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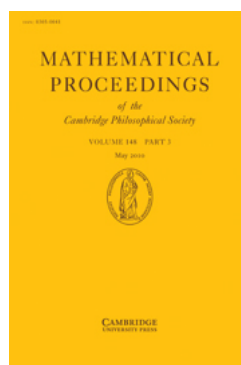
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Characterization of finite type string link invariants of degree < 5

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Abstract

We give a complete set of finite type string link invariants of degree < 5 . In addition to Milnor invariants, these include several string link invariants constructed by evaluating knot invariants on certain closures of (cabled) string links. We show that finite type invariants classify string links up to C_k -moves for $k \leq 5$, which proves, at low degree, a conjecture due to Goussarov and Habiro. We also give a similar classification of string links up to C_k -moves and concordance for $k \leq 6$.

Dedicated to Professor Akio Kawauchi on his 60th birthday

1. Introduction

The notion of Goussarov–Vassiliev finite type link invariants provides a unifying viewpoint on the various quantum link invariants [1, 9, 10, 35]. Denote by $\mathbf{Z}\mathcal{L}(m)$ the free abelian group generated by the set $\mathcal{L}(m)$ of isotopy classes of m -component oriented, ordered links in S^3 . An abelian group-valued link invariant is a finite type invariant of degree at most k if its linear extension to $\mathbf{Z}\mathcal{L}(m)$ vanishes on the $(k + 1)$ th term of the descending filtration

$$\mathbf{Z}\mathcal{L}(m) = J_0(m) \supset J_1(m) \supset \dots \tag{1.1}$$

where each $J_n(m)$ is generated by certain linear combinations of links associated with singular links with n double points. See Subsection 2.1.2 for a definition.

It is a natural question to ask for a topological characterization of finite type invariants. Habiro [14] and Goussarov [11] introduced independently the notion of C_k -move to address this question. A C_1 -move is just a crossing change, and for any integer $k \geq 2$, a C_k -move is a local move on (string) links as illustrated in Figure 1, which can be regarded as a kind of ‘higher order crossing change’.

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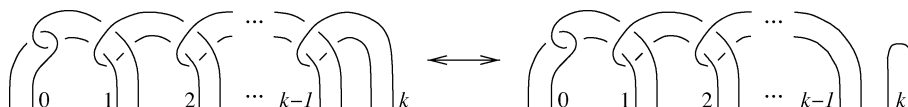


Fig. 1. A C_k -move involves $k + 1$ strands of a link, labelled here by integers between 0 and k .

The C_k -move generates an equivalence relation on links, called C_k -equivalence, which becomes finer as k increases. This notion can also be defined in terms of ‘insertion’ of an element of the k th lower central subgroup of some pure braid group [34], or alternatively by using the theory of clasps (see Section 3). Goussarov and Habiro showed independently the following.

THEOREM 1.1 ([11, 14]). *Two knots cannot be distinguished by any finite type invariant of order less than k if and only if they are C_k -equivalent.*

The ‘if’ part of the statement is easy to check. Actually, it is known to hold for links as well, but explicit examples show that the ‘only if’ part of Theorem 1.1 does not hold for links in general, see [14, section 7.2].

However, Theorem 1.1 may generalize to *string links*. Recall that a string link is a proper tangle without any closed components (see Subsection 2.1.1 for a precise definition).

CONJECTURE (Goussarov–Habiro [11, 14]). *Two string links of the same number of components share all finite type invariants of order less than k if and only if they are C_k -equivalent.*

One nice property of string links, which suggests some analogy with knots, is that they admit a natural composition. Indeed the stacking product \cdot endows the set $\mathcal{SL}(n)$ of n -string links up to isotopy fixing the endpoints with the structure of a monoid. In particular, string links with 1 component are exactly equivalent to knots, and their stacking product is equivalent to the connected sum \sharp of knots. The Goussarov–Habiro Conjecture is also supported by the fact that there are many more finite type invariants for string links than for links. For example, Milnor invariants [27, 28] are defined for both links and string links, but (except for the linking number) they are of finite type only for string links. See Subsection 2.1.4.

As in the link case, it is easy to check that the ‘if’ part of the conjecture is always true. The ‘only if’ part is also true for $k = 1$ (in which case the statement is vacuous) and $k = 2$; the only finite type string link invariant of degree 1 is the linking number, which is known to classify string links up to C_2 -equivalence [29]. (Note that this actually also applies to links.) The Goussarov–Habiro conjecture was then (essentially) proved for $k = 3$ by the first author in [25]. Massuyeau gave a proof for $k = 4$, but it is mostly based on algebraic arguments and thus does not provide any information about the corresponding finite type invariants [24].

In this paper, we classify n -string links up to C_k -equivalence for $k \leq 5$, by explicitly giving a complete set of low degree finite type invariants. In addition to Milnor invariants, these include several ‘new’ string link invariants constructed by evaluating knot invariants on certain closure of the (potentially cabled) string link. See Section 2 for the statements of these main results. As a consequence, we prove the Goussarov–Habiro Conjecture for $k \leq 5$.

We also consider the case of finite type concordance invariants. It is known that, over the rationals, these are all given by Milnor invariants [13]. We introduce the notion of C_k -concordance, which is the equivalence relation on (string) links generated by C_k -moves and concordance. We classify string links up to C_k -concordance for $k \leq 6$ by Milnor invariants and Arf invariant.

Finally, we apply some of the techniques developed in this paper to previous works by the authors [26, 36]. We first give a classification of 2-string links up to self C_3 -moves and concordance, where a self C_k -move is a C_k -move with all $k + 1$ strands in a single component. Next we consider Brunnian string links. Recall that a (string) link is Brunnian if it becomes trivial after removing any positive number of components. We give a classification of Brunnian n -string links up to C_{n+1} -equivalence, thus improving a previous result of the authors [26].

The rest of the paper is organized as follows. In Section 2, we recall the definitions of the main notions of this paper, and state our main results characterizing finite type string link invariants of degree < 5 . In Section 3, we review the main tool used in the proofs, namely the theory of claspers, and provide several key lemmas. The main results are proved in Section 4. In Section 5, we give a classification of string links up to C_k -moves and concordance for $k \leq 6$. Finally, in Section 6 we give the classification of 2-string links up to self C_3 -moves and concordance, and Section 7 contains our (improved) result on Brunnian string links.

2. Statements of the main results

In this section, we state our main results, which provide a complete set of finite type string link invariants of degree < 5 and validate the Goussarov–Habiro conjecture up to this degree.

2.1. Preliminaries

In this subsection we recall the definitions and properties of finite type string link invariants, and review several examples that will be used in our main results.

2.1.1. String links

Let $n \geq 1$, and let D^2 be the standard two-dimensional disk equipped with n marked points x_1, \dots, x_n in its interior. Let I denote the unit interval. An n -string link, or n -component string link, is a proper embedding

$$\sigma : \bigsqcup_{i=1}^n I_i \longrightarrow D^2 \times I,$$

of the disjoint union $\bigsqcup_{i=1}^n I_i$ of n copies of I in $D^2 \times I$, such that for each i , the image σ_i of I_i runs from $(x_i, 0)$ to $(x_i, 1)$. Abusing notation, we will also denote by $\sigma \subset D^2 \times I$ the image of the map σ , and σ_i is called the i th string of σ . Note that each string of an n -string link is equipped with an (upward) orientation induced by the natural orientation of I .

The set $\mathcal{SL}(n)$ of isotopy classes of n -string links fixing the endpoints has a monoidal structure, with composition given by the *stacking product* and with the trivial n -string link $\mathbf{1}_n$ as unit element. We shall sometimes denote the trivial string link by $\mathbf{1}$ when the number of component is irrelevant.

There is a surjective map $\hat{\cdot} : \mathcal{SL}(n) \rightarrow \mathcal{L}(n)$ which sends an n -string link σ to its closure $\hat{\sigma}$ (in the usual sense). For $n = 1$, this map is a monoid isomorphism.

We have a descending filtration

$$\mathcal{SL}(n) = \mathcal{SL}_1(n) \supset \mathcal{SL}_2(n) \supset \mathcal{SL}_3(n) \supset \dots$$

where $\mathcal{SL}_k(n)$ denotes the set of C_k -trivial n -string links, i.e., string links which are C_k -equivalent to $\mathbf{1}_n$. For $1 \leq k \leq l$, let $\mathcal{SL}_k(n)/C_l$ denote the set of C_l -equivalence classes

of C_k -trivial n -string links. This is known to be a finitely generated nilpotent group [14, theorem 5.4] (see also [10]). Furthermore, if $l \leq 2k$, this group is abelian [14, theorem 5.4].

2.1.2. *Finite type invariants of string links*

A *singular n -string link* is a proper immersion $\sqcup_{i=1}^n I_i \rightarrow D^2 \times I$ such that the image of I_i runs from $(x_i, 0)$ to $(x_i, 1)$ ($1 \leq i \leq n$), and whose singularities are transverse double points (in finite number).

Denote by $\mathbf{ZSL}(n)$ the free abelian group generated by $SL(n)$. A singular n -string link σ with k double points gives an element of $\mathbf{ZSL}(n)$ using the following skein formula.

$$\begin{array}{c} \nearrow \searrow \\ \bullet \\ \nwarrow \nearrow \end{array} = \begin{array}{c} \nearrow \nearrow \\ \nwarrow \nwarrow \end{array} - \begin{array}{c} \nearrow \nwarrow \\ \nwarrow \nearrow \end{array}. \tag{2.1}$$

Let A be an abelian group. An n -string link invariant $f: SL(n) \rightarrow A$ is a *finite type invariant of degree $\leq k$* if its linear extension to $\mathbf{ZSL}(n)$ vanishes on every n -string link with (at least) $k + 1$ double points. If f is of degree $\leq k$ but not of degree $k - 1$, then f is called a finite type invariant of degree k .

We recall several classical examples of such invariants in the next two subsections.

The Kontsevich integral [21]

$$Z : SL(n) \longrightarrow \mathcal{A}(\sqcup_n I)$$

is universal among rational-valued finite type string link invariants. The target space $\mathcal{A}(\sqcup_n I)$ of Z is the space of *Jacobi diagrams* on $\sqcup_{i=1}^n I_i$, that is, the vector space over \mathbf{Q} generated by vertex-oriented univalent diagrams whose univalent vertices are identified with distinct points on $\sqcup_{i=1}^n I_i$, modulo the FI (framing independence), AS, IHX and STU relations [1, 2]. Recall that $\mathcal{A}(\sqcup_n I)$ is graded by the degree of Jacobi diagrams, which is defined as half the number of vertices.

2.1.3. *Finite type knot invariants*

In this subsection we recall a few classical results on finite type knot invariants.

The *Conway polynomial* ∇ of oriented links is a polynomial in the variable z defined by the following two properties:

- (i) $\nabla_U(z) = 1$;
- (ii) $\nabla_{L_+}(z) - \nabla_{L_-}(z) = z \nabla_{L_0}(z)$,

where U denotes the unknot and where L_+ , L_- and L_0 are three links that are identical except in a 3-ball where they look as follows:

$$L_+ = \begin{array}{c} \nearrow \nearrow \\ \nwarrow \nwarrow \end{array} \quad ; \quad L_- = \begin{array}{c} \nearrow \nwarrow \\ \nwarrow \nearrow \end{array} \quad ; \quad L_0 = \begin{array}{c} \nearrow \\ \nwarrow \end{array} \begin{array}{c} \nearrow \\ \nwarrow \end{array}.$$

Recall that the Conway polynomial of a knot K has the form

$$\nabla_K(z) = 1 + \sum_{k \geq 1} a_{2k}(K) z^{2k}.$$

It is not hard to show that the z^{2k} -coefficient a_{2k} in the Conway polynomial is a finite type invariant of degree $2k$ [1].

The *HOMFLYPT polynomial* $P(L; t, z) \in \mathbf{Z}[t^{\pm 1}, z^{\pm 1}]$ of an oriented link L is defined by the following formulas (using the same notation as above):

- (i) $P(U; t, z) = 1;$
- (ii) $t^{-1}P(L_+; t, z) - tP(L_-; t, z) = zP(L_0; t, z).$

Recall that the Conway polynomial $\nabla_L(z)$ of L is given from its HOFMLYPT polynomial by $\nabla_L(z) = P(L; 1, z)$. Recall also that the HOMFLYPT polynomial of a knot K is of the form

$$P(K; t, z) = \sum_{k=0}^N P_{2k}(K; t)z^{2k},$$

where $P_{2k}(K; t) \in \mathbf{Z}[t^{\pm 1}]$ is called the $2k$ th coefficient polynomial of K . Denote by $P_{2k}^{(l)}(K)$ the l th derivative of $P_{2k}(K; t)$ evaluated at $t = 1$. It was proved by Kanenobu and Miyazawa that $P_{2k}^{(l)}$ is a finite type invariant of degree $2k + l$ [19].

Note that both the Conway and HOMFLYPT polynomials of knots are invariant under orientation reversal, and that both are multiplicative under the connected sum [22].

In the rest of this paper, we will freely evaluate these invariants on components of an n -string link, via the closure isomorphism $\mathcal{SL}(1) \simeq \mathcal{L}(1)$. For example, $a_2(\sigma_i)$ denotes the invariant a_2 of the closure $\hat{\sigma}_i$.

2.1.4. Milnor invariants

Given an n -component oriented, ordered link L in S^3 , Milnor invariants $\bar{\mu}_L(I)$ of L are defined for each multi-index $I = i_1 i_2 \dots i_m$ (i.e., any sequence of possibly repeating indices) among $\{1, \dots, n\}$ [27, 28]. The number m is called the *length* of Milnor invariant $\bar{\mu}(I)$, and is denoted by $|I|$. Unfortunately, the definition of these $\bar{\mu}(I)$ contains a rather intricate self-recurrent indeterminacy.

Habegger and Lin showed that Milnor invariants are actually well defined integer-valued invariants of string links [12], and that the indeterminacy in Milnor invariants of a link is equivalent to the indeterminacy in regarding it as the closure of a string link. We refer the reader to [12] or [36] for a precise definition of Milnor invariants $\mu(I)$ of string links. The smallest length Milnor invariants $\mu_\sigma(ij)$ of a string link σ coincide with the linking numbers $lk(\hat{\sigma}_i, \hat{\sigma}_j)$. Milnor invariants are thus sometimes referred to as ‘higher order linking numbers’.

It is known that $\mu(I)$ is a finite type invariant of degree $|I| - 1$ for string links [2, 23].

Convention 2.1. As stated above, each Milnor invariant $\mu(I)$ for n -string links is indexed by a sequence I of *possibly repeating* integers in $\{1, \dots, n\}$. In the following, when denoting indices of Milnor invariants, we will always let *distinct* letters denote *distinct* integers, unless otherwise specified. For example, $\mu(iijk)$ ($1 \leq i, j, k \leq n$) stands for all Milnor invariants $\mu(iijk)$ with $i, j, k \in \{1, \dots, n\}$ pairwise distinct.

2.2. Invariants of degree ≤ 2 for string links

We start by recalling the classification of n -string links up to C_3 -equivalence due to the first author.¹

It turns out that, in addition to the z^2 -coefficient a_2 in the Conway polynomial (which is essentially the only finite type knot invariant of degree ≤ 2) and Milnor invariants of length ≤ 3 , this classification requires an additional finite type invariant of degree 2 for

¹ Actually, the present statement is stronger than the one appearing in [25]. However, the proof given in Subsection 4.1 is essentially contained in [25].

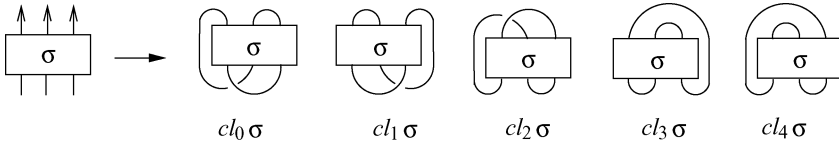


Fig. 2. The five closures $cl_i \sigma$ ($i = 0, \dots, 4$) of a 3-string link σ .

2-string links

$$f_2 : \mathcal{SL}(2) \longrightarrow \mathbf{Z},$$

defined by $f_2(\sigma) = a_2(\bar{\sigma})$. Here $\bar{\sigma}$ denotes the *plat closure* of σ , which is the knot obtained by identifying the two upper (resp. lower) endpoints of σ . More precisely, we have the following.

THEOREM 2.2 ([25]). *Let $\sigma, \sigma' \in \mathcal{SL}(n)$. Then the following assertions are equivalent:*

- (i) σ and σ' are C_3 -equivalent;
- (ii) σ and σ' share all finite type invariants of degree ≤ 2 ;
- (iii) σ and σ' have same Kontsevich integral up to degree 2;
- (iv) σ and σ' share all invariants a_2 and f_2 , and all Milnor invariants $\mu(ij)$ ($1 \leq i < j \leq n$) and $\mu(ijk)$ ($1 \leq i < j < k \leq n$).

