Milnor invariants and the HOMFLYPT Polynomial

JEAN-BAPTISTE MEILHAN
AKIRA YASUHARA

We give formulas expressing Milnor invariants of an n–component link L in the 3–sphere in terms of the HOMFLYPT polynomial as follows. If the Milnor invariant $\overline{\mu}_L(J)$ vanishes for any sequence J with length at most k, then any Milnor $\overline{\mu}$–invariant $\overline{\mu}_L(I)$ with length between 3 and $2k+1$ can be represented as a combination of HOMFLYPT polynomial of knots obtained from the link by certain band sum operations. In particular, the "first nonvanishing" Milnor invariants can be always represented as such a linear combination.

57M25, 57M27

1 Introduction

J Milnor defined in [19; 20] a family of link invariants, known as *Milnor $\overline{\mu}$–invariants*. Here, and throughout the paper, by a link we mean an oriented, ordered link in S^3. Roughly speaking, Milnor invariants encode the behaviour of parallel copies of each link component in the lower central series of the link group. Given an n–component link L in S^3, Milnor invariants are specified by a sequence I of (possibly repeating) indices from $\{1,\dots,n\}$. The length of the sequence is called the *length* of the Milnor invariant. It is known that Milnor invariants of length two are just linking numbers. However in general, Milnor invariant $\overline{\mu}_L(I)$ is only well-defined modulo the greatest common divisor $\Delta_L(I)$ of all Milnor invariants $\overline{\mu}_L(J)$ such that J is obtained from I by removing at least one index and permuting the remaining indices cyclicly. This indeterminacy comes from the choice of the meridian curves generating the link group. Equivalently, it comes from the indeterminacy of representing the link as the closure of a string link; see Habegger and Lin [7]. See Section 4 for the definitions.

Recall that the HOMFLYPT polynomial of a knot K is of the form $P(K;t,z) = \sum_{k=0}^{N} P_{2k}(K;t)z^{2k}$, and denote by $P_0^{(l)}(K)$ the l–th derivative of $P_0(K;t)\in\mathbb{Z}[t^{\pm1}]$ evaluated at $t=1$. Denote by $(\log P_0(K))^{(n)}$ the n–th derivative of $\log P_0(K;t)$ evaluated at $t=1$. Since $P_0(K;1)=1$ and $P_0^{(1)}(K)=0$, we have

$$(\log P_0)^{(n)} = P_0^{(n)} + \sum_{k_1+\dots+k_m=n} n_{(k_1,\dots,k_m)} P_0^{(k_1)}\dots P_0^{(k_m)},$$

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where the sum runs over all $k_1, \dots, k_m$ such that $k_1+\cdots+k_m = n$ ($k_i \geq 2$), and where $n_{(k_1,\dots,k_m)} \in \mathbb{Z}$. For example, one can check that $(\log P_0)^{(n)} = P_0^{(n)}$ for $n = 1, 2, 3$, and that $(\log P_0)^{(4)} = P_0^{(4)} - 3(P_0^{(2)})^2$.

In this paper, we show the following. If Milnor invariant $\overline{\mu}_L(J)$ vanishes for any sequence J with length at most k, then any Milnor $\overline{\mu}$–invariant $\overline{\mu}_L(I)$ with length $m+1$ ($3 \leq m+1 \leq 2k+1$) is given by a linear combination of $(\log P_0)^{(m)}$ invariants of knots obtained from the link by certain band sum operations.

For simplicity, we first state the formula for Milnor link-homotopy invariants $\overline{\mu}(I)$, ie such that the sequence I has no repeated index. Let $L = \bigcup_{i=1}^n L_i$ be an n–component link in S^3. Let $I = i_1 i_2 \dots i_m$ be a sequence of m distinct elements of $\{1, \dots, n\}$. Let B_I be an oriented $(2m)$–gon, and denote by p_j ($j = 1, \dots, m$) a set of m nonadjacent edges of B_I according to the boundary orientation. Suppose that B_I is embedded in S^3 such that $B_I \cap L = \bigcup_{j=1}^m p_j$, and such that each p_j is contained in L_{i_j} with opposite orientation. We call such a disk an *I–fusion disk* for L. For any subsequence J of I, we define the oriented knot L_J as the closure of $\big((\bigcup_{i\in\{J\}} L_i) \cup \partial B_I\big) \setminus \big((\bigcup_{i\in\{J\}} L_i) \cap B_I\big)$, where $\{J\}$ is the subset of $\{1, \dots, n\}$ of all indices appearing in the sequence J.

Theorem 1.1 *Let L be an n–component link in S^3 ($n \geq 3$) with vanishing Milnor link-homotopy invariants of length up to k. Then for any sequence I of $(m+1)$ distinct elements of $\{1, \dots, n\}$ ($3 \leq m+1 \leq 2k+1$) and for any I–fusion disk for L, we have*

$$\overline{\mu}_L(I) \equiv \frac{-1}{m!2^m} \sum_{J<I} (-1)^{|J|} (\log P_0(L_J))^{(m)} \pmod{\Delta_L(I)},$$

where the sum runs over all subsequences J of I, and where $|J|$ denotes the length of the sequence J.

This generalizes widely a result of M Polyak for Milnor's triple linking number $\overline{\mu}(123)$ [22]. There are several other known results relating Milnor invariants of (string) links to the Alexander polynomial; for example see Cochran [3], Levine [13], Masbaum and Vaintrob [16], Meilhan [17], Murasugi [21] and Traldi [24; 25] (note in particular that the results of [13; 16; 17; 22] make use of closure-type operations). The relationship to quantum invariants is also known via the Kontsevich integral; see Habegger and Masbaum [8].

We emphasize that our assumption that the link has vanishing Milnor link-homotopy invariants of length up to k is essential in order to compute its Milnor invariants of length up to $2k+1$ using our formula. See the example at the end of this paper, which shows that Milnor link-homotopy invariants of length 4 are not given by the formula in Theorem 1.1 if there are nonvanishing linking numbers.

As noted above, only the first nonvanishing Milnor invariants of a link are well-defined integer-valued invariants. The following is in some sense a refinement of Theorem 1.1 for the first nonvanishing Milnor invariants.

Theorem 1.2 *Let L be an n–component link in S^3 $(n \geq 3)$ with vanishing Milnor link-homotopy invariants of length up to k (≥ 2). Then for any sequence I of $(k+1)$ distinct elements of $\{1,\dots,n\}$ and for any I–fusion disk for L, we have*

$$\overline{\mu}_L(I) = \frac{-1}{k!2^k} \sum_{J<I} (-1)^{|J|} P_0^{(k)}(L_J) \in \mathbb{Z}.$$

Theorem 1.2 implies that all Milnor link-homotopy invariants of a link L vanish if and only if all linking numbers of L vanish and $\sum_{J<I}(-1)^{|J|}P_0^{(k)}(L_J) = 0$ for all k $(2 \leq k \leq n-1)$ and for all nonrepeated sequences I of length $k+1$.

We remark that since the HOMFLYPT polynomial of knots is preserved by mutation, by Theorem 1.2, the first nonvanishing Milnor link-homotopy invariants are also preserved (cf Cha [2, Theorems 1.4 and 1.5]).

Theorem 1.1 and Theorem 1.2 generalize as follows. Let $L = \bigcup_{i=1}^n L_i$ be an n–component link in S^3, and let $I = i_1 i_2 \dots i_m$ be a sequence of m elements of $\{1,\dots,n\}$, where each element i appears exactly r_i times. Denote by $D_I(L)$ the m–component link obtained from L as follows.

- Replace each string L_i by r_i zero-framed parallel copies of it, labeled from $L_{(i,1)}$ to $L_{(i,r_i)}$. If $r_i = 0$ for some index i, simply delete L_i.
- Let $D_I(L) = L'_1 \cup \dots \cup L'_m$ be the m–string link $\bigcup_{i,j} L_{(i,j)}$ with the order induced by the lexicographic order of the index (i,j). This ordering defines a bijection $\varphi : \{(i,j) \mid 1 \leq i \leq n, 1 \leq j \leq r_i\} \to \{1,\dots,m\}$.

We also define a sequence $D(I)$ of elements of $\{1,\dots,m\}$ without repeated index as follows. First, consider a sequence of elements of $\{(i,j) \mid 1 \leq i \leq n, 1 \leq j \leq r_i\}$ by replacing each i in I with $(i,1),\dots,(i,r_i)$ in this order. For example if $I = 12231$, we obtain the sequence $(1,1),(2,1),(2,2),(3,1),(1,2)$. Next replace each term (i,j) of this sequence with $\varphi((i,j))$. Hence we have $D(12231) = 13452$.

Theorem 1.3 *Let L be an n–component link in S^3 with vanishing Milnor invariants of length up to k. Let I be a sequence of $(m+1)$ possibly repeating elements of $\{1,\dots,n\}$ $(3 \leq m+1 \leq 2k+1)$.*

(i) *For any $D(I)$–fusion disk for $D_I(L)$, we have*

$$\overline{\mu}_L(I) \equiv \frac{-1}{m!2^m} \sum_{J<D(I)} (-1)^{|J|} (\log P_0(D_I(L)_J))^{(m)} \pmod{\Delta_L(I)}.$$

(ii) *If $m = k$ (≥ 2), then for any $D(I)$–fusion disk for $D_I(L)$, we have*

$$\overline{\mu}_L(I) = \frac{-1}{k!2^k} \sum_{J<D(I)} (-1)^{|J|} P_0^{(k)}(D_I(L)_J) \in \mathbb{Z}.$$

Theorem 1.3 follows directly from Theorems 1.1 and 1.2, since Milnor proved in [20, Theorem 7] that $\overline{\mu}_{D_I(L)}(D(I)) = \overline{\mu}_L(I)$ (note that $\Delta_L(I) = \Delta_{D_I(L)}(D(I))$, again as a consequence of [20, Theorem 7]).

