Topology and its Applications 160 (2013) 836–843

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Local moves for links with common sublinks ☆

Jean-Baptiste Meilhan [a], Eri Seida [b], Akira Yasuhara [b,*]

[a] Institut Fourier, Université Grenoble 1, 100 rue des Maths, BP 74, 38402 St Martin d'Hères, France
[b] Tokyo Gakugei University, Department of Mathematics, Koganeishi, Tokyo 184-8501, Japan

ARTICLE INFO

Article history:
Received 9 April 2012
Received in revised form 5 February 2013
Accepted 11 February 2013

MSC:
57M25
57M27

Keywords:
C_k-moves
Brunnian link
Claspers

ABSTRACT

A C_k-move is a local move that involves $k+1$ strands of a link. A C_k-move is called a C_k^d-move if these $k+1$ strands belong to mutually distinct components of a link. Since a C_k^d-move preserves all k-component sublinks of a link, we consider the converse implication: are two links with common k-component sublinks related by a sequence of C_k^d-moves? We show that the answer is yes under certain assumptions, and provide explicit counter-examples for more general situations. In particular, we consider (n,k)-Brunnian links, i.e. n-component links whose k-component sublinks are all trivial. We show that such links can be deformed into a trivial link by C_k^d-moves, thus generalizing a result of Habiro and Miyazawa–Yasuhara, and deduce some results on finite type invariants of (n,k)-Brunnian links.

© 2013 Elsevier B.V. All rights reserved.

1. Introduction

Habiro [4] and Gusarov [3] introduced independently the notion of C_k-move, which is a local move that involves $k+1$ strands of a link as illustrated in Fig. 1.1. A C_1-move is just a crossing change. Alternatively, a C_k-move can be defined in terms of "insertion" of elements of the kth term of the lower central series of the pure braid group [15].

In particular, if all $k+1$ strands involved in a C_k-move belong to pairwise distinct components, we call it a *C_k^d-move*. The C_k-move (resp. C_k^d-move) generates an equivalence relation on links, called *C_k-equivalence* (resp. *C_k^d-equivalence*), which becomes finer as k increases. It is easy to see that if two links are C_k^d-equivalent, then they have *common k-component sublinks*. More precisely, if two ordered links $L=K_1\cup\cdots\cup K_n$ and $L'=K_1'\cup\cdots\cup K_n'$ are C_k^d-equivalent, then for any subset $S\subset\{1,\dots,n\}$ with k elements, $\bigcup_{i\in S}K_i$ and $\bigcup_{i\in S}K_i'$ are ambient isotopic. It seems natural to ask whether the converse implication holds as well.

Question. If two links have common k-component sublinks, then are they C_k^d-equivalent?

Since any link is C_1^d-equivalent to a completely split link, the answer is obviously yes for $k=1$. Hence we may assume that $k\geqslant 2$.

The question can also be given a positive answer for a special class of links. For positive integers n and k with $k\leqslant n$, an *(n,k)-Brunnian link* is an n-component link whose k-component sublinks are trivial [12]. In particular, if $n=k+1$, then it is a Brunnian link in the usual sense.

☆ The first author is supported by the French ANR research project ANR-11-JS01-00201. The third author is partially supported by a Grant-in-Aid for Scientific Research (C) (#23540074) of the Japan Society for the Promotion of Science.
* Corresponding author.
E-mail addresses: jean-baptiste.meilhan@ujf-grenoble.fr (J.-B. Meilhan), yasuhara@u-gakugei.ac.jp (A. Yasuhara).

0166-8641/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.topol.2013.02.006

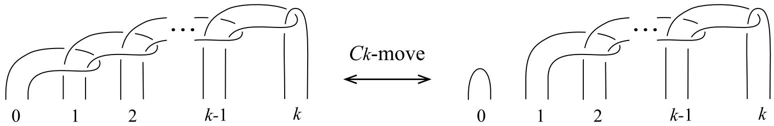

Fig. 1.1. A C_k-move involves $k+1$ strands of a link.

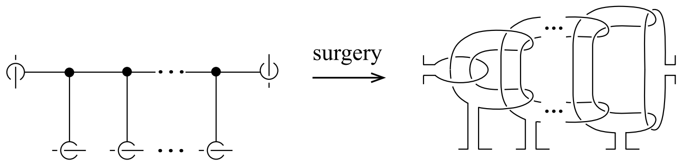

Fig. 2.1. Surgery along a C_k-tree.

Theorem 1.1. *A link is (n,k)-Brunnian if and only if it is C_k^d-equivalent to the n-component trivial link.*

Theorem 1.1 is thus a generalization of the fact that an $(n+1)$-component link is Brunnian if and only if it is C_n^d-equivalent to a trivial link [5,10]. This fact is the key ingredient in further works on finite type invariants of Brunnian links [5–7]. These results generalize in a straightforward way to (n,k)-Brunnian links, see Appendix A.