Convention 2.3. In (iv), by σ and σ' share all invariants a_2 and f_2 , we mean that $\sigma = \bigcup_{i=1}^n \sigma_i$ and $\sigma' = \bigcup_{i=1}^n \sigma'_i$ satisfy $a_2(\sigma_i) = a_2(\sigma'_i)$ and $f_2(\sigma_i \cup \sigma_j) = f_2(\sigma'_i \cup \sigma'_j)$ for all $1 \leq i < j \leq n$. We shall make use of a similar abuse of notation in subsequent statements.

2.3. Invariants of degree 3 for string links

Recall that there is essentially only one finite type knot invariant of degree 3, namely $P_0^{(3)}$. Let

$$f_3 : \mathcal{SL}(2) \longrightarrow \mathbf{Z}$$

be defined by $f_3(\sigma) := P_0^{(3)}(\bar{\sigma})$, where $\bar{\sigma}$ is the plat-closure of σ , and let

$$V_3 : \mathcal{SL}(3) \longrightarrow \mathbf{Z}$$

be defined by $V_3(\sigma) := P_0^{(3)}(cl_3 \sigma)$, where $cl_3 \sigma$ is the closure operation illustrated in Figure 2.

Clearly, f_3 and V_3 are both finite type invariants of degree at most 3.

THEOREM 2.4. *Let $\sigma, \sigma' \in \mathcal{SL}(n)$. Then the following assertions are equivalent:*

- (i) σ and σ' are C_4 -equivalent;
- (ii) σ and σ' share all finite type invariants of degree ≤ 3 ;
- (iii) σ and σ' have same Kontsevich integral up to degree 3;
- (iv) σ and σ' share all invariants $a_2, P_0^{(3)}, f_2, f_3$ and V_3 , and all Milnor invariants $\mu(ij), \mu(iijj)$ ($1 \leq i < j \leq n$), $\mu(ijk)$ ($1 \leq i < j < k \leq n$), $\mu(ijkl)$ ($1 \leq i, j < k < l \leq n$) and $\mu(ijkk)$ ($1 \leq i, j, k \leq n; i < j$).

2.4. Invariants of degree 4 for string links

There are essentially two linearly independent finite type knot invariants of degree 4, namely a_4 and $P_0^{(4)}$. We will use these two knot invariants to define a number of finite type

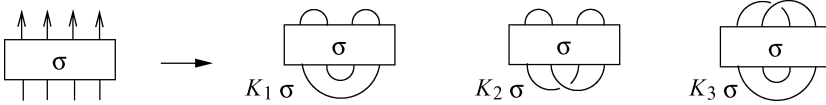


Fig. 3. The three closures $K_i(\sigma)$ ($i = 1, 2, 3$) of a 4-string link σ .

string links invariants of degree 4 by using some cabling and closure operations. We start by setting up some notation.

Given a 3-string link σ , denote by $cl_i\sigma$, $i = 0, \dots, 4$, the five knots obtained from σ by taking the closures illustrated in Figure 2.

Also, given a 4-string link σ , denote by $K_i(\sigma)$, $i = 1, 2, 3$, the knot obtained by the closure operations represented in Figure 3.

Finally, for a 2-string link σ , denote by $\Delta_i\sigma$ ($i = 1, 2$) the 3-string link obtained by taking a 0-framed parallel copy of the i th component σ_i of σ .

We now define five invariants of 2-string links as follows. For $1 \leq i \leq 5$, let

$$f_4^i : \mathcal{SL}(2) \longrightarrow \mathbf{Z}$$

be defined by the following

$$\begin{aligned} f_4^1(\sigma) &:= a_4(\bar{\sigma}), \quad f_4^2(\sigma) := P_0^{(4)}(\bar{\sigma}), \\ f_4^3(\sigma) &:= a_4(cl_0(\Delta_1\sigma)), \quad f_4^4(\sigma) := P_0^{(4)}(cl_0(\Delta_1\sigma)), \\ &\text{and } f_4^5(\sigma) := P_0^{(4)}(cl_1(\Delta_2\sigma)). \end{aligned}$$

We have that f_4^i is a finite type invariant of degree ≤ 4 for $i = 1, \dots, 5$. (It is immediate for $i = 1, 2$, and easy to check for $i = 3, 4, 5$.)

Next we define seven invariants of 3-string links. For $1 \leq i \leq 7$, let

$$V_4^i : \mathcal{SL}(3) \longrightarrow \mathbf{Z}$$

be defined by the following

$$\begin{aligned} V_4^1(\sigma) &:= a_4(cl_1\sigma), \quad V_4^4(\sigma) := P_0^{(4)}(cl_1\sigma), \\ V_4^2(\sigma) &:= a_4(cl_2\sigma), \quad V_4^5(\sigma) := P_0^{(4)}(cl_2\sigma), \\ V_4^3(\sigma) &:= a_4(cl_3\sigma), \quad V_4^6(\sigma) := P_0^{(4)}(cl_3\sigma), \\ &\text{and } V_4^7(\sigma) := P_0^{(4)}(cl_4\sigma). \end{aligned}$$

Clearly, each V_4^i is a finite type invariant of degree ≤ 4 , $i = 1, \dots, 7$.

Finally, we define three finite type invariants of degree ≤ 4 of 4-string links

$$W_4^i : \mathcal{SL}(4) \longrightarrow \mathbf{Z}$$

by setting $W_4^i(\sigma) := P_0^{(4)}(K_i(\sigma))$, $1 \leq i \leq 3$.

These various invariants, together with Milnor invariants of length ≤ 5 , give the following classification of n -string links up to C_5 -equivalence.

THEOREM 2.5. *Let $\sigma, \sigma' \in \mathcal{SL}(n)$. Then the following assertions are equivalent:*

- (i) σ and σ' are C_5 -equivalent;
- (ii) σ and σ' share all finite type invariants of degree ≤ 4 ;
- (iii) σ and σ' have same Kontsevich integral up to degree 4;
- (iv) σ and σ' share all knots invariants of degree ≤ 4 , all invariants $f_2, f_3, V_3, f_4^i, V_4^i$ and W_4^i , and all Milnor invariants of length ≤ 5 ,

where, in (iv), σ and σ' share all Milnor invariants of length ≤ 5 if and only if they share all $\mu(ij)$, $\mu(iijj)$ ($1 \leq i < j \leq n$), $\mu(ijk)$ ($1 \leq i < j < k \leq n$), $\mu(ijkl)$ ($1 \leq i, j < k < l \leq n$), $\mu(ijkk)$ ($1 \leq i, j, k \leq n; i < j$), $\mu(ijklm)$ ($1 \leq i, j, k < l < m \leq n$), $\mu(iiijk)$, $\mu(ijjkk)$ and $\mu(jikll)$ ($1 \leq i, j, k, l \leq n; j < k$).

Remark 2.6. A complete set of finite type link invariants of degree ≤ 3 has been computed in [20] using weight systems and chord diagrams. For 2-component links, this has been done for degree ≤ 4 invariants in [18]. All invariants are given by coefficients of the Conway and HOMFLYPT polynomials of sublinks.

3. Claspers and local moves on links

The primary tool in the proofs of our main results is the theory of claspers. We recall here the main definitions and properties of this theory, and state a couple of additional lemmas that will be useful in later sections.

3.1. A brief review of clasper theory

For convenience, we give all definitions and statements in the context of string links. For a general definition of claspers, we refer the reader to [14].

Definition 3.1. Let σ be a string link. An embedded surface G is called a *graph clasper* for σ if it satisfies the following three conditions:

- (i) G is decomposed into disks and bands, called *edges*, each of which connects two distinct disks;
- (ii) the disks have either 1 or 3 incident edges, and are called *leaves* or *nodes* respectively;
- (iii) G intersects σ transversely, and the intersections are contained in the union of the interiors of the leaves.

In particular, if a connected graph clasper G is simply connected, we call it a *tree clasper*.

A graph clasper for a string link σ is *simple* if each of its leaves intersects σ at one point.

The degree of a connected graph clasper G is defined as half of the number of nodes and leaves. We call a degree k connected graph clasper a C_k -graph. A tree clasper of degree k is called a C_k -tree. A C_k -graph with loop is a C_k -graph which is not a C_k -tree.

Convention 3.1. Throughout this paper, we make use of the following graphical convention. The drawing convention for claspers are those of [14, figure 7], except for the following: a \oplus (resp. \ominus) on an edge represents a positive (resp. negative) half-twist. (This replaces the convention of a circled S (resp. S^{-1}) used in [14].) When representing a clasper c with an edge marked by a $*$, we implicitly also define the clasper c^{-1} which is obtained from c by inserting a *positive half twist* in the $*$ -marked edge. Likewise, when introducing the string link σ obtained from $\mathbf{1}$ by surgery along a clasper c with a $*$ -marked edge, we implicitly also introduce the string link σ^{-1} obtained from $\mathbf{1}$ by surgery along c^{-1} . (This convention/notation is motivated by Lemma 3.2(2).) We will also make use of this convention for knots in S^3 .

Given a graph clasper G for a string link σ , there is a procedure to construct, in a regular neighbourhood of G , a framed link $\gamma(G)$. There is thus a notion of *surgery along G* , which is defined as surgery along $\gamma(G)$. There exists a canonical diffeomorphism between $D^2 \times I$ and the manifold $(D^2 \times I)_{\gamma(G)}$, and surgery along the C_k -graph G can be regarded as an operation on σ in the (fixed) ambient space $D^2 \times I$. We say that the resulting string link σ_G in $D^2 \times I$ is obtained from σ by surgery along G . In particular, surgery along a simple

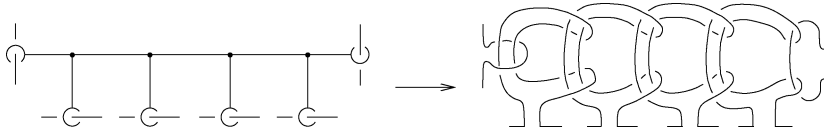


Fig. 4. Surgery along a simple C_5 -tree.

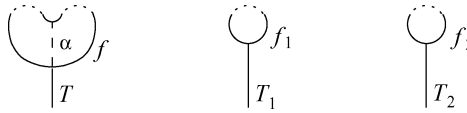


Fig. 5. The 3 claspers are identical outside a small ball, where they are as depicted.

C_k -tree is a local move as illustrated in Figure 4, which is equivalent to a C_k -move as defined in the introduction (Figure 1).

For $k \geq 3$, a C_k -tree G having the shape of the tree clasper in Figure 4 is called *linear*, and the left-most and right-most leaves of G in Figure 4 are called the *ends* of G . Note that there is no ambiguity in the notion of ends of a linear tree as defined here, by referring to the standard situation of Figure 4. Recall indeed that a clasper is an embedded surface: given a linear tree T there is a unique pair (f, f') of leaves joined by the longest path (of edges and nodes) in T such that we can run along the boundary of this underlying surface from ∂f to $\partial f'$ without meeting any other leaf. Note, however, that this notion is ambiguous for C_2 -trees.

The C_k -equivalence (as defined in the introduction) coincides with the equivalence relation on string links generated by surgeries along C_k -graphs and isotopies. In particular, it is known that two links are C_k -equivalent if and only if they are related by surgery along simple C_k -trees [14, theorem 3.17].

3.2. Calculus of Claspers

In this subsection, we summarize several properties of claspers, whose proofs can be found in [14].

LEMMA 3.2 (Calculus of Claspers). (1) Let T be a union of tree claspers for a string link σ , and let C be a component of T which is a C_k -tree. Let T' be obtained from T by passing an edge of C across σ or across another edge of T , or by sliding a leaf of C over a leaf of another component of T .² Then $\sigma_T \stackrel{C_{k+1}}{\sim} \sigma_{T'}$.

(2) Let T be a C_k -tree for $\mathbf{1}_n$ and let \bar{T} be a C_k -tree obtained from T by adding a half-twist on an edge. Then $(\mathbf{1}_n)_T \cdot (\mathbf{1}_n)_{\bar{T}} \stackrel{C_{k+1}}{\sim} \mathbf{1}_n$.

(3) Let T be a C_k -tree for $\mathbf{1}_n$. Let f_1 and f_2 be two disks obtained by splitting a leaf f of T along an arc α as shown in Figure 5. Then, $(\mathbf{1}_n)_T \stackrel{C_{k+1}}{\sim} (\mathbf{1}_n)_{T_1} \cdot (\mathbf{1}_n)_{T_2}$, where T_i denotes the C_k -tree for $\mathbf{1}_n$ obtained from T by replacing f by f_i ($i = 1, 2$), see Figure 5.

In our proofs, we shall use combinations of these relations in many places, and will always refer to them as *Calculus of Claspers*.

Claspers also satisfy relations analogous to the AS, IHX and STU relations for Jacobi diagrams [1].

² For example, the clasper G_U of Figure 6 is obtained from G_T by sliding a leaf over another one.

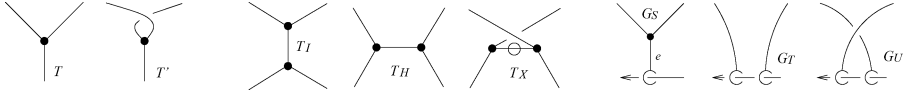


Fig. 6. The AS, IHX and STU relations.

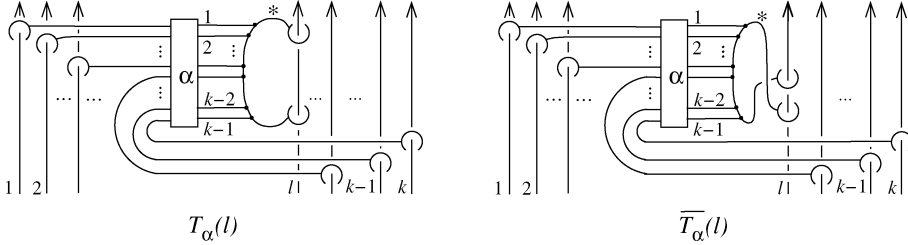


Fig. 7. The C_k -trees $T_\alpha(l)$ and $\overline{T}_\alpha(l)$.

LEMMA 3.3.

(AS) Let T and T' be two C_k -graphs for $\mathbf{1}_n$ which differ only in a small ball as depicted in Figure 6. Then $(\mathbf{1}_n)_T \cdot (\mathbf{1}_n)_{T'} \stackrel{C_{k+1}}{\sim} \mathbf{1}_n$.

(IHX) Let T_I , T_H and T_X be three C_k -graphs for $\mathbf{1}_n$ which differ only in a small ball as depicted in Figure 6. Then $(\mathbf{1}_n)_{T_I} \stackrel{C_{k+1}}{\sim} (\mathbf{1}_n)_{T_H} \cdot (\mathbf{1}_n)_{T_X}$.

(STU) Let G_S , G_T and G_U be three C_k -graphs for $\mathbf{1}_n$ which differ only in a small ball as depicted in Figure 6. Then $(\mathbf{1}_n)_{G_S} \cdot (\mathbf{1}_n)_{G_T} \stackrel{C_{k+1}}{\sim} (\mathbf{1}_n)_{G_U}$.

In the rest of the paper, we will simply refer to Lemma 3.3 as the AS, IHX and STU relations. In some cases, it will be convenient to also use the following terminology. If e denotes the edge of a graph clasper G_S (resp. if f and f' denote the leaves of G_T or G_U) as in Figure 6, we will sometimes say that we apply the STU relation *at the edge e* (resp. at the leaves f and f') when applying Lemma 3.3(STU).

Note that the STU relation stated above differs by a sign from the STU relation for Jacobi diagrams.

We conclude this subsection with an additional lemma which will be used later. We first need some extra notation.

Let $k > 2$ and $l \in \{1, \dots, k\}$ be integers. Denote by $\mathcal{B}_k(l)$ the set of all bijections $\alpha : \{1, \dots, k-1\} \rightarrow \{1, \dots, k\} \setminus \{l\}$ such that $\alpha(1) < \alpha(k-1)$. We denote by $id \in \mathcal{B}_k(l)$ the element which maps i to itself if $1 \leq i < l$, and to $i+1$ otherwise. For each $\alpha \in \mathcal{B}_k(l)$, let $T_\alpha(l)$ and $\overline{T}_\alpha(l)$ denote the C_k -trees for $\mathbf{1}_k$ represented in Figure 7. Denote respectively by $B_\alpha(l)$ and $\overline{B}_\alpha(l)$ the k -string links obtained from $\mathbf{1}_k$ by surgery along $T_\alpha(l)$ and $\overline{T}_\alpha(l)$.

LEMMA 3.4. Let $k > 2$ and $l \in \{1, \dots, k\}$ be integers. For any $\alpha \in \mathcal{B}_k(l)$ and any integer l' ($1 \leq l' \leq k$, $l' \neq l$), there is a bijection $\beta \in \mathcal{B}_k(l')$ such that $B_\alpha(l) \cdot (\overline{B}_\alpha(l))^{-1} \stackrel{C_{k+1}}{\sim} B_\beta(l') \cdot (\overline{B}_\beta(l'))^{-1}$.