Theorem 1.3 implies the following corollary.

Corollary 1.4 *All Milnor invariants of a link L vanish if and only if all linking numbers of L are zero and $\sum_{J<D(I)} (-1)^{|J|} P_0^{(k)}(D_I(L)_J) = 0$ for all k (≥ 2) and for all sequences I with length $k+1$.*

The rest of the paper is organized as follows. In Section 2, we review some elements of the theory of claspers, which is the main tool in proving our main results. In Section 3, we recall some properties of the HOMFLYPT polynomial of knots. In Section 4, we review Milnor invariants and string links and give a few lemmas. Section 5 is devoted to the proof of Theorem 1.1. In Section 6, we prove Theorem 1.2 and show, as a consequence, how to use the HOMFLYPT polynomial to distinguish string links up to link-homotopy. The paper is concluded by a simple example which illustrates the necessity of the assumptions required in our results.

Convention 1.5 In this paper, given a sequence I of elements of $\{1,\dots,n\}$, the notation $J < I$ will be used for any subsequence J of I, possibly empty or equal to I itself. By $J \lneq I$, we mean any subsequence J of I that is not I itself. We will use the notation $\{I\}$ for the subset of $\{1,\dots,n\}$ formed by all indices appearing in the sequence I, and $|I|$ will denote the length of the sequence I.

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This work is dedicated to Professor Shin'ichi Suzuki on his 70th birthday.

2 Some elements of clasper theory

The primary tool in the proofs of our results is the theory of claspers. We recall here the main definitions and properties of this theory that will be useful in subsequent sections. For a general definition of claspers, we refer the reader to Habiro [9].

Definition 2.1 Let L be a (string) link. A surface G embedded in S^3 (or $D^2 \times [0, 1]$) is called a *graph clasper* for L if it satisfies the following three conditions:

(1) G is decomposed into disks and bands, called *edges*, each of which connects two distinct disks.

(2) The disks have either 1 or 3 incident edges, and are called *leaves* or *vertices* respectively.

(3) G intersects L transversely, and the intersections are contained in the union of the interiors of the leaves.

In particular, if a graph clasper G is a disk, we call it a *tree clasper*.

Throughout this paper, the drawing convention for claspers are those of [9, Figure 7], unless otherwise specified.

The degree of a connected graph clasper G is defined as half of the number of vertices and leaves. A tree clasper of degree k is called a C_k*–tree*. Note that a C_k–tree has exactly $(k+1)$ leaves.

A graph clasper for a (string) link L is *simple* if each of its leaves intersects L at exactly one point.

Let G be a simple graph clasper for an n–component (string) link L. The *index* of G is the collection of all integers i such that G intersects the i–th component of L. For example, if G intersects component 3 twice and components 2 and 5 once, and is disjoint from all other components of L, then its index is $\{2, 3, 5\}$.

Given a graph clasper G for a (string) link, there is a procedure to construct a framed link, in a regular neighbourhood of G. There is thus a notion of *surgery along* G, which is defined as surgery along the corresponding framed link. In particular, surgery along a simple C_k–tree is a local move as illustrated in Figure 1.

The C_k–equivalence is the equivalence relation on (string) links generated by surgeries along connected graph claspers of degree k and isotopies. Alternatively, the C_k–equivalence can be defined in term of "insertion" of elements of the k–th term of the lower central series of the pure braid group; see Stanford [23]. We use the notation $L \sim_{C_k} L'$ for C_k–equivalent (string) links L and L'.

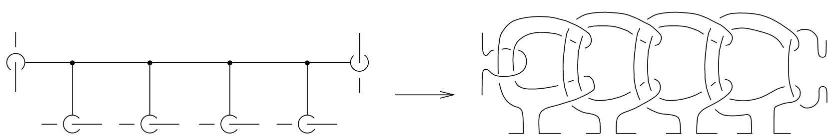


Figure 1. Surgery along a simple C_5–tree

It is known that the C_k–equivalence becomes finer as k increases, and that two links are C_k–equivalent if and only if they are related by surgery along simple C_k–trees [9]. Moreover, it was shown by Goussarov [6] and Habiro [9] that this equivalence relation characterizes the topological information carried by finite type knot invariants. More precisely, it is shown in [6; 9] that two knots are C_k–equivalent if and only if they cannot be distinguished by any finite type invariant of degree $< k$.

2.1 Linear trees and planarity

For $k \geq 3$, a C_k–tree G having the shape of the tree clasper in Figure 1 is called a *linear* C_k–tree. The left-most and right-most leaves of G in Figure 1 are called the *ends* of G.

Now suppose that the G is a linear C_k–tree for some knot K, and denote its ends by f and f'. Then the remaining $(k-1)$ leaves of G can be labelled from 1 to $(k-1)$, by travelling along the boundary of the disk[1] G from f to f' so that all leaves are visited. We say that G is *planar* if, when travelling along K from f to f', either following or against the orientation, the labels of the leaves met successively are strictly increasing. See Figure 2 for an example.

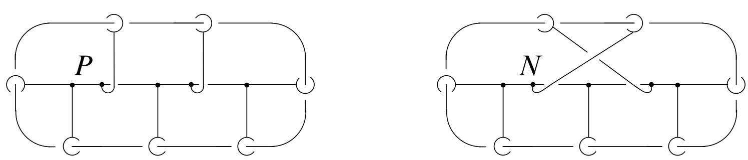


Figure 2. The C_6–tree P is planar for the unknot, while N is nonplanar.

2.2 Calculus of claspers for parallel claspers

We shall need refinements of [9, Propositions 4.4 and 4.6] for parallel tree claspers.

[1]Recall that a clasper is an embedded surface: in particular, since T is a tree clasper, the underlying surface is homeomorphic to a disk.

Here, by *parallel* tree claspers we mean a family of m parallel copies of a tree clasper T, for some $m \geq 1$. We call m the *multiplicity* of the parallel clasper. Note that there is no ambiguity in the notion of parallel copies here, since for a tree clasper the underlying surface is homeomorphic to a disk.

Lemma 2.2 *Let $m, k, k' \geq 1$ be integers. Let T be a parallel C_k–tree with multiplicity m for a (string) link L, and let T' be a $C_{k'}$–tree for L, disjoint from T.*

(1) *Let $\widetilde{T} \cup \widetilde{T}'$ be obtained from $T \cup T'$ by sliding a leaf f' of T' over m parallel leaves of T (see Figure 3(a)). Then $L_{T \cup T'}$ is ambient isotopic to $L_{\widetilde{T} \cup \widetilde{T}' \cup Y \cup C}$, where Y denotes the parallel $C_{k+k'}$–tree with multiplicity m obtained by inserting a vertex v in the edge e of T and connecting v to the edge incident to f' as shown in Figure 3(a), and where C is a disjoint union of $C_{k+k'+1}$–trees for L.*

(2) *Let $\widetilde{T} \cup \widetilde{T}'$ be obtained from $T \cup T'$ by passing an edge of T' across m parallel edges of T (see Figure 3(b)). Then $L_{T \cup T'}$ is ambient isotopic to $L_{\widetilde{T} \cup \widetilde{T}' \cup H \cup C}$, where H denotes the parallel $C_{k+k'+1}$–tree with multiplicity m obtained by inserting vertices in both edges, and connecting them by an edge as shown in Figure 3(b), and where C is a disjoint union of $C_{k+k'+2}$–trees for L.*

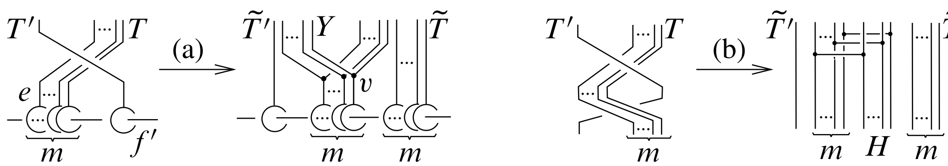


Figure 3. Leaf slide and crossing change involving parallel tree claspers

This result is well-known for $m = 1$. The general case is easily proved using the arguments of the proof of [9, Propositions 4.4 and 4.6], respectively.

Remark 2.3 Notice that, following the proofs of [9, Propositions 4.4 and 4.6], the index of each of the tree claspers involved in Lemma 2.2 can be determined from those of T and T' as follows. We have that the index of $\widetilde{T}$ is equal to the index of T, the index of $\widetilde{T}'$ is equal to the index of T', and the indices of Y, H and each connected component of C are equal to the union of the indices of T and T'.

3 The HOMFLYPT polynomial

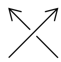
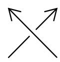
In this section, we recall the definition of the HOMFLYPT polynomial, and mention a few useful examples and properties.

The *HOMFLYPT polynomial* $P(L;t,z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$ of an oriented link L is defined by the formulas

(1) $P(U;t,z) = 1$,

(2) $t^{-1}P(L_+;t,z) - tP(L_-;t,z) = zP(L_0;t,z)$,

where U denotes the unknot and where L_+, L_- and L_0 are three links that are identical except in a 3–ball where they look as follows:

$L_+ =$ , $L_- =$ , $L_0 =$ .

In particular, the HOMFLYPT polynomial of an r–component link K is of the form

$$P(K;t,z) = \sum_{k=1}^{N} P_{2k-1-r}(K;t) z^{2k-1-r},$$

where $P_{2k-1-r}(K;t) \in \mathbb{Z}[t^{\pm 1}]$ is called the $(2k-1-r)$–th coefficient polynomial of K. Furthermore, the lowest degree coefficient polynomial of K is given by

$$P_{1-r}(K;t) = t^{2\,\mathrm{Lk}(L)}(t^{-1}-t)^{r-1} \prod_{i=1}^{r} P_0(K_i;t), \tag{3-1}$$

where K_i denotes the i–th component of K, and where $\mathrm{Lk}(L) := \sum_{i<j} \mathrm{lk}(L_i, L_j)$; see [14, Proposition 22].

