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The universal sl_2 invariant and Milnor invariants

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The universal sl_2 invariant of string links has a universality property for the colored Jones polynomial of links, and takes values in the $\hbar$-adic completed tensor powers of the quantized enveloping algebra of sl_2. In this paper, we exhibit explicit relationships between the universal sl_2 invariant and Milnor invariants, which are classical invariants generalizing the linking number, providing some new topological insight into quantum invariants. More precisely, we define a reduction of the universal sl_2 invariant, and show how it is captured by Milnor concordance invariants. We also show how a stronger reduction corresponds to Milnor link-homotopy invariants. As a byproduct, we give explicit criterions for invariance under concordance and link-homotopy of the universal sl_2 invariant, and in particular for sliceness. Our results also provide partial constructions for the still-unknown weight system of the universal sl_2 invariant.

Keywords: Quantum and finite type invariants; weight system; link concordance; link-homotopy.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

The theory of quantum invariants of knots and links emerged in the middle of the 1980s, after the fundamental work of V. F. R. Jones. Instead of the classical tools of topology, such as algebraic topology, used until then, this new class of invariants was derived from interactions of knot theory with other fields of mathematics, such as operator algebras and representation of quantum groups, and revealed close relationships with theoretical physics. Although this gave rise to a whole new class of powerful tools in knot theory, we still lack a proper understanding of the topological information carried by quantum invariants. One way to attack this fundamental question is to exhibit explicit relationships with classical link invariants. The purpose of this paper is to give such a relation, by showing how a certain reduction of the universal sl_2 invariant is captured by Milnor invariants.

Milnor invariants were originally defined by Milnor for links in S^3 [23, 24]. Their definition contains an intricate indeterminacy, which was shown by Habegger and Lin to be equivalent to the indeterminacy in representing a link as the closure of a *string link*, i.e. of a pure tangle without closed components [8]. Milnor invariants are actually well-defined integer-valued invariants of framed string links, and the first non-vanishing Milnor string link invariants can be assembled into a single *Milnor map* μ_k. See Sec. 2 for a review of Milnor string link invariants.

Milnor invariants constitute an important family of classical (string) link invariants, and as such, their connection with quantum invariants has already been the subject of several works. The first attempt seems to be due to Rozansky, who conjectured a formula relating Milnor invariants to the Jones polynomial [29]. An important step was taken by Habegger and Masbaum, who showed explicitly in [10] how Milnor invariants are related to the Kontsevich integral. More recently, Yasuhara and the first author gave explicit formulas relating Milnor invariants to the HOMFLYPT polynomial [22].

The *universal sl_2 invariant* $J(L)$ for an l-component framed string link L takes values in the l-fold completed tensor power $U_\hbar(sl_2)^{\hat{\otimes} l}$ of the quantized enveloping algebra $U_\hbar(sl_2)$ of sl_2, and has the universal property for the colored Jones polynomial [17, 16, 25, 28]. See Sec. 3 for the definitions of $U_\hbar(sl_2)$ and the universal sl_2 invariant. The second author studied in [31–33] the universal sl_2 invariant of several classes of string links satisfying vanishing properties for Milnor invariants.[a] In this paper, we further explore the relation with the universal sl_2 invariant and Milnor invariants.

Before we proceed with the description of our results, let us recall the relationship between the Kontsevich integral, the universal sl_2 invariant and the colored Jones polynomial. The Kontsevich integral $Z(L)$ for an l-component string links L takes values in the completed space $\mathcal{A}(l)$ of Jacobi diagrams on the disjoint union of l intervals. For the closure link $\mathrm{cl}(L)$ of L, the colored Jones polynomial $J_{V_1,\dots,V_l}(\mathrm{cl}(L))$, with a finite-dimensional representation V_i of $U_\hbar(sl_2)$ attached to the ith component, takes values in $\mathbb{Z}[q^{1/4}, q^{-1/4}] \subset \mathbb{Q}[[\hbar]]$, where $q = \exp \hbar$. The Kontsevich integral has the universal property for finite type invariants, thus for quantum invariants. This implies that there exists an algebra homomorphism $W^U : \mathrm{Im}(Z) \to U_\hbar(sl_2)^{\hat{\otimes} l}$, the so-called *weight system* for the universal sl_2 invariant, such that $W^U \circ Z = J$. (This is well-known, and follows from the fact that the coefficients of the universal invariant have a finite type property, in a strictly similar

[a]More precisely, these results are for *bottom tangles*; but we can identify bottom tangles with string links via a fixed one-to-one correspondence, see [12].

way as in [26, Corollary 7.5].) However, no explicit formula for W^U is known yet. There is, however, a graded algebra homomorphism called *the universal sl_2 weight system*:

$$W : \mathcal{A}(l) \to U(sl_2)^{\otimes l}[[\hbar]],$$

where $U(sl_2)$ is the universal enveloping algebra of sl_2. As a summary, we have the following commutative diagram:

$$\begin{tikzcd}
 & \{l\text{-comp. string links}\} \arrow[r, "\mathrm{cl}"] \arrow[dl, "Z"'] \arrow[d, "J"] & \{l\text{-component links}\} \arrow[d, "J_{V_1,\ldots,V_l}"] \\
\mathcal{A}(l) \supset \mathrm{Im}(Z) \arrow[r, dashed, "W^U"] \arrow[dr, "W"'] & U_\hbar(sl_2)^{\hat{\otimes} l} \arrow[r, "\mathrm{tr}_q^{V_1,\ldots,V_l}"] & \mathbb{Q}[[\hbar]] \\
 & U(sl_2)^{\otimes l}[[\hbar]], \arrow[ur, "\mathrm{tr}_\nu^{V_1,\ldots,V_l}"'] &
\end{tikzcd}$$

where $\mathrm{tr}_q^{V_1,\ldots,V_l}$ and $\mathrm{tr}_\nu^{V_1,\ldots,V_l}$ are variants of the quantum trace map. See [13, Sec. 5] for the commutativity of the upper right square and [18, Sec. 10] for the boundary pentagon. Note that the composition $\mathrm{tr}_\nu^{V_1,\ldots,V_l} \circ W = \mathrm{tr}_q^{V_1,\ldots,V_l} \circ W^U$ is the weight system for the colored Jones polynomial.

Note that the algebras $U_\hbar(sl_2)^{\hat{\otimes} l}$ and $U(sl_2)^{\otimes l}[[\hbar]]$ are isomorphic theoretically, but again, no explicit isomorphism is known, see [14]. In this paper, we will fix a $\mathbb{Q}[[\hbar]]$-linear isomorphism

$$\rho : U_\hbar(sl_2)^{\hat{\otimes} l} \to U(sl_2)^{\otimes l}[[\hbar]], \tag{1.1}$$

with respect to the PBW basis (see Sec. 5.1), so that we can compare the two *different* $\mathbb{Q}$-linear maps $\rho \circ W^U$ and W.

Now, as mentioned above, Habegger and Masbaum showed in [10] that, for an l-component string link L with vanishing Milnor invariants of length $\leq m$, we have

$$Z^t(L) = 1 + \mu_m(L) + (\text{terms of degree} \geq m+1) \in \mathcal{A}^t(l), \tag{1.2}$$

where Z^t is the projection of the Kontsevich integral onto the so-called "tree part" $\mathcal{A}^t(l)$ of $\mathcal{A}(l)$,[b] and where $\mu_m(L)$ is the Milnor map of L regarded as an element of $\mathcal{A}^t(l)$. If we knew the weight system W^U explicitly, we could easily deduce a relation between Milnor invariants and the universal sl_2 invariant by transferring Habegger–Masbaum's result (1.2) via W^U; but, again, this is not the case. Actually, our first main result, Theorem 5.1, implies that, when restricting to the image of Milnor map, we can identify W and W^U via the $\mathbb{Q}$-linear isomorphism ρ. In other words, we give a partial construction of W^U.

[b]The "tree part" is well-defined in the space $\mathcal{B}(l)$ of labeled Jacobi diagrams, which is isomorphic to $\mathcal{A}(l)$ as a $\mathbb{Q}$-module via an analogue of the PBW isomorphism from $S(sl_2)$ to $U(sl_2)$. In this paper, we will use $\mathcal{B}(l)$ rather than $\mathcal{A}(l)$, see Sec. 4.2 for details.

Let us now state the first main result explicitly. The Milnor map μ_m actually takes values in the space of tree Jacobi diagrams, i.e. connected and simply connected Jacobi diagrams. The restriction of W to the space of tree Jacobi diagrams of degree m takes values in $(U(sl_2)^{\otimes l})_{m+1}\hbar^m$, where $(U(sl_2)^{\otimes l})_{m+1}$ is the subspace of $U(sl_2)^{\otimes l}$ of homogeneous elements of degree $m+1$ with respect to the length of the words in sl_2 in the PBW basis, see Lemma 5.1. Thus, if we consider the projection

$$\pi^t : U(sl_2)^{\otimes l}[[\hbar]] \to \prod_{m\geq 1} (U(sl_2)^{\otimes l})_{m+1}\hbar^m,$$

of $\mathbb{Q}$-modules (see Sec. 5.2), then we can compare the maps $W \circ \mu_m$ and $J^t := \pi^t \circ \rho \circ J$. We obtain the following result.

Theorem (Theorem 5.1). *Let $m \geq 1$. If L is a string link with vanishing Milnor invariants of length $\leq m$, then we have*

$$J^t(L) \equiv (W \circ \mu_m)(L) \pmod{\hbar^{m+1}}.$$

Here, and throughout the paper, we simply set $(\mathrm{mod}\ \hbar^k) = (\mathrm{mod}\ \hbar^k U_\hbar^{\hat{\otimes} l})$ for $k \geq 1$ and an appropriate $l \geq 1$.

Theorem 5.1 implies a concordance-invariance property of J^t as follows.

Corollary (Corollary 5.1). *Let L, L' be two concordant string links with vanishing Milnor invariants of length $\leq m$. Then we have*

$$J^t(L') \equiv J^t(L) \pmod{\hbar^{m+1}}.$$

In particular, if L is concordant to the trivial string link, then $J^t(L)$ is trivial.

There is also a variant of Theorem 5.1, using another projection map $\tilde{\pi}^t$ onto a larger quotient of $U_\hbar^{\hat{\otimes} l}$; see Remark 7.1. This provides another criterion for the universal sl_2 invariant, which applies in particular to slice, boundary or ribbon string links as follows.

Theorem (Corollary 7.1). *Let L be an l-component string link with vanishing Milnor invariants. Then we have*

$$\rho(J(L)) \in 1 + \prod_{1\leq i\leq j} (U(sl_2)^{\otimes l})_i \hbar^j.$$

This result strongly supports [32, Conjecture 1.5], where the second author suggests that the universal sl_2 invariant of a bottom tangle with vanishing Milnor invariants is contained in a certain subalgebra of $U_\hbar(sl_2)^{\hat{\otimes} l}$.

As emphasized above, Theorem 5.1 is not a mere consequence of Habegger–Masbaum's work, and the proof will be given by comparing directly the definitions of the Milnor map and the universal sl_2 invariant. One of the main ingredients for the proof is a version for Milnor link-homotopy invariants. Recall that link-homotopy is the equivalence relation generated by self-crossing changes. Habegger and Lin showed that Milnor invariants indexed by sequences with no repetition form a complete set of link-homotopy invariants for string links [8]. We can thus consider the link-homotopy reduction μ_m^h of the Milnor map μ_m, see Sec. 5.2. On the other hand, we consider the projection of $\mathbb{Q}$-modules

$$\pi^h : U(sl_2)^{\otimes l}[[\hbar]] \to \bigoplus_{m=1}^{l-1} \langle sl_2\rangle_{m+1}^{(l)} \hbar^m,$$

where $\langle sl_2\rangle_{m+1}^{(l)} \subset (U(sl_2)^{\otimes l})_{m+1}$ denotes the subspace spanned by tensor products such that each tensorand is of degree ≤ 1, that is, roughly speaking, tensor products of 1's and elements of sl_2.

It turns out that the restriction of the sl_2 weight system W to the space of tree Jacobi diagrams with non-repeated labels takes values in this space $\bigoplus_{m=1}^{l-1} \langle sl_2\rangle_{m+1}^{(l)} \hbar^m$. Thus, similarly as before, we can compare $W \circ \mu_m^h$ and $J^h := \pi^h \circ \rho \circ J$, and obtain the following second main result.

Theorem (Theorem 5.2). *Let $m \geq 1$. If L is a string link with vanishing Milnor link-homotopy invariants of length $\leq m$, then we have*

$$J^h(L) \equiv (W \circ \mu_m^h)(L) \pmod{\hbar^{m+1}}.$$

Note that Theorem 5.2 cannot in general be simply deduced from Theorem 5.1 by a mere link-homotopy reduction process. (This is simply because a string link may in general have nonzero Milnor invariants of length m, yet vanishing Milnor link-homotopy invariants of length m.) In order to prove Theorem 5.2, one of the key results is Proposition 6.1, a link-homotopy invariance property for the map J^h. This reduces the proof to an explicit computation for a link-homotopy representative, given in terms of the lower central series of the pure braid group. In the process of proving Proposition 6.1, we obtain, as in the case of Theorem 5.1 and Remark 7.1, a variant of Theorem 5.2 using another projection map, giving an algebraic criterion detecting link-homotopically trivial string links; see Remark 8.1 and Corollary 8.3.

It is worth mentioning here that the sl_2 weight system W is not injective, and thus we do not expect that the universal sl_2 invariant detects Milnor invariants. This follows from the fact that W takes values in the invariant part of $S(sl_2)^{\otimes l}[[\hbar]]$ and a simple argument comparing the dimensions of the domain and images. We will further study properties of the universal sl_2 weight system in a forthcoming paper [20].