For $k=2$, we have a more general statement as follows.

Theorem 1.2. *Two links with trivial components have common 2-component sublinks if and only if they are C_2^d-equivalent.*

Although the hypotheses in Theorem 1.2, that $k=2$ and that each link component is trivial, may seem restrictive, they turn out to be both necessary to give a positive answer to our question. Indeed, we have the following.

Proposition 1.3.

(1) *For $k \geqslant 3$, there exists a pair of links with trivial components, which have common k-component sublinks but are not C_k^d-equivalent.*
(2) *There exist two links with one nontrivial component, which have common 2-component sublinks and are not C_2^d-equivalent.*

The rest of the paper is organized as follows. In Section 2, we review some elements of the theory of claspers. We prove Theorem 1.1, Theorem 1.2 and Proposition 1.3 in Sections 3, 4 and 5 respectively. The paper is concluded by several straightforward extensions of known results on Brunnian links to (n,k)-Brunnian links, see Appendix A.

2. Claspers

We now recall several notions from clasper theory for links [4]. In this paper, we only need the notion of tree claspers. For a general definition, we refer the reader to [4].

Let L be a link in S^3. An embedded disk F in S^3 is called a *tree clasper* for L if it satisfies the following three conditions:

(1) F is decomposed into disks and bands, called *edges*, each of which connects two distinct disks.
(2) The disks have either 1 or 3 incident edges, called *leaves* or *nodes* respectively.
(3) L intersects F transversely and the intersections are contained in the union of the interior of the leaves.

(In [4], a tree clasper and a leaf are called a *strict tree clasper* and a *disk-leaf* respectively.)

A tree clasper is *simple* if each leaf intersects L at one point. In the following, *we will implicitly assume that all tree claspers are simple.*

The *degree* of a tree clasper is the number of the leaves *minus* 1. A degree k tree clasper is called a *C_k-tree* (or a *C_k-clasper*).

Given a C_k-tree T for a link L, there is a procedure to construct a framed link $\gamma(T)$ in a regular neighborhood of T. *Surgery along T* means surgery along $\gamma(T)$. Since there exists an orientation-preserving homeomorphism, fixing the boundary, from the regular neighborhood $N(T)$ of T to the manifold $N(T)_T$ obtained from $N(T)$ by surgery along T, surgery along the C_k-tree T can be regarded as a local move on L. We say that the resulting link L_T is *obtained from L by surgery along T*. In particular, surgery along a C_k-tree illustrated in Fig. 2.1 is equivalent to band-summing a copy of the $(k+1)$-component Milnor link (see [9, Fig. 7]), and is equivalent to a C_k-move as defined in the introduction (Fig. 1.1). Similarly, for a disjoint union $T_1 \cup \cdots \cup T_m$ of tree claspers for L, we can define $L_{T_1 \cup \cdots \cup T_m}$ as the link obtained by surgery along $T_1 \cup \cdots \cup T_m$.

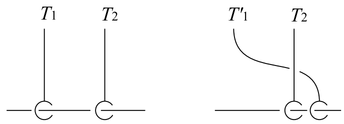

Fig. 2.2. Sliding a leaf over another leaf.

It is known that the C_k-equivalence as defined in Section 1 coincides with the equivalence relation on links generated by surgery along C_k-trees and ambient isotopy [4, Thm. 3.17].

Let $L = K_1 \cup \cdots \cup K_n$ be an n-component link. For a C_k-tree T for L, the set $\{i \mid T \cap K_i \neq \emptyset\}$ is called the *index* of T, and is denoted by $\mathrm{index}(T)$.

A C_k-tree T for L is a C_k^d*-tree* if it satisfies that $|\mathrm{index}(T)| = k+1$, that is, if $|\mathrm{index}(T)|$ is the number of leaves of T.

By arguments similar to those in the proof of [4, Thm. 3.17], we have that the C_k^d-equivalence defined in Section 1 coincides with the equivalence relation on links generated by surgery along C_k^d-trees and ambient isotopy.

Although the following three lemmas follow from results of [16], the main ideas of proofs are due to Habiro [4].

Lemma 2.1 *(Leaf-sliding). ([16, Rem. 2.3]) Let T_1 and T_2 be disjoint C_1^d-trees for a link L. Suppose* $\mathrm{index}(T_1) = \{i, j\}$, $\mathrm{index}(T_2) = \{i, k\}$ *and $j \neq k$. Let T'_1 be obtained from T_1 by sliding the leaf intersecting the ith component of L over the leaf of T_2 intersecting the ith component, see Fig. 2.2. Then $L_{T_1 \cup T_2}$ and $L_{T'_1 \cup T_2}$ are C_2^d-equivalent.*

Lemma 2.2. *([16, Prop. 2.10], cf. [1, Prop. 1.3]) If an n-component link L' is obtained from L by surgery along a C_k-tree T, then for any subset S of* $\mathrm{index}(T)$ *with $|S| \geqslant 2$, there exists a disjoint union of $C_{|S|-1}$-trees for L with index S, such that L' is obtained from L by surgery.*

Combining the latter with [16, Rem. 2.3], we have the following.