Denote by $P_k^{(l)}(L)$ the l–th derivative of $P_k(L;t)$ evaluated at $t = 1$. It was proved by Kanenobu and Miyazawa that $P_k^{(l)}$ is a finite type invariant of degree $k+l$ [12]. In particular, $P_0^{(l)}$ is of degree l, and thus is an invariant of C_{l+1}–equivalence.

It is well-known that the HOMFLYPT polynomial of knots is multiplicative under connected sum. Thus the same holds for the lowest degree coefficient polynomial P_0, and in general, for any integer n and any two oriented knots K and K', we have

$$P_0^{(n)}(K \sharp K') = P_0^{(n)}(K) + P_0^{(n)}(K') + \sum_{k=1}^{n-1} \binom{n}{k} P_0^{(k)}(K) P_0^{(n-k)}(K').$$

If, moreover, we assume that the knot K is C_n–equivalent to the unknot, then we have

$$P_0^{(n)}(K \sharp K') = P_0^{(n)}(K) + P_0^{(n)}(K'), \tag{3-2}$$

since $P_0^{(k)}$ is an invariant of C_{k+1}–equivalence, for all k.

In general, a simple way to derive an additive knot invariant from the coefficient polynomial P_0 is to take its log. (Since $P_0(K;t)$ is in $\mathbb{Z}[t^{\pm 1}]$ and $P_0(K;1) = 1$ for any knot K, $\log P_0(K;t)$ can be regarded as a smooth function defined on an open interval which contains 1.) Indeed, we have that, for any two oriented knots K and K',

$$(\log P_0)(K \sharp K';t) = (\log P_0)(K;t) + (\log P_0)(K';t).$$

Denote by $(\log P_0(K))^{(n)}$ the n–th derivative of $\log P_0(K;t)$ evaluated at $t=1$. As mentioned in the introduction, $(\log P_0(K))^{(n)}$ is equal to $P_0(K)^{(n)}$ plus a sum of products of $P_0(K)^{(k)}$'s with $k<n$. So we see that $(\log P_0)^{(n)}$ is an additive finite type knot invariant of degree n, and thus is an invariant of C_{n+1}–equivalence.

The following simple example shall be useful later.

Lemma 3.1 *Let $n \geq 1$, and let $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}) \in \{-1,1\}^{n+2}$. Let K_n^ε be the knot represented in Figure 4. Then*

$$(\log P_0(K_n^\varepsilon))^{(n+1)} = P_0^{(n+1)}(K_n^\varepsilon) = (-1)^n 2^{n+1}(n+1)! \prod_{i=0}^{n+1} \varepsilon_i.$$

Notice that K_n^ε is C_{n+1}–equivalent to the unknot, and that for all $k \leq n$, we thus have $(\log P_0(K_n^\varepsilon))^{(k)} = P_0^{(k)}(K_n^\varepsilon) = 0$.

Proof Let us prove the second equality. We first prove the formula for the knot $K_n^+ := K_n^{(1,\dots,1)}$, by induction on n. Since K_1^+ is the trefoil, we have $P_0^{(2)}(K_1^+) = -8$. Suppose that $n>1$. Clearly, changing the crossing c of K_n^+ yields the unknot (see Figure 4). Hence

$$P_0(K_n^+;t) = t^2 + tP_{-1}(L_n;t),$$

where $L_n = K_1 \cup K_2$ is the 2–component link represented on the right-hand side of Figure 4. Notice that K_1 is an unknot, while K_2 is a copy of the knot K_{n-1}^+. Hence

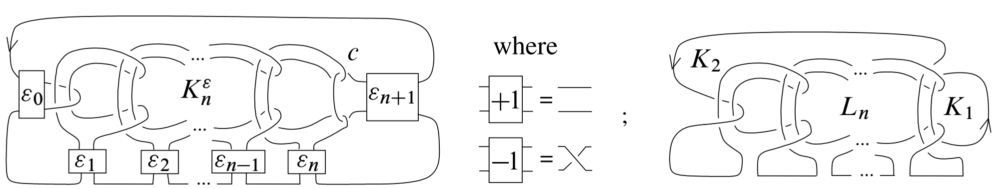


Figure 4. The knot K_n^ε and the 2–component link L_n

by Equation (3-1) we have

$$P_0(K_n^+;t) = t^2 + (1-t^2)P_0(K_{n-1}^+;t).$$

By differentiating this equation $(n+1)$ times and evaluating at $t=1$, we obtain

$$P_0^{(n+1)}(K_n^+) = -2(n+1)P_0^{(n)}(K_{n-1}^+) - n(n+1)P_0^{(n-1)}(K_{n-1}^+).$$

Since K_{n-1}^+ is C_n–equivalent to the unknot, we see that $P_0^{(n-1)}(K_{n-1}^+)=0$. Hence we have

$$P_0^{(n+1)}(K_n^+) = -2(n+1)P_0^{(n)}(K_{n-1}^+).$$

The induction hypothesis implies $P_0^{(n+1)}(K_n^+) = (-1)^n 2^{n+1}(n+1)!$.

Now, notice that in general K_n^ε is obtained from the unknot by surgery along the linear C_{n+1}–tree represented in Figure 5. It follows from [9, Claim on page 36] that

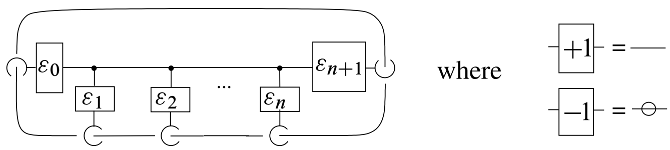


Figure 5. Here, a $\ominus$ on an edge represents a negative half-twist.

$K_n^\varepsilon \sharp K_n^+$ is C_{n+2}–equivalent to the unknot (resp. to $K_n^+ \sharp K_n^+$) if $\prod_{i=0}^{n+1}\varepsilon_i$ is equal to -1 (resp. to $=1$). Since $P_0^{(n+1)}$ is an invariant of C_{n+2}–equivalence and the knot K_n^ε is C_{n+1}–equivalent to the unknot for any $\varepsilon \in \{-1,1\}^{n+2}$, we have

$$\left(1+\prod_{i=0}^{n+1}\varepsilon_i\right)P_0^{(n+1)}(K_n^+) = P_0^{(n+1)}(K_n^\varepsilon \sharp K_n^+) = P_0^{(n+1)}(K_n^\varepsilon) + P_0^{(n+1)}(K_n^+)$$

Hence we have

$$P_0^{(n+1)}(K_n^\varepsilon) = P_0^{(n+1)}(K_n^+)\prod_{i=0}^{n+1}\varepsilon_i.$$

The second equality follows.

Recall that $(\log P_0)^{(n+1)}$ is given by the sum of $P_0^{(n+1)}$ and a combination of $P_0^{(k)}$'s with $k \le n$. Since the knot K_n^ε is C_{n+1}–equivalent to the unknot, the first equality follows. □

We note from the above proof that Lemma 3.1 gives the variation of $(\log P_0)^{(n+1)}$ and $P_0^{(n+1)}$ under surgery along a planar linear C_{n+1}–tree for the unknot. On the other hand, the HOMFLYPT polynomial does not change under surgery along a nonplanar tree clasper, as follows from a formula of Kanenobu [11].

Lemma 3.2 *Let T be a nonplanar linear tree clasper for a knot K. Then $P_0(K_T;t) = P_0(K;t)$.*

Proof We may assume that K_T and K are given by identical diagrams, except in a disk where they differ as illustrated in Figure 6. Let $L[\varepsilon_2, \dots, \varepsilon_n]$ $(\varepsilon_j \in \{-1, 1\},\ j = 2, \dots, n)$

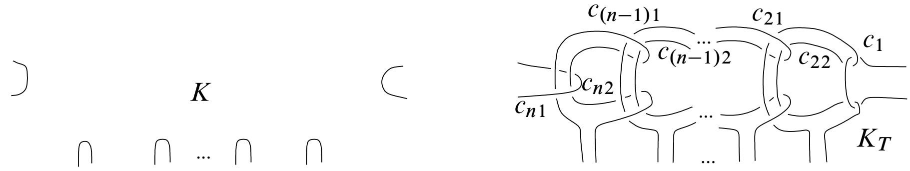


Figure 6. The knots K and K_T

be the link obtained from K_T by smoothing the crossing c_1, and

(i) smoothing the crossing c_{j1} if $\varepsilon_j = 1$, or

(ii) changing the crossing c_{j1} and smoothing the crossing c_{j2} if $\varepsilon_j = -1$.

Kanenobu showed that if all links $L[\varepsilon_2, \dots, \varepsilon_n]$ have less than $n+1$ components, then $P_0(K_T;t) = P_0(K;t)$; see [11, (3.9)]. Moreover, Kanenobu showed how to estimate the number of components of $L[\varepsilon_2, \dots, \varepsilon_n]$ using a kind of a chord diagram which corresponds to the smoothed crossings. More precisely, the chord diagram associated to $L[\varepsilon_2, \dots, \varepsilon_n]$ represents the n–singular knot obtained from K_T by changing the crossing c_1 and each crossing c_{j1} (resp. c_{j2}) such that $\varepsilon_j = 1$ (resp. $\varepsilon_j = -1$) into double points. Kanenobu showed that $L[\varepsilon_2, \dots, \varepsilon_n]$ has $n+1$ components if and only if the associated chord diagram contains no intersection among the chords; see the proof of [11, Lemma (3.7)]. If T is nonplanar, then it is not hard to see that, for any $(\varepsilon_j \in \{-1, 1\},\ j = 1, 2, \dots, n)$, the corresponding chord diagram contains such an intersection. Thus we have the conclusion. □

Remark 3.3 Lemma 3.1 and Lemma 3.2 are related to the main results of Kanenobu [11] and Horiuchi [10].