The rest of the paper is organized as follows. In Sec. 2, we review in detail the definition of Milnor numbers and of the Milnor maps μ_k, and recall some of their properties. In Sec. 3, we recall the definitions of the quantized enveloping algebra $U_\hbar(sl_2)$ and the universal sl_2 invariant, and recall how the framing and linking numbers are simply contained in the latter. Section 4 provides the diagrammatic settings for our paper; we review the definition of Jacobi diagrams, and their close relationships with the material from the previous sections. This allows us to give the precise statements of our main results in Sec. 5. Sections 6–8 are dedicated to the proofs. Specifically, the link-homotopy version of our main result is shown in Sec. 6, while Sec. 7 contains the proof of the general case. Some of the key ingredients of these proofs require the theory of claspers, which we postponed to Sec. 8.

2. Milnor Invariants

Throughout the paper, let $l \geq 1$ be some fixed integer.

Let D^2 denote the standard 2-disk equipped with l marked points $p_1, \ldots, p_l$ in its interior as shown in Fig. 1. Fix also a point e on the boundary of the disk D^2, and for each $i = 1, \ldots, l$, pick an oriented loop α_i in D^2 based at this e and winding around p_i in the trigonometric direction. See Fig. 1.

An l-component string link is a proper embedding of l disjoint copies of the unit interval $[0, 1]$ in $D^2 \times [0, 1]$, such that for each i, the image L_i of the ith copy of $[0, 1]$ runs from $(p_i, 1)$ to $(p_i, 0)$. The arc L_i is called the ith component of L. An l-component string link is equipped with the *downwards* orientation induced by the natural orientation of $[0, 1]$.

In this paper, by a string link we will implicitly mean a *framed* string link, that is, equipped with a trivialization of its normal tangent bundle. (Here, it is required that this trivialization agrees with the positive real direction at the boundary points.) In the various figures of this paper, we make use of the blackboard framing convention.

The (0-framed) l-component string link $\{p_1, \ldots, p_l\} \times [0, 1]$ in $D^2 \times [0, 1]$ is called the *trivial l-component string link* and is denoted by $\mathbf{1}_l$, or sometimes simply $\mathbf{1}$ when the number of components is implicit.

Let $SL(l)$ denote the set of isotopy classes of l-component string links fixing the endpoints. The stacking product endows $SL(l)$ with a structure of monoid, with the trivial l-component string link $\mathbf{1}_l$ as unit element. In this paper, we use the notation $\cdot$ for the stacking product, with the convention that the rightmost factor is *above*. Note that the group of units of $SL(l)$ is precisely the pure braid group on l strands $P(l)$ [9].

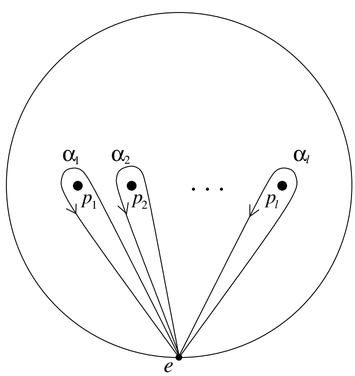


Fig. 1. The disk D^2 with l marked points p_i, and the arcs α_i; $i = 1, \ldots, l$.

2.1. *Artin representation and the Milnor map μ_k for string links*

In this subsection, we review Milnor invariants for string links, following [8, 9].

For an l-component string link $L = L_1 \cup \cdots \cup L_l$ in $D^2 \times [0,1]$, denote by $Y = (D^2 \times [0,1]) \setminus N(L)$ the exterior of an open tubular neighborhood $N(L)$ of L, and set $Y_0 = (D^2 \times \{0\}) \setminus N(L)$ and $Y_1 = (D^2 \times \{1\}) \setminus N(L)$. For $i = 0, 1$, the fundamental group of Y_i based at (e, i) identifies with the free group F_l on generators $\alpha_1, \ldots, \alpha_l$.

Recall that the lower central series of a group G is defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = [G, \Gamma_k G]$. By a theorem of Stallings [30], the inclusions $\iota_i : Y_i \to Y$ induce isomorphisms $(\iota_i)_k : \pi_1(Y_t)/\Gamma_{k+1}\pi_1(Y_t) \to \pi_1(Y)/\Gamma_{k+1}\pi_1(Y)$ for any positive integer k. Hence for each k, the string link L induces an automorphism $(\iota_0)_k^{-1} \circ (\iota_1)_k$ of $\mathrm{F}_l/\Gamma_{k+1}\mathrm{F}_l$. Actually, this assignment defines a monoid homomorphism

$$A_k : SL(l) \to \mathrm{Aut}_0(\mathrm{F}_l/\Gamma_{k+1}\mathrm{F}_l),$$

called the *kth Artin representation*, where $\mathrm{Aut}_0(\mathrm{F}_l/\Gamma_{k+1}\mathrm{F}_l)$ denotes the group of automorphisms of $\mathrm{F}_l/\Gamma_{k+1}\mathrm{F}_l$ sending each generator α_j to a conjugate of itself and preserving the product $\prod_j \alpha_j$. More precisely, for each component j, consider the *preferred jth longitude* of L, which is a f_j-framed parallel copy of L_j, where f_j denotes the framing of component j. This defines an element l_j in $\pi_1(Y)/\Gamma_{k+1}\pi_1(Y)$, and for any positive integer k, we set $l_j^k := (\iota_0)_k^{-1}(l_j) \in \mathrm{F}_l/\Gamma_{k+1}\mathrm{F}_l$. Then we have that $A_k(L)$ maps each generator α_j to its conjugate

$$A_k(L) : \alpha_j \mapsto l_j^k \alpha_j (l_j^k)^{-1}.$$

(Here, we denoted the image of α_j in the lower central series quotient $\mathrm{F}_l/\Gamma_{k+1}\mathrm{F}_l$ again by α_j.)

Denote by $SL_k(l)$ the set of l-component string links whose longitudes are all trivial in $\mathrm{F}_l/\Gamma_k\mathrm{F}_l$. We have a descending filtration of monoids

$$SL(l) = SL_1(l) \supset SL_2(l) \supset \cdots \supset SL_k(l) \supset \cdots$$

called the *Milnor filtration*, and we can consider the map

$$\mu_k : SL_k(l) \to \frac{\mathrm{F}_l}{\Gamma_2\mathrm{F}_l} \otimes \frac{\Gamma_k\mathrm{F}_l}{\Gamma_{k+1}\mathrm{F}_l}$$

for each $k \geq 1$, which maps L to the sum

$$\mu_k(L) := \sum_{i=j}^{l} \alpha_j \otimes l_j^k,$$

called the *degree k Milnor map*.

2.2. *Milnor numbers for string links*

As mentioned in Sec. 1, Milnor invariants were originally defined as numerical invariants. Let us briefly review their definition and connection to the Milnor map.

Let $\mathbb{Z}\langle\langle X_1, \ldots, X_l \rangle\rangle$ denote the ring of formal power series in the non-commutative variables $X_1, \ldots, X_l$. The *Magnus expansion* $E : \mathrm{F}_l \to \mathbb{Z}\langle\langle X_1, \ldots, X_l \rangle\rangle$ is the injective group homomorphism which maps each generator α_j of F_l to $1 + X_j$ (and thus maps each α_j^{-1} to $1 - X_j + X_j^2 - X_j^3 + \cdots$).

Since the Magnus expansion E maps $\Gamma_k \mathrm{F}_l$ to terms of degree $> k$, the coefficient $\mu_{i_1 i_2 \cdots i_m j}(L)$ of $X_{i_1} \cdots X_{i_m}$ in the Magnus expansion $E(l_j^k)$ is a well-defined invariant of L for any $m \leq k$,[c] and it is called a *Milnor μ-invariant*, or Milnor number, of length $m+1$. Milnor invariants are sometimes referred to as higher order linking numbers, since $\mu_{ij}(L)$ is merely the linking number of components i and j, while $\mu_{ii}(L)$ is just the framing of the ith component.

For each $k \geq 1$, the kth term $SL_k(l)$ of the Milnor filtration coincides with the submonoid of $SL(l)$ of string links with vanishing Milnor μ-invariants of length $\leq k$, and the Milnor map μ_k is strictly equivalent to the collection of all Milnor μ-invariants of length $k+1$.

Recall that two l-component string links L and L' are concordant if there is an embedding

$$f : \left(\bigsqcup_{i=1}^{l} [0,1]_i \right) \times I \to (D^2 \times I) \times I,$$

where $\bigsqcup_{i=1}^{l} [0,1]_i$ is the disjoint union of l copies of the unit interval $[0,1]$, such that $f((\bigsqcup_{i=1}^{l}[0,1]_i) \times \{0\}) = L \times \{0\}$ and $f((\bigsqcup_{i=1}^{l}[0,1]_i) \times \{1\}) = L' \times \{1\}$, and such that $f(\partial(\bigsqcup_{i=1}^{l}[0,1]_i) \times I) = (\partial L) \times I$. It is well-known that Milnor numbers, hence Milnor maps, are not only isotopy invariants, but also concordance invariants: this is for example shown by Casson in [4], although it is already implicit in Stallings' paper [30].

2.3. *Link-homotopy and the lower central series of the pure braid group*

Recall that the link-homotopy is an equivalence relation on knotted objects generated by isotopies and self-crossing changes. Using the properties of Magnus expansion, Milnor proved that, if I is a sequence *with no repeated index*, then the corresponding invariant μ_I is a link-homotopy invariant, see [24, Theorem 8]. Habegger and Lin subsequently proved that string links are classified up to link-homotopy by Milnor invariants with no repeated indices [8].

More precisely, Habegger and Lin showed that the set $\bigcup_{m=2}^{l} \{\mu_I \mid I \in \mathcal{I}_m\}$ forms a complete set of link-homotopy invariants for string links [8, 9], where for each $m \in \{2, \ldots, l\}$,

$$\mathcal{I}_m := \left\{ j_{\tau(1)} \cdots j_{\tau(m-2)} j_{m-1} j_m \;\middle|\; \begin{array}{r} 1 \leq j_1 < \cdots < j_{m-2} < j_{m-1} < j_m \leq l, \\ \tau \in S_{m-2} \end{array} \right\}.$$

[c]Note that the integer k can be chosen arbitrarily large, so this condition is not restrictive.

In other words, $\mathcal{I}_m$ is the set of all sequences $j_1 \cdots j_m$ of m non-repeating integers from $\{1, \ldots, l\}$ such that $j_i < j_{m-1} < j_m$ for all $i \leq m-2$.

In this subsection, we use this result to give an explicit representative for the link-homotopy class of any string link in terms of the lower central series of the pure braid group.

Recall that the pure braid group on l strands $P(l)$ is generated by elements

$$A_{i,j} = \sigma_{j-1} \cdot \ldots \cdot \sigma_{i+1} \cdot \sigma_i^2 \cdot \sigma_{i+1}^{-1} \cdot \ldots \cdot \sigma_{j-1}^{-1}, \quad \text{for } 1 \leq i < j \leq l,$$

which may be represented geometrically as the pure braid where the ith string overpasses the strings $(i+1), \ldots, (j-1)$ and j, underpasses the jth string, then goes back to the ith position by overpassing all strings. For convenience, we also define $A_{i,j}$ for $i > j$, by the convention $A_{i,j} := A_{j,i}$.

Given a sequence $J = j_1 \cdots j_m$ in $\mathcal{I}_m$, we define the pure braid

$$B_J^{(l)} = [[\cdots[[A_{j_1,j_2}, A_{j_2,j_3}], A_{j_3,j_4}], \ldots], A_{j_{m-1},j_m}], \tag{2.1}$$

which lies in the $(m-1)$th term $\Gamma_{m-1}P(l)$ of the lower central series. We simply write $B_J = B_J^{(l)}$ when there is no risk of confusion.

The pure braids B_J $(J \in \mathcal{I}_m)$ can be used to construct an explicit representative of the link-homotopy class of any string link as follows.

Lemma 2.1. *Any l-component string link L is link-homotopic to $b_1^L \cdots b_{l-1}^L$, where*

$$b_i^L = \prod_{J \in \mathcal{I}_{i+1}} (B_J)^{\mu_J(b_i^L)}, \quad \textit{where } \mu_J(b_i^L) = \begin{cases} \mu_J(L) & \textit{if } i = 1, \\ \mu_J(L) - \mu_J(b_1^L \cdots b_{i-1}^L) & \textit{if } i \geq 2. \end{cases} \tag{2.2}$$

Remark 2.1. This lemma is to be compared with [34, Theorem 4.3; 22, Theorem 4.1], where similar results are given in terms of tree claspers — see Sec. 8.

Proof of Lemma 2.1. In view of the link-homotopy classification result of Habegger and Lin recalled above, the lemma simply follows from a computation of Milnor invariants of the pure braids B_J $(J \in \mathcal{I}_m)$. Specifically, using the additivity property of Milnor string link invariants (see e.g. [21, Lemma 3.3]), it suffices to show that, for any m and any two sequences J and J' in $\mathcal{I}_m$, we have

$$\mu_{J'}(B_J) = \begin{cases} 1 & \text{if } J = J', \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

(See [22]). Fixing a sequence $J = j_1 \cdots j_m$ in $\mathcal{I}_m$, set

$$B_k = [[\cdots[[A_{j_1,j_2}, A_{j_2,j_3}], A_{j_3,j_4}], \ldots], A_{j_{k-1},j_k}] \in \Gamma_{k-1}P(l),$$

for all $k = 2, \ldots, m$. (In particular, $B_2 = A_{j_1,j_2}$, while $B_k = B_J$.) Using the skein formula for Milnor invariants due to Polyak [27], one can easily check that, for any

$k = 3, \ldots, m$, we have

$$\mu_{j_1 \cdots j_{k-1} j_k}(B_k) = \mu_{j_1 \cdots j_{k-1}}(B_{k-1}).$$

It follows that $\mu_J(B_J) = \mu_{j_1 j_2}(A_{j_1,j_2}) = 1$, as desired. The fact that $\mu_{J'}(B_J) = 0$ for any $J' \neq J$ in $\mathcal{I}_m$ follows easily from similar arguments. $\square$

The following notation will be useful in the next sections. Let $SL^h_m(l)$ be the set of l-component string links with vanishing Milnor link-homotopy invariant of length $\leq m$, that is, $L \in SL^h_m(l)$ if and only if L is link-homotopic to $b^L_m b^L_{m+1} \cdots b^L_{l-1}$ as in Lemma 2.1. Note that we have a descending filtration

$$SL(l) \supset SL^h_1(l) \supset SL^h_2(l) \supset \cdots \supset SL^h_k(l) \supset \cdots \supset SL^h_l(l).$$

3. The Universal sl_2 Invariant

In the rest of this paper, we use the following q-integer notation:

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q \{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$

$$[i]_q = \{i\}_q / \{1\}_q, \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q, \quad \begin{bmatrix} i \\ n \end{bmatrix}_q = \{i\}_{q,n} / \{n\}_q!,$$

for $i \in \mathbb{Z}, n \geq 0$.

