SURGERY ON BRUNNIAN LINKS AND FINITE TYPE INVARIANTS OF INTEGRAL HOMOLOGY SPHERES

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Abstract. We use surgery along Brunnian links to relate, via a certain isomorphism, the Goussarov-Vassiliev theory for Brunnian links and the finite type invariants of integral homology spheres. To do so, we show that no finite type invariant of degree $< 2n-2$ can vary under surgery along an $(n+1)$-component Brunnian link in a compact connected oriented 3-manifolds, where the framing of the components is in $\{\frac{1}{k}\ ;\ k \in \mathbf{Z}\}$.

1. Introduction

The notion of finite type invariants of integral homology spheres, introduced by Ohtsuki in [12], provides a unified point of view on the topological invariants of these objects. A similar notion for compact connected oriented 3-manifolds was developed (independently) by Goussarov and Habiro [3, 5]. Their theory coincides with the Ohtsuki theory in the case of integral homology spheres [2], and involves a new and powerful tool called *clasper*, which is a kind of embedded graph carrying some surgery instruction.

Surgery moves along claspers define a family of (finer and finer) equivalence relations among 3-manifolds, called Y_k*-equivalence*, which gives a characterization of the topological information carried by finite type invariants of integral homology spheres: two integral homology spheres are not distinguished by invariants of degree $< k$ if and only if they are Y_k-equivalent [4, 5]. In the case of compact connected oriented 3-manifolds, the 'if' part of the statement is still always true. Denote by $\mathcal{S}_k$ the set of integral homology spheres which are Y_k-equivalent to S^3. We have a filtration

$$\mathbb{Z}HS = \mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset ...$$

where $\mathbb{Z}HS$ denotes the set of all integral homology spheres. For all $k \geq 1$, the quotient $\overline{\mathcal{S}}_k := \mathcal{S}_k / \sim_{Y_{k+1}}$ forms an abelian group under the connected sum [5]. It is a important problem to understand the structure of $\overline{\mathcal{S}}_k$ [13, Sec. 10.3]. For all $k \geq 1$, $\overline{\mathcal{S}}_{2k+1} = 0$, and it is well known that $\overline{\mathcal{S}}_{2k}$ is isomorphic, when tensoring by $\mathbb{Q}$, to the $\mathbb{Z}$-module $\mathcal{A}^c_k(\emptyset)$ of connected trivalent diagrams with $2k$ vertices [14].

In a recent joint work with K. Habiro, we used this space of diagrams $\mathcal{A}^c_k(\emptyset)$ to describe the so-called *Brunnian part* of the Goussarov-Vassiliev filtration. Let $\mathbb{Z}\mathcal{L}(n)$ be the free $\mathbb{Z}$-module generated by the set of isotopy classes of n-component links in S^3. The definition of Goussarov-Vassiliev invariants of links involves a descending filtration

$$\mathbb{Z}\mathcal{L}(n) = J_0(n) \supset J_1(n) \supset J_2(n) \supset ...$$

The *Brunnian part* $\mathrm{Br}(\overline{J}_{2n}(n+1))$ of $J_{2n}(n+1)/J_{2n+1}(n+1)$ is defined as the $\mathbb{Z}$-submodule generated by elements $[L-U]_{J_{2n+1}}$ where L is an $(n+1)$-component Brunnian link and U is the $(n+1)$-component unlink [7]. Habiro and the author

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constructed a linear map

$$h_n : \mathcal{A}^c_{n-1}(\emptyset) \longrightarrow \mathrm{Br}(\overline{J}_{2n}(n+1)),$$

and showed that h_n is an isomorphism over $\mathbb{Q}$ for $n \geq 2$.

As a consequence of this result, one observes that the abelian groups $\mathrm{Br}(\overline{J}_{2n}(n+1))$ and $\overline{\mathcal{S}}_{2n-2}$ are isomorphic over $\mathbb{Q}$, for $n \geq 2$. The following theorem states that this isomorphism is induced by $(+1)$-framed surgery. For a Brunnian link L in S^3, denote by $S^3_{(L,+1)}$ the 3-manifold obtained by surgery along the link L with all components having framing $+1$ (in fact, $S^3_{(L,+1)}$ is always an integral homology sphere, see Sec. 3.1).