Proof. Observe that $T_\alpha(l)$ and $\overline{T}_\alpha(l)$ are identical except in a 3-ball where they look exactly like G_T and G_U in Figure 6. So by the STU relation we have $B_\alpha(l) \cdot (\overline{B}_\alpha(l))^{-1} \stackrel{C_{k+1}}{\sim} \mathbf{1}_G$, where G is a C_k -graph intersecting each component of $\mathbf{1}_k$ once. Note that G has one loop, and that each leaf of G is connected to the loop by a single edge. So for each $1 \leq l' \leq k$ we

can apply the STU relation at the edge of G which is attached to the leaf intersecting the l 'th component of $\mathbf{1}_k$. This gives the desired formula.

3.3. k -additivity

We now introduce the notion of k -additivity of a string link invariant.

Definition 3.2. Let $k, n \geq 1$ be integers. We say that an invariant $v: \mathcal{SL}(n) \rightarrow \mathbf{Z}$ is k -additive if for every $\sigma \in \mathcal{SL}(n)$ and every $\sigma' \in \mathcal{SL}_k(n)$, we have $v(\sigma \cdot \sigma') = v(\sigma) + v(\sigma')$.

Note that a string link invariant is additive if and only if it is 1-additive. Note also that for $k > l$, the l -additivity implies the k -additivity. We now show that all the invariants involved in our classification results are k -additive for some k .

First, Milnor invariants $\mu(I)$ of length $|I| = k$ are $(k - 1)$ -additive. This follows from Milnor invariants' additivity property [26, lemma 3.3] and the fact that Milnor invariants of length k are C_k -equivalence invariants [14, theorem 7.2].

Now, observe that the plat closure of the product of two 2-string links $\sigma \cdot \sigma'$ is just the connected sum of their plat closures. So it follows, by the multiplicativity of the Conway and HOMFLYPT polynomial (see Subsection 2.1.3), that f_2 is 2-additive, f_3 is 3-additive, and f_4^1 and f_4^2 are both 4-additive.

Alternatively, the above can be shown using the following observation, which was pointed out to us by the referee.

Claim 3.5. Let v be a finite type invariant of n -string links of degree $\leq k$. If $v(\mathbf{1}_n) = 0$, then v is k -additive.

Proof of Claim 3.5 Let $\sigma \in \mathcal{SL}(n)$ and $\sigma' \in \mathcal{SL}_k(n)$. There exists a disjoint union $G = G_1 \cup \dots \cup G_r$ ($r \geq 1$) of C_1 -trees for $\mathbf{1}_n$ such that $\sigma = (\mathbf{1}_n)_G$, and a disjoint union $H = H_1 \cup \dots \cup H_s$ ($s \geq 1$) of C_k -trees for $\mathbf{1}_n$ such that $\sigma' = (\mathbf{1}_n)_H$. One can check that in $\mathbf{ZSL}(n)$ we have

$$\sum_{i=0}^{r-1} \sum_{j=0}^{s-1} ((\sigma_i)_{G_{i+1}} \cdot (\sigma'_j)_{H_{j+1}} - (\sigma_i)_{G_{i+1}} \cdot (\sigma'_j) - (\sigma_i) \cdot (\sigma'_j)_{H_{j+1}} + (\sigma_i) \cdot (\sigma'_j)) = \sigma \cdot \sigma' - \sigma - \sigma' + \mathbf{1}_n,$$

where $\sigma_i = (\mathbf{1}_n)_{G_1 \cup \dots \cup G_i}$ ($i \neq 0$), $\sigma'_j = (\mathbf{1}_n)_{H_1 \cup \dots \cup H_j}$ ($j \neq 0$), and $\sigma_0 = \sigma'_0 = \mathbf{1}_n$. Now, since v is a finite type invariant of degree $\leq k$, by [14, theorem 6.7 (1)] we have

$$v((\sigma_i)_{G_{i+1}} \cdot (\sigma'_j)_{H_{j+1}} - (\sigma_i)_{G_{i+1}} \cdot (\sigma'_j) - (\sigma_i) \cdot (\sigma'_j)_{H_{j+1}} + (\sigma_i) \cdot (\sigma'_j)) = 0$$

for any pair i, j . This implies the desired statement.

It follows that V_3 is 3-additive and that each invariant V_4^i ($i = 1, \dots, 7$), W_4^i ($i = 1, 2, 3$) and f_4^j ($j = 3, 4, 5$) are 4-additive invariants.

3.4. The clasper index

Let G be a simple C_k -graph for an n -string link σ . We call a leaf of G an i -leaf if it intersects the i th component of σ . The *index* of G is the collection of all integers i such that G contains an i -leaf, counted with multiplicities. For example, a simple C_3 -tree of index $\{2, 3^{(2)}, 5\}$ for σ intersects component 3 twice and components 2 and 5 once (and is disjoint from all other components of σ).

We will need the following lemma.



Fig. 8.

LEMMA 3.6. For $k \geq 3$, let T be a simple C_k -tree of index $\{i, j^{(k)}\}$ for an n -string link σ , $1 \leq i, j \leq n$. Then σ_T is C_{k+1} -equivalent to a string link σ' which is obtained from σ by surgery along C_k -trees with index $\{i^{(2)}, j^{(k-1)}\}$.

Proof. For simplicity we prove the lemma for $\sigma = \mathbf{1}_n$. For an arbitrary σ , the proof is strictly similar (using the fact that there exists a tree clasper C such that $\sigma = (\mathbf{1}_n)_C$). Pick a node of T which is connected to two j -leaves f and f' . Travelling along the j th component of $\mathbf{1}_n$ from f to f' , we meet in order m j -leaves f_1, \dots, f_m . The proof is by induction on the number m of leaves separating f and f' .

If $m = 0$, then using Calculus of Claspers we may assume that there exists a 3-ball which intersects T as on the left-hand side of Figure 8. By the IHX and STU relations, we have $\mathbf{1}_T \stackrel{C_{k+1}}{\sim} \mathbf{1}_G$, where G is a simple C_k -graph with one loop and with index $\{i, j^{(k-1)}\}$ as illustrated in Figure 8. We now prove that any simple C_k -graph C for $\mathbf{1}_n$ with one loop and with index $\{i, j^{(k-1)}\}$ satisfies

$$(\mathbf{1}_n)_C \stackrel{C_{k+1}}{\sim} (\mathbf{1}_n)_F, \tag{3.1}$$

where F is a disjoint union of simple C_k -trees for $\mathbf{1}_n$ with index $\{i^{(2)}, j^{(k-1)}\}$. In order to prove (3.1), observe that the unique i -leaf l of C is connected to the loop γ of C by a path P of edges and nodes. We proceed by induction on the number n of nodes in P . For $n = 0$, applying the STU relation at the edge connecting l to γ proves the claim. For an arbitrary $n \geq 1$, applying the IHX relation at the edge of P which is incident to γ gives $\mathbf{1}_C \stackrel{C_{k+1}}{\sim} \mathbf{1}_{C'} \cdot \mathbf{1}_{C''}$, where C' and C'' are two simple C_k -graphs with a unique i -leaf connected to a loop by a path with $(n - 1)$ nodes. Equation (3.1) then follows from the induction hypothesis.

Now suppose that f and f' are separated by m j -leaves f_1, \dots, f_m ($m \geq 1$). We can apply the STU relation at the leaves f_m and f' to obtain that $\mathbf{1}_T \stackrel{C_{k+1}}{\sim} \mathbf{1}_{T'} \cdot \mathbf{1}_G$, where T' is the C_k -tree obtained by sliding f_m over f' (so that the j -leaves f and f' of T' are separated by $m - 1$ leaves)³ and where G is a simple C_k -graph with one loop and with index $\{i, j^{(k-1)}\}$. The result thus follows from (3.1) and the induction hypothesis.

4. Proofs of the main results

In this section we give the proofs of Theorems 2.2, 2.4 and 2.5. The plan of proof is always the same and is as follows. That (1) \Rightarrow (2) \Rightarrow (4) is clear, so the core of the proof consists in showing that (4) \Rightarrow (1). This is done by giving an explicit representative for the C_k -equivalence class ($k = 3, 4, 5$) of an arbitrary n -string link, in terms of the invariants listed in (4). That (3) \Leftrightarrow (2) follows from the fact that the group $\mathcal{SL}(n)/C_k$ is torsion-free for $k = 3, 4, 5$, which comes as a consequence of the fact that no torsion element appears in our representatives.

Before proceeding to the proofs, we summarize in Figure 9, for the reader's convenience, the various knots that will be used throughout the rest of this section. (We implicitly define

³ Abusing notation, we still call f and f' the corresponding j -leaves of T' .

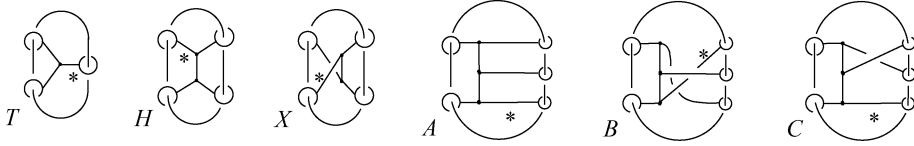


Fig. 9. The knots T, H, X, A, B and C .

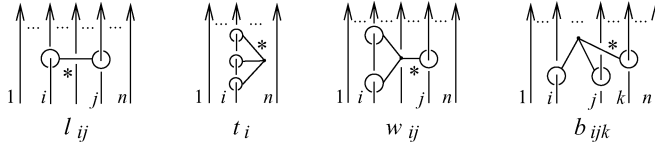


Fig. 10. The C_1 -tree l_{ij} and the C_2 -trees t_i, w_{ij} and b_{ijk} .

the knots in Figure 9 as the results of surgery along the represented tree claspers for the unknot U .) We will sometimes identify these knots with their images by the monoid isomorphism $\mathcal{L}(1) \simeq \mathcal{SL}(1)$. Also, for each knot K in Figure 9 and for any $1 \leq i \leq n$, we will denote by K_i the n -string link obtained from $\mathbf{1}_n$ by connected sum of a copy of K on the i th component.

4.1. Proof of Theorem 2.2

Let $\sigma \in \mathcal{SL}(n)$. By Murakami–Nakanishi’s characterization of C_2 -equivalence [29], we have that σ is C_2 -equivalent to

$$\sigma_{(1)} := \prod_{1 \leq i < j \leq n} L_{ij}^{\mu_\sigma(ij)}, \tag{4.1}$$

where $L_{ij} \in \mathcal{SL}(n)$ is obtained by surgery along the C_1 -tree l_{ij} represented in Figure 10.⁴ Here, the product is taken using the lexicographic order. (We shall make implicitly use of this convention in the rest of this paper.) So σ is obtained from $\sigma_{(1)}$ by surgery along C_k -trees ($k \geq 2$). By Calculus of Claspers, this implies that $\sigma \overset{C_3}{\sim} \sigma_{(1)} \cdot \sigma_{(2)}$, where

$$\sigma_{(2)} := \prod_{1 \leq i \leq n} T_i^{\alpha_i} \cdot \prod_{1 \leq i < j \leq n} W_{ij}^{\beta_{ij}} \cdot \prod_{1 \leq i < j < k \leq n} B_{ijk}^{\gamma_{ijk}}, \tag{4.2}$$

for some integers α_i, β_{ij} and γ_{ijk} , where T_i, W_{ij} and B_{ijk} are n -string links obtained from $\mathbf{1}_n$ by surgery along the C_2 -trees t_i, w_{ij} and b_{ijk} represented in Figure 10 respectively. Note that the closure of T_i, W_{ij} and B_{ijk} are the trefoil, Whitehead link and Borromean rings respectively. Note also that $W_{ij} = W_{ji}$ (see for example [25, figure 6]). For $\gamma \in \mathcal{SL}(n)$, set $(a_2)_i(\gamma) = a_2(\gamma_i)$ ($1 \leq i \leq n$) and $(f_2)_{ij}(\gamma) = f_2(\gamma_i \cup \gamma_j)$ ($1 \leq i, j \leq n$). We have $(a_2)_i(T_j) = \delta_{i,j}$, $(f_2)_{ij}(T_k) = \delta_{i,k} + \delta_{j,k}$, $(f_2)_{ij}(W_{kl}) = \delta_{(i,j),(k,l)}$ and $\mu_{B_{abc}}(ijk) = \delta_{(i,j,k),(a,b,c)}$, where δ denotes the Kronecker delta. Using the fact that a_2, f_2 and $\mu(ijk)$ are all 2-additive, it follows that in (4.2) we have

$$\begin{aligned} \alpha_i &= (a_2)_i(\sigma_{(2)}) = (a_2)_i(\sigma), \\ \beta_{ij} &= (f_2)_{ij}(\sigma_{(2)}) - (a_2)_j(\sigma_{(2)}) - (a_2)_i(\sigma_{(2)}) = (f_2)_{ij}(\sigma) - (a_2)_j(\sigma) - (a_2)_i(\sigma), \\ \text{and } \gamma_{ijk} &= \mu_{\sigma_{(2)}}(ijk) = \mu_\sigma(ijk) - \mu_{\sigma_{(1)}}(ijk). \end{aligned}$$

This concludes the proof.

⁴ Recall from Convention 3.1 that this also implicitly defines the n -string link L_{ij}^{-1} obtained by surgery along the C_1 -tree l_{ij}^{-1} that differs from l_{ij} by a positive half twist on its edge.

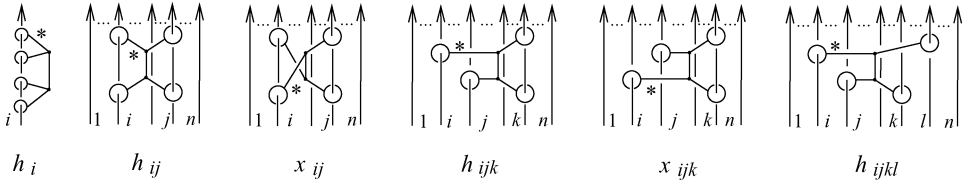


Fig. 11. The C_3 -trees $h_i, h_{ij}, x_{ij}, h_{ijk}, x_{ijk}$ and h_{ijkl} .

4.2. Proof of Theorem 2.4

Let $\sigma \in \mathcal{SL}(n)$. From the proof of Theorem 2.2, we have that $\sigma \stackrel{C_3}{\sim} \sigma_{(1)} \cdot \sigma_{(2)}$ where $\sigma_{(1)}$ and $\sigma_{(2)}$ are given by (4.1) and (4.2) respectively, and the exponents α_i, β_{ij} and γ_{ijk} in (4.2) are uniquely determined by the invariants a_2, f_2 and $\mu(ijk)$ of σ and $\sigma_{(1)}$.

It follows, by Calculus of Claspers, that $\sigma \stackrel{C_4}{\sim} \sigma_{(1)} \cdot \sigma_{(2)} \cdot \sigma_{(3)}$ with

$$\sigma_{(3)} := (\mathbf{1}_n)_{G_1} \cdot (\mathbf{1}_n)_{G_2} \cdot \dots \cdot (\mathbf{1}_n)_{G_N},$$

where, for each k, G_k is a simple C_3 -tree for $\mathbf{1}_n$. Set $G = \sqcup_k G_k$. By Lemma 3.6, we may assume that each G_k in G has index $\{i^{(4)}\}, \{i^{(2)}, j^{(2)}\}, \{i, j, k^{(2)}\}$ or $\{i, j, k, l\}$, for some indices $i, j, k, l \in \{1, \dots, n\}$. Let us consider each of these four cases successively.

Index $\{i^{(4)}\}$: let $F_i \subset G$ denote the union of C_3 -trees with index $\{i^{(4)}\}$, for each i . By Calculus of Claspers we may assume that F_i is contained in a tubular neighbourhood of the i th strand of $\mathbf{1}_n$. Let H_i denote the n -string link obtained from $\mathbf{1}_n$ by surgery along the C_3 -tree h_i represented in Figure 11. The knot obtained by closing the i th strand of H_i is the knot H of Figure 9. By [17] we know that $\mathbf{h} := P_0^{(3)}(H)$ is nonzero. It thus follows from Theorem 1.1, and the fact that $P_0^{(3)}$ is the only degree 3 knot invariant, that $(\mathbf{1}_n)_{F_i} \stackrel{C_4}{\sim} (H_i)^{P_0^{(3)}((\sigma_{(3)})_i)/\mathbf{h}}$.

Index $\{i^{(2)}, j^{(2)}\}$: Fix $i < j \in \{1, \dots, n\}$, and let $F_{ij} \subset G$ denote the union of C_3 -trees with index $\{i^{(2)}, j^{(2)}\}$. By the AS and IHX relations we may assume that both ends of each C_3 -tree in F_{ij} are j -leaves. Hence we have

$$(\mathbf{1}_n)_{F_{ij}} \stackrel{C_4}{\sim} H_{ij}^{a_{ij}} \cdot X_{ij}^{b_{ij}} \tag{4.3}$$

for some integers a_{ij} and b_{ij} , where H_{ij} and X_{ij} denote the n -string links obtained from $\mathbf{1}_n$ by surgery along the C_3 -trees h_{ij} and x_{ij} shown in Figure 11 respectively.