3.1. *Quantized enveloping algebra* $U_\hbar(sl_2)$

We first recall the definition of the quantized enveloping algebra $U_\hbar(sl_2)$, following the notation in [13, 32].

We denote by $U_\hbar = U_\hbar(sl_2)$ the $\hbar$-adically complete $\mathbb{Q}[[\hbar]]$-algebra, topologically generated by H, E, and F, defined by the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},$$

where we set

$$q = \exp \hbar, \quad K = q^{H/2} = \exp \frac{\hbar H}{2}.$$

We equip $U_\hbar$ with a topological $\mathbb{Z}$-graded algebra structure with $\deg F = -1$, $\deg E = 1$, and $\deg H = 0$.

There is a unique complete ribbon Hopf algebra structure on $U_\hbar$ such that

$$\Delta_\hbar(H) = H \otimes 1 + 1 \otimes H, \quad \varepsilon_\hbar(H) = 0, \quad S_\hbar(H) = -H,$$

$$\Delta_\hbar(E) = E \otimes 1 + K \otimes E, \quad \varepsilon_\hbar(E) = 0, \quad S_\hbar(E) = -K^{-1}E,$$

$$\Delta_\hbar(F) = F \otimes K^{-1} + 1 \otimes F, \quad \varepsilon_\hbar(F) = 0, \quad S_\hbar(F) = -FK.$$

The *universal R-matrix* and its inverse are given by

$$R = D\left(\sum_{n\ge 0} q^{\frac{1}{2}n(n-2)}\frac{(q-1)^n}{[n]_q!}F^n \otimes E^n\right), \tag{3.1}$$

$$R^{-1} = \left(\sum_{n\ge 0}(-1)^n q^{-\frac{n}{2}}\frac{(q-1)^n}{[n]_q!}F^n \otimes E^n\right)D^{-1}, \tag{3.2}$$

where $D = q^{\frac{1}{4}H\otimes H} = \exp(\frac{\hbar}{4}H\otimes H) \in U_\hbar^{\hat{\otimes}2}$. For simplicity, we set $R^{\pm 1} = \sum_{n\ge 0}\alpha_n^{\pm}\otimes \beta_n^{\pm}$ with

$$\alpha_n\otimes\beta_n (= \alpha_n^+\otimes\beta_n^+) = D\left(q^{\frac{1}{2}n(n-2)}\frac{(q-1)^n}{[n]_q!}F^n\otimes E^n\right),$$

$$\alpha_n^-\otimes\beta_n^- = D^{-1}\left((-1)^n q^{-\frac{n}{2}}\frac{(q-1)^n}{[n]_q!}F^nK^n\otimes K^{-n}E^n\right).$$

Note that the right-hand sides above are sums of infinitely many tensors of the form $x\otimes y$ with $x, y \in U_\hbar$, which we denote by $\alpha_n^{\pm}\otimes\beta_n^{\pm}$ formally.

3.2. *Universal sl_2 invariant for string links*

In this section, we recall the definition of the universal sl_2 invariant of string links.

For an n-component string link $L = L_1 \cup \cdots \cup L_n$, we define the universal sl_2 invariant $J(L) \in U_\hbar^{\hat{\otimes}n}$ in three steps as follows. We follow the notation in [32].

Step 1. Choose a diagram. We choose a diagram $\tilde{L}$ of L which is obtained by pasting, horizontally and vertically, copies of the fundamental tangles depicted in Fig. 2. We call such a diagram *decomposable*.

Step 2. Attach labels. We attach labels on the copies of the fundamental tangles in the diagram, following the rule described in Fig. 3, where $S'_\hbar$ should be replaced with $S_\hbar$ if the string is oriented upward, and with the identity otherwise. We do not attach any label to the other copies of fundamental tangles, i.e. to a straight strand and to a local maximum or minimum oriented from right to left. See Fig. 5 for an (elementary) example.

Step 3. Read the labels. We define the ith tensorand of $J(L)$ as the product of the labels on the ith component of $\tilde{L}$, where the labels are read off along L_i

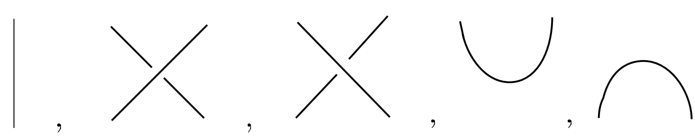


Fig. 2. Fundamental tangles, where the orientations of the strands are arbitrary.

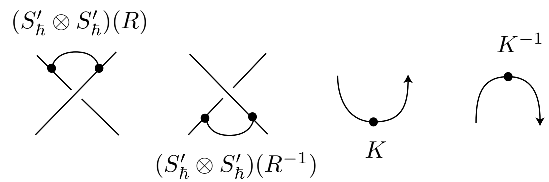


Fig. 3. How to place labels on the fundamental tangles.

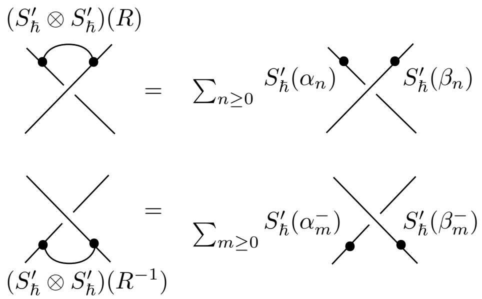


Fig. 4. How to read the labels on crossings.

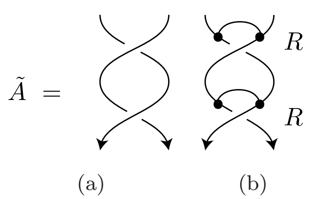


Fig. 5. (a) A diagram $\tilde{A}$ of the string link A; (b) The label put on $\tilde{A}$.

reversing the orientation, and written from left to right. Here, the labels on the crossings are read as in Fig. 4.

As is well-known [25], $J(L)$ does not depend on the choice of the diagram, and thus defines an isotopy invariant of string links.

For example, for the string link A shown in Fig. 5, we have

$$\begin{aligned} J(A) &= \sum_{m,n\geq 0} \beta_m\alpha_n \otimes \alpha_m\beta_n \\ &= D\left(\sum_{m\geq 0} q^{\frac{1}{2}m(m-2)} \frac{(q-1)^m}{[m]_q!} E^m \otimes F^m\right) D\left(\sum_{n\geq 0} q^{\frac{1}{2}n(n-2)} \frac{(q-1)^n}{[n]_q!} F^n \otimes E^n\right) \\ &= D^2\left(\sum_{m,n\geq 0} q^{\frac{1}{2}m(m-2)+\frac{1}{2}n(n-2)+m^2} \frac{(q-1)^{m+n}}{[m]_q![n]_q!} E^m K^m F^n \otimes F^m K^{-m} E^n\right), \end{aligned} \tag{3.3}$$

where the last identity follows from

$$D(1 \otimes x) = (K^{|x|} \otimes x)D,$$

for $x \in U_\hbar$ an homogeneous element of degree $|x|$.

Note that

$$J(A) \equiv 1 + c\hbar \pmod{\hbar^2},$$

where c denotes the symmetric element

$$c = \frac{1}{2} H \otimes H + F \otimes E + E \otimes F. \tag{3.4}$$

3.3. *Universal sl_2 invariant and linking number*

We now recall how the linking number and framing can be simply derived from the "coefficient" of $\hbar$ in the universal sl_2 invariant. Before giving a precise statement (Proposition 3.1), we need to introduce a few extra notation, which will be used throughout the paper.

For $1 \le i \le n$, and for $x \in U_\hbar$, we define $x_i^{(l)} \in U_\hbar^{\hat{\otimes} l}$ by

$$x_i^{(l)} = 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1,$$

where x is at the ith position.

More generally, for $1 \le j_1, \ldots, j_m \le l$ and $y = \sum y_1 \otimes \cdots \otimes y_m \in U_\hbar^{\hat{\otimes} m}$, we define $y_{j_1 \cdots j_m}^{(l)} \in U_\hbar^{\hat{\otimes} l}$ by

$$y_{j_1 \cdots j_m}^{(l)} = \sum (y_1)_{j_1}^{(l)} \cdots (y_m)_{j_m}^{(l)}.$$

For $x \in U_\hbar^{\hat{\otimes} l}$ such that $x \equiv 1 \pmod{\hbar}$, set

$$\mathrm{coeff}_\hbar(x) = \frac{x - 1}{\hbar} \in U_\hbar^{\hat{\otimes} l} / \hbar U_\hbar^{\hat{\otimes} l},$$

i.e. we have $x \equiv 1 + \mathrm{coeff}_\hbar(x)\hbar \pmod{\hbar^2}$.

Note that $J(L) \equiv 1 \pmod{\hbar}$ for any string link L, by definition.

Proposition 3.1. *For $L \in SL(l)$ with linking matrix $(m_{ij})_{1 \le i,j \le l}$, we have*

$$\begin{aligned} \mathrm{coeff}_\hbar(J(L)) &= \frac{1}{2} \sum_{1 \le i,j \le l} m_{ij} c_{ij}^{(l)} \\ &= \sum_{1 \le i < j \le l} m_{ij} c_{ij}^{(l)} + \frac{1}{2} \sum_{1 \le i \le l} m_{ii} c_{ii}^{(l)}. \end{aligned}$$

Remark 3.1. This result is similar to the well-known formula expressing the degree one part of the (framed) Kontsevich integral in terms of the linking matrix, which is generalized by Habegger and Masbaum [10, Theorem 6.1] with respect to Milnor invariants, where the case $n = 1$ corresponds to the formula for the linking matrix:

$$Z^t(L) = 1 + \mu_1(L) + (\text{terms of degree} \geq 2) \in \mathcal{A}^t(l), \tag{3.5}$$

noting that Milnor numbers of length 2 are the coefficients of the linking matrix (see Example 5.1). Our main result in this paper generalizes Proposition 3.1 with respect to Milnor invariants, in a similar way to [10, Theorem 6.1].

In the rest of this section, we prove Proposition 3.1 in an elementary way.

Proof of Proposition 3.1. Let $L \in SL(l)$, and choose a decomposable diagram $\tilde{L} = \tilde{L}_1 \cup \cdots \cup \tilde{L}_l$ such that each crossing has both strands oriented downwards. Denote by $C(\tilde{L})$ the set of the crossings, and by $M(\tilde{L})$ the set of local maxima and minima oriented from left to right. For $a \in C(\tilde{L}) \cup M(\tilde{L})$, let $J(a) \in U_\hbar^{\hat{\otimes} l}$ be the element obtained by reading only the labels on a, as indicated in Step 2 of the definition of $J(L)$. Note that $J(a) \equiv 1 \pmod{\hbar}$ for each $a \in C(\tilde{L}) \cup M(\tilde{L})$, and

$$\operatorname{coeff}_\hbar(J(L)) = \sum_{a \in C(\tilde{L}) \cup M(\tilde{L})} \operatorname{coeff}_\hbar(J(a)). \tag{3.6}$$

Now, for $1 \leq i < j \leq l$, let $C_{i,j}(\tilde{L}) \subset C(\tilde{L})$ be the subset of crossings between $\tilde{L}_i$ and $\tilde{L}_j$. Set $R_{21} = R_{21}^{(2)}$. Since we have

$$\operatorname{coeff}_\hbar(R^\varepsilon) + \operatorname{coeff}_\hbar(R_{21}^\varepsilon) = \varepsilon c,$$

for $\varepsilon = \pm 1$, it follows that

$$\sum_{a \in C_{i,j}(\tilde{L})} \operatorname{coeff}_\hbar(J(a)) = m_{ij} c_{ij}^{(l)}. \tag{3.7}$$

