casson knot invariant.

When considering the case of string links, and using the invariants listed in Thm. 2.3, we obtain the following.

Theorem 3.5. *Let σ and σ' be two n-component algebraically split string links. Then the following assertions are equivalent:*
1. σ and σ' are clasp-pass equivalent;
2. σ and σ' are not distinguished by Milnor's triple linking numbers, nor the invariant V_2 and the Casson knot invariant.

We will not give the proof in this note. Actually, the author was not aware of clasp-pass equivalence and Thm. 3.4 when he proved Thm. 3.5, and thus used different techniques, namely the theory of claspers (a clasp-pass move is indeed equivalent to a C_3-move, that is surgery along a degree 3 clasper), which we will not introduce here.
The idea of the proof is as follows. Consider the monoid $\mathcal{SL}^{as}(n)$ of algebraically split n-string links. The quotient $\overline{\mathcal{SL}}^{as}(n) := \frac{\mathcal{SL}^{as}(n)}{\text{clasp-equ.}}$ is an Abelian group (this is clear from the theoy of claspers). We can define an isomorphism

$$\overline{\mathcal{SL}}^{as}(n) \xrightarrow{\simeq} \Lambda^3 H \oplus S^2 H,$$

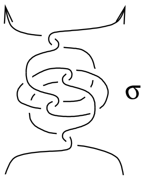
where $H = H_1(D^2 \setminus \{x_1, ..., x_n\}; \mathbf{Z})$, as a linear combination of all possible μ_{ijk}, V_2 and φ's.

Remark 3.6. By comparing theorems 3.4 and 3.5, it is tempting to guess that V_2 is related in some way to the Sato-Levine invariant. We can actually prove that the *modulo 2* reduction of V_2 coincides on $\mathcal{SL}^{as}(2)$ with the *modulo 2* reduction of the Sato-Levine invariant β (defined by taking the closure $\hat{\sigma}$ of a 2-string link $\sigma \in \mathcal{SL}^{as}(2)$) [Me2].

Lemma 3.7. *For all $\sigma \in \mathcal{SL}^{as}(2)$,*

$$V_2(\sigma) \equiv \beta(\sigma) \ (mod\ 2).$$

However, V_2 differs from the Sato-Levine invariant. For example, the 2-string link σ depicted below is mapped to 0 by V_2, whereas $\beta(\sigma) = 2$.



σ

References

[BN] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995), 423-472.

[BN2] ———, *Vassiliev homotopy string link invariants*, J. Knot Theory Ram. **4** (1995), 13-32.

[B] J.S. Birman, *On the combinatorics of Vassiliev invariants*, in "Braid group, knot theory and statistical mechanics II" (M.L. Ge and C.N. Yang, Eds.), Adv. Ser. Math. Phys. **17**, World Scientific (1994) 1-19.

[H] K. Habiro, *Claspers and finite type invariants of links*, Geometry and Topology, **4** (2000), 1–83.

[HL] N. Habegger, X.S. Lin, *The classification of links up to link-homotopy*, J. Amer. Math. Soc. **3** (1990), 389-419.

[KMT] T. Kanenobu, Y. Miyazawa, A. Tani, *Vassiliev link invariants of order three*, J. Knot Theory Ram. **7** (1998), n°4, 433–462.

[Lin] X.S. Lin, *Power series expansions and invariants of links*, in "Geometric topology" (W. Kazez, Ed.), Proc. Georgia Int. Topology Conf.1993, AMS/IP Studies in Adv. Math. (1997).

[Me1] J.B. Meilhan, *On Vassiliev invariants of order two for string links*, to appear in J. Knot Theory Ram.

[Me2] J.B. Meilhan, *Invariants de type fini des cylindres d'homologie et des string links*, Thèse de Doctorat (2003), Université de Nantes.

[M] H. Murakami, *Vassiliev invariants of type two for a link*, Proc. Amer. Math. Soc. **124** n°12 (1996), 3889-3896.

[TY] K. Taniyama, A. Yasuhara, *Clasp-pass moves on knots, links and spatial graphs*, Topology Appl. **122** (2002), no. 3, 501–529.

Commutative diagrams were drawn with Paul Taylor's package.

Research Institute for Mathematical Sciences (RIMS), Kyoto University, Oiwake-cho, Kitashirakawa, Sakyo-ku,, Kyoto 606-8502, Japan

E-mail address: `meilhan@kurims.kyoto-u.ac.jp`