A direct computation shows that $\mu_{H_{ij}}(iijj) = \mu_{X_{ij}}(iijj) = 2$, and clearly we have $\mu_{(\mathbf{1}_n)_{G_p}}(iijj) = 0$ for any $G_p \subset G$ with index $\neq \{i^{(2)}, j^{(2)}\}$. Now, for $\gamma \in \mathcal{SL}(n)$ and $1 \leq k < l \leq n$, set $(f_3)_{k,l}(\gamma) := f_3(\gamma_k \cup \gamma_l) = P_0^{(3)}(\gamma_k \cup \gamma_l)$. By [17], we have

$$(f_3)_{k,l}(H_{ij}) = \delta_{(i,j),(k,l)} \cdot \mathbf{h} \text{ and } (f_3)_{k,l}(X_{ij}) = 0.$$

Also, we have $(f_3)_{k,l}(H_i) = (\delta_{i,k} + \delta_{i,l}) \cdot \mathbf{h}$, and $(f_3)_{k,l}((\mathbf{1}_n)_{G_p}) = 0$ for any $G_p \subset G$ with index other than $\{k^{(4)}\}, \{l^{(4)}\}$ or $\{k^{(2)}, l^{(2)}\}$. It follows that the integers a_{ij} and b_{ij} in (4.3) are uniquely determined by the invariants $P_0^{(3)}, f_3$ and $\mu(iijj)$ of $\sigma_{(3)}$.

Index $\{i, j, k^{(2)}\}$: Fix $i, j, k \in \{1, \dots, n\}$ with $i < j$, and let $F_{ijk} \subset G$ denote the union of C_3 -trees with index $\{i, j, k^{(2)}\}$. By the AS and IHX relation we may assume that

$$(\mathbf{1}_n)_{F_{ijk}} \stackrel{C_4}{\sim} H_{ijk}^{\alpha_{ijk}} \cdot X_{ijk}^{\beta_{ijk}}$$

for some integers α_{ijk} and β_{ijk} , where H_{ijk} and X_{ijk} denote the n -string links obtained

Table 1.

σ	$cl_3^{ijk}(\sigma)/C_4$	$(V_3)_{ijk}(\sigma)/\mathbf{h}$
H_a	$H^{\delta_{a,i}} \cdot H^{\delta_{a,j}} \cdot H^{\delta_{a,k}}$	$\delta_{a,i} + \delta_{a,j} + \delta_{a,k}$
$H_{ab} (a < b)$	$H^{\delta_{(a,b),(i,j)}} \cdot H^{\delta_{(a,b),(j,k)}} \cdot X^{\delta_{(a,b),(i,k)}}$	$\delta_{(a,b),(i,j)} + \delta_{(a,b),(j,k)}$
$X_{ab} (a < b)$	$H^{\delta_{(a,b),(i,k)}} \cdot X^{\delta_{(a,b),(i,j)}} \cdot X^{\delta_{(a,b),(j,k)}}$	$\delta_{(a,b),(i,k)}$
$H_{abc} (a < b < c)$	$X^{\delta_{(a,b,c),(i,j,k)}}$	0
$X_{abc} (a < b)$	$H^{\delta_{(a,b,c),(i,j,k)}}$	$\delta_{(a,b,c),(i,j,k)}$

respectively from $\mathbf{1}_n$ by surgery along the C_3 -trees h_{ijk} and x_{ijk} represented in Figure 11. Note that H_{ijk} and X_{ijk} correspond to the string links $B_{id}(k)$ and $\overline{B}_{id}(k)$ defined for Lemma 3-4 respectively (using Lemma 3-2 (2) for the second one). Thus by Lemma 3-4, the union $F_{(3)} := \bigcup_{i,j,k} F_{ijk}$ of all C_3 -trees in G intersecting 3 strands of $\mathbf{1}_n$ satisfies

$$(\mathbf{1}_n)_{F_{(3)}} \stackrel{C_4}{\sim} \prod_{1 \leq i < j < k \leq n} (H_{jki})^{a_{ijk}} \cdot (H_{ikj})^{b_{ijk}} \cdot (H_{ijk})^{c_{ijk}} \cdot (X_{ijk})^{d_{ijk}} \tag{4.4}$$

for some integers $a_{ijk}, b_{ijk}, c_{ijk}$ and d_{ijk} .

We have $\mu_{H_{ijk}}(ijkk) = \mu_{X_{ijk}}(ijkk) = 1$ for all $1 \leq i, j, k \leq n$ with $i < j$, and $\mu_{(\mathbf{1}_n)_{G_p}}(ijkk) = 0$ for any $G_p \subset G$ with index $\neq \{i, j, k^{(2)}\}$.

For $L \in \mathcal{SL}(n)$ and $1 \leq i < j < k \leq n$, set $cl_3^{ijk}(L) := cl_3(L_i \cup L_j \cup L_k)$ and $(V_3)_{ijk}(L) := P_3^{(0)}(cl_3^{ijk}(L))$. We have

$$cl_3^{ijk}(X_{ijk}) \stackrel{C_5}{\sim} H \text{ and } cl_3^{ijk}(H_{ikj}) \stackrel{C_5}{\sim} cl_3^{ijk}(H_{jki}) \stackrel{C_5}{\sim} cl_3^{ijk}(H_{ijk}) \stackrel{C_5}{\sim} X,$$

where H and X are the two knots represented in Figure 9. Note that by the IHX relation we have $X \stackrel{C_5}{\sim} U$. More generally, we compute the closures cl_3^{ijk} and invariants $(V_3)_{ijk}$ of the relevant n -string links in Table 1.

It follows that all exponents in (4.4) are uniquely determined by the invariants $P_0^{(3)}, f_3, V_3$ and $\mu(ijkk)$ ($1 \leq i, j, k \leq n; i < j$) of $\sigma_{(3)}$.

Index $\{i, j, k, l\}$: by the IHX and AS relations, we may assume that the k -leaf and l -leaf of any C_3 -tree $C \subset G$ with index $\{i, j, k, l\}$ ($i < j < k < l$) are its two ends. More precisely, the union $F_{(4)} \subset G$ of all C_3 -trees intersecting 4 distinct components of $\mathbf{1}_n$ satisfies

$$(\mathbf{1}_n)_{F_{(4)}} \stackrel{C_4}{\sim} \prod_{1 \leq i < j < k < l \leq n} (H_{ijkl})^{\mu_{\sigma_{(3)}}(ijkl)} \cdot (H_{jikl})^{\mu_{\sigma_{(3)}}(jikl)},$$

where H_{ijkl} denotes the n -string link obtained from $\mathbf{1}_n$ by surgery along the C_3 -trees h_{ijkl} represented in Figure 11. This follows from the fact that $\mu_{H_{ijkl}}(i'j'k'l') = \delta_{(i,j,k,l),(i',j',k',l')}$ [27, 16]. So we have shown that $\sigma_{(3)}$ is C_4 -equivalent to

$$\prod_i (H_i)^{a_i} \cdot \prod_{i < j} (H_{ij})^{a_{ij}} \cdot \prod_{\substack{i < j \\ k}} (X_{ij})^{b_{ij}} \cdot \prod_{\substack{i < j \\ k}} (H_{ijk})^{a_{ijk}} \cdot \prod_{i < j < k} (X_{ijk})^{b_{ijk}} \cdot \prod_{\substack{i < j \\ k < l}} (H_{ijkl})^{a_{ijkl}}, \tag{4.5}$$

where the exponents are integers determined uniquely by the invariants $P_0^{(3)}, f_3$ and V_3 , and Milnor invariants $\mu(iijj)$ ($1 \leq i < j \leq n$), $\mu(ijkk)$ ($1 \leq i, j, k \leq n; i < j$) and $\mu(ijkl)$ ($1 \leq i, j < k < l \leq n$) of $\sigma_{(3)}$. The result follows from the fact that all the above-listed invariants are 3-additive.

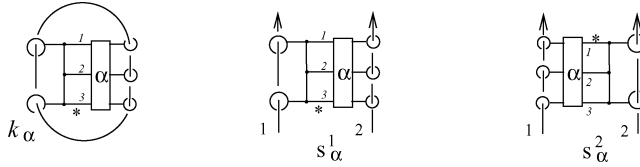


Fig. 12. The C_4 -trees k_α , s_α^1 and s_α^2 ; $\alpha \in S_3$.

Remark 4.1. From the proof of Theorem 2.4 (for the case of index $\{i, j, k^{(2)}\}$ trees), it appears that the statement of Theorem 2.4 can be modified to replace the invariants V_3 of σ with only the invariants $(V_3)_{ijk}$ with $1 \leq i < j < k \leq n$, as these are the only values used to determine the exponents in (4.4).

4.3. *Proof of Theorem 2.5*

Before proving Theorem 2.5 we investigate individually the case of n -string links for $n = 1, 2, 3$ and 4. We start by reviewing briefly the case $n = 1$, that is, the knot case.

4.3.1. *The knot case*

It is well known that there exist essentially two linearly independent finite type knots invariants of degree 4, namely a_4 and $P_0^{(4)}$.

For an element α of the symmetric group S_3 , denote by K_α the knot obtained from the unknot U by surgery along the C_4 -tree k_α represented in Figure 12. Note that K_{id} , $K_{(13)}$ and $K_{(12)}$ are the three knots A , B and C illustrated in Figure 9.⁵ By the AS and IHX relations, the abelian group $\mathcal{SL}_4(1)/C_5$ is generated by these six elements K_α , $\alpha \in S_3$. Further, by using the IHX and STU relations we observe that

$$K_{(12)} \stackrel{C_5}{\sim} K_{(23)} \stackrel{C_5}{\sim} U \quad \text{and} \quad K_{(13)} \stackrel{C_5}{\sim} K_{(123)} \stackrel{C_5}{\sim} K_{(132)}. \tag{4.6}$$

(In particular, we have that the knot C of Figure 9 satisfies $C \stackrel{C_5}{\sim} U$.) This shows that $\mathcal{SL}_4(1)/C_5$ is generated by the two knots $A = K_{id}$ and $B = K_{(13)}$ of Figure 9. By using [31] and [17], we have that

$$a_4(A) = 0 \text{ and } a_4(B) = \pm 2, \\ P_0^{(4)}(A) = \pm 4!2^4 \text{ and } P_0^{(4)}(B) = 0.$$

Set $\mathbf{a} := P_0^{(4)}(A)$ and $\mathbf{b} := a_4(B)$. The C_5 -equivalence class of a knot K is thus determined by its degree ≤ 4 invariants as follows

$$K \stackrel{C_5}{\sim} K_{(1)} \cdot K_{(2)} \cdot K_{(3)},$$

where $K_{(1)} = T^{a_2(K)}$, $K_{(2)} = H^{(P_0^{(3)}(K) - P_0^{(3)}(K_{(1)}))/\mathbf{b}}$ and $K_{(3)} = A^{(P_0^{(4)}(K) - P_0^{(4)}(K_{(1)} \cdot K_{(2)}))/\mathbf{a}}$. $B^{(a_4(K) - a_4(K_{(1)} \cdot K_{(2)}))/\mathbf{b}}$, where T and H are given in Figure 9.

4.3.2. *The 2-component case*

We aim to prove the following particular case of Theorem 2.5.

LEMMA 4.2. *Let $\sigma, \sigma' \in \mathcal{SL}_4(2)$. Then σ and σ' are C_5 -equivalent if and only if they share all knots invariants of degree 4 and the five invariants f_4^i ($i = 1, \dots, 5$).*

Note that there is no nontrivial Milnor invariant of length 5 for 2-string links [28].

⁵ Here, and in the rest of the paper, we denote by id the identity element of the symmetric group.

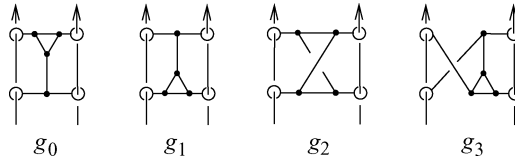


Fig. 13. The C_4 -graphs $g_i ; i = 0, 1, 2, 3$.

Proof of Lemma 4.2 Let $\sigma \in \mathcal{SL}_4(2)$. By Calculus of claspers and Subsection 4.3.1,

$$\sigma \stackrel{C_5}{\sim} \prod_{i=1}^2 \left((A_i)^{P_0^{(4)}(\sigma_i)/a} \cdot (B_i)^{a_4(\sigma_i)/b} \right) \cdot (\mathbf{1}_2)_F$$

where F is a disjoint union of simple C_4 -trees for $\mathbf{1}_2$ with at least one 1-leaf and one 2-leaf, and where A_i (resp. B_i) is the n -string link obtained from $\mathbf{1}_n$ by connected sum of a copy of the knot A (resp. B) of Figure 9 on the i th component. Note that by Lemma 3.6, we may assume that each C_4 -tree in F has index $\{1^{(2)}, 2^{(3)}\}$ or $\{1^{(3)}, 2^{(2)}\}$. It follows, by the IHX relation, that the abelian group $\mathcal{SL}_4(2)/C_5$ is generated by $A_i, B_i (i = 1, 2)$ and the 2-string links σ_α^1 and σ_α^2 obtained from $\mathbf{1}_2$ by surgery along the C_4 -trees s_α^1 and s_α^2 represented in Figure 12 ($\alpha \in S_3$).

We can use the AS and STU relations to prove the following relations ($i = 1, 2$):

$$\begin{aligned} \sigma_{(12)}^i &\stackrel{C_5}{\sim} \sigma_{id}^i \cdot (\mathbf{1}_2)_{g_0}, & \sigma_{(23)}^i &\stackrel{C_5}{\sim} \sigma_{id}^i \cdot (\mathbf{1}_2)_{g_1}, \\ \sigma_{(123)}^i &\stackrel{C_5}{\sim} \sigma_{(23)}^i \cdot (\mathbf{1}_2)_{g_2}, & \sigma_{(132)}^i &\stackrel{C_5}{\sim} \sigma_{(12)}^i \cdot (\mathbf{1}_2)_{g_2}, \\ & & \sigma_{(13)}^i &\stackrel{C_5}{\sim} \sigma_{(123)}^i \cdot (\mathbf{1}_2)_{g_3}, \end{aligned}$$

where $g_k (k = 0, 1, 2, 3)$ is the C_4 -graph represented in Figure 13. By applying the STU relation at an edge of g_k that connects its loop to a 2-leaf, we can express $(\mathbf{1}_2)_{g_k}$ as a product of s_α^1 's. So the relations above imply that for any $\alpha \in S_3 \setminus \{id\}$, the string link s_α^2 is generated in $\mathcal{SL}_4(2)/C_5$ by s_{id}^2 and the s_α^1 's. Further, one can easily check using the IHX relation that $(\mathbf{1}_2)_{g_0} \stackrel{C_5}{\sim} (\mathbf{1}_2)_{g_1}$. This implies that $\sigma_{(12)}^1 \stackrel{C_5}{\sim} \sigma_{(23)}^1$, and thus (by the above relations) that $\sigma_{(123)}^1 \stackrel{C_5}{\sim} \sigma_{(132)}^1$.

So $\mathcal{SL}_4(2)/C_5$ is generated by the five elements $\sigma_{id}^1, \sigma_{(12)}^1, \sigma_{(123)}^1, \sigma_{(13)}^1$ and σ_{id}^2 .

In Subsection 2.3 we introduced three different ways of closing a 2-string link σ into a knot, namely by taking the plat closure $\bar{\sigma}$ and by taking the closure cl_0 (resp. cl_1) in Figure 2 of the 3-string link $\Delta_1\sigma$ (resp. $\Delta_2\sigma$) obtained from σ by doubling the first (resp. second) component. In particular if $\sigma \in \mathcal{SL}_4(2)/C_5$, the resulting knot is an element of $\mathcal{SL}_4(1)/C_5$, and can be expressed in terms of the generators A and B given in Subsection 4.3.1. We collect the results in Table 2. This is straightforward for the plat closure case, and uses the fact that the knot C given in Figure 9 satisfies $C \stackrel{C_5}{\sim} U$. (This fact is also used for Table 4.) For the two latter cases, the computations make use of Calculus of Claspers, and in particular they make use of Lemma 3.2(3). For example, let us explain here the computation for $cl_0(\Delta_1\sigma_{id}^1)$. Observe that σ_{id}^1 is just a copy of the C_4 -tree s represented on the left-hand side of Figure 14. Starting with s , doubling the first component of $\mathbf{1}_2$ yields a C_4 -tree s' for $\mathbf{1}_3$ as illustrated in Figure 14. We apply Lemma 3.2(3) repeatedly, to decompose s' into simple C_4 -trees. This gives $(\mathbf{1}_3)_{s'} \stackrel{C_5}{\sim} \prod_{1 \leq i \leq 4} (\mathbf{1}_3)_{s_i}$, where s_i is a simple C_4 -tree for $\mathbf{1}_3$ as illustrated in Figure 14, $i = 1, 2, 3, 4$. We need the following.

Table 2.