4 Milnor invariants

4.1 A short definition

Given an n–component link L in S^3, denote by π the fundamental group of $S^3 \setminus L$, and by π_q the q–th subgroup of the lower central series of π. We have a presentation of π/π_q with n generators, given by a choice of meridian m_i of the i–th component of L, $i = 1, \dots, n$. So the longitude λ_j of the j–th component of L $(1 \le j \le n)$ is expressed modulo π_q as a word in the m_i's (abusing notation, we still denote this word by λ_j). The *Magnus expansion* $E(\lambda_j)$ of λ_j is the formal power series in noncommuting variables $X_1, \dots, X_n$ obtained by substituting $1 + X_i$ for m_i and $1 - X_i + X_i^2 - X_i^3 + \cdots$ for m_i^{-1}, $1 \le i \le n$.

Let $I = i_1 i_2 \dots i_{k-1} j$ be a sequence of elements of $\{1, \dots, n\}$. Denote by $\mu_L(I)$ the coefficient of $X_{i_1} \cdots X_{i_{k-1}}$ in the Magnus expansion $E(\lambda_j)$. The *Milnor invariant* $\overline{\mu}_L(I)$ is the residue class of $\mu_L(I)$ modulo the greatest common divisor of all $\mu_L(J)$ such that J is obtained from I by removing at least one index and permuting the remaining indices cyclicly [19; 20]. The indeterminacy comes from the choice of the meridians m_i. Equivalently, it comes from the indeterminacy of representing the link as the closure of a string link [7]. Let us recall below the definition of these objects.

4.2 String links

Let $n \ge 1$, and let $D^2 \subset R^2$ be the unit disk equipped with n marked points $x_1, \dots, x_n$ in its interior, lying on the diameter on the x–axis of R^2. An *n–string link*, or n–component string link, is the image of a proper embedding $\bigsqcup_{i=1}^n [0,1]_i \to D^2 \times [0,1]$ of the disjoint union $\bigsqcup_{i=1}^n [0,1]_i$ of n copies of $[0,1]$ in $D^2 \times [0,1]$, such that for each i, the image of $[0,1]_i$ runs from $(x_i, 0)$ to $(x_i, 1)$. Each string of an n–string link is equipped with an (upward) orientation. The n–string link $\{x_1, \dots, x_n\} \times [0,1]$ in $D^2 \times [0,1]$ is called the *trivial n–string link* and is denoted by $\mathbf{1}_n$.

For each marked point $x_i \in D^2$, there is a point y_i on ∂D^2 in the upper half of R^2 such that the segment $p_i = x_i y_i$ is vertical to the x–axis, as illustrated in Figure 8. Given an n–string link $L = \bigcup_{i=1}^n L_i$ in $D^2 \times [0,1] \subset R^2 \times [0,1]$, the *closure* $\widehat{L}$ of L is the n–component link defined by $\widehat{L} = \bigcup_{i=1}^n \widehat{L}_i = L \cup (\bigcup_{i=1}^n (p_i \times \{0,1\}) \cup (y_i \times [0,1]))$.

The set of isotopy classes of n–string links fixing the endpoints has a monoid structure, with composition given by the *stacking product* and with the trivial n–string link $\mathbf{1}_n$ as unit element. Given two n–string links L and L', we denote their product by $L \cdot L'$, which is obtained by stacking L' *above* L and reparametrizing the ambient cylinder $D^2 \times [0,1]$.

Habegger and Lin showed that Milnor invariants are actually well defined integer-valued invariants of string links [7]. (We refer the reader to [7] or Yasuhara [26] for a precise definition of Milnor invariants $\mu(I)$ of string links.) Furthermore, Milnor invariants of length k are known to be finite-type invariants of degree $k-1$ for string links by Bar-Natan [1] and Lin [15]. As a consequence, Milnor invariants of length k for string links are invariants of C_k–equivalence. Habiro showed that the same actually holds for Milnor invariants of links [9].

4.3 Some results

It was shown by Habegger and Lin that Milnor invariants without repeated indices classify string links up to link-homotopy [7]. Here, the link-homotopy is the equivalence relation generated by self-crossing changes. In [27], the second author gave an explicit representative for the link-homotopy class of any n–string link in terms of linear tree claspers. We shall make use of this representative in this paper, and recall its definition below.

Let $\mathcal{J}_k$ denote the set of all sequences $j_0 j_1 \dots j_k$ of $k+1$ nonrepeating integers from $\{1,\dots,n\}$ such that $j_0 < j_m < j_k$ for all m. Let $i_0 i_1 \dots i_k$ be a sequence of $(k+1)$ integers from $\{1,\dots,n\}$ such that $i_0 < i_1 < \cdots < i_{k-1} < i_k$, and let a_J be a permutation of $\{i_1,\dots,i_{k-1}\}$. Then $J = i_0 a_J(i_1)\cdots a_J(i_{k-1}) i_k$ is in $\mathcal{J}_k$ (and all elements of $\mathcal{J}_k$ can be realized in this way). Let T_J be the simple linear C_k–tree for $\mathbf{1}_n$ as illustrated in Figure 7. Here, a_J is the unique positive k–braid which defines the permutation a_J and such that every pair of strings crosses at most once. In the figure, we also implicitly assume that all edges of T_J *overpass* all components of $\mathbf{1}_n$. (This assumption is crucial in the computation of Milnor invariants.) Let $\overline{T}_J$ be the

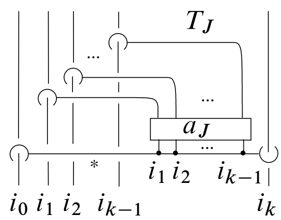


Figure 7. The C_k–trees T_J and $\overline{T}_J$

C_k–tree obtained from T_J by inserting a positive half-twist in the $*$–marked edge; see Figure 7. Denote respectively by V_J and V_J^{-1} the n–string links obtained from $\mathbf{1}_n$ by surgery along T_J and $\overline{T}_J$. This notation is justified by the fact that, for any J in $\mathcal{J}_k$, the string link $V_J \cdot V_J^{-1}$ is C_{k+1}–equivalent to the trivial one [9].

Theorem 4.1 [27] *Any n–string link L is link-homotopic to $l = l_1 \cdots l_{n-1}$, where*

$$l_i = \prod_{J\in\mathcal{J}_i} V_J^{x_J}, \text{ where } \quad x_J = \mu_{l_i}(J) = \begin{cases} \mu_L(J) & \text{if } i=1, \\ \mu_L(J) - \mu_{l_1\cdots l_{i-1}}(J) & \text{if } i \geq 2. \end{cases}$$

Remark 4.2 The above statement slightly differs from the one in [27]. There, another family of C_k–trees is used in place of T_J and $\overline{T}_J$. However, the present statement is shown by the exact same arguments as for [27, Theorem 4.3], since for any $J, J' \in \mathcal{J}_k$ ($k \geq 1$), we have

$$\mu_{V_J}(J') = \begin{cases} 1 & \text{if } J = J', \\ 0 & \text{otherwise} \end{cases} \tag{4-1}$$

(compare with [27, Lemma 4.1]).

For the representative $l = l_1 \cdots l_{n-1}$ above, we have the following lemmas.

Lemma 4.3 *Let $I < 12\dots n$ with $|I| = m \leq n$. Then,*

$$\mu_l(I) \equiv x_I \ (\text{mod } \gcd\{x_J \mid J \lneq I\}). \tag{4-2}$$

Moreover, for all $k < m-1$, we have

$$\mu_{l_1 \cdots l_k}(I) \equiv 0 \ (\text{mod } \gcd\{x_J \mid J \lneq I,\ |J| \leq k+1\}). \tag{4-3}$$

Proof By [18, Lemma 3.3] and Theorem 4.1, we have $\mu_l(I) = \mu_{l_1 \cdots l_{m-2}}(I) + \mu_{l_{m-1}}(I) = \mu_{l_1 \cdots l_{m-2}}(I) + x_I$. Hence (4-3) implies (4-2), and it suffices to prove Equation (4-3).

Note that, for an n–string link $L = \bigcup_{i=1}^n L_i$, we have $\mu_I(L) = \mu_I(\bigcup_{i \in \{I\}} L_i)$. Hence we may assume that $I = 12\dots m$ and that l is an m–string link.

The result is shown by an analysis of the Magnus expansion of a longitude of each "building block" $V_J^{x_J}$, for all $k < m-1$ and all $J \in \mathcal{J}_k$. Since we are aiming at computing Milnor invariant $\mu(I)$, we compute up to terms $O(I)$ involving monomials $X_{i_1} X_{i_2} \cdots X_{i_p}$ such that $i_1 i_2 \dots i_p$ is not a subsequence of $12\dots(m-1)$. Note that $O(I)$ includes any monomial where some variable appears at least twice, as well as any monomial involving X_m. For a subset $\{K\}$ of $\{I\}$, we will also use the notation $M(\{K\})$ for a sum of terms involving monomials such that all X_j ($j \in \{K\}$) appear exactly once in each monomial.