Lemma 2.3 *(Crossing change). Let T be a C_k-tree for a link L, and let T' (resp. T'') be obtained from T by changing a crossing of an edge of T and the ith component of L (resp. an edge of a C_1-tree G intersecting the ith component). Then L_T is C_{k+1}-equivalent to $L_{T'}$ (resp. $L_{T\cup G}$ is C_{k+1}-equivalent to $L_{T''\cup G}$), and the C_{k+1}-equivalence is realized by surgery along C_{k+1}-trees with index* $\mathrm{index}(T) \cup \{i\}$.

In the following, we will freely use the terms *leaf-sliding* and *crossing change* when performing the operations of Lemmas 2.1 and 2.3 respectively.

3. Proof of Theorem 1.1

As noted in the introduction, the 'if' part of the statement is obvious, so we only need to prove the 'only if' part.

Let $L = K_1 \cup \cdots \cup K_n$ be an n-component link with the same k-component sublinks as the trivial n-component link $O = O_1 \cup \cdots \cup O_n$. Set $l = n - k$. We will show that L is C_k^d-equivalent to O by induction on l.

If $l = 0$, then L is trivial and the result is obviously true.

Suppose that $l > 0$. Let m be the maximum integer so that L is C_m^d-equivalent to O, i.e., so that there is a disjoint union F_m of C_m^d-trees for O such that the link O_{F_m} is ambient isotopic to L. If $m = k$, then we have the result. Hence we assume that $m < k$ and show that this leads to a contradiction.

Observe that O_{F_m} can be deformed into the split union of O_1 and $L \setminus K_1$ by deleting all C_m^d-trees in F_m intersecting O_1 and performing several crossing changes between O_1 and edges of C_m^d-trees disjoint from O_1. By Lemma 2.3, these crossing changes are realized by surgery along C_{m+1}^d-trees intersecting O_1. Since $L \setminus K_1$ is an $(n-1,k)$-Brunnian link, by induction hypothesis, it is C_k^d-equivalent to a trivial link. It follows by Lemma 2.2 that L is obtained from O by surgery along a disjoint union F_m^1 of C_m^d-trees intersecting O_1 and C_k^d-trees.

Similarly, $O_{F_m^1}$ can be deformed into the split union of O_2 and $L \setminus K_2$ by deleting all tree claspers in F_m^1 intersecting O_2 and performing several crossing changes between O_2 and edges of tree claspers in F_m^1 disjoint from O_2. Since $L \setminus K_2$ is an $(n-1,k)$-Brunnian link, by induction hypothesis, it is C_k^d-equivalent to a trivial link. It follows by Lemmas 2.3 and 2.2 that L is obtained from O by surgery along a disjoint union F_m^2 of C_m^d-trees, each intersecting both O_1 and O_2, and C_k^d-trees.

Repeating this argument inductively, we show that L is obtained from O by surgery along a disjoint union F_m^{m+1} of C_m^d-trees with index $\{1, 2, \ldots, m+1\}$, and C_k^d-trees. Since $m \leqslant k-1 \leqslant n-2$, L has an $(m+2)$th component, which is disjoint from the C_m^d-trees in F_m^{m+1}. We can deform $O_{F_m^{m+1}}$ into the split union of O_{m+2} and $L \setminus K_{m+2}$ by deleting all C_k^d-trees in F_m^{m+1} intersecting O_{m+2} and several crossing changes between O_{m+2} and edges of trees clasper in F_m^{m+1}. Since $L \setminus K_{m+2}$ is an $(n-1,k)$-Brunnian link, it is C_k^d-equivalent to a trivial link. Lemmas 2.3 and 2.2 then imply that there is disjoint union of C_{m+1}^d-trees F' for O such that $O_{F'}$ is ambient isotopic to L. This contradicts the definition of m, and thus proves that L is C_k^d-equivalent to O.

4. Proof of Theorem 1.2

The ‘if’ part of the statement is obvious, so we only need to prove the ‘only if’ part.

Let L be an n-component link with trivial components. By [8], there is a diagram of L in $\mathbb{R}^2 \times \{0\}$ such that each component has no self crossing.