Similarly, for $1 \leq i \leq l$, let $C_i(\tilde{L}) \subset C(\tilde{L})$ be the subset of self-crossings of $\tilde{L}_i$, and $M_i(\tilde{L}) \subset M(\tilde{L})$ the subset of local maxima and minima oriented from left to right in $\tilde{L}_i$. Let us consider $J(a)$ for $a \in C_i(\tilde{L}) \cup M_i(\tilde{L})$, for $l = i = 1$ for simplicity. Notice that each crossing in $C_1(\tilde{L})$ is either left-connected or right-connected, where a downward oriented crossing is called left-connected (respectively, right-connected) if its left (respectively, right) outgoing strand is connected to the left (respectively, right) ingoing strand in $\tilde{L}$. For a left-connected (respectively, right-connected) positive crossing $a \in C_1(\tilde{L})$, we have $J(a) = R_{11}^{(1)}$ (respectively, $J(a) = (R_{21})_{11}^{(1)}$), and on a left-connected (respectively, right-connected) negative crossing $b \in C_1(\tilde{L})$, we have $J(b) = (R_{21}^{-1})_{11}^{(1)}$ (respectively, $J(b) = (R^{-1})_{11}^{(1)}$). Recall that we put K (respectively, K^{-1}) on a local maximum (respectively, minimum) oriented left to right. For these labels we have

$$\operatorname{coeff}_\hbar(R_{11}^{(1)}) = \frac{c_{11}^{(1)} - H}{2}, \quad \operatorname{coeff}_\hbar((R^{-1})_{11}^{(1)}) = \frac{-c_{11}^{(1)} + H}{2},$$

$$\mathrm{coeff}_\hbar((R_{21})^{(1)}_{11}) = \frac{c^{(1)}_{11} + H}{2}, \quad \mathrm{coeff}_\hbar((R^{-1}_{21})^{(1)}_{11}) = \frac{-c^{(1)}_{11} - H}{2},$$
$$\mathrm{coeff}_\hbar(K) = \frac{H}{2}, \quad \mathrm{coeff}_\hbar(K^{-1}) = \frac{-H}{2}.$$

We consider the sum of these coefficients over all labels on $C_1(\tilde{L}) \cup M_1(\tilde{L})$. Actually, if l (respectively, r) denotes the number of left-connected (respectively, right-connected) crossings in $C_1(\tilde{L})$, and if M (respectively, m) denotes the number of local maximum (resp. minimum) in $M_1(\tilde{L})$, then it is not difficult to check that

$$l - r - M + m = 0. \tag{3.8}$$

(By [14, Theorem XII.2.2], and since $l - r - M + m$ is clearly invariant under a crossing change, it suffices to prove that this quantity is invariant under each of the moves of [14, Figures 2.2–2.9]: this is easily checked by a case-by-case study of all possible types of crossings involved in the moves.) This implies that

$$\sum_{a \in C_1(\tilde{L})} \mathrm{coeff}_\hbar(J(a)) + \sum_{b \in M_1(\tilde{L})} \mathrm{coeff}_\hbar(J(b)) = \frac{1}{2} m_{11} c^{(1)}_{11}.$$

This, together with Eqs. (3.7) and (3.6), implies the desired formula. □

4. Diagrammatic Approach

4.1. *Jacobi diagrams*

We mostly follow the notation in [10].

A *Jacobi diagram* is a finite uni-trivalent graph, such that each trivalent vertex is equipped with a cyclic ordering of its three incident half-edges. In this paper we require that each connected component of a Jacobi diagram has at least one univalent vertex. The *degree* of a Jacobi diagram is half its number of vertices.

Let X be a compact oriented 1-manifold. A *Jacobi diagram on X* is a Jacobi diagram whose univalent vertices are disjointly embedded in X. Let $\mathcal{A}(X)$ denote the $\mathbb{Q}$-vector space spanned by Jacobi diagrams on X, subject to the AS, IHX and STU relations depicted in Fig. 6. Here as usual [1], we use bold lines to depict the 1-manifold X and dashed ones to depict the Jacobi diagram, and the cyclic ordering at a vertex is given by the counterclockwise orientation in the plane of the figure. We denote by $\mathcal{A}_k(X)$ the subspace spanned by Jacobi diagrams of degree k. Abusing

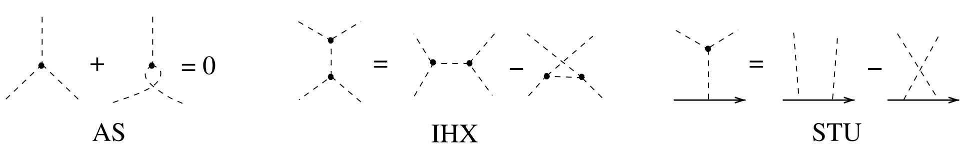


Fig. 6. The relations AS, IHX and STU.

notation, we still denote by $\mathcal{A}(X)$ its completion with respect to the degree, i.e. $\mathcal{A}(X) = \prod_{k\geq 0} \mathcal{A}_k(X)$.

In this paper, we shall restrict our attention to the case $X = \coprod_{j=1}^{l} I_j$, where each I_j is a copy of the interval $I = [0,1]$. For simplicity, set $\mathcal{A}(l) = \mathcal{A}(\coprod_{j=1}^{l} I_j)$. Note that $\mathcal{A}(l)$ has an algebra structure with multiplication defined by stacking.

We denote by $\mathcal{B}(l)$ the completed $\mathbb{Q}$-vector space spanned by Jacobi diagrams whose univalent vertices are labeled by elements of the set $\{1,\ldots,l\}$, subject to the AS and IHX relations. Here completion is given by the degree as before. Note that $\mathcal{B}(l)$ has an algebra structure with multiplication defined by disjoint union.

There is a natural graded $\mathbb{Q}$-linear isomorphism [1]

$$\chi : \mathcal{B}(l) \to \mathcal{A}(l),$$

which maps a diagram to the average of all possible combinatorially distinct ways of attaching its i-colored vertices to the ith interval, for $i = 1,\ldots,l$. Note that χ is not an algebra homomorphism.

In what follows, we focus only on the subspace $\mathcal{A}^t(l)$ of $\mathcal{A}(l)$, which is the graded quotient of $\mathcal{A}(l)$ by the space spanned by Jacobi diagrams containing non-simply connected diagrams. It follows that $\mathcal{B}^t(l) = \chi^{-1}(\mathcal{A}^t(l))$ is the commutative polynomial algebra on the subspace $\mathcal{C}^t(l)$ spanned by *trees*, that is, by connected and simply connected Jacobi diagrams.

Let us also denote by $\mathcal{A}^h(l)$ the graded quotient of $\mathcal{A}^t(l)$ by the space spanned by Jacobi diagrams containing a chord between the same components of $\coprod_{j=1}^{l} I_j$. Similarly, denote $\mathcal{B}^h(l) := \chi^{-1}(\mathcal{A}^h(l))$. Then $\mathcal{B}^h(l)$ is the commutative polynomial algebra on the subspace $\mathcal{C}^h(l)$ spanned by trees with distinct labels [1].

As above, we denote by $\mathcal{C}^t_k(l)$ and $\mathcal{C}^h_k(l)$ the respective subspaces of $\mathcal{C}^t(l)$ and $\mathcal{C}^h(l)$ spanned by Jacobi diagrams of degree k.

For any sequence $I = (i_1,\ldots,i_m)$ of integers in $\{1,\ldots,l\}$, let $T_I^{(l)}$ be the tree Jacobi diagram of degree $(m-1)$ labeled by I as shown in Fig. 7.

It is not difficult to see, by the AS and IHX relations, that $\mathcal{C}^t_m(l)$ is spanned by diagrams $T_I^{(l)}$ indexed by sequences $I = (i_1,\ldots,i_m)$ of integers in $\{1,\ldots,l\}$, while $\mathcal{C}^h_m(l)$ is spanned by those with distinct integers.

4.2. *Kontsevich integral and Milnor map*

A *tangle* is a proper embedding in $D^2 \times [0,1]$ of a compact, oriented 1-manifold X, whose boundary points are on the two parallel lines $[-1,1]\times\{0\}\times\{0\}$ and

$T_I =$

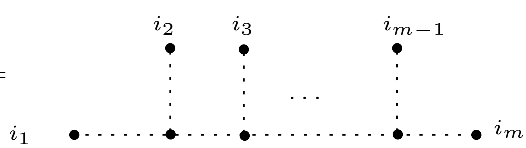


Fig. 7. The tree Jacobi diagram $T_I^{(l)}$ for $I = (i_1,\ldots,i_m)$.

$[-1,1] \times \{0\} \times \{1\}$, where we use the parametrization $D^2 = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} \leq 1\}$. We further require that a tangle is equipped with a framing and with a q-structure, i.e. a consistent collection of parentheses on each of the two sets of boundary points. In the case where T is a string link, we assume that the q-structure is the same on both sets of boundary points.[d]

The *Kontsevich integral* $Z(T)$ of a framed q-tangle T lives in the space $\mathcal{A}(X)$ of Jacobi diagrams on X [15]. We shall not review the definition here, but refer the reader to [1, 5, 26] for surveys. In this paper, we use the combinatorial definition as in [26, Section 6.4].[e] Although this invariant depends on the choice of an associator, this choice will not be relevant in this paper.

A fundamental property of the Kontsevich integral is its universality, over $\mathbb{Q}$, among finite type (or Vassiliev) invariants and among quantum invariants, in the sense that any such invariant can be recovered from the Kontsevich integral by post-composition with an appropriate map, called *weight system*.

Bar-Natan [2] and Lin [19] proved that Milnor invariants for string links are finite type invariants, and thus can be recovered from the Kontsevich integral. This connection was made completely explicit by Habegger and Masbaum, who showed that Milnor invariants determine and are determined by the so-called tree-part of the Kontsevich integral [10]. In order to state this result, we first need the following diagrammatic formulation for the image of the Milnor map defined in Sec. 2.1.

Denote by H the abelianization $\mathrm{F}_l/\Gamma_2\mathrm{F}_l$ of the free group F_l, and denote by $L(H) = \bigoplus_k L_k(H)$ the free $\mathbb{Q}$-Lie algebra on H. Note that $L_k(H)$ is isomorphic to $(\Gamma_k\mathrm{F}_l/\Gamma_{k+1}\mathrm{F}_l) \otimes \mathbb{Q}$, so that μ_k can be regarded as taking values in $H \otimes L_k(H)$. It turns out that the Milnor map μ_k actually takes values in the kernel $D_k(H)$ of the Lie bracket map $H \otimes L_k(H) \to L_{k+1}(H)$, and that $D_k(H)$ identifies with the space $\mathcal{C}^t_k(l)$, as we now explain. Let T be a tree Jacobi diagram in $\mathcal{C}^t_k(l)$. To each univalent vertex v_0 of T, we associate an element T_{v_0} of $L_k(H)$ as follows. For any univalent vertex $v \neq v_0$, label the incident edge of T by $c_v = \alpha_j \in \mathrm{F}_l$, where j is the label of v. Next, label all edges of T by recursively assigning the label $[a,b]$ to any edge which meets an a-labeled and a b-labeled edge at a trivalent vertex (following the cyclic ordering). The last step of this process assigns a label to the edge incident to v_0: this final label is the desired element T_{v_0} of $L_k(H)$. Using this, we can define a $\mathbb{Q}$-linear isomorphism

$$\mathcal{C}^t_k(l) \to D_k(H) \tag{4.1}$$

by sending a tree T to the sum $\sum_v c_v \otimes T_v$, where the sum ranges over the set of all univalent vertices of T.

[d] As we will see below, we only consider here the first non-vanishing term of $Z(T) - 1$, which does not depend on this q-structure.

[e] Note in particular that what is denoted here by $Z(T)$ is denoted by $\hat{Z}(T)$ in [26].

Example 4.1. A single chord with vertices labeled i and j is mapped to $\alpha_i \otimes \alpha_j + \alpha_j \otimes \alpha_i$, which is an element of $D_1(H)$ by antisymmetry. (Note in particular that for $i = j$, the corresponding diagram is mapped to $2 \cdot \alpha_i \otimes \alpha_i$.)

Similarly, a Y-shaped diagram with univalent vertices labeled i, j and k (following the cyclic ordering) is mapped to the sum $\alpha_i \otimes [\alpha_j, \alpha_k] + \alpha_j \otimes [\alpha_k, \alpha_i] + \alpha_k \otimes [\alpha_i, \alpha_j]$: clearly, this is an element of $D_2(H)$ by the Jacobi identity.

In the rest of this paper, we implicitly identify the image of the Milnor map μ_k with $\mathcal{C}_k^t(l)$ via the isomorphism (4.1).

Now, Habegger–Masbaum's result can be simply formulated as follows. Let $L \in SL_m(l)$ be an l-component string link with vanishing Milnor invariants of length up to m. The tree-part of the Kontsevich integral of L, which is defined as $Z^t = p^t \circ \chi^{-1} \circ Z$, where $p^t : \mathcal{B}(l) \to \mathcal{B}^t(l)$ is the natural projection, is then given by

$$Z^t(L) = 1 + \mu_m(L) + \text{terms of degree} \geq m+1,$$

where 1 denotes the empty Jacobi diagram. In particular, the leading term of $Z^t - 1$ does not depend on the choice of q-structure, and lives in the space $\mathcal{C}_m^t(l)$ of degree m tree Jacobi diagrams.

In [10], it is also proved that Z^t is the universal finite type concordance invariant over $\mathbb{Q}$, which implies in particular that it determines Milnor invariants.

Furthermore, Habegger and Masbaum showed that, for $L \in SL_m^h(l)$, we have

$$Z^h(L) = 1 + \mu_m^h(L) + \text{terms of degree} \geq m+1, \tag{4.2}$$

where Z^h is the Kontsevich integral Z composed with the projection $\mathcal{B}(l) \to \mathcal{B}^h(l)$ [10], and where μ_m^h is the link-homotopy reduction of the Milnor map μ_m, which is defined as $\mu_m^h = p^h \circ \mu_m$, with $p^h : \mathcal{C}^t(l) \to \mathcal{C}^h(l)$ the natural projection. (Note in particular that the leading term of $Z^h - 1$ lives in $\mathcal{C}_m^h(l)$.)