Let $J = j_0 j_1 \dots j_k \in \mathcal{J}_k$, for some $k < m-1$. Let j be an index in I, and denote by λ_j the j–th longitude of $V_J^{\pm 1}$. Notice that all monomials appearing in the Magnus expansion $E(\lambda_j)$ are in the variables X_i such that $i \in \{J\}$, since all edges of both T_J and $\overline{T}_J$ overpass all components of $\mathbf{1}_n$. There are three cases:

(i) If $j < j_0$ or $j_k < j$, then clearly we have $E(\lambda_j) = 1$.

(ii) If $j \in \{J\}$, since all Milnor invariants of $V_J^{\pm 1}$ with length at most k vanish,

$$E(\lambda_j) = 1 + M(\{J\} \setminus \{j\}) + M(\{J\}) + O(I).$$

(iii) If $j \notin \{J\}$ and $j_0 < j < j_k$, then since $V_J^{\pm 1} \setminus (i\text{th component})$ is trivial for any $i \in \{J\}$, all (nontrivial) monomials appearing in $E(\lambda_j)$ contain all variables X_i $(i \in \{J\})$. Hence we have a Magnus expansion of the form

$$E(\lambda_j) = 1 + M(\{J\}) + O(I).$$

Summarizing all three cases, we have

$$E(\lambda_j) = \begin{cases} 1 & \text{if } j < j_0 \text{ or } j_k < j, \\ 1 + M(\{J\} \setminus \{j\}) + M(\{J\}) + O(I) & \text{if } j \in \{J\}, \\ 1 + M(\{J\}) + O(I) & \text{if } j \notin \{J\} \text{ and } j_0 < j < j_k. \end{cases} \tag{4-4}$$

Now let us consider the stacking product $V_J^{\pm 1} \cdot V_{\tilde{J}}^{\pm 1}$. The Magnus expansion of the j–th longitude of $V_J^{\pm 1} \cdot V_J^{\pm 1}$ is given by $E(\lambda_j)E(\widetilde{\lambda_j})$, where λ_j is the j–th longitude of $V_J^{\pm 1}$ and $E(\widetilde{\lambda_j})$ is obtained from $E(\lambda_j)$ by replacing X_i with $E(\lambda_i)^{-1} X_i E(\lambda_i)$ for each $i \in \{I\}$. By (4-4), we have

$$E(\lambda_i)^{-1} X_i E(\lambda_i) = \begin{cases} X_i & \text{if } i < j_0 \text{ or } j_k < i, \\ X_i + M(\{J\}) + O(I) & \text{if } i \in \{J\}, \\ X_i + M(\{J\} \cup \{i\}) + O(I) & \text{if } i \notin \{J\} \text{ and } j_0 < i < j_k. \end{cases}$$

This implies that the Magnus expansion of the j–th longitude of $V_J^{\pm 1} \cdot V_J^{\pm 1}$ is given by

$$E(\lambda_j)E(\widetilde{\lambda_j}) = \begin{cases} 1 & \text{if } j < j_0 \text{ or } j_k < j, \\ E(\lambda_j)E(\lambda_j) + M(\{J\}) + O(I) & \text{if } j \in \{J\}, \\ 1 + M(\{J\}) + O(I) & \text{if } j \notin \{J\} \text{ and } j_0 < j < j_k. \end{cases}$$

Generalizing this argument, we obtain that the Magnus expansion of the j–th longitude $\lambda_{J,j}$ in $V_J^{x_J}$ is given by

$$E(\lambda_{J,j}) = \begin{cases} 1 & \text{if } j < j_0 \text{ or } j_k < j, \\ E(\lambda_j)^{|x_J|} + M(\{J\}) + O(I) & \text{if } j \in \{J\}, \\ 1 + M(\{J\}) + O(I) & \text{if } j \notin \{J\} \text{ and } j_0 < j < j_k. \end{cases} \tag{4-5}$$

Let us now focus on the case $j = m$. There are two cases to consider.

- If m is not in $\{J\}$, then necessarily $j_k < m$ and by (4-5) we have that the Magnus expansion of the m–th longitude $\lambda_{J,m}$ in $V_J^{x_J}$ is 1.
- If m is in $\{J\}$, then necessarily $j_k = m$. Since j_0 is the smallest integer in $\{J\}$, any monomial in $M(\{J\} \setminus \{m\})$ whose leftmost variable is not X_{j_0} belongs to $O(I)$. Moreover, each term in $M(\{J\})$ involves the variable X_m, and hence belongs to $O(I)$. So (4-1) and (4-4) show that the Magnus expansion of the m–th longitude of $V_J^{\pm 1}$ is $E(\lambda_m) = 1 \pm X_{j_0} \cdots X_{j_{k-1}} + O(I)$. Equation (4-5) then implies that the Magnus expansion of the m–th longitude $\lambda_{J,m}$ in $V_J^{x_J}$ is given by $1 + x_J X_{j_0} \cdots X_{j_{k-1}} + O(I)$.

Summarizing, for $j = m$ we obtain that

$$E(\lambda_{J,m}) = \begin{cases} 1 & \text{if } m \notin \{J\}, \\ 1 + O(I) \pmod{x_J} & \text{if } m \in \{J\} \text{ and } J < I, \\ 1 + O(I) & \text{otherwise.} \end{cases}$$

We can now complete the computation of $\mu_{l_1 \cdots l_k}(I)$. Since the Magnus expansion of the m–th longitude of $l_1 \cdots l_k$ is obtained from a product of $E(\lambda_{J,m})$'s ($\{J\} \subset \{I\}$, $J \in \bigcup_{s=1}^{k} \mathcal{J}_s$) by replacing each variable X_i with $X_i +$ (monomials involving X_i), it is of the form

$$1 + O(I) \pmod{\gcd\{x_J \mid J \lneq I,\ |J| \le k+1\}}.$$

This implies that $\mu_{l_1 \cdots l_k}(I) \equiv 0 \pmod{\gcd\{x_J \mid J \lneq I,\ |J| \le k+1\}}$, as desired. □

Lemma 4.4 *Let $I < 12 \dots n$ with $|I| = m \le n$. Then*

$$\Delta_l(I) = \gcd\{x_J \mid J \lneq I\}.$$

Proof The proof is by induction on m. For $m = 3$, the result is clear since $x_{ij} = \mu_{l_1}(ij) = \mu_l(ij)$ for any i, j. Now, let $m \ge 4$. It will be convenient to use the notation $\delta_k(I)$ for the set of all sequences of length $(m-k)$ obtained from I by removing k indices and permuting cyclicly. By definition,

$$\Delta_l(I) = \gcd(\{\mu_l(J) \mid J \in \delta_k(I),\ k > 1\} \cup \{\mu_l(J) \mid J \in \delta_1(I)\}).$$

By the induction hypothesis, we have that

$$\begin{aligned} \gcd\{\mu_l(J) \mid J \in \delta_k(I),\ k > 1\} &= \gcd\{\Delta_l(J) \mid J \in \delta_1(I)\} \\ &= \gcd\{\Delta_l(J) \mid J \in \delta_1(I),\ J < I\} \\ &= \gcd\{x_{J'} \mid J' \lneq J,\ J \in \delta_1(I),\ J < I\} \\ &= \gcd\{x_{J'} \mid J' < I,\ |J'| < m-1\}. \end{aligned}$$

On the other hand, by Lemma 4.3, for all $J \in \delta_1(I)$ $(J < I)$ and for any sequence $\tau(J)$ obtained from J by permuting cyclicly, we have

$$\mu_l(\tau(J)) \equiv \mu_l(J) \equiv x_J \pmod{\gcd\{x_{J'} \mid J' \lneq J\}\ (= \Delta_l(J))}.$$

It follows that $\Delta_l(I) = \gcd\{x_J \mid J < I,\ |J| \le m-1\}$, as desired. □

5 Proof of Theorem 1.1

Let $L = \bigcup_{i=1}^{n} L_i$ be an n–component link in S^3, and let I be a sequence of $(m+1)$ distinct elements of $\{1, \dots, n\}$. It is sufficient to consider here the case $m + 1 = n$, since, if $m+1<n$, we have that $\overline{\mu}_L(I)=\overline{\mu}_{\bigcup_{i\in\{I\}}L_i}(I)$. We may further assume that $I=12\dots n$ without loss of generality. Indeed, for any permutation I' of $12\dots n$, we have that $\overline{\mu}_L(I')=\overline{\mu}_{L'}(12\dots n)$, where L' is obtained from L by reordering the components appropriately.

We first show how to reformulate the problem in terms of string links.

5.1 Closing string links into knots

Let B_I be an I–fusion disk for L, as defined in the introduction. Up to isotopy, we may assume that the $2n$–gon B_I lies in the unit disk D^2 as shown in Figure 8, where the edges p_j are defined by $p_j=x_j y_j$, $1\le j\le n$. We may furthermore assume that $L\cup B_I$ lies in the cylinder $D^2\times[0,1]$, such that $B_I\subset(D^2\times\{0\})$, and such that

$$L\cap\partial(D^2\times[0,1])=\bigcup_{j=1}^{n}\big((p_j\times\{0\})\cup(\{y_j\}\times[0,1])\cup(p_j\times\{1\})\big).$$

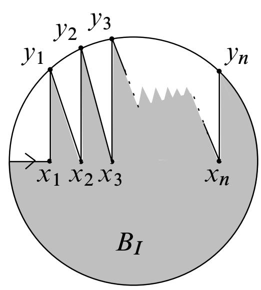


Figure 8. The $2n$–gon B_I lying in the unit disk D^2

In this way, we obtain an n–string link σ whose closure $\hat{\sigma}$ is the link L, by setting

$$\sigma:=\overline{L\setminus(L\cap\partial(D^2\times[0,1]))}. \tag{5-1}$$

For example, Figure 9 represents a 3–string link β whose closure is the Borromean rings (there, the dotted part represents the intersection of the Borromean rings and $\partial(D^2\times[0,1])$).