By a sequence of crossing changes, we can deform L into a trivial link $O = O_1 \cup \cdots \cup O_n$ such that O_i lies in $\mathbb{R}^2 \times \{i\}$ ($i = 1, \dots, n$) and the projections of L and O coincide. Hence, for each pair i, j of distinct integers in $\{1, \dots, n\}$ such that $i < j$, there is a disjoint union F_{ij}, possibly $F_{ij} = \emptyset$, of C_1^d-trees for O with index $\{i, j\}$ such that L is ambient isotopic to $O_{\bigcup_{i<j} F_{ij}}$ (since each crossing in L where the jth component passes under the ith one is achieved by surgery on O along such a C_1^d-tree). Let D_i be the disk in $\mathbb{R}^2 \times \{i\}$ with $\partial D_i = O_i$. For each pair i, j ($i < j$), we choose an arc α_{ij} in $R^2 \times [i, j]$ which connects a point in O_i and a point in O_j, such that $\alpha_{ij} \cap (R^2 \times \{t\})$ is a single point for each $t \in [i, j]$, and α_{ij} is disjoint from D_k for all $k \neq i, j$. Since the edge of each C_1^d-tree in F_{ij} is contained in $\mathbb{R}^2 \times (i, j)$, we have that $O_i \cup O_j \cup F_{ij}$ is ambient isotopic to $O_i \cup O_j \cup E_{ij}$, where E_{ij} is a disjoint union of C_1^d-trees with index $\{i, j\}$ and contained in a regular neighborhood of α_{ij}. Now, $O \cup (\bigcup_{i<j} F_{ij})$ can be deformed into $O \cup (\bigcup_{i<j} E_{ij})$ by a sequence of isotopies and the following three types of moves:

(i) leaf-sliding between two C_1^d-trees with distinct indices;
(ii) crossing change between edges of two C_1^d-trees with distinct indices;
(iii) crossing change between a component of O and the edge of a C_1^d-tree disjoint from this component.

It follows, by Lemmas 2.1, 2.2 and 2.3, that $O_{\bigcup_{i<j} F_{ij}}$ is C_2^d-equivalent to $O_{\bigcup_{i<j} E_{ij}}$.

Now, let L' be an n-component link with trivial components and with the same 2-component sublinks as L. As above, there exists disjoint unions E'_{ij} of C_1^d-trees with index $\{i, j\}$ for a trivial link O', contained in a regular neighborhood of an arc α'_{ij} in $\mathbb{R}^2 \times [i, j]$ ($1 \leqslant i < j \leqslant n$), such that L' is C_2^d-equivalent to $O'_{\bigcup_{i<j} E'_{ij}}$. Since O and O' are both n-component trivial links, there is an ambient isotopy mapping O' to O. (Note that this isotopy may be assumed to preserve each level $\mathbb{R}^2 \times \{t\}$, for $t \in \mathbb{R}$.) The image of $\bigcup_{i<j} E'_{ij}$ under this ambient isotopy can be deformed into a union of C_1^d-tree such that the union of C_1^d-trees with index $\{i, j\}$ is contained in a regular neighborhood of the arc α_{ij} by a sequence of isotopies and the moves (i) and (ii) above. Hence, by Lemmas 2.1, 2.2 and 2.3, the results of surgery on O along $\bigcup_{i<j} E'_{ij}$ and along the image of $\bigcup_{i<j} E'_{ij}$ under these deformations, are C_2^d-equivalent. Thus, for simplicity, we may assume in the following that $O = O'$ and that $\alpha_{ij} = \alpha'_{ij}$ ($1 \leqslant i < j \leqslant n$).

Suppose that there is a pair of integers s, t such that $E_{st} \neq E'_{st}$. Since L and L' have common 2-component sublinks, so do $O_{\bigcup_{i<j} E_{ij}}$ and $O_{\bigcup_{i<j} E'_{ij}}$. In particular, $(O_s \cup O_t)_{E'_{st}}$ is ambient isotopic to $(O_s \cup O_t)_{E_{st}}$. Since $E_{ij} \cap D_k = E'_{ij} \cap D_k = \emptyset$ for any i, j, k with $k \notin \{i, j\}$, we have that $O_{E'_{st}}$ is ambient isotopic to $O_{E_{st}}$. Moreover, this ambient isotopy can be performed in a regular neighborhood N_{st} of $D_s \cup D_t \cup \alpha_{st}$. It follows that $O_{E'_{st}} \cup (\bigcup_{(i,j)\neq(s,t);\, i<j} E'_{ij})$ is ambient isotopic to $O_{E_{st}} \cup (\bigcup_{(i,j)\neq(s,t);\, i<j} \widetilde{E'_{ij}})$, where $\widetilde{E'_{ij}}$ denotes the image of E'_{ij} under the above isotopy of N_{st}. Note that $\widetilde{E'_{kl}} \neq E'_{kl}$ only if $\{k, l\} \cap \{s, t\} \neq \emptyset$, since $E'_{kl} \cap N_{st} = \emptyset$ otherwise. Below we prove the following

Claim 4.1. *$O_{E_{st} \cup (\bigcup_{(i,j)\neq(s,t);\, i<j} \widetilde{E'_{ij}})}$ and $O_{E_{st} \cup (\bigcup_{(i,j)\neq(s,t);\, i<j} E'_{ij})}$ are C_2^d-equivalent.*

It follows that $O_{\bigcup_{i<j} E'_{ij}}$ is C_2^d-equivalent to $O_{E_{st} \cup (\bigcup_{(i,j)\neq(s,t);\, i<j} E'_{ij})}$. Since this argument holds for an arbitrary pair of integers s, t such that $s < t$, we eventually obtain that $O_{\bigcup_{i<j} E'_{ij}}$ is C_2^d-equivalent to $O_{\bigcup_{i<j} E_{ij}}$. Hence L and L' are C_2^d-equivalent.