4.3. *Weight system associated to* sl_2

Recall that the Lie algebra sl_2 is the three-dimensional Lie algebra over $\mathbb{Q}$ generated by h, e, and f with Lie bracket

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Let $U = U(sl_2)$ denote the universal enveloping algebra of sl_2, and $S = S(sl_2)$ the symmetric algebra of sl_2. There is a well-known commutative diagram [3]

$$\begin{array}{ccc} \mathcal{A}(l) & \xrightarrow{W} & U^{\otimes l}[[\hbar]] \\ \uparrow{\scriptstyle\chi} & & \uparrow{\scriptstyle\beta} \\ \mathcal{B}(l) & \xrightarrow{W} & S^{\otimes l}[[\hbar]], \end{array}$$

where χ is the isomorphism defined in Sec. 4.1, β is the $\mathbb{Q}$-linear isomorphism induced by the Poincaré–Birkhoff–Witt isomorphism $S \cong U$, sending a monomial

$v_1 \cdots v_m \in S$ to $\sum_{\sigma \in S(m)} \frac{1}{m!} v_{\sigma(1)} \cdots v_{\sigma(m)} \in U$, and where W is the weight system associated to sl_2.

In this paper, we will make use of the map W defined on the space $\mathcal{B}(l)$ of labeled Jacobi diagrams, and more precisely of its restriction to $\mathcal{B}^t(l)$, and thus recall its definition below. More precisely, we first define a map $w_m : \mathcal{C}^t_m(l) \to S^{\otimes l}$, and then define $W : \mathcal{B}^t(l) \to S^{\otimes l}[[\hbar]]$ as the $\mathbb{Q}$-algebra homomorphism such that $W(D) = w_m(D)\hbar^m$ for $D \in \mathcal{C}^t_m(l)$.

For $m = 1$, we simply define w_1 by

$$w_1(D_{ij}) = c^{(l)}_{ij} \in S^{\otimes l}, \tag{4.3}$$

where D_{ij} is a single chord with vertices labeled by i and j (possibly $i = j$), and where[f]

$$c = \frac{1}{2} h \otimes h + f \otimes e + e \otimes f \in sl_2^{\otimes 2}. \tag{4.4}$$

Now let $m \geq 2$, and let $D \in \mathcal{C}^t_m(l)$. Set

$$\begin{aligned} b &= \sum_{\sigma \in \mathfrak{S}(3)} (-1)^{|\sigma|} \sigma(h \otimes e \otimes f) \\ &= h \otimes e \otimes f + e \otimes f \otimes h + f \otimes h \otimes e - h \otimes f \otimes e - f \otimes e \otimes h - e \otimes h \otimes f \\ &\in sl_2^{\otimes 3}, \end{aligned} \tag{4.5}$$

where σ acts by permutation of the tensorands. Consider a copy of b for each trivalent vertex of D, where each tensorand of b is associated to one of the half-edges incident to the trivalent vertex, following the cyclic ordering. Each internal edge (i.e. each edge between two trivalent vertices) comprises a pair of half-edges, and we contract the two corresponding copies of sl_2 using the symmetric bilinear form

$$\langle -, - \rangle : sl_2 \otimes sl_2 \to \mathbb{Q}$$

defined by $\langle a, b \rangle = \mathrm{Tr}(ab)$, that is given by

$$\langle h, h \rangle = 2, \quad \langle e, f \rangle = 1, \quad \langle h, e \rangle = \langle h, f \rangle = \langle e, e \rangle = \langle f, f \rangle = 0.$$

Fix an arbitrary total order on the set of univalent vertices of D; we get in this way an element $\sum x_1 \otimes \cdots \otimes x_{m+1}$ of $sl_2^{\otimes m+1}$, the ith tensorand corresponding to the ith univalent vertex of D. We then define $w_m(D) \in S^{\otimes l}$ by

$$w_m(D) = \sum y_1 \otimes \cdots \otimes y_l, \tag{4.6}$$

where y_j is the product of all $x_i \in sl_2$ such that the ith vertex is labeled by j.

It is known that w_m is well-defined, i.e. is invariant under AS and IHX relations; see e.g. [6, Section 6.2].

[f] Abusing notation, we denote by the same letter c the element of $U_\hbar^{\otimes 2}$ defined in (3.4) and the corresponding element in $sl_2^{\otimes 2}$.

4.4. *Computing w_m on trees*

There is another formulation of w_m for tree Jacobi diagrams, which we will use later. Recall that $\mathcal{C}_m^t(l)$ is spanned by the trees $T_I^{(l)}$ indexed by sequences I of (possibly repeating) integers in $\{1,\ldots,l\}$, introduced in Sec. 4.1. For convenience, we only give this alternative definition of w_m on the trees $T_I^{(l)}$.

Recall the elements $c \in sl_2^{\otimes 2}$ and $b \in sl_2^{\otimes 3}$ defined in (4.4) and (4.5), respectively. Let

$$s : sl_2 \to sl_2^{\otimes 2}$$

be the $\mathbb{Q}$-linear map defined by

$$s(a) = (\mathrm{ad} \otimes 1)(a \otimes c) = \frac{1}{2}[a,h] \otimes h + [a,f] \otimes e + [a,e] \otimes f$$

for $a \in sl_2$, where $\mathrm{ad}(x \otimes y) = [x,y]$ for $x, y \in sl_2$. On the basis elements, we have

$$s(e) = h \otimes e - e \otimes h, \quad s(h) = 2(e \otimes f - f \otimes e), \quad s(f) = f \otimes h - h \otimes f.$$

Set $\varsigma_2 = c \in sl_2^{\otimes 2}$. For $m \geq 3$, set

$$\varsigma_m = (1^{\otimes m-2} \otimes s)(1^{\otimes m-3} \otimes s) \cdots (1 \otimes s)(c) \in sl_2^{\otimes m}. \tag{4.7}$$

For example, one can easily check that $\varsigma_3 = b$.

Proposition 4.1. *For $m \geq 1$, we have*

$$w_m(T^{(m+1)}_{(1,\ldots,m+1)}) = \varsigma_{m+1}. \tag{4.8}$$

Proof. This is easily shown by induction on $m \geq 1$. For $m = 1$, we have $w_1(T^{(2)}_{(1,2)}) = c = \varsigma_2$. Now let $m \geq 2$, and let $X_h, X_e, X_f \in sl_2^{\otimes m-2}$ such that

$$\begin{aligned}\varsigma_m &= X_h \otimes h + X_e \otimes e + X_f \otimes f \\ &= w_{m-1}(T^{(m)}_{(1,\ldots,m)}).\end{aligned}$$

Then we have

$$\begin{aligned}\varsigma_{m+1} &= (1^{\otimes m-1} \otimes s)(\varsigma_{m-1}) \\ &= (1^{\otimes m-1} \otimes s)(X_h \otimes h + X_e \otimes e + X_f \otimes f) \\ &= X_h \otimes 2(e \otimes f - f \otimes e) + X_e \otimes (h \otimes e - e \otimes f) + X_f \otimes (f \otimes h - h \otimes f) \\ &= X_h \otimes \left(\sum \langle h, b_1\rangle b_2 \otimes b_3\right) + X_e \otimes \left(\sum \langle e, b_1'\rangle b_2' \otimes b_3'\right) \\ &\quad + X_f \otimes \left(\sum \langle f, b_1''\rangle b_2'' \otimes b_3''\right) \\ &= w_m(T^{(m+1)}_{(1,\ldots,m+1)}),\end{aligned}$$

where $\sum b_1 \otimes b_2 \otimes b_3 = \sum b_1' \otimes b_2' \otimes b_3' = \sum b_1'' \otimes b_2'' \otimes b_3'' = b$.

Hence we have the assertion. □

For an arbitrary sequence $I = (i_1, \ldots, i_{m+1})$ of indices in $\{1, \ldots, l\}$, set

$$\varsigma_I^{(l)} = (\varsigma_{m+1})_{i_1,\ldots,i_{m+1}}^{(l)},$$

that is, if we write formally $\varsigma_{m+1} = \sum x_1 \otimes \cdots \otimes x_{m+1}$, the jth tensorand of $\varsigma_I^{(l)}$ is the product of all $x_p \in U$ such that $i_p = j$. Then by Proposition 4.1 and the definition of w_m, it immediately follows that

$$w_m(T_I^{(l)}) = \varsigma_I^{(l)} \in S^{\otimes l}. \tag{4.9}$$

5. Milnor Map and the Universal sl_2 Invariant

In this section, we give the main results of this paper, which relate Milnor invariants to the universal sl_2 invariant via the sl_2 weight system W.

5.1. *The quantized enveloping algebra $U_\hbar$ and formal power series $S[[\hbar]]$ over the symmetric algebra*

The symmetric algebra S of sl_2 has a graded structure $S = \bigoplus_{m\geq 0} S_m$, where S_m is the $\mathbb{Q}$-subspace spanned by elements of homogeneous degree m. Likewise, its l-fold tensor product $S^{\otimes l} = \bigoplus_{m\geq 0}(S^{\otimes l})_m$ is graded, with

$$(S^{\otimes l})_m = \bigoplus_{\substack{m_1+\cdots+m_l=m\\ m_1,\ldots,m_l\geq 0.}} S_{m_1} \otimes \cdots \otimes S_{m_l}.$$

Consider the $\mathbb{Q}$-subspace $\langle sl_2\rangle_m^{(l)}$ of $(S^{\otimes l})_m$ defined by

$$\langle sl_2\rangle_m^{(l)} = \bigoplus_{\substack{m_1+\cdots+m_l=m\\ 0\leq m_1,\ldots,m_l\leq 1}} S_{m_1} \otimes \cdots \otimes S_{m_l}.$$

Roughly speaking, $\langle sl_2\rangle_m^{(l)}$ is spanned by tensors in $S^{\otimes l}$ with exactly m nontrivial tensorands, each of which being of degree one. For example, the tensor ς_m defined in (4.7) is an element of $\langle sl_2\rangle_m^{(l)}$.

By the definition of W given in Secs. 4.3 and 4.4, we immediately have the following.

Lemma 5.1. *For $m \geq 1$, we have*

$$W(\mathcal{C}_m^t(l)) \subset (S^{\otimes l})_{m+1}\hbar^m \quad \textit{and} \quad W(\mathcal{C}_m^h(l)) \subset \langle sl_2\rangle_{m+1}^{(l)}\hbar^m.$$

In what follows, we will identify $U_\hbar^{\hat{\otimes} l}$ and $U^{\otimes l}[[\hbar]]$ as $\mathbb{Q}[[\hbar]]$-modules via the isomorphism

$$\rho : U_\hbar^{\hat{\otimes} l} \to U^{\otimes l}[[\hbar]]$$

defined by

$$\rho\left(\sum_{s_1,\dots,s_l,t_1,\dots,t_l,u_1,\dots,u_l\geq 0} a_{s_1,\dots,s_l,t_1,\dots,t_l,u_1,\dots,u_l} F^{s_1}H^{t_1}E^{u_1}\otimes\cdots\otimes F^{s_l}H^{t_l}E^{u_l}\right)$$
$$= \sum_{s_1,\dots,s_l,t_1,\dots,t_l,u_1,\dots,u_l\geq 0} a_{s_1,\dots,s_l,t_1,\dots,t_l,u_1,\dots,u_l} f^{s_1}h^{t_1}e^{u_1}\otimes\cdots\otimes f^{s_l}h^{t_l}e^{u_l},$$

for $a_{s_1,\dots,s_l,t_1,\dots,t_l,u_1,\dots,u_l}\in\mathbb{Q}[[\hbar]]$. We also identify $U_\hbar^{\hat{\otimes}l}$ and $S^{\otimes l}[[\hbar]]$ as $\mathbb{Q}$-modules similarly.

5.2. *Main results*

We can now give the precise statements of our main results.

Set

$$J^t := \pi^t\circ J : SL(l)\to\prod_{m\geq 1}(S^{\otimes l})_{m+1}\hbar^m,$$

where

$$\pi^t : U_\hbar^{\hat{\otimes}l}\to\prod_{m\geq 1}(S^{\otimes l})_{m+1}\hbar^m$$

denotes the projection as $\mathbb{Q}$-modules, that is,

$$\pi^t\left(\sum_{m\geq 0}\sum_{i\geq 0}\left(\sum_{\sum_{i=1}^l s_i+t_i+u_i=m} b^{(i)}_{s_1,\dots,s_l,t_1,\dots,t_l,u_1,\dots,u_l} F^{s_1}H^{t_1}E^{u_1}\otimes\cdots\otimes F^{s_l}H^{t_l}E^{u_l}\right)\hbar^i\right)$$
$$=\sum_{m\geq 1}\left(\sum_{\sum_{i=1}^l s_i+t_i+u_i=m+1} b^{(m)}_{s_1,\dots,s_l,t_1,\dots,t_l,u_1,\dots,u_l} f^{s_1}h^{t_1}e^{u_1}\otimes\cdots\otimes f^{s_l}h^{t_l}e^{u_l}\right)\hbar^m,$$

for $b^{(i)}_{s_1,\dots,s_l,t_1,\dots,t_l,u_1,\dots,u_l}\in\mathbb{Q}$.

The first main result in this paper is as follows.

Theorem 5.1. *Let $m\geq 1$. If $L\in SL_m(l)$, then we have*

$$J^t(L)\equiv(W\circ\mu_m)(L)\pmod{\hbar^{m+1}}.$$

Example 5.1. Let L be a string link with nonzero linking matrix $(m_{ij})_{1\leq i,j\leq l}$. Then the ith longitude l_i^1 in $\mathrm{F}_l/\Gamma_2\mathrm{F}_l$, defined in Sec. 2.1, reads $l_i^1=\sum_{j=1}^l m_{ij}\alpha_j$ $(1\leq i\leq l)$, so that the degree 1 Milnor map of L is given by

$$\begin{aligned}\mu_1(L)&=\sum_{i=1}^l\alpha_i\otimes l_i^1\\&=\sum_{1\leq i,j\leq l} m_{ij}\alpha_i\otimes\alpha_j\end{aligned}$$

$$= \sum_{1\le i<j\le l} m_{ij}(\alpha_i \otimes \alpha_j + \alpha_j \otimes \alpha_i) + \sum_{1\le i\le l} m_{ii}(\alpha_i \otimes \alpha_i)$$

$$= \sum_{1\le i<j\le l} m_{ij} \cdot D_{\alpha_i,\alpha_j} + \sum_{1\le i\le l} m_{ii} \cdot \frac{1}{2} D_{\alpha_i,\alpha_i},$$

where D_{α_i,α_j} denotes a single chord with vertices labeled α_i and α_j, and where the last equality uses isomorphism (4.1), see Example 4.1. Applying the sl_2 weight system W then yields

$$(W \circ \mu_1)(L) = \sum_{1\le i<j\le l} m_{ij} c_{ij}^{(l)} + \frac{1}{2} \sum_{1\le i\le l} m_{ii} c_{ii}^{(l)},$$

as predicted by Proposition 3.1.