Given an n–string link $K=\bigcup_{i=1}^{n}K_i$ and any subsequence J of $I=12\dots n$, we will denote by $K(J)$ the knot

$$K(J):=\overline{\Big(\Big(\bigcup_{j\in\{J\}}\hat{K}_j\Big)\cup\partial B_I\Big)\setminus\Big(\Big(\bigcup_{j\in\{J\}}\hat{K}_j\Big)\cap B_I\Big)}.$$

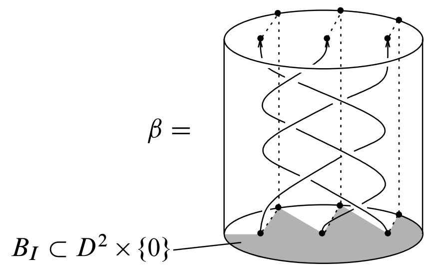


Figure 9. The string link β, whose closure is the Borromean rings

For example, Figure 10 represents the knots $\beta(13)$ and $\beta(123)$ obtained from the 3–string link β shown in Figure 9.

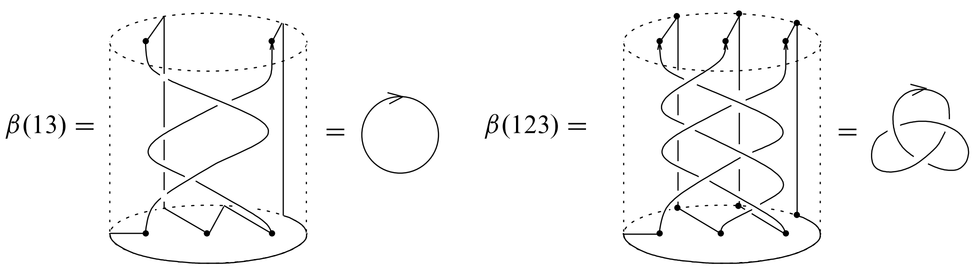


Figure 10. The knots $\beta(13)$ and $\beta(123)$ for the 3–string link β

Note that $K(J)$ coincides with the knot $(\widehat{K})_J$ defined in the introduction for the choice of I–fusion disk B_I specified above.

Recall from Section 4.3 that, for any k and any $J \in \mathcal{J}_k$, V_J (resp. V_J^{-1}) denotes the n–string link obtained from $\mathbf{1}_n$ by surgery along the C_k–tree T_J (resp. $\overline{T}_J$); see Figure 7. Denote by t_J (resp. $\overline{t}_J$) the image of the C_k–tree T_J (resp. $\overline{T}_J$) for $\mathbf{1}_n$ under taking the closure $V_J^{\pm 1}(I)$. We observe that t_J (resp. $\overline{t}_J$) is a planar tree clasper for the unknot if and only if $J < I$. In this case, note that $V_J(I)$ (resp. $V_J^{-1}(I)$) is the knot K_{k-1}^{ε} of Figure 4 with $\varepsilon = (-,+,\dots,+)$ (resp. for $\varepsilon = (+,+,\dots,+)$). In particular, observe that $V_I^{x_I}(J)$ is the unknot for all $J \lneq I$ and that, by Lemma 2.2 (for $m=1$), the knot $V_I^{x_I}(I)$ is C_n–equivalent to the connected sum of $|x_I|$ copies of $V_I^{\varepsilon_I}(I)$, where ε_I denotes the sign of x_I. By Lemmas 3.1 and 3.2 we thus have, for all $J \in \mathcal{J}_{n-1}$,

$$(\log P_0(V_J^{x_J}(I)))^{(n-1)} = P_0^{(n-1)}(V_J^{x_J}(I)) = \begin{cases} (-1)^{n-1} x_I (n-1)! 2^{n-1} & \text{if } J = I, \\ 0 & \text{otherwise.} \end{cases} \tag{5-2}$$

5.2 Proof of Theorem 1.1

Let σ be the n–string link with closure L defined in Section 5.1. By Theorem 4.1, σ is link-homotopic to $l_k \cdots l_{n-1}$, where $l_i = \prod_{J \in \mathcal{J}_i} V_J^{x_J}$ is defined in Section 4.3, and with $n \leq 2k+1$ (by our vanishing assumption on Milnor invariants). Hence σ is obtained from $l_k \cdots l_{n-1}$ by surgery along a disjoint union R_1 of simple C_1–trees whose leaves intersect a single component of $l_k \cdots l_{n-1}$.

By Lemma 2.2, for all $J < I$, we have that

$$\sigma(J) \sim_{C_n} l_{n-1}(J) \sharp (l_k \cdots l_{n-2})_{R_1}(J).$$

Since $(\log P_0)^{(n-1)}$ is an invariant of C_n–equivalence for all n, it follows from the additivity property of $(\log P_0)$ that

$$\big(\log P_0(\sigma(J))\big)^{(n-1)} = \big(\log P_0(l_{n-1}(J))\big)^{(n-1)} + \big(\log P_0\left((l_k \cdots l_{n-2})_{R_1}(J)\right)\big)^{(n-1)}.$$

The proof of the next lemma is postponed to Section 5.3.

Lemma 5.1

$$\frac{-1}{(n-1)!2^{n-1}} \sum_{J<I} (-1)^{|J|} \big(\log P_0\left((l_k \cdots l_{n-2})_{R_1}(J)\right)\big)^{(n-1)} \equiv 0 \pmod{\Delta_L(I)}.$$

It follows from Equation (5-2) that

$$\begin{aligned}
&\frac{-1}{(n-1)!2^{n-1}} \sum_{J<I} (-1)^{|J|} \big(\log P_0(\sigma(J))\big)^{(n-1)} \\
&\qquad \equiv \frac{-1}{(n-1)!2^{n-1}} \sum_{J<I} (-1)^{|J|} \big(\log P_0(l_{n-1}(J))\big)^{(n-1)} \pmod{\Delta_L(I)} \\
&\qquad \equiv \frac{(-1)^{n-1}}{(n-1)!2^{n-1}} \big(\log P_0(V_I^{x_I}(I))\big)^{(n-1)} \pmod{\Delta_L(I)} \\
&\qquad \equiv x_I \pmod{\Delta_L(I)}.
\end{aligned}$$

On the other hand, by Lemma 4.3 and Lemma 4.4, we have

$$\mu_L(I) = \mu_\sigma(I) = \mu_{l_k \cdots l_{n-1}}(I) \equiv x_I \pmod{\Delta_L(I)},$$

which completes the proof.

5.3 Proof of Lemma 5.1

First, it is convenient to slightly modify the string link $(l_k \cdots l_{n-2})_{R_1}$. For that purpose, we regard it as obtained from $\mathbf{1}_n$ by surgery along the disjoint union of tree claspers $G \cup R_1$, with

$$G := \bigcup_{i=k}^{n-2} \Big(\bigcup_{J \in \mathcal{J}_i} T_J^{x_J} \Big),$$

where $T_J^{x_J}$ denotes $|x_J|$ parallel copies of T_J (resp. $\overline{T}_J$) if $x_J > 0$ (resp. if $x_J < 0$).

A tree clasper for $\mathbf{1}_n$ is said to be *in good position* if, in the usual diagram of $\mathbf{1}_n$, each component of $\mathbf{1}_n$ underpasses all edges of the tree clasper. For example, each component of G is in good position (see Figure 7), whereas the components of R_1 may not be. However, by repeated applications of [9, Proposition 4.5] we have

$$(\mathbf{1}_n)_{G \cup R_1} \sim_{C_n} (\mathbf{1}_n)_{G \cup \widetilde{R}},$$

where $\widetilde{R}$ is a disjoint union, disjoint from G, of simple tree claspers for $\mathbf{1}_n$ in good position and intersecting some component of $\mathbf{1}_n$ more than once.

We now close the string link $(\mathbf{1}_n)_{G \cup \widetilde{R}}$ using the sequence $I = 12 \dots n$, as explained in Section 5.1. It follows from Lemma 3.2 that, for all $J < I$, we have

$$\big(\log P_0((\mathbf{1}_n)_{G \cup \widetilde{R}}(J))\big)^{(n-1)} = \big(\log P_0((\mathbf{1}_n)_{\widetilde{G} \cup \widetilde{R}}(J))\big)^{(n-1)},$$

where

$$\widetilde{G} := \bigcup_{i=k}^{n-2} \Big(\bigcup_{J \in \mathcal{J}_i \,;\, J < I} T_J^{x_J} \Big).$$

In other words, we only need to consider those tree claspers T_J and $\overline{T}_J$ with $J < I$, since only those become planar under closure. Moreover, since $\Delta_L(I)$ divides all x_J with $J < I$, we can express each $T_J^{x_J}$ as a disjoint union of parallel tree claspers with multiplicity $\Delta_L(I)$.[2] The knot $(\mathbf{1}_n)_{\widetilde{G} \cup \widetilde{R}}(I)$ is obtained from the unknot U by surgery along a disjoint union of tree claspers

$$F := t \cup r, \quad t := \bigcup_{i=1}^{q} t_i, \quad r := \bigcup_{j=1}^{p} r_j,$$

for some integers q, p, where r is the image of $\widetilde{R}$ under closure and where, for each $i = 1, \dots, p$, the clasper t_i is a parallel family of $\Delta_L(I)$ copies of (the image under closure of) some C_m–tree T_J or $\overline{T}_J$ with $J < I$ $(k \le m \le n-1)$.

[2] Recall from Section 2.2 that a parallel tree clasper with multiplicity $k \ge 1$ is a family of k parallel copies of a tree clasper.