Proof of Claim 4.1. Since surgery along E'_{st} takes place in a regular neighborhood of α_{st}, we may consider the arcs α_{ij}, where $(i, j) \neq (s, t)$ and $i < j$, is attached to the link $O_{E'_{st}}$. Up to isotopy, we may freely assume that N_{st} intersects the union of $O_{E'_{st}}$ and $(\bigcup_{i<j;\, (i,j)\neq(s,t)} \alpha_{ij})$ as shown in Fig. 4.1. In particular, for h being either s or t, we have that

$$\alpha_h := \Bigg(\bigcup_{\substack{(i,j);\, i<j \\ \{i,j\}\cap\{s,t\}=\{h\}}} \alpha_{ij} \Bigg) \cap N_{st}$$

is a parallel family of arcs. Denote by $\widetilde{\alpha_h}$ the image of α_h under the isotopy of N_{st} that sends $O_{E'_{st}}$ to $O_{E_{st}}$. Observe that, for each pair i, j ($i < j$) with $\{i, j\} \cap \{s, t\} = \{h\}$, $\widetilde{E'_{ij}} \cap N_{st}$ is contained in a regular neighborhood of a certain arc in $\widetilde{\alpha_h}$.

Note that $\partial(\widetilde{\alpha_s} \cup \widetilde{\alpha_t}) \cap \partial N_{st} = \partial(\alpha_s \cup \alpha_t) \cap \partial N_{st}$, and that we may also freely assume that $(\widetilde{\alpha_s} \cup \widetilde{\alpha_t}) \cap (O_s \cup O_t)_{E_{st}} = (\alpha_s \cup \alpha_t) \cap (O_s \cup O_t)_{E'_{st}}$. It follows that $\widetilde{\alpha_s} \cup \widetilde{\alpha_t} \cup (O_s \cup O_t)_{E_{st}}$ can be deformed into $\alpha_s \cup \alpha_t \cup (O_s \cup O_t)_{E_{st}}$ by a sequence of isotopies and the following moves:

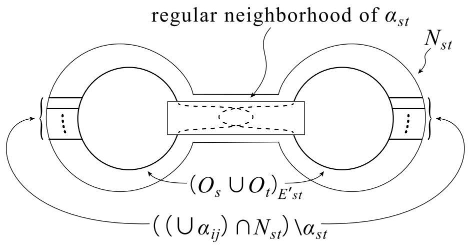

Fig. 4.1. The intersection of N_{st} and $O_{E'_{st}} \cup (\bigcup_{i<j;\,(i,j)\neq(s,t)} \alpha_{ij})$.

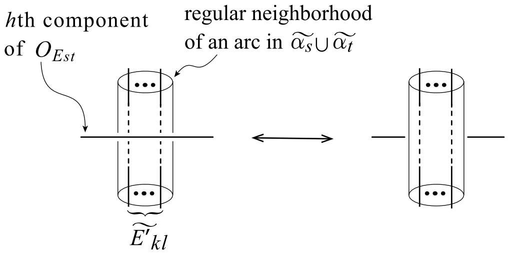

Fig. 4.2. The move takes place in a 3-ball of N_{st} that intersects a union of hth component of $O_{E_{st}}$ and $\widetilde{E'_{kl}}$ as shown, where $h \in \{s,t\}$, $(k,l) \neq (s,t)$ and $\{k,l\} \cap \{s,t\} \neq \emptyset$.

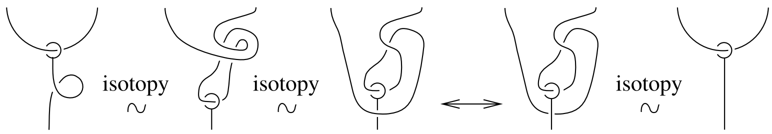

Fig. 4.3. Here, the clasper may be replaced with a parallel family of claspers.