Since Milnor maps are concordance invariants, we obtain the following topological property for J^t as an immediate consequence of Theorem 5.1.

Corollary 5.1. *Let $L, L' \in SL_m(l)$ be two concordant string links. Then we have*

$$J^t(L') \equiv J^t(L) \pmod{\hbar^{m+1}}.$$

In particular, if L is concordant to the trivial string link, then $J^t(L)$ is trivial.

Theorem 5.1 is proved in Sec. 7. The proof relies on the fact that Milnor concordance invariants are related to Milnor link-homotopy invariants via some cabling operation, so that Theorem 5.2 is actually used as a tool for proving Theorem 5.1.

In order to state the second main result in this paper, set

$$J^h := \pi^h \circ J : SL(l) \to \bigoplus_{m=1}^{l-1} \langle sl_2 \rangle_{m+1}^{(l)} \hbar^m,$$

where $\pi^h : U_\hbar^{\hat{\otimes} l} \to \bigoplus_{m=1}^{l-1} \langle sl_2 \rangle_{m+1}^{(l)} \hbar^m$ is the projection as $\mathbb{Q}$-modules.

We have the following.

Theorem 5.2. *Let $m \geq 1$. If $L \in SL_m^h(l)$, then we have*

$$J^h(L) \equiv (W \circ \mu_m^h)(L) \pmod{\hbar^{m+1}},$$

where μ_m^h is the link-homotopy reduction of the Milnor map μ_m, defined in Sec. 4.2.

Theorem 5.2 is equivalent to the following theorem, formulated in terms of Milnor numbers and the tensors $\varsigma_I^{(l)}$ defined in Sec. 4.4.

Theorem 5.3. *For $m \geq 1$, if $L \in SL_m^h(l)$, then we have*

$$J^h(L) \equiv \left(\sum_{I \in \mathcal{I}_{m+1}} \mu_I(L) \varsigma_I^{(l)} \right) \hbar^m \pmod{\hbar^{m+1}},$$

where the sum runs over the set $\mathcal{I}_{m+1}$ defined in Sec. 2.

Proof of equivalency of Theorems 5.2 and 5.3. We need to prove that

$$(w_m \circ \mu_m^h)(L) = \sum_{I \in \mathcal{I}_{m+1}} \mu_I(L)\varsigma_I^{(l)}. \tag{5.1}$$

Recall from Lemma 2.1 that if $L \in SL_m^h(l)$, then L is link-homotopic to $b_m^L b_{m+1}^L \cdots b_{l-1}^L$, where the pure braids b_i^L are defined in (2.2). Actually, it follows directly from Eq. (2.2) that if $L \in SL_m^h(l)$, then we have $\mu_m^h(L) = \mu_m^h(b_m^L)$. We thus have that

$$\begin{aligned}
\mu_m^h(L) &= \mu_m^h(b_m^L) \\
&= \mu_m^h\left(\prod_{I \in \mathcal{I}_{m+1}} (B_I)^{\mu_I(b_m^L)}\right) \\
&= \mu_m^h\left(\prod_{I \in \mathcal{I}_{m+1}} (B_I)^{\mu_I(L)}\right) \\
&= \sum_{I \in \mathcal{I}_{m+1}} \mu_I(L) \cdot \mu_m^h(B_I),
\end{aligned}$$

where the last equality uses the additivity of the first non-vanishing Milnor string link invariants. The result then follows from (4.9) and the fact that $\mu_m^h(B_I) = T_I$ for $I \in \mathcal{I}_{m+1}$, which can be easily checked either by a direct computation or using (4.2). □

We prove Theorems 5.3 in the next section.

6. Proof of Theorem 5.3: The Link-Homotopy Case

We reduce Theorem 5.3 to the following two propositions. The first one shows that the invariant J^h is well-behaved with respect to link-homotopy.

Proposition 6.1. *Let $L, L' \in SL_m^h(l)$ be two link-homotopic string links. Then we have*

$$J^h(L') \equiv J^h(L) \pmod{\hbar^{m+1}}.$$

In particular, if L is link-homotopic to the trivial string link, then $J^h(L)$ is trivial.

For the second proposition, recall from Sec. 2.3 that for each sequence $J \in \mathcal{I}_{m+1}$ we defined a pure braid $B_J^{(l)}$ which lies in the mth term of the lower central series of $P(l)$.

Proposition 6.2. *For any $J \in \mathcal{I}_{m+1}$, we have*

$$J(B_J^{(l)}) \equiv 1 + \varsigma_J^{(l)}\hbar^m \pmod{\hbar^{m+1}},$$

where $\varsigma_J^{(l)} \in$ was defined in Sec. 4.4.

Proof of Theorem 5.3 assuming Propositions 6.1 and 6.2. We first note that, as an immediate consequence of Proposition 6.2, for any $J \in \mathcal{I}_{m+1}$ we have

$$J^h(B_J^{(l)}) \equiv \varsigma_J^{(l)} \hbar^m \pmod{\hbar^{m+1}}. \tag{6.1}$$

Now, let $L \in SL_m^h(l)$, for some $m \geq 1$. We have

$$\begin{aligned} J^h(L) &\equiv J^h(b_m^L \cdot b_{m+1}^L \cdot \dots \cdot b_{l-1}^L) \\ &\equiv J^h(b_m^L) \\ &\equiv J^h\left(\prod_{I \in \mathcal{I}_{m+1}} B_I^{\mu_I(L)}\right) \\ &\equiv \sum_{I \in \mathcal{I}_{m+1}} \mu_I(L) \varsigma_I^{(l)} \hbar^m \pmod{\hbar^{m+1}}, \end{aligned}$$

where the first equality uses Lemma 2.1 and Proposition 6.1, while the last three equalities follow from the definition of the pure braids b_i^L and from (6.1). Thus we have the assertion. $\square$

The proof of Proposition 6.1, which makes use of the theory of claspers, is postponed to Sec. 8. Proposition 6.2 is proved by a direct computation, as shown below.

Proof of Proposition 6.2. Set $J = j_1 j_2 \cdots j_{m+1}$. The result is shown by induction on m. For $m = 1$, then $B_{j_1 j_2}^{(l)}$ is the pure braid $A_{j_1,j_2}^{(l)}$, so by Proposition 3.1 we have

$$\begin{aligned} J(B_{j_1 j_2}^{(l)}) &\equiv 1 + c_{j_1,j_2}^{(l)} \hbar \\ &\equiv 1 + \varsigma_{j_1 j_2}^{(l)} \hbar \pmod{\hbar^2}, \end{aligned}$$

as desired.

For $m > 1$, by the induction hypothesis we have

$$\begin{aligned} J(B_J^{(l)}) &= J([B_{j_1 \cdots j_m}^{(l)}, A_{j_m,j_{m+1}}^{(l)}]) \\ &= [J(B_{j_1 \cdots j_m}^{(l)}), J(A_{j_m,j_{m+1}}^{(l)})] \\ &\in [1 + \varsigma_{j_1 \cdots j_m}^{(l)} \hbar^{m-1} + \hbar^m U_\hbar^{\hat{\otimes} l}, 1 + c_{j_m,j_{m+1}}^{(l)} \hbar + \hbar^2 U_\hbar^{\hat{\otimes} l}] \\ &\subset 1 + [\varsigma_{j_1 \cdots j_m}^{(l)}, c_{j_m,j_{m+1}}^{(l)}] \hbar^m + \hbar^{m+1} U_\hbar^{\hat{\otimes} l}, \end{aligned}$$

and on the other hand we have

$$\begin{aligned} [\varsigma_{j_1 \cdots j_m}^{(l)}, c_{j_m,j_{m+1}}^{(l)}] &= ([\varsigma_m \otimes 1, c_{m,m+1}^{(m+1)}])_{j_1 \cdots j_{m+1}}^{(l)} \\ &= ((1^{\otimes m-2} \otimes \mathrm{ad} \otimes 1)(\varsigma_m \otimes c))_{j_1 \cdots j_{m+1}}^{(l)} \end{aligned}$$

$$= ((1^{\otimes m-2} \otimes s)(\varsigma_m))^{(l)}_{j_1\cdots j_{m+1}}$$
$$= (\varsigma_{m+1})^{(l)}_{j_1\cdots j_{m+1}}$$
$$= \varsigma^{(l)}_J.$$

This completes the proof. □

7. Proof of Theorem 5.1: The General Case

In this section, we show how to deduce Theorem 5.1 from Theorem 5.2.

First, let us set some notation for the various projection maps that will be used throughout this section. For $i, j \geq 1$, let

$$\pi^t_{i,j} : S^{\otimes l}[[\hbar]] \to (S^{\otimes l})_i \hbar^j,$$

where $(S^{\otimes l})_i$ was defined in Sec. 5.1, and set also $\pi^t_i := \pi^t_{i+1,i}$, so that $\pi^t = \prod_{i\geq 1} \pi^t_i$. Likewise, let

$$\pi^h_{i,j} : S^{\otimes l}[[\hbar]] \to \langle sl_2\rangle^{(l)}_i \hbar^j,$$

and set $\pi^h_i := \pi^h_{i+1,i}$.

Next, recall that there are two (completed) coalgebra structures on $S[[\hbar]]$ as follows. The first one is defined by $\bar{\Delta}(x) = x \otimes 1 + 1 \otimes x$ and $\varepsilon(x) = 0$ for $x \in sl_2$ as algebra morphisms. The second one is induced by the coalgebra structure of $U_\hbar$ defined in Sec. 3, via the $\mathbb{Q}$-module isomorphism $\rho : U_\hbar \to S[[\hbar]]$ seen in Sec. 5.1. For $\Delta = \Delta_\hbar, \bar{\Delta}$ and $p \geq 0$, define

$$\Delta^{[p]} : S[[\hbar]] \to S^{\otimes p}[[\hbar]]$$

by $\Delta^{[0]} = \varepsilon$, $\Delta^{[1]} = \mathrm{id}$, $\Delta^{[2]} = \Delta$, and $\Delta^{[p]} = (\Delta \otimes 1^{\otimes p-2}) \circ \Delta^{[p]}$ for $p \geq 3$. Abusing notation, for $l \geq 0$, we write $\Delta^{(p)} := (\Delta^{[p]})^{\otimes l} : S^{\otimes l}[[\hbar]] \to S^{\otimes pl}[[\hbar]]$.

Since we have $\Delta_\hbar(y) \equiv \bar{\Delta}(y) \pmod{\hbar}$ for any $y \in S$, the restriction of $\pi^h_{i,j} \circ \bar{\Delta}^{(p)}$ to $(S^{\otimes l})_i \hbar^j$ is equal to that of $\pi^h_{i,j} \circ \Delta^{(p)}_\hbar$ for any $1 \leq i, j < p$, that is, we have

$$\pi^h_{i,j} \circ \bar{\Delta}^{(p)}|_{(S^{\otimes l})_i \hbar^j} = \pi^h_{i,j} \circ \Delta^{(p)}_\hbar|_{(S^{\otimes l})_i \hbar^j}. \tag{7.1}$$

Actually, the injectivity of these maps is one of the key points in this section.

Lemma 7.1. *For any $1 \leq i, j < p$, the restriction to $(S^{\otimes l})_i \hbar^j$ of the $\mathbb{Q}$-linear map $\pi^h_{i,j} \circ \bar{\Delta}^{(p)}$ is injective.*

Proof. This simply follows from the fact that the map $\nabla^{(p)} \circ \pi^h_{i,j} \circ \bar{\Delta}^{(p)}$ is a scalar map on $(S^{\otimes l})_i \hbar^j$, where $\nabla^{(p)} : U_\hbar^{\hat{\otimes} pl} \to U_\hbar^{\hat{\otimes} l}$ is the tensor power of p-fold multiplications, i.e. the map sending $x_1 \otimes \cdots \otimes x_{pl} \in U_\hbar^{\hat{\otimes} pl}$ to $x_1 \cdots x_p \otimes \cdots \otimes x_{p(l-1)+1} \cdots x_{pl} \in U_\hbar^{\hat{\otimes} l}$. □

Now, for $p \geq 1$, let $D^{(p)} : SL(l) \to SL(pl)$ be the cabling map, which sends a string link $L \in SL(l)$ to the string link $D^{(p)}(L) \in SL(pl)$ obtained by replacing each component with p parallel copies. Recall from [10] that, for $m \geq 1$ and $p > m$, we have that $L \in SL_m(l)$ if and only if $D^{(p)}(L) \in SL^h_m(pl)$. We have the following.