We need the following additional definition. A disjoint union $C_1 \cup \cdots \cup C_s$ of $s \geq 1$ (possibly parallel) tree claspers of degree $< n$ for U is called *balanced* if each tree C_j is being assigned a subset $w(C_j)$ of $\{1, \ldots, n\}$, called *weight*, such that

$$(\mathbf{1}_n)_{\widetilde{G} \cup \widetilde{R}}(J) \sim_{C_n} U_{(\bigcup_{w(C_i) \subset \{J\}} C_i)}, \tag{5-3}$$

for all $J < I$. For example, F is balanced if we assign the index of each tree as weight. We say that a C_k–tree in a balanced family is *repeated* if its weight has at most k elements, that is, if its weight is smaller than the number of leaves. For instance, all tree claspers r_j $(j = 1, \ldots, p)$ are repeated.

Now, *up to C_n–equivalence*, we deform U_F into a connected sum of knots obtained from U by surgery along a single (possibly parallel) tree clasper. In other words, we will deform F into a balanced union of *localized* tree claspers for U, ie tree claspers sitting in a 3–ball that intersects U at a single strand and is disjoint from all other tree claspers. Since U is the unknot, this deformation can be achieved, starting from F, by a sequence of only leaf slides and edge crossing changes; see Figure 3. By Lemma 2.2, performing such operations may introduce additional tree claspers up to C_n–equivalence. However, the following is easily verified.

Fact 5.2 Let $M \cup T \cup T'$ be a balanced union of tree claspers for U, where $T \cup T'$ is as in Lemma 2.2. If, in the statement of Lemma 2.2(1) (resp. of Lemma 2.2(2)), we assign the weights $w(T)$ and $w(T')$ to $\widetilde{T}$ and $\widetilde{T}'$ respectively, and the weight $w(T) \cup w(T')$ to Y (resp. H) and each connected component of C, then $M \cup \widetilde{T} \cup \widetilde{T}' \cup Y \cup C$ (resp. $M \cup \widetilde{T} \cup \widetilde{T}' \cup H \cup C$) is balanced. In particular, if the degrees of T and T' are at least $(n-1)/2$, where n is the number of strands of L, then all tree claspers in C are repeated.

We now start our localization process, which goes in three steps.

The first step consists in localizing all parallel tree claspers t_i. Consider, say, the parallel C_{k_1}–tree t_1. Then by assumption we have that $k_1 \geq (n-1)/2$, and Lemma 2.2 and Fact 5.2 imply that $U_F \sim_{C_n} U_{t_1} \sharp U_{(F \setminus t_1) \cup F_1}$, where F_1 is a disjoint union of tree claspers of degree $> k_1$, which are either repeated trees or parallel tree claspers with multiplicity $\Delta(I)$. Using this argument repeatedly, we see that F can be deformed into a balanced union of tree claspers $F' = h_1 \cup \cdots \cup h_l \cup r'$, for some integer l, where r' is a disjoint union of repeated trees and each h_i is a parallel tree clasper with multiplicity $\Delta_L(I)$, such that

$$(\mathbf{1}_n)_{\widetilde{G} \cup \widetilde{R}}(I) = U_F \sim_{C_n} U_{F'} = U_{h_1} \sharp \cdots \sharp U_{h_l} \sharp U_{r'}.$$

In the second step, we "split" each parallel tree clasper into $\Delta_L(I)$ localized ones. Indeed, since each h_i is a parallel family of $\Delta_L(I)$ copies of some tree clasper h'_i, we can apply Lemma 2.2 (with $m=1$) and Fact 5.2 to deform F' into a balanced family

$$F'' = \bigcup_{i=1}^{l} (\underbrace{h'_i \cup \cdots \cup h'_i}_{\Delta_L(I) \text{ times}}) \cup r'',$$

where for each i the tree clasper h'_i has weight $w(h'_i) = w(h_i)$ and where r'' is a disjoint union of repeated trees, such that

$$U_{F'} \sim_{C_n} U_{F''} = \left(\Delta_L(I) \times U_{h'_1}\right) \sharp \cdots \sharp \left(\Delta_L(I) \times U_{h'_l}\right) \sharp\, U_{r''}.$$

(Here $\Delta_L(I) \times U_{h'_i}$ denotes the connected sum of $\Delta_L(I)$ copies of $U_{h'_i}$, $(i = 1, \ldots, l)$.

In the third and last step, we localize all repeated trees in r''. Note that, by Fact 5.2, performing a leaf slide or an edge crossing change between two repeated tree claspers only introduces new tree claspers that are also repeated. Hence F'' can be deformed into a balanced disjoint union of tree claspers

$$X = \bigcup_{i=1}^{l} (\underbrace{h'_i \cup \cdots \cup h'_i}_{\Delta_L(I) \text{ times}}) \cup \bigcup_{j=1}^{l'} x_j,$$

for some integer l', where each x_j is a repeated tree clasper, such that

(5-4) $$U_{F''} \sim_{C_n} U_X = \left(\Delta_L(I) \times U_{h'_1}\right) \sharp \cdots \sharp \left(\Delta_L(I) \times U_{h'_l}\right) \sharp\, U_{x_1} \sharp \cdots \sharp\, U_{x_{l'}}.$$

This concludes the localization process.

Now, since X is balanced, and since $\log P_0$ is additive under connected sum (see Section 3), for any $J < I$ we have

$$(\log P_0((\mathbf{1}_n)_{\tilde{G} \cup \tilde{R}}(J)))^{(n-1)} = \Delta_L(I) \sum_{w(h'_i) \subset \{J\}} (\log P_0(U_{h'_i}))^{(n-1)} + \sum_{w(x_j) \subset \{J\}} (\log P_0(U_{x_j}))^{(n-1)},$$

where the first (resp. second) sum is over all tree claspers h'_i (resp. x_j) whose weight is contained in $\{J\}$. On the other hand, we have the following:

Claim 5.3 Let g be a connected component of X.

(1) If $|w(g)| < n (= |I|)$, then

$$\sum_{J < I, w(g) \subset \{J\}} (-1)^{|J|} (\log P_0(U_g))^{(n-1)} = 0.$$

(2) If $|w(g)| = n$ (ie $g = h'_i$ for some i and g is a C_{n-1}–tree), then

$$(\log P_0(U_g))^{(n-1)} \equiv 0 \pmod{(n-1)!2^{n-1}}.$$

Note that, since any connected component g of X has degree $< n$, we have that $|w(g)| < n$ if g is repeated. Hence it follows from Claim 5.3 that

$$\frac{-1}{(n-1)!2^{n-1}} \sum_{J<I} (-1)^{|J|} (\log P_0((\mathbf{1}_n)_{\tilde{G}\cup\tilde{R}}(J)))^{(n-1)} \equiv 0 \pmod{\Delta_L(I)},$$

which concludes the proof of Lemma 5.1.

Proof of Claim 5.3 (1) Since $w(g) < n$, there is an element $a \in \{I\}$ such that $a \notin w(g)$. We denote by $I \setminus a$ the sequence obtained from I by deleting a. Then we have that

$$\sum_{J<I, w(g)\subset\{J\}} (-1)^{|J|} = \sum_{J<I\setminus a, w(g)\subset\{J\}} (-1)^{|J|} + \sum_{J<I\setminus a, w(g)\subset\{J\}\cup\{a\}} (-1)^{|J|+1} = 0,$$

which implies the desired equality.

(2) Using the AS and IHX relations for tree claspers (see [6; 9]), one can check that the knot U_g is C_n–equivalent to a connected sum of knots U_{g_i}, where each g_i is a linear C_{n-1}–tree which is either nonplanar or of the form shown in Figure 5. Since $(\log P_0)^{(n-1)}$ is an invariant of C_n–equivalence, the result then follows from Lemmas 3.1 and 3.2. □

6 First nonvanishing Milnor invariants and link-homotopy of string links

We begin this section by proving Theorem 1.2. Most of the arguments follow very closely the proof of Theorem 1.1, and we therefore freely use the notions and results of the previous section.

6.1 Proof of Theorem 1.2

Let $L = \bigcup_{i=1}^n L_i$ be an n–component link in S^3 with vanishing Milnor link-homotopy invariants of length up to k $(3 \le k+1 \le n)$. Let I be a sequence of $(k+1)$ distinct elements of $\{1,\dots,n\}$. As in Section 5, we may assume without loss of generality that $k+1 = n$ and that $I = 12\dots n$. Following Section 5.1, we may also assume that the $2n$–gon B_I is chosen so that $L \cup B_I$ lies in the cylinder $D^2 \times [0,1]$, such that $B_I \subset (D^2 \times \{0\})$ is as shown in Figure 8. Hence (5-1) defines an n–string link σ whose closure is L.

By Theorem 4.1, the n–string link σ is link-homotopic to $l_{n-1} = \prod_{J \in \mathcal{J}_{n-1}} V_J^{x_J}$ defined in Section 4.3. By applying the exact same arguments as in Section 5.3, there exists a disjoint union of tree claspers $R = r_1 \cup \cdots \cup r_p$, with each being assigned a weight $w(r_i) \subset \{1, \dots, n\}$, such that

- for each i, we have $|w(r_i)| \leq \deg(r_i)$,
- $L_J \sim_{C_n} l_{n-1}(J) \sharp U_{R_J}$ for all $J < I$, where $R_J = \bigcup_{w(r_i) \subset \{J\}} r_i$. (In particular, $R_I = R$ and L_I is C_n–equivalent to $l_{n-1}(I) \sharp U_R$.)

Since l_{n-1} is C_{n-1}–equivalent to $\mathbf{1}_n$, we have by Equation (3-2) that, for all $J < I$,

$$P_0^{(n-1)}(L_J) = P_0^{(n-1)}(l_{n-1}(J)) + P_0^{(n-1)}(U_{R_J}).$$

The following lemma is proved below.