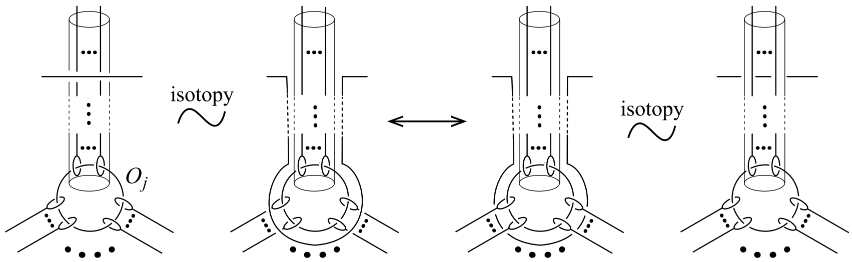

Fig. 4.4. Passing the hth component of $O_{E_{st}}$ across edges of C_1^d-trees with index $\{j,l\}$ $(l \neq h)$. Here j, l and h are mutually distinct.

- crossing change between an arc in $\widetilde{\alpha_s} \cup \widetilde{\alpha_t}$ and a component of $(O_s \cup O_t)_{E_{st}}$;
- crossing change between two arcs in $\widetilde{\alpha_s} \cup \widetilde{\alpha_t}$.

Note that the second kind of move can be achieved by isotopy and moves of the first kind. We thus have that $\widetilde{\alpha_s} \cup \widetilde{\alpha_t} \cup (O_s \cup O_t)_{E_{st}}$ can be deformed into $\alpha_s \cup \alpha_t \cup (O_s \cup O_t)_{E_{st}}$ by a sequence of isotopies and crossing changes between $\widetilde{\alpha_s} \cup \widetilde{\alpha_t}$ and $(O_s \cup O_t)_{E_{st}}$. This deformation induces a deformation from $O_{E_{st}} \cup (\bigcup_{(i,j)\neq(s,t);\, i<j} \widetilde{E'_{ij}})$ to $O_{E_{st}} \cup (\bigcup_{(i,j)\neq(s,t);\, i<j} E'_{ij})$, which can be realized by a sequence of the move in Fig. 4.2. Note that, although insertion of a full-twist in some arc α_{kl} of $\alpha_s \cup \alpha_t$ can be achieved by an isotopy, this is no longer true for the induced insertion of a full-twist in E_{kl}. However, it is easily achieved by isotopy and the move in Fig. 4.2, as illustrated in Fig. 4.3.

It remains to check that such a move among claspers yields C_2^d-equivalent results of surgery.

In general, this move involves the hth component of $O_{E_{st}}$ and the union of C_1^d-trees with index $\{i,j\}$, for some $i, h \in \{s,t\}$ and some $j \notin \{s,t\}$. If $i \neq h$, the result follows directly from Lemma 2.3. Otherwise, Fig. 4.4 shows how the desired deformation can be achieved by passing the hth component of $O_{E_{st}}$ across edges of C_1^d-trees with index $\{j,l\}$ with $l \neq h$. The result then follows from Lemma 2.3. □

Remark 4.2. The following was suggested by the referee:

Conjecture. *Two $(n, k-1)$-Brunnian links have common k-component sublinks if and only if they are C_k^d-equivalent.*

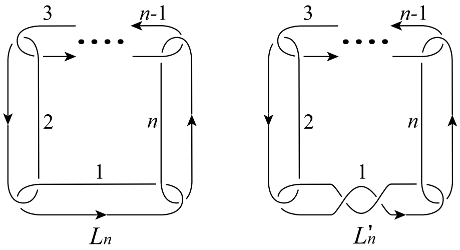

Fig. 5.1. L_n and L_n' are n-component links with trivial components and common k-component sublinks for $k \leqslant n-1$.

For $k=2$, the answer is positive by Theorem 1.2, and since C_k^d-equivalent links have common k-component sublinks, a positive answer to this question would be a generalization of this theorem. Unfortunately, the arguments in the above proof of Theorem 1.2 do not seem to be easily adapted to this more general settings. Nevertheless, the authors could not come up with any counter-example, and are therefore confident in stating the above as a conjecture.

5. Proof of Proposition 1.3

In this section, we show that each of the hypotheses imposed in Theorem 1.2 is necessary for the conclusion to hold.

5.1. The case $k \geqslant 3$: proof of Proposition 1.3 (1)

We first observe that Theorem 1.2 does not hold for $k \geqslant 3$.

Let L_n and L_n' be two n-component links as illustrated in Fig. 5.1. Clearly, both links have trivial components, and have common k-component sublinks for $k \leqslant n-1$.

On the other hand, we notice that $\text{Arf}(L_n)=0$ and $\text{Arf}(L_n')=1$, where Arf denotes the Arf invariant [14]. Note that a C_k-move preserves the Arf invariant when $k \geqslant 3$, since it can be achieved by a pass-move, which preserves the Arf invariant [11]. This implies that L_n and L_n' are not C_k-equivalent, and hence not C_k^d-equivalent.

5.2. An invariant of C_k^d-equivalence: proof of Proposition 1.3 (2)

We now consider the case $k=2$, but without the assumption that all components are trivial. For that purpose, we first introduce an invariant of C_k^d-equivalence derived from the linking number in the double branched cover of S^3 branched over a knot.