σ	σ_{id}^1	$\sigma_{(12)}^1$	$\sigma_{(123)}^1$	$\sigma_{(13)}^1$	σ_{id}^2
$\bar{\sigma}/C_5$	A	U	B	B	A
$cl_0(\Delta_1\sigma)/C_5$	$A^3 \cdot B^{-1}$	B^{-1}	B^3	$A^{-1} \cdot B^3$	$A^4 \cdot B^2$
$cl_1(\Delta_2\sigma)/C_5$	$A^4 \cdot B^2$	$A \cdot B^2$	$A \cdot B^6$	B^6	$A^3 \cdot B^{-1}$

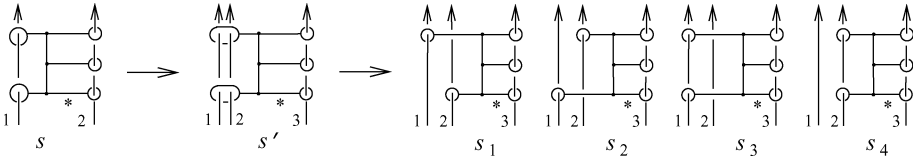


Fig. 14. Doubling the first component of $\mathbf{1}_2$ and splitting the leaves.

Claim 4.3. Let $\sigma \in \mathcal{SL}(3)$, and $\sigma' \in \mathcal{SL}_k(3)$ for an integer $k \geq 1$. Then for each $i = 0, \dots, 4$, the closure cl_i of $\sigma \cdot \sigma'$ satisfies $cl_i(\sigma \cdot \sigma') \stackrel{C^{k+1}}{\sim} (cl_i\sigma)\sharp(cl_i\sigma')$.

Proof of Claim 4.3. By [14, theorem 3-17], we have $\sigma' = (\mathbf{1}_3)_G$, where G is a disjoint union of simple C_k -trees for $\mathbf{1}_3$. Let $i \in \{0, \dots, 4\}$. Using Calculus of Claspers, we have $cl_i(\sigma \cdot \sigma') = cl_i(\sigma \cdot (\mathbf{1}_3)_G) \stackrel{C^{k+1}}{\sim} cl_i(\sigma \cdot (\mathbf{1}_3)_{G'})$, where G' is a union of C_k -trees for $\mathbf{1}_3$ which is contained in a tubular neighbourhood of the first strand. Clearly, we have $cl_i(\sigma \cdot (\mathbf{1}_3)_{G'}) = (cl_i\sigma)\sharp(cl_i(\mathbf{1}_3)_{G'})$. On the other hand, it can be easily checked that $cl_i\sigma' = cl_i(\mathbf{1}_3)_G \stackrel{C^{k+1}}{\sim} cl_i(\mathbf{1}_3)_{G'}$. This completes the proof.

It follows that

$$cl_0 \left(\prod_{i=1}^4 (\mathbf{1}_3)_{s_i} \right) \stackrel{C_5}{\sim} \prod_{i=1}^4 cl_0((\mathbf{1}_3)_{s_i}).$$

By an isotopy, we see that $cl_0((\mathbf{1}_3)_{s_1}) = U_{k'}$, where k' is a C_4 -tree for U represented in Figure 15. As shown there, we have $U_{k'} \stackrel{C_5}{\sim} (U_{s_{(13)}})^{-1} = B^{-1}$ using Lemma 3-2(2). For $i = 2, 3, 4$, a simple isotopy shows that $cl_0((\mathbf{1}_3)_{s_i}) = A$.

Recall that $\mathbf{a} = P_0^{(4)}(A)$ and $\mathbf{b} = a_4(B)$. Table 3 follows immediately from the definitions of the invariants f_4^i ($i = 1, \dots, 5$) and the computations given in Table 2. The 5×5 matrix given by the entries of Table 3 having rank 5, we obtain that the five invariants f_4^i ($i = 1, \dots, 5$) (together with the knot invariants a_4 and $P_0^{(3)}$) do classify the abelian group $\mathcal{SL}_4(2)/C_5$, thus completing the proof of the lemma.

Remark 4.4. The number of linearly independent finite type 2-string link invariants of degree 4 has been computed by Bar-Natan [3]. In particular, there are 10 linearly independent such invariants which do not have a factor coming from a single knot component, see

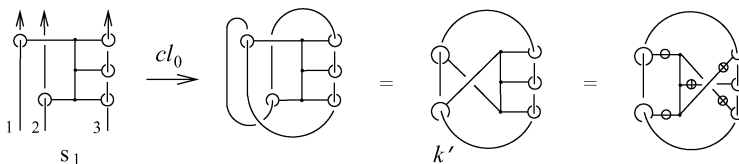


Fig. 15. The proof that $cl_0((\mathbf{1}_3)_{s_1}) \stackrel{C_5}{\sim} B^{-1}$.

Table 3.

σ	σ_{id}^1	$\sigma_{(12)}^1$	$\sigma_{(123)}^1$	$\sigma_{(13)}^1$	σ_{id}^2
$f_4^1(\sigma)/\mathbf{b}$	0	0	1	1	0
$f_4^2(\sigma)/\mathbf{a}$	1	0	0	0	1
$f_4^3(\sigma)/\mathbf{b}$	-1	-1	3	3	2
$f_4^4(\sigma)/\mathbf{a}$	3	0	0	-1	4
$f_4^5(\sigma)/\mathbf{a}$	4	1	1	0	3

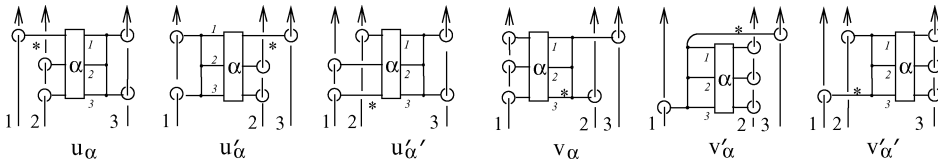


Fig. 16. The C_4 -trees $u_\alpha, u'_\alpha, u''_\alpha, v_\alpha, v'_\alpha$ and v''_α .

[3, §2.3-4]. Half of them come from products of lower degree invariants (namely $\mu(12)^4, \mu(12)^2 f_2, \mu(12) f_3, \mu(12)\mu(1122)$ and $(f_2)^2$) and the remaining five are the invariants f_4^i ($i = 1, \dots, 5$).

4.3.3. The 3-component case

In this subsection we prove the following lemma.

LEMMA 4.5. Let $\sigma, \sigma' \in \mathcal{SL}_4(3)$. Then σ and σ' are C_5 -equivalent if and only if they share all knots invariants of degree 4, all invariants f_4^i ($1 \leq i \leq 5$), all invariants V_4^j ($1 \leq j \leq 7$), and all Milnor invariants $\mu(iijjk)$ and $\mu(ijjkk)$ ($1 \leq i, j, k \leq n; j < k$).

Proof of Lemma 4.5. Let $\sigma \in \mathcal{SL}_4(3)$. By Calculus of Claspers and Subsections 4.3.1 and 4.3.2 above,

$$\sigma \stackrel{C_5}{\sim} \tilde{\sigma} \cdot (\mathbf{1}_3)_E \cdot (\mathbf{1}_3)_F, \tag{4.7}$$

where $\tilde{\sigma}$ is determined uniquely by the invariants $a_4, P_0^{(4)}$ and f_4^i ($1 \leq i \leq 5$) of σ , and where E (resp. F) is a disjoint union of simple C_4 -trees for $\mathbf{1}_3$ with index $\{i, j^{(2)}, k^{(2)}\}$ (resp. $\{i, j, k^{(3)}\}$) ($1 \leq i, j, k \leq n$).

For $\alpha \in S_3$, denote by $U_\alpha, U'_\alpha, U''_\alpha, V_\alpha, V'_\alpha$ and V''_α the 3-string links obtained from $\mathbf{1}_3$ by surgery along the C_4 -trees $u_\alpha, u'_\alpha, u''_\alpha, v_\alpha, v'_\alpha$ and v''_α represented in Figure 16 respectively.

Set $\mathbb{U} := \{U_\alpha, U'_\alpha, U''_\alpha \mid \alpha \in S_3\}$ and $\mathbb{V} := \{V_\alpha, V'_\alpha, V''_\alpha \mid \alpha \in S_3\}$. By Calculus of Claspers and the AS and IHX relations, we have that the C_5 -equivalence class of $(\mathbf{1}_3)_E$, resp. of $(\mathbf{1}_3)_F$, is generated by \mathbb{U} , resp. by \mathbb{V} . So (4.7) can be rewritten as

$$\sigma \stackrel{C_5}{\sim} \tilde{\sigma} \cdot \prod_{\alpha \in S_3} (U_\alpha)^{m_\alpha} \cdot (U'_\alpha)^{m'_\alpha} \cdot (U''_\alpha)^{m''_\alpha} \cdot (V_\alpha)^{n_\alpha} \cdot (V'_\alpha)^{n'_\alpha} \cdot (V''_\alpha)^{n''_\alpha} \tag{4.8}$$

for some integers $m_\alpha, m'_\alpha, m''_\alpha, n_\alpha, n'_\alpha$ and n''_α .

We first consider the set $\mathbb{V} \subset \mathcal{SL}_4(3)$. We have the following.

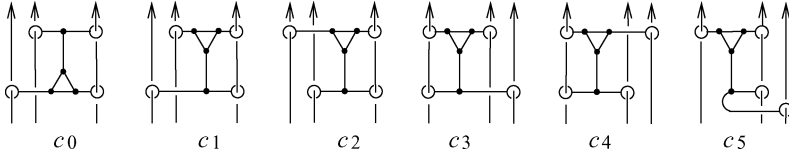


Fig. 17. The C_4 -graphs c_i , $1 \leq i \leq 5$.

Claim 4.6. Any element L of \mathbb{V} satisfies $L \stackrel{C_5}{\approx} L^{(1)} \cdot L^{(2)}$, where $L^{(1)}$ is obtained from $\mathbf{1}_3$ by surgery along C_4 -trees of index $\{i, j^{(2)}, k^{(2)}\}$ and where $L^{(2)}$ is generated by the elements

$$I_1 := V_{id}, I_2 := V'_{id}, I_3 := V''_{id}.$$

Proof of Claim 4.6. Let us consider the case of the 6 elements V_α ($\alpha \in S_3$). For $\alpha = (12)$, the STU relation gives $V_{(12)} \stackrel{C_5}{\approx} I_1 \cdot (\mathbf{1}_3)_{c_4}$, where c_4 is the C_4 -graph represented in Figure 17. The claim thus follows, by using the STU relation to express $(\mathbf{1}_3)_{c_4}$ as a product $(\mathbf{1}_3)_T \cdot (\mathbf{1}_3)_{T'}$ for two C_4 -trees T and T' with index $\{1^{(2)}, 2, 3^{(2)}\}$. The same argument can be applied for any $\alpha \in S_3$, as V_α is related to V_{id} by successive applications of the STU relation. By symmetry, the case of elements V'_α and V''_α is also strictly similar.

Now, observe that the 3-string links I_l , $l = 1, 2, 3$, are distinguished by Milnor invariants. More precisely, for all $1 \leq i, j, k \leq 3$ with $j < k$, we have $\mu_l(iijjk) = \pm\delta_{i,l}$. Note also that $\mu_l(ijjkk) = 0$, and that $\mu_u(iijjk) = 0$ for any $u \in \mathbb{U}$.

So it remains to classify the 18 elements of $\mathbb{U} \subset \mathcal{SL}_4(3)$. We will show that \mathbb{U} is generated by the 10 elements $U_{id}, U_{(12)}, U_{(123)}, U'_{id}, U'_{(12)}, U''_{id}, U''_{(12)}, U''_{(123)}, U''_{(23)}$ and $U''_{(13)}$. The following relations among elements of \mathbb{U} can be proved using the AS, IHX and STU relations:

$$\begin{aligned} &U_{(123)} \cdot (U_{(13)})^{-1} \stackrel{C_5}{\approx} U''_{id} \cdot (U''_{(23)})^{-1}; \\ &U_{id} \cdot (U_{(23)})^{-1} \stackrel{C_5}{\approx} U''_{(123)} \cdot (U''_{(13)})^{-1}; \\ &U'_{(123)} \cdot (U'_{(13)})^{-1} \stackrel{C_5}{\approx} (U''_{id})^{-1} \cdot U''_{(12)} \cdot U''_{(132)} \cdot (U''_{(13)})^{-1}; \\ &U'_{id} \cdot (U'_{(23)})^{-1} \stackrel{C_5}{\approx} (U''_{(123)})^{-1} \cdot U''_{(12)} \cdot U''_{(132)} \cdot (U''_{(23)})^{-1}; \\ &U_{id} \cdot (U_{(12)})^{-1} \cdot (U_{(132)})^{-1} \cdot U_{(13)} \stackrel{C_5}{\approx} U'_{(123)} \cdot (U'_{(12)})^{-1} \cdot (U'_{(132)})^{-1} \cdot U'_{(23)}. \end{aligned}$$

More precisely, the first relation is obtained as follows. Consider the C_4 -graph c_0 represented in Figure 17. By applying the STU relation to the edge incident to the 1-leaf of c_0 , we have $(\mathbf{1}_n)_{c_0} \stackrel{C_5}{\approx} (U''_{id})^{-1} \cdot U''_{(23)}$. Now, it follows from the IHX relation that $(\mathbf{1}_n)_{c_0} \stackrel{C_5}{\approx} (\mathbf{1}_n)_{c_1}$, where c_1 is shown in Figure 17, and on the other hand the STU relation can be used to show that $(\mathbf{1}_n)_{c_1} \stackrel{C_5}{\approx} (U_{(123)})^{-1} \cdot U_{(13)}$, which implies the desired relation. The next four relations are proved strictly similarly by using respectively the C_4 -graphs c_2, c_3, c_4 and c_5 of Figure 17 in place of c_1 .

Also, we can use the C_4 -graphs d_1 and d_2 of Figure 18 in a similar way (that is, by applying the STU relation in two different ways) to obtain the additional two relations.

$$\begin{aligned} &U''_{(12)} \cdot U''_{(132)} \stackrel{C_5}{\approx} U_{(12)} \cdot U_{(132)} \\ &U''_{(12)} \cdot (U''_{(132)})^{-1} \stackrel{C_5}{\approx} U'_{(12)} \cdot (U'_{(132)})^{-1}. \end{aligned}$$

Table 4.

σ	U''_{id}	$U''_{(12)}$	$U''_{(123)}$	$U''_{(23)}$	$U''_{(13)}$	U_{id}	$U_{(12)}$	$U_{(123)}$	U'_{id}	$U'_{(12)}$
$cl_1(\sigma)/C_5$	A^{-1}	U	B^{-1}	U	B^{-1}	B	B	U	B	U
$cl_2(\sigma)/C_5$	B	U	A	B	U	A	U	B	U	B
$cl_3(\sigma)/C_5$	U	B	B	A	B	B^{-1}	U	A^{-1}	A^{-1}	U
$cl_4(\sigma)/C_5$	B	B	U	B	A	A^{-1}	U	B^{-1}	B^{-1}	U

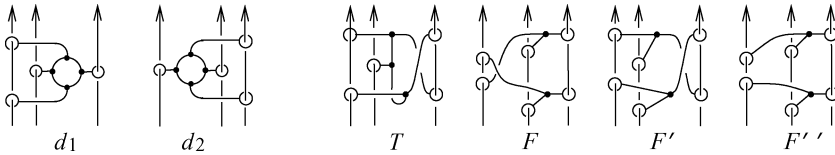


Fig. 18.

Finally let us show that

$$U''_{(132)} \stackrel{C_5}{\sim} (U_{(12)})^{-1} \cdot U'_{(12)}. \tag{4.9}$$

To prove (4.9), we need the following lemma, which can be easily derived from the proof of [14, proposition 4.4].

LEMMA 4.7. *Let G_S be a C_k -graph for $\mathbf{1}_n$, and let G_T and G_U be the unions of two tree claspers which differ from G_S only in a small ball as depicted in Figure 6, where the two leaves of G_T , resp. G_U , are from different components. Then $(\mathbf{1}_n)_{G_S} \stackrel{C_{k+1}}{\sim} ((\mathbf{1}_n)_{G_T})^{-1} \cdot (\mathbf{1}_n)_{G_U}$, where $((\mathbf{1}_n)_{G_T})^{-1}$ denotes the inverse of $(\mathbf{1}_n)_{G_T}$ in the abelian group $\mathcal{SL}_k(n)/C_{k+1}$.*

Observe that, by the AS relation and Calculus of Claspers, we have $U''_{(132)} \stackrel{C_5}{\sim} \mathbf{1}_T$, where T is the C_4 -tree represented in Figure 18. We have

$$\mathbf{1}_T \stackrel{C_5}{\sim} \mathbf{1}_F \cdot (\mathbf{1}_{F'})^{-1} \stackrel{C_5}{\sim} \mathbf{1}_F \cdot (\mathbf{1}_{F''})^{-1} \cdot \mathbf{1}_{F''} \cdot (\mathbf{1}_{F'})^{-1} \stackrel{C_5}{\sim} (U_{(12)})^{-1} \cdot U'_{(12)},$$

where F, F' and F'' are as shown in Figure 18. Here, the first and third equivalence follow from Lemma 4.7 and isotopies, and $(\mathbf{1}_{F'})^{-1}$, resp. $(\mathbf{1}_{F''})^{-1}$, denotes the inverse of $\mathbf{1}_{F'}$, resp. $\mathbf{1}_{F''}$, in $\mathcal{SL}(4)/C_5$.