Theorem 1.1. *For $n \geq 2$, the assignment*

$$[L-U]_{J_{2n+1}} \mapsto [S^3_{(L,+1)}]_{Y_{2n-1}}$$

defines an isomorphism

$$\kappa_n : \mathrm{Br}(\overline{J}_{2n}(n+1)) \otimes \mathbb{Q} \longrightarrow \overline{\mathcal{S}}_{2n-2} \otimes \mathbb{Q}.$$

Remark 1. Theorem 1.1 means that, *rationally*, any integral homology sphere M can be seen as a finite connected sum $\oplus_{1\leq i\leq m} S^3_{(L_{k_i},+1)}$ where each L_{k_i} is a Brunnian link in S^3. Consequently, M can be seen as a (finite) connected sum of integral homology spheres, each obtained from S^3 by $(+1)$-surgery along a single *knot*.

We organize the rest of this note as follows. In Section 2 we introduce the basic tools used in this work, namely the notions of clasper and finite type invariant. In Section 3 we state two results on surgery along Brunnian links in 3-manifolds, which are the main ingredients for the proof of Theorem 1.1. We conclude Section 3 with a sketch of this proof.

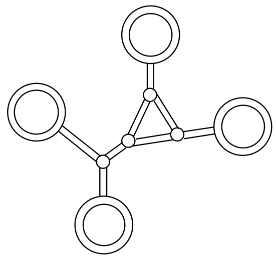
Note 1. This note is based on a talk given at the conference *Intelligence of low dimensional Topology 2006*, held in Hiroshima in July 2006. All results are taken from [9], where the reader is referred to for proofs and details.

2. Claspers and finite type invariants

In this section, we outline the main tools used for proving Theorem 1.1, namely the notions of claspers and finite type invariants.

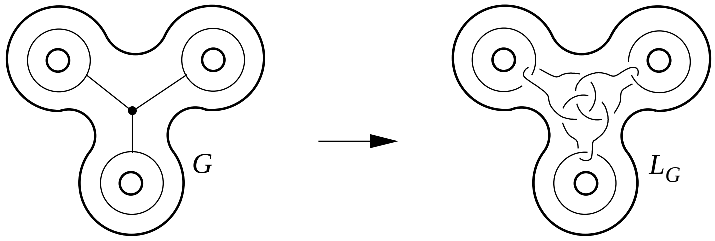
2.1. Clasper theory in a nutshell.

Roughly speaking, a clasper is essentially a surgery link of a special type. More precisely, a clasper G in a 3-manifold M is defined as a non-oriented embedding $G : F \hookrightarrow M$ of a surface F decomposed into disjoint discs and annuli connected by bands, such that each annulus (resp. disk) has one (resp. three) band(s) attached. An example is given below. (See [5] for a precise definition).



In this note, a clasper will always be supposed to be connected. The *degree* of a clasper is defined as the number of disks in the underlying surface.

Given a clasper G in a 3-manifold M, there is a procedure to construct, in a regular neighborhood of G, an associated framed link L_G in M. Though this procedure will not be explained here, it is well illustrated by the degree 1 case below.



Surgery along G simply means surgery along the associated framed link L_G.

For $k \geq 1$, the Y_k*-equivalence* is the equivalence relation on 3-manifolds generated by surgeries along degree k claspers and orientation-preserving diffeomorphisms. This equivalence relation is finer and finer as k increases, *i.e.* if $k > l$, the Y_k-equivalence implies the Y_l-equivalence.

Matveev showed that two closed oriented 3-manifolds are Y_1-equivalent if and only if they have isomorphic homology groups and linking form [8]. In particular, any two integral homology spheres are related by a sequence of surgeries along claspers.

2.1.1. *Finite type invariants.* Consider a Y_1-equivalence class $\mathcal{M}_0$ of compact connected oriented 3-manifolds, and let $k \geq 0$ be an integer.