Let $K \cup K_1 \cup \cdots \cup K_m$ $(m \geqslant 1)$ be an oriented $(m+1)$-component link in S^3. If the linking number $\text{lk}(K, K_i)$ is even for all i $(=1,\dots,m)$, then there is a possibly nonorientable surface F bounded by K disjoint from $K_1 \cup \cdots \cup K_m$. Let G_α be the *Goeritz matrix* [2] with respect to a basis $\alpha=(a_1,\dots,a_n)$ of $H_1(F)$, i.e., the (i,j)-entry of G_α is equal to $\text{lk}(a_i, \tau a_j)$, where τa_j is a 1-cycle in $S^3 - F$ obtained by pushing off $2a_j$ in both normal directions. Let $V_\alpha(K_i) = (\text{lk}(K_i, a_1), \dots, \text{lk}(K_i, a_n))$. In [13] J.H. Przytycki and the last author define, for i, j $(1 \leqslant i, j \leqslant m)$,

$$\lambda_F(K_i, K_j) = V_\alpha(K_i) G_\alpha^{-1} V_\alpha(K_j)^T,$$

and $\lambda_F(K_i, K_j) = 0$ when F is a 2-disk. It follows directly from [13, Thm. 2.3] that for the double branched cover M of S^3 branched over K and for lifts $\widetilde{K_i}$ and $\widetilde{K_j}$ of K_i and K_j respectively, we have

$$\text{lk}_M(\widetilde{K_i}, \widetilde{K_j}) \equiv \pm\lambda_F(K_i, K_j) \bmod 1.$$

If two links $L = K \cup K_1 \cup \cdots \cup K_m$ and $L' = K' \cup K_1' \cup \cdots \cup K_m'$ are C_k^d-equivalent for some k $(2 \leqslant k \leqslant m)$, then $K_i \cup K_j$ and $K_i' \cup K_j'$ are homotopic in the complement of K. This implies that there is a lift $\widetilde{K_i} \cup \widetilde{K_j}$ (resp. $\widetilde{K_i'} \cup \widetilde{K_j'}$) of $K_i \cup K_j$ (resp. $K_i' \cup K_j'$) such that

$$\text{lk}_M(\widetilde{K_i}, \widetilde{K_j}) \equiv \text{lk}_M\big(\widetilde{K_i'}, \widetilde{K_j'}\big) \bmod 1.$$

It follows that we have the following proposition.

Proposition 5.1. *For any $k \geqslant 2$, $\pm\lambda_F(K_i, K_j)$ (mod 1) is an invariant of C_k^d-equivalence.*

We can now complete the proof of Proposition 1.3.

Proof of Proposition 1.3 (2). Let $L = K \cup K_1 \cup K_2$ and $L' = K \cup K_1 \cup K_2'$ be links as illustrated in Fig. 5.2. Note that L and L' have common 2-component sublinks. Let F be a nonorientable surface, and let a_1, a_2 be a basis of $H_1(F)$ as il-

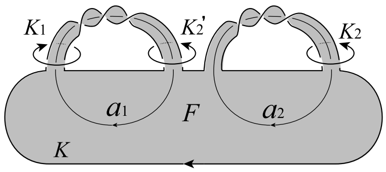

Fig. 5.2. $L = K \cup K_1 \cup K_2$ and $L' = K \cup K_1 \cup K_2'$ are 3-component links with common 2-component sublinks.

lustrated in Fig. 5.2. Then we have $\lambda_F(K_1, K_2) = 0$ and $\lambda_F(K_1, K_2') = -1/3$. Proposition 5.1 implies that L and L' are not C_2^d-equivalent. □

Acknowledgement

The authors would like to thank the referee for his/her thorough and constructive comments which helped to improve the quality of this article, and for pointing out a possible direction for extending our result, as mentioned in Remark 4.2.

Appendix A. Finite type invariants of (n, k)-Brunnian links

Theorem 1.1 states that a link is (n, k)-Brunnian if and only if it is C_k^d-equivalent to the n-component trivial link. As recalled in the introduction, the case $k = n - 1$, i.e. the case of Brunnian links, was shown in [5,10], and is a key ingredient in proving several results and Brunnian links and their finite type invariants. Using Theorem 1.1, we can easily generalize these to (n, k)-Brunnian links. We only provide statements here, since the proofs are straightforward generalizations of [5, 10,6,7], and require no new idea.

In [5], Habiro shows that for $n \geqslant 3$, an n-component Brunnian links cannot be distinguished from the trivial link by any finite type invariant of order less than $2(n-1)$. (Note that for $n = 2$, this does not hold since the Hopf link and the 2-component trivial link can be distinguished by the linking number, which is of order 1.) By the same arguments as those in [5, §4], we have that if a link is C_k^d-equivalent to a trivial link for $k \geqslant 2$, then these links cannot be distinguished by any finite type invariant of order less than $2k$. Hence we obtain the following result.