Lemma 7.2. *For $1 \leq m < p$, the following diagram commutes*

$$\begin{tikzcd} SL_m(l) \arrow[r, "W\circ\mu_m"] \arrow[d, "D^{(p)}"'] & (S^{\otimes l})_{m+1}\hbar^m \arrow[d, "\pi^h_m\circ\bar{\Delta}^{(p)}"] \\ SL^h_m(pl) \arrow[r, "W\circ\mu^h_m"'] & \langle sl_2\rangle^{(pl)}_{m+1}\hbar^m. \end{tikzcd}$$

Proof. Denote by $D^{(p)} : \mathcal{C}^t_m(l) \to \mathcal{C}^t_m(pl)$ the map defined by sending a tree Jacobi diagram $\xi \in \mathcal{C}^t_m(l)$ to the sum of all diagrams obtained from ξ by replacing each label $i \in \{1, \ldots, l\}$ by one of $(i-1)p+1, (i-1)p+2, \ldots, ip$. Then the lemma follows from the following two commutative diagrams

$$\begin{tikzcd} SL_m(l) \arrow[r, "\mu_m"] \arrow[d, "D^{(p)}"'] & \mathcal{C}^t_m(l) \arrow[d, "p^h\circ D^{(p)}"] \\ SL^h_m(pl) \arrow[r, "\mu^h_m"'] & \mathcal{C}^h_m(pl), \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C}^t_m(l) \arrow[r, "W"] \arrow[d, "p^h\circ D^{(p)}"'] & (S^{\otimes l})_{m+1}\hbar^m \arrow[d, "\pi^h_m\circ\bar{\Delta}^{(p)}"] \\ \mathcal{C}^h_m(pl) \arrow[r, "W"'] & \langle sl_2\rangle^{(pl)}_{m+1}\hbar^m. \end{tikzcd}$$

The fact that the left-hand side diagram commutes is due to Habegger and Masbaum [10], while the commutativity of the right-hand side diagram is a direct consequence of the definitions. □

The next technical lemma will be shown in Sec. 8.

Lemma 7.3. *Let $L \in SL^h_m(l)$, and $1 \leq j \leq i-2 \leq m$. We have $\pi^h_{i,j}(J(L)) = 0$.*

We use Lemma 7.3 to establish the following.

Lemma 7.4. *For $1 \leq m < p$, the following diagram commutes*

$$\begin{tikzcd} SL_m(l) \arrow[r, "\pi^t_m\circ J"] \arrow[d, "D^{(p)}"'] & (S^{\otimes l})_{m+1}\hbar^m \arrow[d, "\pi^h_m\circ\Delta^{(p)}_\hbar"] \\ SL^h_m(pl) \arrow[r, "\pi^h_m\circ J"'] & \langle sl_2\rangle^{(pl)}_{m+1}\hbar^m. \end{tikzcd}$$

Proof. The diagram in the statement decomposes as

$$\begin{tikzcd} SL_m(l) \arrow[r, "J"] \arrow[d, "D^{(p)}"'] & J(SL_m(l)) \arrow[r, "\pi^t_m"] \arrow[d, "\Delta^{(p)}_\hbar"'] & (S^{\otimes l})_{m+1}\hbar^m \arrow[d, "\pi^h_m\circ\Delta^{(p)}_\hbar"] \\ SL^h_m(pl) \arrow[r, "J"'] & J(SL^h_m(pl)) \arrow[r, "\pi^h_m"'] & \langle sl_2\rangle^{(pl)}_{m+1}\hbar^m, \end{tikzcd}$$

where the left-hand side square commutes as a general property of the universal sl_2 invariant. In order to prove that the right-hand square commutes as well, we first show that, given a string link $L \in SL_m(l)$, we have

$$J(L) \in 1 + \bigoplus_{1\le i\le j\le m} (S^{\otimes l})_i \hbar^j + (S^{\otimes l})_{m+1}\hbar^m + \hbar^{m+1} U_\hbar^{\hat{\otimes} l}. \tag{7.2}$$

In other words, we show that

(a) $\pi^t_j(J(L)) = 0$ for $1 \le j < m-1$,
(b) $\pi^t_{i,j}(J(L)) = 0$ for $1 \le j \le i-2 \le m$.

By Lemma 7.1, if $\pi^t_{i,j}(J(L)) \neq 0$ for $i, j \ge 1$, then for $p > i, j$, we have

$$\pi^h_{i,j}(J(D^{(p)}(L))) = (\pi^h_{i,j} \circ \Delta^{(p)}_\hbar)(J(L)) \neq 0. \tag{7.3}$$

However, as already recalled above, the fact that $L \in SL_m(l)$ implies that $J(D^{(p)}(L))$ is in $SL^h_m(pl)$. So (7.3) above can neither hold in case $1 \le j = i-1 < m-1$ by Theorem 5.3(ii), which implies (a), nor in case $1 \le j \le i-2 \le m$ by Lemma 7.3, which implies (b).

Let us now proceed with the proof that the right-hand square of the diagram above is commutative. In view of (7.2), we only need to show the following two claims:

(i) $\pi^h_m \circ \Delta^{(p)}_\hbar\big((S^{\otimes l})_i \hbar^j\big) = 0$ for any $1 \le i \le j \le m$, and
(ii) $\pi^h_m \circ \Delta^{(p)}_\hbar(\hbar^{m+1} U_\hbar^{\hat{\otimes} l}) = 0$.

Claim (ii) is obvious, from the fact that $\Delta^{(p)}_\hbar(\hbar^i U_\hbar^{\hat{\otimes} l}) \subset \hbar^i U_\hbar^{\hat{\otimes} pl}$ for $i \ge 0$. In order to prove Claim (i), it is enough to show for $0 \le i \le j$ that

$$\Delta^{(p)}_\hbar((S^{\otimes l})_i \hbar^j) \subset \prod_{0\le u\le v} (S^{\otimes pl})_u \hbar^v.$$

Recall from [13] that for $s, n, r \ge 0$, $\Delta_\hbar(F^s H^n E^r)$ is equal to

$$\sum_{0\le j_1\le s, 0\le j_2\le n, 0\le j_3\le r} \begin{bmatrix} s \\ j_1 \end{bmatrix}_q \begin{bmatrix} n \\ j_2 \end{bmatrix}_q \begin{pmatrix} r \\ j_3 \end{pmatrix} F^{s-j_1} H^{n-j_2} K^{j_3} E^{r-j_3} \otimes F^{j_1} K^{s-j_1} H^{j_2} E^{j_3}.$$

Since $K = \exp \frac{\hbar H}{2} \in \prod_{t\ge 0} \mathbb{Q} H^t \hbar^t$, the above formula implies

$$\Delta_\hbar(F^s H^n E^r) \in \prod_{0\le t\le k} (S^{\otimes 2})_{s+n+r+t} \hbar^k.$$

Thus we have

$$\Delta^{(p)}_\hbar((S^{\otimes l})_i \hbar^j) \subset \prod_{0\le t\le k} (S^{\otimes pl})_{i+t} \hbar^{j+k} \subset \prod_{0\le u\le v} (S^{\otimes pl})_u \hbar^v.$$

This concludes the proof of Lemma 7.4. □

We can finally proceed with the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $L \in SL_m(l)$ for some $m \geq 1$. By (7.2), we have

$$J^t(L) \in (S^{\otimes l})_{m+1}\hbar^m + \hbar^{m+1}U_\hbar^{\hat{\otimes} l},$$

and thus we only need to prove that $\pi_m^t \circ J^t = W \circ \mu_{m+1}$. By Lemma 7.1, it suffices to show that this equality holds after post-composing with $\pi_m^h \circ \bar{\Delta}^{(p)}$, that is, it suffices to prove that

$$\pi_m^h \circ \bar{\Delta}^{(p)} \circ \pi_m^t \circ J^t = \pi_m^h \circ \bar{\Delta}^{(p)} \circ W \circ \mu_{m+1}.$$

But according to the commutative diagrams of Lemmas 7.2 and 7.4, this is equivalent to proving that

$$(\pi_m^h \circ J^h)(D^{(p)}(L)) = (\pi_m^h \circ W \circ \mu_{m+1})(D^{(p)}(L)),$$

which follows immediately from Theorem 5.2. Thus we have the assertion. $\square$

Remark 7.1. We have in particular shown that the universal sl_2 invariant for a string link $L \in SL_m(l)$ satisfies (7.2), and the proof of Theorem 5.1 given above relies on the fact that, when applying the projection map π^t to the above equation, we obtain that $J^t(L) \in (S^{\otimes l})_{m+1}\hbar^m + \hbar^{m+1}U_\hbar^{\hat{\otimes} l}$. So we could consider an alternative version of the reduction J^t of the universal sl_2 invariant for the statement of our main result, by setting

$$\tilde{J}^t := \tilde{\pi}^t \circ J,$$

where $\tilde{\pi}^t$ denotes the quotient map as $\mathbb{Q}$-modules

$$\tilde{\pi}^t : (S^{\otimes l})[[\hbar]] \to \frac{(S^{\otimes l})[[\hbar]]}{\prod_{1\leq i\leq j}(S^{\otimes l})_i\hbar^j}.$$

Clearly, it appears from the above proof, that Theorem 5.1 still holds when replacing J^t with this alternative version $\tilde{J}^t$.

The above observation gives the following, which in particular applies to slice and boundary string links.

Corollary 7.1. *Let L be an l-component string link with vanishing Milnor invariants. Then we have*

$$J(L) \in 1 + \prod_{1\leq i\leq j}(S^{\otimes l})_i\hbar^j.$$

8. Universal sl_2 Invariant and Clasper Surgery

This section contains the proof of Lemma 7.3 and Proposition 6.1. In order to prove these results, we will make use of the theory of claspers, and more precisely we will study the behavior of the universal sl_2 invariant under clasper surgery.

8.1. *A quick review of clasper theory*

We recall here only the definition and a few properties of claspers for string links, and refer the reader to [11] for more details.

Let L be a string link. A *clasper* for L is an embedded surface in $D^2 \times [0,1]$, which decomposes into disks and bands, called *edges*, each of which connects two distinct disks. The disks have either 1 or 3 incident edges, and are called *leaves* or *nodes*, respectively, and the clasper intersects L transversely at a finite number of points, which are all contained in the interiors of the leaves. A clasper is called a *tree clasper* if it is connected and simply connected. In this paper, we make use of the drawing convention of [11, Figure 7] for representing claspers.

The *degree* of a tree clasper is defined to the number of nodes plus 1, i.e. the number of leaves minus 1.

Given a clasper G for a string link L, we can modify L using the local moves 1 and 2 of Fig. 8 as follows. If G contains one or several nodes, pick any leaf of G that is connected to a node by an edge, and apply the local move 1. Keep applying this first move at each node, until none remains: this produces a disjoint union of degree 1 claspers for the string link L (note indeed that erasing these degree 1 claspers gives back the string link L). Now apply the local move 2 at each degree 1 clasper. We say that the resulting string link L_G in $D^2 \times [0,1]$ is obtained from L by *surgery along* G. Note that the isotopy class of L_G does not depend on the order in which the moves were performed.

The C_k*-equivalence* is the equivalence relation on string links generated by surgeries along tree claspers of degree k and isotopies.

A clasper for a string link L is called *simple* if each of its leaves intersects L at one point. Habiro showed that two string links are C_k-equivalent if and only if they are related by surgery along a disjoint union of simple degree k tree claspers.

In the following, *we will implicitly assume that all tree claspers are simple.*

A tree clasper G for a string link L is called *repeated* if more than one leaf of G intersects the same component of L. An important property of repeated tree claspers is the following, see for example [7].

Lemma 8.1. *Surgery along a repeated tree clasper preserves the link-homotopy class of* (*string*) *links.*

We conclude this subsection with a couple of standard lemmas in clasper theory. Proofs are omitted, since they involve the same techniques as in [11, Section 4], where similar statements appear.

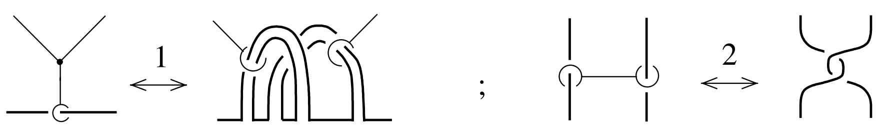


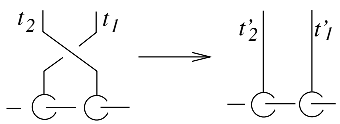
Fig. 8. Constructing the image of a string link under clasper surgery. Here, bold lines represent a bunch of parallel strands from the string link.

Lemma 8.2. *Let C be a union of tree claspers for a string link L, and let t be a component of C which is a tree clasper of degree k. Let C' be obtained from C by passing an edge of t across L or across another edge of C. Then we have*

$$L_C \underset{\sim}{C_{k+1}} L_{C'}. \tag{8.1}$$

Moreover, if t is repeated, then the C_{k+1}-equivalence in (8.1) *is realized by surgery along repeated tree claspers.*

Lemma 8.3. *Let $t_1 \cup t_2$ be a disjoint union of a degree k_1 and a degree k_2 clasper for a string link L. Let $t_1' \cup t_2'$ be obtained from $t_1 \cup t_2$ by sliding a leaf of t_1 across a leaf of t_2, as shown below*:



Then we have

$$L_{t_1 \cup t_2} \underset{\sim}{C_{k_1+k_2}} L_{t_1' \cup t_2'}. \tag{8.2}$$

Moreover, if one of t_1 and t_2 is repeated, then the $C_{k_1+k_2}$-equivalence in (8.2) *is realized by surgery along repeated tree claspers.*

8.2. *Proofs of Lemma* 7.3 *and Proposition* 6.1

In this section, we prove Lemma 7.3 and Proposition 6.1. The proofs rely on two results (Corollary 8.1 and Lemma 8.5) which describe the behavior of the universal sl_2 invariant with respect to clasper surgery.

We need an additional technical notion to state these results. Recall from Sec. 2 that the trivial string link is defined as $\mathbf{1}(= \mathbf{1}_l) = \{p_1, \dots, p_l\} \times [0,1]$. We assume that the points p_i are on the line $\{(x,y) \in D^2 \mid y = 0\}$. A tree clasper T for the trivial string link $\mathbf{1}$ is called *overpassing*, if all edges and nodes of T are contained in $\{(x,y) \in D^2 \mid y \leq 0\} \times [0,1] \subset D^2 \times [0,1]$. In other words, T is overpassing if there is a diagram of $\mathbf{1} \cup T$ which restricts to the standard diagram of $\mathbf{1}$, where the strands do not cross, and where the edges of T overpass $\mathbf{1}$ at all crossings.