A *finite type invariant of degree* k on $\mathcal{M}_0$ is a map $f : \mathcal{M}_0 \to A$, where A is an abelian group, such that for all $M \in \mathcal{M}_0$ and all family $F = \{G_1, ...G_l\}$ of disjoint claspers in M such that $\sum_i deg(G_i) \geq k+1$, we have

$$\sum_{F' \subseteq F} (-1)^{|F'|} f(M_{F'}) = 0,$$

where the sum runs over all subsets F' of F, and $|F'|$ denote the number of elements of F'. For $\mathcal{M}_0 = \mathbb{Z}HS$, this definition essentially coincides with Ohtsuki's notion of finite type invariant of integral homology spheres (up to a shift of index).

One can easily check from the definition that two Y_k-equivalent 3-manifolds cannot be distinguished by any finite type invariant f of degree $< k$. A fundamental result, due independently to Goussarov [3] and Habiro [5], is that the converse is also true for $\mathcal{M}_0 = \mathbb{Z}HS$.

3. Surgery on Brunnian links in 3-manifolds

In this section, we provide two results which are the main new ingredients for proving Theorem 1.1.

For a fixed number of components, we consider those compact connected oriented 3-manifolds which are obtained by surgery along a Brunnian link, and study which finite type invariants (*i.e.* of which degree) are preserved under such an operation.

3.1. **$\frac{1}{m}$-surgery along Brunnian links.** Let $m = (m_1, ..., m_n) \in \mathbb{Z}^n$ be a collection of n integers. Given a null-homologous, ordered link L in a compact connected oriented 3-manifold M, denote by (L, m) the link L with framing $\frac{1}{m_i}$ on the i^{th} component ; $1 \leq i \leq n$. We denote by $M_{(L,m)}$ the 3-manifold obtained from M by surgery along the framed link (L, m). We say that $M_{(L,m)}$ is obtained from M by $\frac{1}{m}$*-surgery* along the link L.

Theorem 3.1. *Let $n \geq 2$ and $m \in \mathbb{Z}^n$. Let L be an $(n+1)$-component Brunnian link in a compact, connected, oriented 3-manifold M.*

For $n = 2$, $M_{(L,m)}$ and M are Y_1-equivalent.

For $n \geq 3$, $M_{(L,m)}$ and M are Y_{2n-2}-equivalent. Consequently, they cannot be distinguished by any finite type invariant of degree $< 2n-2$.

In particular, $\frac{1}{m}$-surgery along a Brunnian link always preserves the homology. Note that, for any Brunnian link L in M, we have $M_{(L,m)} \cong M$ if $m_i = 0$ for some $1 \leq i \leq n$. In this case, the statement is thus vacuous.

Two links are *link-homotopic* if they are related by a sequence of isotopies and self-crossing changes, *i.e.* crossing changes involving two strands of the same component. This equivalence relation is natural to consider among Brunnian links because the set of link-homotopy classes of Brunnian links has a natural abelian group structure, induced by the band-sum. We obtain the following.

Theorem 3.2. *Let $n \geq 2$ and $m \in \mathbb{Z}^n$. Let L and L' be two link-homotopic $(n+1)$-component Brunnian links in a compact, connected, oriented 3-manifold M. Then $M_{(L,m)}$ and $M_{(L',m)}$ are Y_{2n-1}-equivalent. Consequently, they cannot be distinguished by any finite type invariant of degree $< 2n-1$.*

The proofs of Theorems 3.1 and 3.2 are very similar. We start with a characterization, in terms of claspers, of Brunnian links and the link-homotopy relation [6, 11]. Then, we only use one technical result on claspers (Theorem 3.2 of [9]). The proof of this technical result, which is in some sense the core of [9], involves rather advanced clasper theory.

3.2. **An important remark.** A natural question, in view of Theorems 3.1 and 3.2, is to ask whether finite type invariants (of appropriate degree) *can* indeed distinguish a manifold obtained by surgery along a Brunnian link from the initial manifold.