Lemma 6.1 $\displaystyle\sum_{J<I} (-1)^{|J|} P_0^{(n-1)}(U_{R_J}) = 0.$

This lemma and Equation (5-2) imply that

$$\begin{aligned}\frac{-1}{(n-1)!2^{n-1}} \sum_{J<I} (-1)^{|J|} P_0^{(n-1)}(L_J) &= \frac{-1}{(n-1)!2^{n-1}} \sum_{J<I} (-1)^{|J|} P_0^{(n-1)}(l_{n-1}(J)) \\ &= \frac{-1}{(n-1)!2^{n-1}} \sum_{J<I} (-1)^{|J|} P_0^{(n-1)}(V_I^{x_I}(J)) \\ &= \frac{(-1)^{n-1}}{(n-1)!2^{n-1}} P_0^{(n-1)}(V_I^{x_I}(I)) = x_I.\end{aligned}$$

Lemma 4.3 completes the proof.

Proof of Lemma 6.1 We will show that the alternate sum

$$\sum_{J<I} (-1)^{|J|} U_{R_J}$$

is a linear combination of singular knots with n double points. Since $P_0^{(n-1)}$ is a finite type invariant of degree $n-1$, this implies Lemma 6.1.

We may assume without loss of generality that $\bigcup_i w(r_i) = \{1, \dots, n\}$. Indeed, if there exists some $j \in \{1, \dots, n\}$ such that $j \notin w(r_i)$ for all i, we can freely add a C_1–tree c_j with weight $\{j\}$ such that $U_{R \cup c_j} = U_R \sharp U_{c_j} = U_R \sharp U$.

For each $i = 1, \dots, p$, let $\deg(r_i) = d_i$. Consider the (d_i+1)–component trivial tangle which is the intersection of U with a regular neighborhood of r_i. Then surgery along r_i yields a (d_i+1)–component tangle $\beta^i = \beta^i_0 \cup \cdots \cup \beta^i_{d_i}$. Note that β^i is a Brunnian tangle [9]. Since $\beta^i \setminus \beta^i_0$ is trivial, there is a diagram of β^i such that, for all $u = 1, \dots, d_i$, the component β^i_u is a trivial arc that only crosses component β^i_0. Fix a diagram of U_R that satisfies this condition for all $i = 1, \dots, p$. Now, let $w(r_i) = \{j_1, \dots, j_{m_i}\} \subset \{1, \dots, n\}$, with $m_i \le d_i$, and for all $u \in \{1, .., m_i\}$. Set

$$\mathcal{S}_i(j_u) := \text{the set of all crossings where } \beta^i_0 \text{ underpasses } \beta^i_u\ .$$

Note that this is only possible because r_i satisfies $m_i = |w(r_i)| \le d_i$. For all $j \in \{1, .., n\}$, set

$$\mathcal{S}(j) = \bigcup_i \mathcal{S}_i(j).$$

For any $J < I$, denote by $U_R[J]$ the knot obtained from U_R by switching all crossings in $\bigcup_{j\in\{J\}} \mathcal{S}(j)$. Then $U_R[J]$ is obtained from U by surgery along all r_i such that $w(r_i) \cap \{J\} = \varnothing$, ie,

$$U_R[J] = U_{R_{I\setminus J}}$$

for any $J < I$, where $I \setminus J$ denotes the sequence obtained from I by deleting all $j \in \{J\}$. (In particular, we have $U_R[\varnothing] = U_R$.) Hence we have

$$\sum_{J<I} (-1)^{|J|} U_{R_J} = \sum_{J<I} (-1)^{|J|} U_R[I \setminus J] = \sum_{J<I} (-1)^{n-|J|} U_R[J],$$

Clearly the alternate sum on the right-hand side, which involves knots that differ from one another by crossing changes on n sets of crossings, can be written as a linear combination of singular knots with n double points. This completes the proof of Lemma 6.1. □

6.2 Link-homotopy of string links

In this section, we give several interesting consequences of Theorem 1.2 for Milnor invariants of string links.

We first define an analogue for string links of the band sum operations on links given in the introduction. Let L be an n–string link. Recall from Section 4.2 and Section 5.1 that, for each $i = 1, \dots, n$, we pick a point $y_i \in \partial D^2$ and thus have a segment $p_i = x_i y_i \subset D^2$ (see Figure 8). Recall also that the closure of L is defined by $\hat{L} = \bigcup_{i=1}^n \hat{L}_i = L \cup (\bigcup_{i=1}^n (p_i \times \{0, 1\}) \cup (y_i \times I))$.

Let $I = i_1 i_2 \dots i_{m+1}$ be a sequence of $m+1$ distinct integers in $\{1, \dots, n\}$. We choose a $2(m+1)$–gon B_I in $R^2 \times (-\infty, 0]$ such that $B_I \cap (R^2 \times \{0\}) = \bigcup_{i\in I} (p_i \times \{0\})$ is a set of $m+1$ nonadjacent edges and $p_{i_1}\times\{0\},\dots,p_{i_{m+1}}\times\{0\}$ appear in this order along the oriented boundary of B_I. As in the introduction, for any subsequence J of I, we can define an oriented knot $\widehat{L}_J$ as the closure of $((\bigcup_{j\in\{J\}}\widehat{L}_j)\cup\partial B_I)\setminus((\bigcup_{j\in\{J\}}\widehat{L}_j)\cap B_I)$.

Set

$$f_{B_I}(L)=\frac{-1}{m!2^m}\sum_{J<I}(-1)^{|J|}P_0^{(m)}(\widehat{L}_J).$$

This function depends on the choice of B_I. Hence, for every nonrepeated sequence I, we choose B_I and fix it, to obtain an invariant of string links f_{B_I}.

The following is a string link version of Theorem 1.2.

Theorem 6.2 *Let L be an n–string link with vanishing Milnor link-homotopy invariants of length $\leq k$ $(3\leq k+1\leq n)$. Then for any sequence I of length $k+1$ without repeated indices, we have $\mu_L(I)=f_{B_I}(L)$.*

Proof Let $\widehat{L}$ be the closure of L. Then $\overline{\mu}_{\widehat{L}}(J)=0$ for all sequence J of length $\leq k$ without repeated indices, and $\overline{\mu}_{\widehat{L}}(I)=\mu_L(I)$. The result then follows immediately from Theorem 1.2. □

We now show how to use the P_0 polynomial to distinguish string links up to link-homotopy.

Corollary 6.3 *Two n–string links L and L' are link-homotopic if and only if they have same linking numbers and $f_{B_I}(L\cdot\overline{L'})=0$ for all nonrepeated sequences I, where $\overline{L'}$ denotes the horizontal mirror image of L' with the orientation reversed.*

Proof The string link $\overline{L'}$ is the inverse of L' under concordance, ie $L'\cdot\overline{L'}$ is concordant to the trivial string link. Since concordance of string links implies link-homotopy [4; 5], the two string links L and L' are link-homotopic if and only if $L\cdot\overline{L'}$ is link-homotopic to $\mathbf{1}_n$. (The result of [4; 5] is given for links in S^3. However, it still holds for string links.) Corollary 6.3 follows from Theorem 6.2 and the fact that a string-link is link-homotopic to the trivial one if and only if all Milnor link-homotopy invariants of the link vanish [7]. □

For an n–string link L and a sequence I of possibly repeating elements of $\{1,\dots,n\}$, we can define a nonrepeated sequence $D(I)$ and a string link $D_I(L)$ with $|D(I)|$ components, in a strictly similar way as for links in the introduction. By combining Corollary 6.3 and [26, Proposition 3.3] we have the following.

Corollary 6.4 *Two string links L and L' cannot be distinguished by Milnor invariants if and only if they have same linking numbers and $f_{B_{D(I)}}(D_I(L \cdot \overline{L'})) = 0$ for all sequences I.*

7 Example

In this last section, we give a simple example illustrating the necessity of our hypothesis in Theorem 1.1.

Consider a link L which is the split union of two positive Hopf links, with components labelled by 1, 2 and 3, 4 respectively. Then, for the sequence $I = 1324$, $\overline{\mu}_L(I)$ vanishes since $\Delta(I) = 1$.

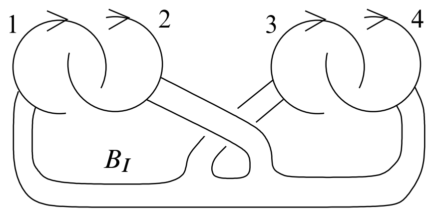


Figure 11

On the other hand, for the choice of I–fusion disk B_I illustrated in Figure 11, we have

$$P_0(L_J) = \begin{cases} 2t^2 - t^4 & \text{if } J = I, \\ 1 & \text{if } J \lneq I. \end{cases}$$

Hence we notice that the alternate sum

$$\sum_{J<I} (-1)^{|J|} (\log P_0(L_J))^{(3)} = \sum_{J<I} (-1)^{|J|} (P_0(L_J))^{(3)} = 24$$

is not divisible by $3!2^3 = 48$.

This divisibility issue is the main obstruction for our formula to hold in general.

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Institut Fourier, Université Grenoble 1
100 rue des Maths, BP 74, 38402 Saint-Martin d'Hères, France

Department of Mathematics, Tokyo Gakugei University
4-1-1 Nukuikita-Machi, Koganei-shi, Tokyo 184-8501, Japan

jean-baptiste.meilhan@ujf-grenoble.fr, yasuhara@u-gakugei.ac.jp

http://www-fourier.ujf-grenoble.fr/~meilhan/,
http://www.u-gakugei.ac.jp/~yasuhara/

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