Theorem A.1. *For $n > k \geqslant 2$, (n, k)-Brunnian links and the n-component trivial link cannot be distinguished by any finite type invariant of order less than $2k$.*

In [6,7], the study of finite type invariants of Brunnian links is continued, by expressing the restriction of an invariant of degree $2n - 1$ to n-component Brunnian links as a quadratic form on the Milnor link-homotopy invariants of length n, see [9]. The arguments used in [6] (and [7]) can be generalized in a straightforward way to (n, k)-Brunnian links to prove the following.

Theorem A.2. *Let f be any finite type link invariant of degree $2k + 1$ taking values in an abelian group A. Then there are (non-unique) elements $f_I^{\sigma,\sigma'} \in A$ for σ, σ' in the symmetric group S_{k-1} on the set $\{1, \dots, k-1\}$ and for any subsequence I of $12\dots n$ of length $k + 1$, such that, for any (n, k)-Brunnian link L, the difference $f(L) - f(O)$ is equal to*

$$\sum_{\substack{I=i_1 i_2 \dots i_{k+1} \\ \text{subseq. of } 12\dots n}} \sum_{\sigma,\sigma' \in S_{k-1}} f_I^{\sigma,\sigma'} \bar{\mu}_L(i_{\sigma(1)} \dots i_{\sigma(k-1)} i_k i_{k+1}) \bar{\mu}_L(i_{\sigma'(1)} \dots i_{\sigma'(k-1)} i_k i_{k+1}).$$

Here O denotes the n-component trivial link and $\bar{\mu}_L$ denotes Milnor invariants of L.

In other words, the restriction of an invariant of degree $2k + 1$ to (n, k)-Brunnian links can be expressed as a quadratic form on the Milnor link-homotopy invariants of its $(k + 1)$-component sublinks, and, in particular, is determined by the $(k + 1)$-component sublinks.

Milnor invariants are useful not only for understanding finite type invariants of (n, k)-Brunnian links, but also for providing classification results. It is indeed known that n-component Brunnian links are classified up to C_n-equivalence by Milnor link-homotopy invariants [10,6]. Again, strictly similar arguments can be used to extend this classification result as follows.

Theorem A.3. *Two (n, k)-Brunnian links are C_{k+1}-equivalent if and only if they cannot be distinguished by any Milnor link-homotopy invariant of length $k + 1$.*

References

[1] T. Fleming, A. Yasuhara, Milnor's invariants and self C_k-equivalence, Proc. Amer. Math. Soc. 137 (2009) 761–770.
[2] L. Goeritz, Knoten und quadratische Formen, Math. Z. 36 (1933) 647–654.

[3] M.N. Gusarov, Variations of knotted graphs. The geometric technique of n-equivalence, Algebra i Analiz 12 (2000) 79–125 (in Russian); translation in St. Petersburg Math. J. 12 (2001) 569–604.
[4] K. Habiro, Claspers and finite type invariants of links, Geom. Topol. 4 (2000) 1–83.
[5] K. Habiro, Brunnian links, claspers and Goussarov–Vassiliev finite type invariants, Math. Proc. Cambridge Philos. Soc. 142 (2007) 459–468.
[6] K. Habiro, J.B. Meilhan, Finite type invariants and Milnor invariants for Brunnian links, Int. J. Math. 19 (2008) 747–766.
[7] K. Habiro, J.B. Meilhan, On the Kontsevich integral of Brunnian links, Algebr. Geom. Topol. 6 (2006) 1399–1412.
[8] J.H. Lee, G.T. Jin, Link diagrams realizing prescribed subdiagram partitions, Kobe J. Math. 18 (2001) 199–202.
[9] J. Milnor, Link groups, Ann. of Math. (2) 59 (1954) 177–195.
[10] H.A. Miyazawa, A. Yasuhara, Classification of n-component Brunnian links up to C_n-move, Topology Appl. 153 (2006) 1643–1650.
[11] H. Murakami, Y. Nakanishi, On a certain move generating link-homology, Math. Ann. 283 (1989) 75–89.
[12] D.E. Penney, Generalized Brunnian links, Duke Math. J. 36 (1969) 31–32.
[13] J.H. Przytycki, A. Yasuhara, Linking numbers in rational homology 3-spheres, cyclic branched covers and infinite cyclic covers, Trans. Amer. Math. Soc. 356 (2004) 3669–3685.
[14] R.A. Robertello, An invariant of knot cobordism, Comm. Pure Appl. Math. 18 (1965) 543–555.
[15] T. Stanford, Braid commutators and Vassiliev invariants, Pacific J. Math. 174 (1996) 269–276.
[16] A. Yasuhara, Self delta-equivalence for links whose Milnor's isotopy invariants vanish, Trans. Amer. Math. Soc. 361 (2009) 4721–4749.