So we obtain that \mathbb{U} is generated by the following 10 elements: $U''_{id}, U''_{(12)}, U''_{(123)}, U''_{(23)}, U''_{(13)}, U_{id}, U_{(12)}, U_{(123)}, U'_{id}$ and $U'_{(12)}$. In order to show that they are linearly independent, we make use of Milnor invariants $\mu(ijkk)$ ($1 \leq i, j, k \leq 3; j < k$) and the invariants V_4^i ($i = 1, \dots, 7$) defined in Subsection 2.4. We compute the C_5 -equivalence classes of the closures cl_j ($j = 1, \dots, 4$) of the 10 elements listed above in a similar way as for Table 2. These computations are summarized in Table 4.

From Table 4 and the definitions of the invariants, we obtain the desired computations, as given in Table 5. The matrix given by this table has rank 10, which shows that any element generated by \mathbb{U} is uniquely determined by the invariants listed in the table. It follows that Milnor invariants $\mu_\sigma(iijk)$ and $\mu_\sigma(ijkk)$ ($1 \leq i, j, k \leq 3, j < k$), and the invariants V_4^i ($1 \leq i \leq 7$) of σ determine uniquely all the exponents in (4.8). The lemma then follows from the 4-additivity of these invariants.

Table 5.

σ	U''_{id}	$U''_{(12)}$	$U''_{(123)}$	$U''_{(23)}$	$U''_{(13)}$	U_{id}	$U_{(12)}$	$U_{(123)}$	U'_{id}	$U'_{(12)}$
$\mu_\sigma(12233)$	0	0	0	0	0	1	0	-1	0	0
$\mu_\sigma(32211)$	0	0	0	0	0	0	0	0	-1	0
$\mu_\sigma(21133)$	1	0	-1	1	-1	0	0	0	0	0
$V_4^1(\sigma)/\mathbf{b}$	0	0	-1	0	-1	1	1	0	1	0
$V_4^2(\sigma)/\mathbf{b}$	1	0	0	1	0	0	0	1	0	1
$V_4^3(\sigma)/\mathbf{b}$	0	1	1	0	1	-1	0	0	0	0
$V_4^4(\sigma)/\mathbf{a}$	-1	0	0	0	0	0	0	0	0	0
$V_4^5(\sigma)/\mathbf{a}$	0	0	1	0	0	1	0	0	0	0
$V_4^6(\sigma)/\mathbf{a}$	0	0	0	1	0	0	0	-1	-1	0
$V_4^7(\sigma)/\mathbf{a}$	0	0	0	0	1	-1	0	0	0	0

4.3.4. The 4-component case

Let $\sigma \in \mathcal{SL}_4(4)$. We proceed as in the previous subsections to construct a representative of the C_5 -equivalence class of σ .

By Calculus of Claspers and Subsections 4.3.1 to 4.3.3, we have

$$\sigma \stackrel{C_5}{\sim} \tilde{\sigma} \cdot (\mathbf{1}_4)_P$$

where $\tilde{\sigma}$ is uniquely determined by the invariants of σ listed in Lemma 4.5, and where P is a disjoint union of simple C_4 -trees for $\mathbf{1}_4$ with index $\{i, j, k, l^{(2)}\}$ ($1 \leq i, j, k, l \leq 4$). By the IHX relation, we may assume that each C_4 -tree in P is linear and that its ends are the two l -leaves. Recall from Subsection 3.2 that $\mathcal{B}_4(k)$ ($1 \leq k \leq 4$) is the set of all bijections τ from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\} \setminus \{k\}$ such that $\tau(1) < \tau(3)$. By Lemma 3.4, there exists integers m_α ($\alpha \in \mathcal{B}_4(4)$) and $m_{\alpha,k}$ ($1 \leq k \leq 4$ and $\alpha \in \mathcal{B}_4(k)$) such that

$$(\mathbf{1}_4)_P \stackrel{C_5}{\sim} \prod_{\alpha \in \mathcal{B}_4(4)} (\overline{B}_\alpha(4))^{m_\alpha} \cdot \prod_{k=1}^4 \prod_{\alpha \in \mathcal{B}_4(k)} (B_\alpha(k))^{m_{\alpha,k}}, \tag{4.10}$$

where the string links $B_\alpha(k)$ and $\overline{B}_\alpha(k)$ are defined in Figure 7.

For $\tau \in \mathcal{B}_4(k)$, set $\mu_\tau := \mu(\tau(1), \tau(2), \tau(3), k, k)$. Then for any $1 \leq l \leq 4$ and $\alpha \in \mathcal{B}_4(l)$, we have (see [26, §4]):

$$\mu_\tau(B_\alpha(k)) = \mu_\tau(\overline{B}_\alpha(k)) = \delta_{\alpha,\tau}.$$

Observe that, by definition, $\mathcal{B}_4(4)$ is just the subgroup $\{id, (12), (23)\}$ of S_3 . One can check that the closures K_1, K_2 and K_3 (see Figure 3) of the six 4-string links $B_\alpha(4), \overline{B}_\alpha(4)$ ($\alpha \in \mathcal{B}_4(4)$) are as given in Table 6 ($1 \leq i, j \leq 3$). We thus obtain the values of the invariants μ_τ ($\tau \in \mathcal{B}_4(4)$) and W_4^i ($i = 1, 2, 3$) as in Table 7. Clearly, the matrix given by the entries of Table 7 has rank 6. This implies that all exponents in (4.10) are uniquely determined by Milnor invariants $\mu_\sigma(jikll)$ ($1 \leq i, j, k, l \leq 4$; $j < k$) and all invariants W_4^i of σ ($i = 1, 2, 3$), and thus proves the result by the 4-additivity of these invariants.

Table 6.

σ	$B_{id}(4)$	$B_{(12)}(4)$	$B_{(23)}(4)$	$\overline{B_{id}}(4)$	$\overline{B_{(12)}}(4)$	$\overline{B_{(23)}}(4)$
$K_1(\sigma)/C_5$	B^{-1}	B^{-1}	B^{-1}	A	U	U
$K_2(\sigma)/C_5$	B^{-1}	B^{-1}	U	U	A	B
$K_3(\sigma)/C_5$	B^{-1}	U	B^{-1}	U	B	A

Table 7.

σ	$B_{id}(4)$	$B_{(12)}(4)$	$B_{(23)}(4)$	$\overline{B_{id}}(4)$	$\overline{B_{(12)}}(4)$	$\overline{B_{(23)}}(4)$
$\mu_\sigma(12344)$	1	0	0	1	0	0
$\mu_\sigma(21344)$	0	1	0	0	1	0
$\mu_\sigma(13244)$	0	0	1	0	0	1
$W_4^1(\sigma)/\mathbf{a}$	0	0	0	1	0	0
$W_4^2(\sigma)/\mathbf{a}$	0	0	0	0	1	0
$W_4^3(\sigma)/\mathbf{a}$	0	0	0	0	0	1

4.3.5. Proof of Theorem 2.5

We now prove Theorem 2.5 in the general case.

Given $\sigma \in \mathcal{SL}(n)$, we know from the proof of Theorem 2.5 that σ is C_4 -equivalent to $\sigma_{(1)} \cdot \sigma_{(2)} \cdot \sigma_{(3)}$, where $\sigma_{(1)}, \sigma_{(2)}$ and $\sigma_{(3)}$ are given by (4.1), (4.2) and (4.5) respectively.

By Calculus of Claspers, $\sigma \stackrel{C_5}{\sim} \sigma_{(2)} \cdot \sigma_{(3)} \cdot \sigma_{(4)}$, where $\sigma_{(4)}$ is obtained from $\mathbf{1}_n$ by surgery along a union of C_4 -trees. More precisely,

$$\sigma_{(4)} \stackrel{C_5}{\sim} \prod_{i=1}^5 \sigma_{(4)}^i,$$

where for each $i = 1, \dots, 5$, the n -string link $\sigma_{(4)}^i$ is obtained from $\mathbf{1}_n$ by surgery along a union of C_4 -trees that each intersect i distinct components of $\mathbf{1}_n$.

By Subsections 4.3.1 to 4.3.4, we can determine explicitly $\sigma_{(4)}^i$ for $1 \leq i \leq 4$ using all invariants $a_4, P_0^{(4)}, f_4^i (1 \leq i \leq 5), V_4^j (1 \leq j \leq 7)$ and $W_4^k (1 \leq k \leq 3)$ of σ , and all Milnor invariants $\mu_\sigma(iijjk), \mu_\sigma(ijjkk)$ and $\mu_\sigma(jikll) (1 \leq i, j, k, l \leq n; j < k)$. (Using the fact that all these invariants are 4-additive.)

Now, it is easy to see that Milnor invariants $\mu(ijklm) (1 \leq i, j, k < l < m \leq n)$ do classify n -string links of the form $(\mathbf{1}_n)_T$ for T a C_4 -tree intersecting 5 distinct components of $\mathbf{1}_n$. Indeed, if T has index $I = \{i, j, k, l, m\} (1 \leq i, j, k < l < m \leq n)$, we may assume by the IXH relation that the ends of T (which is linear, since it is a C_4 -tree) are the l -leaf and m -leaf. Then for every multi-index $I' = i'j'k'l'm' (1 \leq i', j', k' < l' < m' \leq n)$ we have $\mu_{\mathbf{1}_T}(I') = \pm \delta_{I,I'}$, see [16, 28]. Since these Milnor invariants are 4-additive, the proof is completed.

5. Finite type concordance invariants

In this section, we define the equivalence relation on string links generated by C_k -moves and concordance, called C_k -concordance. We classify string links up to C_k -concordance for $k \leq 6$.

5.1. C_k -concordance

Recall that two n -string links σ, σ' are *concordant* if there is an embedding

$$f : (\sqcup_{i=1}^n I_i) \times I \longrightarrow (D^2 \times I) \times I$$

such that $f((\sqcup_{i=1}^n I_i) \times \{0\}) = \sigma \times \{0\}$ and $f((\sqcup_{i=1}^n I_i) \times \{1\}) = \sigma' \times \{1\}$, and such that $f(\partial(\sqcup_{i=1}^n I_i) \times I) = (\partial\sigma) \times I$. String link concordance is an equivalence relation, and is denoted by $\overset{c}{\sim}$.

In order to study finite type concordance invariants, it is natural to consider the following.

Definition 5.1. Let $k, n \geq 1$ be integers. Two n -string links σ, σ' are C_k -concordant if there is a sequence $\sigma = \sigma_0, \sigma_1, \dots, \sigma_m = \sigma'$ such that for each $i \geq 1$, either $\sigma_i \overset{C_k}{\sim} \sigma_{i+1}$ or $\sigma_i \overset{c}{\sim} \sigma_{i+1}$. We denote the C_k -concordance relation by $\overset{C_k+c}{\sim}$.

Clearly, two C_k -concordant string links share all finite type concordance invariants of degree less than k . It is thus natural to ask the following.

Question 1. Let $\sigma, \sigma' \in \mathcal{SL}(n)$. Is it true that $\sigma \overset{C_k+c}{\sim} \sigma'$ if and only if they share all finite type concordance invariants of degree $< k$?

It is known that Milnor invariants are concordance invariants [4]. So by [14, theorem 7.1], $\mu(J)$ is a C_k -concordance invariant for any J with $|J| \leq k$. Habegger and Masbaum showed that all *rational* finite type concordance invariants of string links are given by Milnor invariants via the Kontsevich integral [13].

5.2. The ordered index

In order to study C_k -concordance for string links, we use the notion of ordered index of a C_k -tree.

Definition 5.2. Let t be a linear C_k -tree with ends f_0, f_k . Since t is a disk, we can travel from f_0 to f_k along ∂t so that we meet all other leaves f_1, \dots, f_{k-1} in this order. If f_s is an i_s -leaf ($s = 0, \dots, k$), we can consider two vectors (i_0, \dots, i_k) and (i_k, \dots, i_0) and may assume that $(i_0, \dots, i_k) \leq (i_k, \dots, i_0)$, where ' \leq ' is the lexicographic order in \mathbb{Z}^{k+1} . We call (i_0, \dots, i_k) the *ordered index* of t and denote it by $\text{o-index}(t)$.

By Calculus of Claspers and AS, IHX, STU relations, we have the following.

LEMMA 5.1. (1) Let t and t' be linear C_k -trees for $\mathbf{1}_n$ with the same ordered index. Then there are C_k -graphs g_1, \dots, g_m with loops such that $(\mathbf{1}_n)_{t'} \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_t^\varepsilon \cdot \prod_i (\mathbf{1}_n)_{g_i}$, for some $\varepsilon = \pm 1$.

(2) Let t be a linear C_k -tree ($k \geq 3$) for $\mathbf{1}_n$ with $\text{o-index}(t) = (i_0, \dots, i_k)$. If $i_0 = i_1$ or $i_{k-1} = i_k$, then there are C_k -graphs g_1, \dots, g_m with loops such that $(\mathbf{1}_n)_t \overset{C_{k+1}}{\sim} \prod_i (\mathbf{1}_n)_{g_i}$.

(3) Let t be a linear C_k -tree ($k \geq 2$) for $\mathbf{1}_n$ with $\text{o-index}(t) = (i_0, \dots, i_k)$. If $(i_0, \dots, i_k) = (i_k, \dots, i_0)$ and k is even, then there are C_k -graphs g_1, \dots, g_m with loops such that $((\mathbf{1}_n)_t)^2 \overset{C_{k+1}}{\sim} \prod_i (\mathbf{1}_n)_{g_i}$.

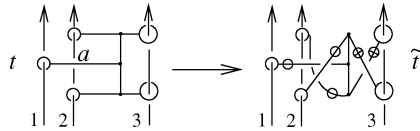


Fig. 19. A planar C_4 -tree t with o-index $\{3, 2, 1, 2, 3\}$, and the clasper \tilde{t} obtained by a 180-degree rotation around its edge a .

Before proving this lemma, we need the following definition. A C_k -tree for $\mathbf{1}_n$ is *planar* if it can be represented, in the usual diagram of $\mathbf{1}_n$, by a tree clasper T without any crossing among the edges of T and with all edges of T overpassing all components of $\mathbf{1}_n$ up to isotopy.

Proof of Lemma 5.1. Statements (1) and (2) follow from similar arguments as for Lemma 3.6. For (1), observe that t can be deformed into t' by crossing changes and sliding leaves. By the STU relation, if c' is obtained from a C_k -tree c for $\mathbf{1}_n$ by a sliding a leaf, we have $(\mathbf{1}_n)_c \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_{c'} \cdot (\mathbf{1}_n)_g$ for some C_k -graph g with a loop. For (2), use the IHX and STU relation as in the proof of Lemma 3.6.

For simplicity, we show (3) in the case where t is planar and both ends of t are n -leaves. See Figure 19 for an example. By assumption the o-index(t) has the form $(i_0, \dots, i_{(k/2)-1}, i_{k/2}, i_{(k/2)-1}, \dots, i_0)$. We may assume that the axis a of the edge incident to the $i_{k/2}$ -leaf of t is transverse to each component of $\mathbf{1}_n$ up to isotopy. Let \tilde{t} be obtained by a 180-degree rotation of t around a fixing the leaves. See Figure 19. By sliding the leaves of \tilde{t} repeatedly, we can deform it into a planar C_k -tree \bar{t} which only differs from t by a half-twist on each edge incident to a leaf. By the observation above, the STU relation gives that $(\mathbf{1}_n)_{\tilde{t}} \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_{\bar{t}} \prod_i (\mathbf{1}_n)_{g_i}$ for some union g_1, \dots, g_m of C_k -graphs with loops. On the other hand, by Lemma 3.2(2) we have $(\mathbf{1}_n)_t \cdot (\mathbf{1}_n)_{\bar{t}} \overset{C_{k+1}}{\sim} \mathbf{1}_n$. The result follows.

It is known that surgery along graphs with loops implies concordance.

LEMMA 5.2. ([6, 8]) *Let g be a C_k -graph with a loop for $\mathbf{1}_n$. Then $(\mathbf{1}_n)_g \overset{c}{\sim} \mathbf{1}_n$.*

For each o-index (i_0, \dots, i_k) , we choose one string link $T(i_0, \dots, i_k)$ obtained from $\mathbf{1}_n$ by surgery along a linear C_k -tree with o-index (i_0, \dots, i_k) , and fix it. We note that by Lemmas 5.1 (1) and 5.2, there are essentially two choices in $SL(n)/(C_{k+1} + c)$ for each o-index, namely $T(i_0, \dots, i_k)$ and $T(i_0, \dots, i_k)^{-1}$.