Lemma 8.4. *Let L and L_0 be two link-homotopic l-component string links. Then for any $m \geq 1$, there exists $n \geq 0$ overpassing repeated tree claspers $R_1, \dots, R_n$ of degree $\leq m$ for $\mathbf{1}$ such that*

$$L \underset{\sim}{C_{m+1}} L_0 \cdot \prod_{j=1}^{n} \mathbf{1}_{R_j}.$$

Proof. By the definition of link-homotopy, L can be obtained from L_0 a finite sequence of self-crossing changes, i.e. by surgery along a disjoint union R of n_1 repeated degree 1 tree claspers.

Pick a connected component R_1 of R. By a sequence of crossing changes and leaf slides, we can "pull down" R_1 in $D^2 \times [0,1]$ so that it leaves in a small neighborhood of $D^2 \times \{0\}$, which is disjoint from $R \setminus R_1$ and intersects L_0 at n trivial arcs. Apply further crossing changes to ensure that the image $\tilde{R}_1$ of R_1 under this deformation is overpassing. By Lemmas 8.2 and 8.3, we have

$$L \overset{C_2}{\sim} (L_0)_{R \setminus R_1} \cdot \mathbf{1}_{\tilde{R}_1},$$

and the C_2-equivalence is realized by surgery along repeated tree claspers of degree 2. Applying this procedure to each of the n_1 connected components of R successively, we eventually obtain that

$$L = \left(L_0 \cdot \prod_{1 \le i \le n_1} \mathbf{1}_{\tilde{R}_i} \right)_{R^{(2)}},$$

where each $\tilde{R}_i$ is an overpassing tree clasper of degree 1, and $R^{(2)}$ is a disjoint union of repeated tree claspers of degree 2.

Next, we apply the same "pull down" procedure to each connected component of $R^{(2)}$ successively. Using the same lemmas, we then have that

$$L = \left(L_0 \cdot \prod_{1 \le i_2 \le n_2} \mathbf{1}_{\tilde{R}^{(2)}_{i_2}} \right)_{R^{(3)}},$$

where each $R^{(2)}_i$ is an overpassing repeated tree clasper of degree at most 2, and $R^{(3)}$ is a disjoint union of repeated tree claspers of degree ≥ 3.

Iterating this procedure, we obtain that, for any integer $m \ge 1$, we have

$$L = \left(L_0 \cdot \prod_{1 \le i_m \le n_m} \mathbf{1}_{\tilde{R}^{(m)}_{i_m}} \right)_{R^{(m+1)}},$$

where $\tilde{R}^{(m)}_i$ is an overpassing repeated tree clasper of degree at most m, and $R^{(m+1)}$ is a disjoint union of repeated tree claspers of degree $\ge m+1$.

This completes the proof. □

Since the universal sl_2 invariant modulo the ideal $\hbar^k U_\hbar^{\hat{\otimes} l}$ is a finite type invariant of degree $< k$, hence is an invariant of C_k-equivalence [11], the multiplicativity of the universal sl_2 invariant implies the following.

Corollary 8.1. *Let L and L_0 be two link-homotopic l-component string links. Then for any $m \ge 1$, there exists $n \ge 0$ overpassing repeated tree claspers $R_1, \ldots, R_n$ of degree $\le m$ for $\mathbf{1}$ such that*

$$J(L) \equiv J(L_0) \cdot \prod_{j=1}^{n} J(\mathbf{1}_{R_j}) \pmod{\hbar^{m+1}}.$$

We will apply this result to the case where L_0 is the explicit representative for the link-homotopy class of L given in Lemma 2.1, whose universal sl_2 invariant was studied in detail in Sec. 6. By Corollary 8.1, we are thus lead to studying the universal sl_2 invariant of string links obtained from $\mathbf{1}$ by surgery along an overpassing repeated tree clasper: this is the subject of Lemma 8.5.

For $x = F^{s_1}H^{n_1}E^{r_1} \otimes F^{s_2}H^{n_2}E^{r_2} \otimes \cdots \otimes F^{s_l}H^{n_l}E^{r_l} \in S^{\otimes l}$, set

$$\mathrm{supp}(x) = \sharp\{1 \le i \le l \,|\, s_i + n_i + r_i \ne 0\},$$

that is, roughly speaking, the number of nontrivial tensorands. We denote by $(S^{\otimes l})_{\mathrm{supp} \le n}$ the $\mathbb{Q}$-submodule of $S^{\otimes l}$ spanned by all monomials x such that $\mathrm{supp}(x) \le n$.

Lemma 8.5. *Let C be an overpassing repeated tree clasper for $\mathbf{1} \in SL(l)$. We have*

$$J(\mathbf{1}_C) \in 1 + \prod_{j\ge 1} (S^{\otimes l})_{\mathrm{supp}\le j}\hbar^j.$$

Proof. Since C is an overpassing repeated tree clasper for $\mathbf{1}_l$, there exists an l-component braid B such that

$$(\mathbf{1}_l)_C = B \cdot ((\mathbf{1}_k)_{C'} \otimes \mathbf{1}_{l-k}) \cdot B^{-1}, \tag{8.3}$$

where k denotes the number of strands of $\mathbf{1}_l$ intersecting C, and where C' denotes the image of C under this isotopy. (Recall that $\otimes$ denotes the horizontal juxtaposition of string links.)

Let m denote the degree of C (and C'). Since $J((\mathbf{1}_k)_{C'}) \equiv 1 \pmod{\hbar^m}$ and $k \le \sharp\{\text{leaves of } C\} - 1 = m$, we have

$$J((\mathbf{1}_k)_{C'}) \in 1 + \prod_{j\ge m} S^{\otimes k}\hbar^j \subset 1 + \prod_{j \ge k} S^{\otimes k}\hbar^j,$$

which implies that

$$J((\mathbf{1}_k)_{C'} \otimes \mathbf{1}_{l-k}) \in 1 + \prod_{j\ge k} (S^{\otimes l})_{\mathrm{supp}\le k}\hbar^j.$$

So, by Eq. (8.3), in order to obtain the desired result it only remains to show that $\prod_{j\ge 1}(S^{\otimes l})_{\mathrm{supp}\le j}\hbar^j$ is invariant under the braid group action. Here, the braid group acts on $U_\hbar^{\hat{\otimes} l}$ by quantized permutation: the action of Artin generator σ_n on an element x is given by $R^{(l)}_{n+1,n}(\bar\sigma_n(x))(R^{-1})^{(l)}_{n+1,n}$, where $\bar\sigma_n(x)$ denotes the permutation of the nth and $(n+1)$th tensorands of x.

Hence it suffices to prove that, for any monomial $x = (x_1 \otimes \cdots \otimes x_k \otimes 1^{\otimes l-k})\hbar^j$ with $x_1, \ldots, x_k \in S$, $j \ge k$, and any $n \in \{1, \ldots, l-1\}$, we have

$$R^{(l)}_{n+1,n} \cdot x \cdot (R^{-1})^{(l)}_{n+1,n} \in \prod_{j\ge 1}(S^{\otimes l})_{\mathrm{supp}\le j}\hbar^j,$$

and

$$(R^{-1})^{(l)}_{n,n+1} \cdot x \cdot R^{(l)}_{n,n+1} \in \prod_{j \ge 1}(S^{\otimes l})_{\mathrm{supp}\le j}\hbar^j.$$

We prove the first inclusion. The second one is similar. This is clear when $1 \leq n \leq k-1$ and $k+1 \leq n$. When $n = k$, since $R^{\pm 1} \equiv 1 \pmod{\hbar}$, we have

$$\begin{aligned} R^{(l)}_{k+1,k} \cdot x \cdot (R^{-1})^{(l)}_{k+1,k} &= (x_1 \otimes \cdots \otimes R_{21}(x_k \otimes 1)(R^{-1})_{21} \otimes 1^{\otimes l-k-1})\hbar^j \\ &\in (x_1 \otimes \cdots \otimes x_k \otimes 1^{\otimes l-k})\hbar^j + \prod_{i \geq j+1} (S^{\otimes l})_{\mathrm{supp} \leq k+1} \hbar^i \\ &\subset (x_1 \otimes \cdots \otimes x_k \otimes 1^{\otimes l-k})\hbar^j + \prod_{i \geq j+1} (S^{\otimes l})_{\mathrm{supp} \leq j+1} \hbar^i. \end{aligned}$$

This completes the proof. □

Corollary 8.2. *If $L \in SL^h_m(l)$, then we have*

$$J(L) \in 1 + (W \circ \mu^h_m)(L) + \bigoplus_{j=1}^{m} (S^{\otimes l})_{\mathrm{supp} \leq j} \hbar^j + \hbar^{m+1} U_{\hbar}^{\hat{\otimes} l}. \tag{8.4}$$

Proof. Recall from Lemma 2.1 that $L \in SL^h_m(l)$ is link-homotopic to $b = b^L_m b^L_{m+1} \cdots b^L_{l-1}$. By Proposition 6.2 we have

$$J(b) \equiv 1 + \left(\sum_{J \in \mathcal{I}_{m+1}} \mu_J(L) \cdot \varsigma^{(l)}_J \right) \hbar^m \equiv 1 + (W \circ \mu^h_m)(L) \pmod{\hbar^{m+1}}, \tag{8.5}$$

where the second equality follows from Eq. (5.1). Since $\prod_j (S^{\otimes l})_{\mathrm{supp} \leq j} \hbar^j$ is closed under multiplication, Corollary 8.1 and Lemma 8.5 imply that

$$\begin{aligned} J(L) &\in J(b) \cdot \left(1 + \prod_j (S^{\otimes l})_{\mathrm{supp} \leq j} \hbar^j \right) + \hbar^{m+1} U_{\hbar}^{\hat{\otimes} l} \\ &\subset (1 + (W \circ \mu^h_m)(L)) \cdot \left(1 + \prod_j (S^{\otimes l})_{\mathrm{supp} \leq j} \hbar^j \right) + \hbar^{m+1} U_{\hbar}^{\hat{\otimes} l} \\ &\subset 1 + (W \circ \mu^h_m)(L) + \bigoplus_{j=1}^{m} (S^{\otimes l})_{\mathrm{supp} \leq j} \hbar^j + \hbar^{m+1} U_{\hbar}^{\hat{\otimes} l}. \end{aligned}$$

This completes the proof. □

We can now prove Lemma 7.3 and Proposition 6.1.

Proof of Lemma 7.3. Let $L \in SL^h_m(l)$ and $1 \leq j \leq i-2 \leq m$. By Corollary 8.2, we have

$$\begin{aligned} \pi^h_{i,j}(J(L)) &\in \pi^h_{i,j} \left(1 + (W \circ \mu^h_m)(L) + \bigoplus_{k=1}^{m} (S^{\otimes l})_{\mathrm{supp} \leq k} \hbar^k + \hbar^{m+1} U_{\hbar}^{\hat{\otimes} l} \right) \\ &= \pi^h_{i,j} \left((W \circ \mu^h_m)(L) \right) + \pi^h_{i,j} \left(\bigoplus_{k=1}^{m} (S^{\otimes l})_{\mathrm{supp} \leq k} \hbar^k \right). \end{aligned}$$

But the right-hand side is equal to 0 since we have

$$(W \circ \mu_m^h)(L) \in \langle sl_2 \rangle_{m+1}^{(l)} \hbar^m,$$

and

$$\bigoplus_{k=1}^{m} (S^{\otimes l})_{\mathrm{supp} \leq k} \hbar^k \cap \langle sl_2 \rangle_i^{(l)} \hbar^j = \emptyset,$$

since for any monomial $x \in \langle sl_2 \rangle_i^{(l)}$ we have $\mathrm{supp}(x) = i \geq j + 2$.

This completes the proof. □

Proof of Proposition 6.1. Let $L, L' \in SL_m^h(l)$ be two link-homotopic string links. By Corollary 8.1 and Lemma 8.5, together with the fact that $\prod_j (S^{\otimes l})_{\mathrm{supp} \leq j} \hbar^j$ is closed under multiplication, we have

$$J(L) \in J(L') \cdot \left(1 + \prod_j (S^{\otimes l})_{\mathrm{supp} \leq j} \hbar^j \right) + \hbar^{m+1} U_\hbar^{\hat{\otimes} l}.$$

Then Corollary 8.2 implies that $J^h(L) \equiv J^h(L') \pmod{\hbar^{m+1}}$, as desired. □

In a similar spirit as Remark 7.1, we have the following.

Remark 8.1. By Corollary 8.2, the universal sl_2 invariant for a string link L in $SL_m^h(l)$ satisfies (8.4). So we have a variant of Theorem 5.2, using an alternative version of the reduction J^h of the universal sl_2 invariant J, by setting

$$\tilde{J}^h := \tilde{\pi}^h \circ J,$$

where $\tilde{\pi}^h$ denotes the quotient map as $\mathbb{Q}$-modules

$$\tilde{\pi}^h : (S^{\otimes l})[[\hbar]] \to \frac{(S^{\otimes l})[[\hbar]]}{\prod_j (S^{\otimes l})_{\mathrm{supp} \leq j} \hbar^j}.$$

Indeed, it follows immediately from Corollary 8.2 that if $L \in SL_m^h(l)$, then

$$\tilde{J}^h(L) \equiv (W \circ \mu_m^h)(L) \pmod{\hbar^{m+1}}.$$

In particular, we obtain the following.

Corollary 8.3. *Let L be a link-homotopically trivial l-component string link. Then we have*

$$J(L) \in 1 + \bigoplus_{j=1}^{l-1} (S^{\otimes l})_{\mathrm{supp} \leq j} \hbar^j + \hbar^l U_\hbar^{\hat{\otimes} l}.$$

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