The answer is (predictably) yes. Consider for example the $(n+1)$-component Milnor link L_{n+1} (see [10, Fig. 7]) in S^3. It follows from [5, Sec. 7.25], and [9, Prop. 3.8] that

$$S^3_{(L,+1)} \sim_{Y_{2n+1}} S^3_{\Theta_n},$$

where Θ_n is the degree $2n$ clasper depicted on the left-hand side of Fig. 3.1.
So the LMO invariant of $S^3_{(L,+1)}$ is of the form

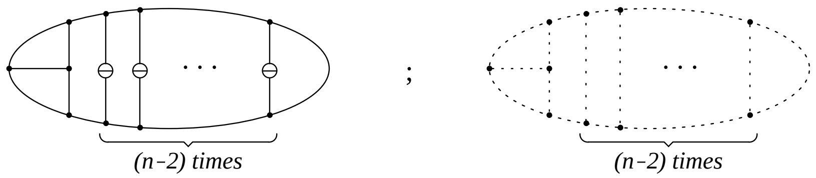


FIGURE 3.1. Here a $\ominus$ on an edge denotes a negative half-twist.

$$Z^{LMO}(S^3_{(L,+1)}) = 1 \pm \theta_n + \text{terms of higher degree},$$

where θ_n is the degree n trivalent diagram depicted on the right-hand side of Fig. 3.1. (See for example [14] for the computation of the LMO invariant of an integral homology sphere obtained from S^3 by surgery along a clasper). So the degree n part of the LMO invariant — which is a finite type invariant of degree $2n$ — distinguishes $S^3_{(L,+1)}$ from S^3.

3.3. **Sketch of the proof of Theorem 1.1.** We conclude this note by giving a brief (and rough) sketch of the proof of our main result.

For $k \geq 0$, let $\mathcal{A}_k(\emptyset)$ denote the $\mathbb{Z}$-module generated by trivalent diagrams (*i.e.* finite graphs with cyclicly ordered trivalent vertices) with $2k$ vertices, subject to the usual AS and IHX relations (see [1]). Denote by $\mathcal{A}^c_k(\emptyset)$ the $\mathbb{Z}$-submodule of $\mathcal{A}_k(\emptyset)$ generated by *connected* trivalent diagrams.

On one hand, there is a well-defined, surjective homomorphism of abelian groups

$$\phi_k : \mathcal{A}^c_k(\emptyset) \longrightarrow \overline{\mathcal{S}}_{2k},$$

and ϕ_k is an isomorphism over the rationals [14].

On the other hand, Habiro and the author showed that there is a well-defined map

$$h_n : \mathcal{A}^c_{n-1}(\emptyset) \longrightarrow \mathrm{Br}(\overline{J}_{2n}(n+1))$$

which also produces an isomorphism when tensoring by $\mathbb{Q}$, for $n \geq 2$ [7].

As said above, Theorems 3.1 and 3.2 are the main new tools for proving Theorem 1.1. Indeed, these theorems (together with some further arguments) provide a well defined map

$$\kappa_n : \mathrm{Br}(\overline{J}_{2n}(n+1)) \otimes \mathbb{Q} \longrightarrow \overline{\mathcal{S}}_{2n-2} \otimes \mathbb{Q}.$$

The result is then a consequence of the commutativity of the following diagram over the rationals, which follows quite easily from the definitions of the various maps

$$\begin{tikzcd} \mathcal{A}^c_{n-1}(\emptyset) \arrow[rr, "h_n"] \arrow[dr, "\phi_{n-1}"'] & & \mathrm{Br}(\overline{J}_{2n}(n+1)) \arrow[dl, "\kappa_n"] \\ & \overline{\mathcal{S}}_{2n-2} & \end{tikzcd}$$

Remark 2. To be more precise, it should be mentioned that we make use, in the proof of Theorem 1.1, of the *connected part* of the $2n^{th}$ graded quotient of the Ohtsuki filtration. The latter is naturally isomorphic to $\overline{\mathcal{S}}_{2n-2}$, and plays a central role in our proof. But this is outside the scope of this note.

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