The next lemma can be obtained using the calculation method in [37, remark 5.3].

LEMMA 5.3. *Let σ be an n -string link obtained from $\mathbf{1}_n$ by surgery along a linear C_k -tree with o-index $I = (i_0, \dots, i_k)$. If $\{i_0, i_k\} \cap \{i_1, \dots, i_{k-1}\} = \emptyset$, then we have:*

- (1) *If $(i_0, \dots, i_k) \neq (i_k, \dots, i_0)$, then for any $J = i_0 j_1 \dots j_{k-1} i_k$, $\mu_\sigma(J) = \pm \delta_{I,J}$;*
- (2) *if $(i_0, \dots, i_k) = (i_k, \dots, i_0)$ and k is an odd number $2m + 1$, i.e., $(i_0, \dots, i_k) = (i_0, \dots, i_m, i_m, \dots, i_0)$, then for any $J = i_0 j_1 \dots j_{k-1} i_0$, $\mu_\sigma(J) = \pm 2\delta_{I,J}$;*
- (3) *if $(i_0, \dots, i_k) = (i_k, \dots, i_0)$ and k is an even number $2m$, that is if $(i_0, \dots, i_k) = (i_0, \dots, i_{m-1}, i_m, i_{m-1}, \dots, i_0)$, then the Milnor invariants of σ with length $\leq k$ vanish, and for any $J = i_0 j_1 \dots j_{2m} i_0$,*

$$\mu_\sigma(J) = \begin{cases} \pm 1 & \text{if } (j_1, \dots, j_{2m}) = (i_1, \dots, i_{m-1}, i_m, i_m, i_{m-1}, \dots, i_1) \\ 0 & \text{otherwise} \end{cases}$$

Remark 5.4. It follows in particular from (2) that Milnor invariants $\mu(I) \pmod{2}$ with $I = (i_0, \dots, i_m, i_m, \dots, i_0)$ are C_k -equivalence invariants ($k = 2m + 1$).

5.3. C_k -concordance for knots

In this subsection we give a classification of knots up to C_k -concordance.

Recall that $T(iii)$ is a fixed n -string link obtained from the trivial 1-string link $\mathbf{1}$ by surgery along a linear C_2 -tree with o-index (i, i, i) . Note that this tree can be chosen to be the C_2 -tree t_i represented in Figure 10, in which case the closure of the i th component of $T(iii)$ is the right-handed trefoil. For $n = 1$, we simply denote $T(111)$ by T .

LEMMA 5.5. *Let σ be a 1-string link. For any integer $k \geq 3$, there is a union G of disjoint graph claspers with loops for $\mathbf{1}_n$ such that $\sigma \overset{C_k}{\sim} T^\varepsilon \cdot (\mathbf{1}_n)_G$ for some $\varepsilon \in \{0, 1\}$.*

Proof. We proceed by induction on k . For $k = 3$, by Lemma 5.1 (1) (or Theorem 2.2), we have $\sigma \overset{C_3}{\sim} T^x$ for some integer x . By Lemma 5.1 (3), there is a union g of C_2 -graph with loop such that $T^2 \overset{C_3}{\sim} (\mathbf{1}_n)_g$. (Actually, it is easy to check using the AS and STU relations that in this case g is connected.) Hence we have

$$\sigma \overset{C_3}{\sim} \begin{cases} T \cdot ((\mathbf{1}_n)_g)^{(x+x/|x|)/2} & \text{if } x \text{ is odd,} \\ ((\mathbf{1}_n)_g)^{x/2} & \text{if } x \text{ is even.} \end{cases}$$

Now suppose that there is a union g_1, \dots, g_m of disjoint graph claspers with loops for $\mathbf{1}_n$ such that $\sigma \overset{C_k}{\sim} T^\varepsilon \cdot \prod_i (\mathbf{1}_n)_{g_i}$. Hence σ is obtained from $T^\varepsilon \cdot \prod_i (\mathbf{1}_n)_{g_i}$ by surgery along linear C_k -trees. Since any linear C_k -tree for a 1-string link has o-index $(1, \dots, 1)$, by Lemma 5.1 (2), we have that there are C_k -graphs h_1, \dots, h_l with loops for $\mathbf{1}_n$ such that $\sigma \overset{C_{k+1}}{\sim} T^\varepsilon \cdot \prod_i (\mathbf{1}_n)_{g_i} \cdot \prod_j (\mathbf{1}_n)_{h_j}$.

The next result is well-known to experts, and was first essentially proved by Ng [30]. We include a short, direct proof for completeness. Another proof can be found in [33].

THEOREM 5.6 (Ng). *For an integer $k \geq 3$, two knots K and K' are C_k -concordant if and only if $\text{Arf}(K) = \text{Arf}(K')$.*

Recall that any knot is C_2 -equivalent to the trivial one [29].

Proof. Let σ and σ' be 1-string links whose closures are K and K' respectively. By Lemma 5.5, there are graph claspers g_1, \dots, g_m and g'_1, \dots, g'_l with loops for $\mathbf{1}_n$ such that $\sigma \overset{C_k}{\sim} T^\varepsilon \cdot \prod_i (\mathbf{1}_n)_{g_i}$ and $\sigma' \overset{C_k}{\sim} T^{\varepsilon'} \cdot \prod_j (\mathbf{1}_n)_{g'_j}$ for some $\varepsilon, \varepsilon' \in \{0, 1\}$. So by Lemma 5.2, $\sigma \overset{C_{k+c}}{\sim} T^\varepsilon$ and $\sigma' \overset{C_{k+c}}{\sim} T^{\varepsilon'}$. Since the Arf invariant is a C_k -concordance invariant [32], and since $\text{Arf}(T) = 1$, we have $\text{Arf}(K) = \varepsilon$ and $\text{Arf}(K') = \varepsilon'$. This completes the proof.

5.4. C_k -concordance for string links

In this section, we give classifications for n -string links up to C_k -concordance ($k = 3, 4, 5, 6$). For each $k > 0$, the set of C_k -concordance classes forms a group. In order to give these classifications, we give a representative of the C_k -concordance class of an arbitrary n -string link in terms of the generators $T(iii)$ ($1 \leq i \leq n$) and $T(I)$'s, where I contains at least 2 distinct integers. More precisely, we will show that any string link is C_k -concordant to $\prod_i T(iii)^{a(i)} \cdot \prod_I T(I)^{b(I)}$ where $a(i)$ and $b(I)$ are determined by the Arf invariant and (mod 2) Milnor invariants respectively. For $k = 3, 4, 5$, we already have generators for the C_k -equivalence classes, by the proofs of Theorems 2.2, 2.4 and 2.5, and we can choose the desired generators among them. We will introduce similar generators for $k = 6$.

We will give the classification results successively, as consequences of each step of our construction of a representative of the $(C_6 + c)$ -equivalence class of a string link. In particular, the various proofs are contained in this construction.

Before starting the construction, we fix the convention below.

Convention 5.7. (1) By Lemma 5.3 (1) and (2), we see that for each o-index $I = (i_0, \dots, i_k)$ with $\{i_0, i_k\} \cap \{i_1, \dots, i_{k-1}\} = \emptyset$, we have $\mu_{T(I)}(I) = \pm 1$ or ± 2 . As mentioned before, we have essentially two choices for $T(I)$ and $T(I)^{-1}$ up to C_{k+1} -concordance. In this section, we chose $T(I)$ so that $\mu_{T(I)}(I)$ is positive whenever I satisfies Lemma 5.3 (1) or (2). (Note that for such a multi-index I we have $\mu_{T(I)^{-1}}(I) = -\mu_{T(I)}(I)$.) For example, $T(ij)$ is the n -string link L_{ij} obtained from $\mathbf{1}_n$ by surgery along the C_1 -tree l_{ij} of Figure 10 ($1 \leq i < j \leq n$).

(2) When denoting o-indices, we will let distinct letters denote distinct integers unless otherwise specified. For example the set $\{(ijk) | 1 \leq i, j, k \leq n\}$ of o-indices does not contain (iii) ($1 \leq i \leq n$).

Let σ be an n -string link. By [29], we have that σ is C_2 -equivalent to a string link

$$\sigma(1) = \prod_{1 \leq i < j \leq n} T(ij)^{\mu(ij)}.$$

So σ is obtained from $\sigma(1)$ by surgery along linear C_2 -trees. So, by Lemmas 5.1 and 5.5, there is a disjoint union G_1 of C_2 -graphs with loops such that

$$\sigma \stackrel{C_3}{\sim} \sigma(1) \cdot \sigma(2) \cdot (\mathbf{1}_n)_{G_1}, \tag{5.1}$$

where

$$\sigma(2) = \prod_{1 \leq i \leq n} T(iii)^{\varepsilon(iii)} \cdot \prod_{1 \leq i < j \leq n} T(jij)^{\varepsilon(jij)} \cdot \prod_{1 \leq i < j < k \leq n} T(ijk)^{x(ijk)}$$

for some $\varepsilon(iii), \varepsilon(jij) \in \{0, 1\}$ and some integers $x(ijk)$. Note that $T(iii)$, $T(jij)$ and $T(ijk)$ are the n -string links T_i , W_{ji} and B_{ijk} introduced in Subsection 4.1, obtained respectively from $\mathbf{1}_n$ by surgery along the C_2 -trees t_i , w_{ji} and b_{ijk} of Figure 10. In particular, we have $T(jij) \stackrel{C_3}{\sim} T(jji)$. By Lemma 5.2, it follows that $\sigma \stackrel{C_3+C}{\sim} \sigma(1) \cdot \sigma(2)$.

We denote by $\text{Arf}_i(\sigma)$ the Arf invariant for (the closure of) the i th component of σ . By Lemma 5.3, we have

$$\begin{aligned} \mu_\sigma(ij) &= \mu_{\sigma(1)}(ij), \\ \text{Arf}_i(\sigma) &= \text{Arf}_i(\sigma(1)) + \text{Arf}_i(\sigma(2)) = \text{Arf}_i(\sigma(2)) = \varepsilon(iii), \\ \mu_\sigma(ijk) &= \mu_{\sigma(1)}(ijk) + \mu_{\sigma(2)}(ijk) = \mu_{\sigma(1)}(ijk) + x(ijk), \\ \mu_\sigma(jiij) &\equiv \mu_{\sigma(1)}(jiij) + \mu_{\sigma(2)}(jiij) \\ &\equiv \mu_{\sigma(1)}(jiij) + \sum_{1 \leq p < q < r \leq n} x(pqr) \mu_{T(pqr)}(jiij) + \varepsilon(jij) \\ &\equiv \mu_{\sigma(1)}(jiij) + \varepsilon(jij) \pmod{2}. \end{aligned}$$

Since these invariants are C_3 -concordance invariants, we have the following.

THEOREM 5.8. *Two n -string links are C_3 -concordant if and only if they share all invariants Arf_i , $\mu(ij)$ ($1 \leq i < j \leq n$), $\mu(ijk)$ ($1 \leq i < j < k \leq n$) and $\mu(jiij) \pmod{2}$ ($1 \leq i < j \leq n$).*

Since, by (5.1), the n -string link σ is obtained from $\sigma(1) \cdot \sigma(2) \cdot (\mathbf{1}_n)_{G_1}$ by surgery along linear C_3 -trees, by Lemmas 5.1 and 5.5, there is a disjoint union G_2 of C_3 -graphs with loops such that

$$\sigma \stackrel{C_4}{\sim} \sigma(1) \cdot \sigma(2) \cdot (\mathbf{1}_n)_{G_1} \cdot \sigma(3) \cdot (\mathbf{1}_n)_{G_2}, \tag{5.2}$$

where

$$\sigma(3) := \prod_{1 \leq i < j \leq n} T(jiij)^{y(jiij)} \cdot \prod_{\substack{1 \leq i, j, k \leq n \\ i < j}} T(kijk)^{y(kijk)} \cdot \prod_{1 \leq i, j < k < l \leq n} T(kijl)^{y(kijl)}$$

for some integers $y(jiij)$, $y(kijk)$, $y(kijl)$. Observe that $T(jiij)$, $T(kijk)$ and $T(kijl)$ correspond respectively to the string links obtained by surgery along the C_3 -trees h_{ij} , h_{ijk} and h_{ijkl} of Figure 11. (Actually, (5.2) can also be derived from the proof of Theorem 2.4.)

By Lemma 5.3, we have

$$\begin{aligned} \mu_\sigma(jiij) &= \mu_{\sigma(1)\cdot\sigma(2)}(jiij) + \mu_{\sigma(3)}(jiij) = \mu_{\sigma(1)\cdot\sigma(2)}(jiij) + 2y(jiij), \\ \mu_\sigma(kijk) &= \mu_{\sigma(1)\cdot\sigma(2)}(kijk) + \sum_{1 \leq p < q \leq n} y(qppq)\mu_{T(qppq)}(kijk) + y(kijk), \\ \mu_\sigma(kijl) &= \mu_{\sigma(1)\cdot\sigma(2)}(kijl) + \sum_{1 \leq p < q \leq n} y(qppq)\mu_{T(qppq)}(kijl) \\ &\quad + \sum_{\substack{1 \leq p, q, r \leq n \\ p < q}} y(rpqr)\mu_{T(rpqr)}(kijl) + y(kijl). \end{aligned}$$

Since these invariants are C_4 -concordance invariants and $\sigma \stackrel{C_4+c}{\sim} \sigma(1) \cdot \sigma(2) \cdot \sigma(3)$, we have the following.

THEOREM 5.9. *Two n -string links are C_4 -concordant if and only if they are C_3 -concordant and they share all invariants $\mu(jiij)$ ($1 \leq i < j \leq n$), $\mu(kijk)$ ($1 \leq i, j, k \leq n$; $i < j$) and $\mu(kijl)$ ($1 \leq i, j < k < l \leq n$).*

Now, by (5.2), there is a disjoint union G_3 of C_4 -graphs with loops such that

$$\sigma \stackrel{C_5}{\sim} \sigma(1) \cdot \sigma(2) \cdot (\mathbf{1}_n)_{G_1} \cdot \sigma(3) \cdot (\mathbf{1}_n)_{G_2} \cdot \sigma(4) \cdot (\mathbf{1}_n)_{G_3}, \tag{5.3}$$

where $\sigma(4)$ is given by

$$\sigma(4) := \prod_{i,j} T(jiij)^{\varepsilon(jiij)} \cdot \prod_{i,j < k} T(kjik)^{\varepsilon(kjik)} \cdot \prod_{s=1}^4 \prod_{I \in \mathcal{I}_s} T(I)^{z(I)} \tag{5.4}$$

for some $\varepsilon(jiij) \in \{0, 1\}$ and some integers $\varepsilon(kjik)$ and $z(I)$, where

$$\begin{aligned} \mathcal{I}_1 &= \{ikkkj \mid 1 \leq i, j, k \leq n, i < j\}, \\ \mathcal{I}_2 &= \{kiijk \mid 1 \leq i, j, k \leq n, i < j, i < k\} \cup \{kjiiik \mid 1 \leq j < i < k \leq n\}, \\ \mathcal{I}_3 &= \{kijpk \mid 1 \leq i, j, k, p \leq n, i < p\}, \mathcal{I}_4 = \{pijkq \mid 1 \leq i, j, k < p < q \leq n\}. \end{aligned}$$

In particular, the second product in (5.4) is obtained from the following two observations. On one hand, for $1 \leq i, j < k \leq n$, we have by (4.9) that

$$T(ijkji) \stackrel{C_5}{\sim} T(kjijk)^\delta \cdot T(kjik)^\delta \quad (\delta, \delta' \in \{-1, 1\})$$

(noting that $T(ijkij)$, $T(kjijk)$ and $T(kjik)$ correspond to the string links obtained by surgery along the C_4 -trees $u''_{(12)}$, $u_{(12)}$ and $u'_{(12)}$ of Figure 16 respectively, and that $u''_{(12)} \stackrel{C_3+c}{\sim} u'_{(132)}$). On the other hand, by Lemma 5.1 (3) the C_4 -trees above are 2-torsion elements in $\mathcal{SL}(n)/(C_5 + c)$.

Note also that $T(jiii)$ in (5.4) corresponds to the string links obtained by surgery along the C_4 -tree s_{id}^k of Figure 12 ($k = 1, 2$), and that similarly for $I \in \mathcal{I}_1, \mathcal{I}_2$, the various $T(I)$ correspond to C_5 -concordance classes of the string links obtained by surgery along the C_4 -trees of Figure 16.

By Lemma 5.3, we have

$$\mu_\sigma(jiiii) \equiv \mu_{\sigma(1)\cdot\sigma(2)\cdot\sigma(3)}(jiiii) + \varepsilon(jiii) \pmod 2,$$

$$\mu_\sigma(kijjik) \equiv \mu_{\sigma(1)\cdot\sigma(2)\cdot\sigma(3)}(kijjik) + \sum_{p,q} \varepsilon(qpppq)\mu_{T(qpppq)}(kijjik) + \varepsilon(kijik) \pmod 2,$$

and for each $I \in \mathcal{I}_s$ ($1 \leq s \leq 4$),

$$\mu_\sigma(I) = \mu_{\sigma(1)\cdot\sigma(2)\cdot\sigma(3)}(I) + \sum_{p,q} \varepsilon(qpppq)\mu_{T(qpppq)}(I) + \sum_{\substack{W \in \mathcal{I}_I \\ t < s}} z(W)\mu_{T(W)}(I) + z(I).$$

Since these invariants are C_5 -concordance invariants and $\sigma \stackrel{C_5^+c}{\sim} \sigma(1) \cdot \sigma(2) \cdot \sigma(3) \cdot \sigma(4)$, we have the following.

THEOREM 5.10. *Two n -string links are C_5 -concordant if and only if they are C_4 -concordant and share all invariants $\mu(jiiii) \pmod 2$ ($1 \leq i, j \leq n$), $\mu(kijjik) \pmod 2$ ($1 \leq i, j < k \leq n$) and $\mu(I)$ ($I \in \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3 \cup \mathcal{I}_4$).*

Moreover, from (5.3) we have that there is a string link $\sigma(5)$ and a disjoint union G_4 of C_5 -graphs with loops such that

$$\sigma \stackrel{C_6}{\sim} \sigma(1) \cdot \sigma(2) \cdot (\mathbf{1}_n)_{G_1} \cdot \sigma(3) \cdot (\mathbf{1}_n)_{G_2} \cdot \sigma(4) \cdot (\mathbf{1}_n)_{G_3} \cdot \sigma(5) \cdot (\mathbf{1}_n)_{G_4}, \quad (5.5)$$

where

$$\sigma(5) = \prod_{s=0}^7 \prod_{J \in \mathcal{J}_s} T(J)^{w(J)}$$

for some integers $w(J)$, where

$$\mathcal{J}_0 = \{ijjij \mid 1 \leq i < j \leq n\}, \quad \mathcal{J}_1 = \{ijjji \mid 1 \leq i, j \leq n\},$$

$$\mathcal{J}_2 = \{ikkkk \mid 1 \leq i, j, k \leq n, i < j\},$$

$$\mathcal{J}_3 = \{kiiijk, kiijik, kijjjk, kjijjk \mid 1 \leq i, j, k \leq n, i < j\},$$

$$\mathcal{J}_4 = \{kiiijk, kijijk, kijjik, kjiijk \mid 1 \leq i < j < k \leq n\},$$

$$\begin{aligned} \mathcal{J}_5 = \{pikkjp, pkijkp \mid 1 \leq i, j, k, p \leq n, i < j, k < p\} \\ \cup \{pijkkp, pikjpk \mid 1 \leq i, j < k < p \leq n\} \\ \cup \{pkkijp, pkikjp \mid 1 \leq k < i, j, p \leq n\} \\ \cup \{pkkijp, pkikjp \mid 1 \leq i, j, k, p \leq n, i < k < j\}, \end{aligned}$$

$$\mathcal{J}_6 = \{qkijpq \mid 1 \leq i, j, k, p, q \leq n, k < p\}, \quad \mathcal{J}_7 = \{qijkpr \mid 1 \leq i, j, k, p < q < r \leq n\}.$$

Let us briefly explain how to determine these \mathcal{J}_s 's. First, separate C_5 -trees by their indices. By Lemmas 3.6, 5.1 (2) and 5.2, we have eight cases : $\{i^{(3)}, j^{(3)}\}$, $\{i^{(2)}, j^{(4)}\}$, $\{i, j, k^{(4)}\}$, $\{i^{(3)}, j, k^{(2)}\}$, $\{i^{(2)}, j^{(2)}, k^{(2)}\}$, $\{i, j, k^{(2)}, p^{(2)}\}$, $\{i, j, k, p, q^{(2)}\}$ and $\{i, j, k, p, q, r\}$, which correspond to $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_6$ and \mathcal{J}_7 respectively. By the IHX relation, we may

assume that each C_5 -tree is linear, and we may choose any pair of leaves as ends. Hence for each of the eight cases above, we may choose the ends of any C_5 -tree having the corresponding index. Then we enumerate all possible o-indices, using Lemmas 5.1 (2) and 5.2. For example, we may choose that the ends of any linear C_5 -tree with index $\{i^{(3)}, j^{(3)}\}$ are an i -leaf and a j -leaf, so we enumerate all o-indices starting with i and ending with j . By Lemma 5.1 (2), we are left with only two cases, namely $ijijij$ and $ijjiiij$. Now, it follows from two applications of the AS relation that $T(ijijij) \stackrel{C_6}{\sim} T(ijjiiij)$. So $ijijij$ is essentially the only o-index for C_5 -trees with index $\{i^{(3)}, j^{(3)}\}$.

By combining a similar method as in [37, remark 5.3] and the IHX relation, we have that for each $J \in \mathcal{J}_0$

$$\mu_\sigma(J) = \mu_{\sigma(1)\cdot\sigma(2)\cdot\sigma(3)\cdot\sigma(4)}(J) + 12 \cdot w(J).$$

By Lemma 5.3, we have that for each $J \in \mathcal{J}_s$ ($1 \leq s \leq 7$)

$$\mu_\sigma(J) = \mu_{\sigma(1)\cdot\sigma(2)\cdot\sigma(3)\cdot\sigma(4)}(J) + \sum_{0 \leq i < s} \sum_{V \in \mathcal{J}_i} w(V)\mu_{T(V)}(J) + c_J \cdot w(J),$$

where $c_J = 2$ if $J \in \mathcal{J}_1 \cup \{kijjik, kjiijk \mid 1 \leq i < j < k \leq n\}$ and $c_J = 1$ otherwise. Since these invariants are C_6 -concordance invariants and $\sigma \stackrel{C_6+c}{\sim} \sigma(1) \cdot \sigma(2) \cdot \sigma(3) \cdot \sigma(4) \cdot \sigma(5)$, we have the following.

THEOREM 5.11. *Two n -string links are C_6 -concordant if and only if they are C_5 -concordant and they share all invariants $\mu(J)$ for $J \in \mathcal{J}_i$ ($i = 0, 1, \dots, 6$).*

Remark 5.12. Theorem 5.8, as well as the 2-component cases of Theorems 5.9, 5.10 and 5.11, are also proved in [33], using different methods.

Note that for $k \leq 6$, we meet torsion elements in the group $S\mathcal{L}(n)/(C_k + c)$. This is a remarkable difference with the group $S\mathcal{L}(n)/C_k$, where no torsion element exists for $k < 6$ as proved in Section 4. The torsion elements we meet are all 2-torsion elements of the form $T(iii)$ and $T(i_0, i_1, \dots, i_p, i_{p+1}, i_p, \dots, i_1, i_0)$, possibly with $i_j = i_k$ for $j, k \neq 0$. By Lemma 5.1 (3), there are 2-torsion elements which are not of the form $T(iii)$ in $S\mathcal{L}(n)/(C_k + c)$ for any odd k . This implies the following.

PROPOSITION 5.13. *Rational concordance finite type invariants of even degree k , together with the invariants Arf_i , do not suffice to determine the classes $S\mathcal{L}(n)/(C_{k+1} + c)$.*

Remark 5.14. It is known that, for a knot K , $a_2(K) \equiv \text{Arf}(K) \pmod 2$. Since a_2 is a finite type invariant of degree 2 and the Arf invariant is a concordance invariant, $a_2 \pmod 2$ and $f_2 \pmod 2$ are both concordance finite type invariants of degree 2, where f_2 is the invariant defined in Subsection 2.2. In the definition of the string link $\sigma(2)$ involved in the proof of Theorem 5.8, $\varepsilon(iii)$ is modulo 2 equivalent to a_2 of the closure of the i th component of $\sigma(2)$, and $\varepsilon(jij)$ is modulo 2 equivalent to f_2 of the union of the i th and j th components of $\sigma(2)$. It follows that two n -string links are C_3 -concordant if and only if they share all invariants $a_2 \pmod 2$, $\mu(ij)$ ($1 \leq i < j \leq n$), $\mu(ijk)$ ($1 \leq i < j < k \leq n$) and $f_2 \pmod 2$. This implies that the classification of string links up to C_3 -concordance is given by concordance finite type invariants of degree ≤ 2 , and hence that rational concordance finite type invariants of degree 2, together with the invariants Arf_i , do not suffice to determine all concordance finite type invariants of degree 2.

6. 2-string links up to self C_3 -moves and concordance

Given a multi-index I , let $r(I)$ denote the maximum number of times that any index appears. For example, $r(1123) = 2$, $r(1231223) = 3$. It is known that if $r(I) = 1$, then Milnor invariant with index I is a link-homotopy invariant [12, 27], where link-homotopy is an equivalence relation on links generated by self crossing changes. Milnor invariants give a link-homotopy classification of string links [12].

Although Milnor invariants with $r(I) \geq 2$ are not necessarily link-homotopy invariants, Fleming and the second author showed that μ -invariants with $r(I) \leq k$ are self C_k -equivalence invariants for string links, where the self C_k -equivalence is an equivalence relation on (string) links generated by self C_k -moves, which are C_k -moves with all $k + 1$ strands in a single component. See [7, theorem 3.1] and [36].

Two string links σ and σ' are self- C_k concordant if there is a sequence $\sigma = \sigma_1, \dots, \sigma_m = \sigma'$ of string links such that for each $i \in \{1, \dots, m - 1\}$, σ_i and σ_{i+1} are either concordant or self C_k -equivalent.

Since Milnor invariants are concordance invariants, any Milnor invariant indexed by I with $r(I) \leq k$ is a self- C_k concordance invariant. The second author showed that Milnor invariants $\mu(I)$ with $r(I) \leq 2$ classify string links up to self- C_2 concordance [36]. Here we give a self- C_3 concordance classification for 2-string links.

THEOREM 6.1. *Two 2-string links are self- C_3 concordant if and only if they share all invariants Arf_i ($i = 1, 2$), $\mu(12)$, $\mu(2112)$, $\mu(121212)$, and $\mu(jiiiij) \bmod 2$ ($\{i, j\} = \{1, 2\}$).*

Remark 6.2. Let L be the Hopf link with both components Whitehead doubled. In [36, Remark], the second author mentioned that L is neither self- C_3 equivalent nor concordant to the trivial 2-component link. It follows that a 2-string link σ with closure $\hat{\sigma} = L$ is neither self- C_3 equivalent nor concordant to the trivial 2-string link. On the other hand, since L is a boundary link, all Milnor invariants of L (and of σ) vanish. This and Theorem 6.1 imply that the 2-string link σ is self- C_3 concordant to the trivial 2-string link.

Proof. By [7, lemma 1.2], C_6 -concordance implies self- C_3 concordance for 2-string links. Starting with a representative of the C_6 -concordance class of a 2-string link σ , as given by (5.5), we can eliminate the generators $T(I)$ such that I contains the same index at least four times to obtain a self- C_3 concordance representative. We obtain that σ is self- C_3 concordant to

$$T(12)^x \cdot T(111)^{\varepsilon_1} \cdot T(222)^{\varepsilon_2} \cdot T(212)^{\varepsilon_3} \cdot T(2112)^y \cdot T(21112)^{\varepsilon_4} T(12221)^{\varepsilon_5} \cdot T(121212)^z,$$

for some integers x, y, z and for some $\varepsilon_i \in \{0, 1\}$ ($1 \leq i \leq 5$). By Theorem 5.11, x, y, z, ε_i ($1 \leq i \leq 5$) are determined by the invariants Arf_i ($i = 1, 2$), $\mu(12)$, $\mu(2112)$, $\mu(121212)$, $\mu(211112) \bmod 2$, and $\mu(122221) \bmod 2$. Since all these invariants except for $\mu(jiiiij) \bmod 2$ ($\{i, j\} = \{1, 2\}$) are self- C_3 concordance invariants, there only remains to show that $\mu(jiiiij) \bmod 2$ is a self- C_3 equivalence invariant.

Suppose that σ' is a string link obtained from $\sigma \in \mathcal{SL}(2)$ by surgery along a C_3 -tree with index $\{1^{(4)}\}$. It is enough to show that $\mu_{\sigma'}(jiiiij) \equiv \mu_{\sigma}(jiiiij) \bmod 2$. By Calculus of Clasper,

$$\sigma' \overset{C_4}{\sim} \sigma \cdot (\mathbf{1}_2)_{t'},$$

where t' is a C_3 -tree with index $\{1^{(4)}\}$ and is in a tubular neighbourhood of the 1st strand of

$\mathbf{1}_2$. By [37, lemma 2.1], we may assume that the C_4 -equivalence above is realized by surgery along a disjoint union of C_4 -trees with indices $\{1^{(5)}\}$ or $\{1^{(4)}, 2\}$. So by Lemma 5.1 (2), we have

$$\sigma' \stackrel{C_5^+c}{\sim} \sigma \cdot (\mathbf{1}_2)_{t'}.$$

Hence, by Remark 5.4, we have

$$\mu_{\sigma'}(jiiiij) \equiv \mu_{\sigma}(jiiiij) + \mu_{(\mathbf{1}_2)_{t'}}(jiiiij) \equiv \mu_{\sigma}(jiiiij) \pmod{2}.$$

This completes the proof.

7. C_{n+1} -moves for n -component Brunnian string links

An n -string link is Brunnian if every proper substring link of it is trivial. In this section, we use tools developed in the present paper to classify Brunnian n -string links up to C_{n+1} equivalence, thus improving a previous result of the authors [26].

Let B be a Brunnian n -string link. An explicit formula for a representative B_0 of the C_n -equivalence class of B was given in [16] (see also [26, proposition 4.2]), and can be formulated as follows (using the notation of Section 5):

$$B_0 := \prod_{\eta \in \mathcal{S}_{n-2}} T(n-1, \eta(1), \dots, \eta(n-2), n)^{\mu_B(n-1, \eta(1), \dots, \eta(n-2), n)}. \tag{7.1}$$

Recall from Subsection 3.2 that for an integer k in $\{1, \dots, n\}$, $\mathcal{B}_n(k)$ denotes the set of all bijections τ from $\{1, \dots, n-1\}$ to $\{1, \dots, n\} \setminus \{k\}$ such that $\tau(1) < \tau(n-1)$, and that $B_{\alpha}(k)$, resp. $\overline{B}_{\alpha}(k)$, is the n -string link obtained from $\mathbf{1}_n$ by surgery along the C_n -tree $T_{\alpha}(l)$, resp. $\overline{T}_{\alpha}(l)$ represented in Figure 7. For $\tau \in \mathcal{B}_n(k)$, set $\mu_{\tau}(B) := \mu_B(\tau(1), \dots, \tau(n-1), k, k)$. It was proved in [26, proposition 4.5] that

$$B \stackrel{C_{n+1}}{\sim} B_0 \cdot B_{(1)} \cdot \dots \cdot B_{(n)},$$

where, for each k ($1 \leq k \leq n$), $B_{(k)}$ is the Brunnian n -string link

$$\prod_{\tau \in \mathcal{B}_n(k)} (B_{\tau}(k))^{n_{\tau}(k)} \cdot (\overline{B}_{\tau}(k))^{n'_{\tau}(k)}, \tag{7.2}$$

such that, for any $\tau \in \mathcal{B}_n(k)$ ($1 \leq k \leq n$), the exponents $n_{\tau}(k)$ and $n'_{\tau}(k)$ are two integers satisfying

$$n_{\tau}(k) + n'_{\tau}(k) = \mu_{\tau}(B_{(1)} \cdot \dots \cdot B_{(n)}) = \mu_{\tau}(B) - \mu_{\tau}(B_0).$$

This uses the fact that, for any $k \in \{1, \dots, n\}$ and $\alpha, \tau \in \mathcal{B}_n(k)$, we have

$$\mu_{\alpha}(B_{\tau}(k)) = \mu_{\alpha}(\overline{B}_{\tau}(k)) = \delta_{\alpha, \tau}.$$

Given an n -string link σ and $\tau \in \mathcal{B}_n(1)$, we can construct a knot $K_{\tau}(\sigma)$ in S^3 as follows. Connect the upper endpoints of the first and the $\tau(1)$ th components of σ by an arc a_1 in $S^3 \setminus (D^2 \times I)$. Next, connect the lower endpoints of the $\tau(1)$ th and the $\tau(2)$ th components by an arc a_2 in $S^3 \setminus (D^2 \times I)$ disjoint from a_1 , then the upper endpoints of the $\tau(2)$ th and $\tau(3)$ th components by an arc a_3 in $S^3 \setminus (D^2 \times I)$ disjoint from $a_1 \cup a_2$. Repeat this construction until reaching the $\tau(n-1)$ th component, and connect its lower or upper endpoint (depending on the parity of n) to the lower endpoint of the first component by an arc a_n in $S^3 \setminus (D^2 \times I)$ disjoint from $\bigcup_{1 \leq i \leq n-1} a_i$. The arcs are chosen so that, if a_i and a_j ($i < j$) meet in the diagram of L , then a_i overpasses a_j . It follows from the construction of $K_{\tau}(\sigma)$ and [17] that

for any $\tau \in \mathcal{B}_n(1)$, we have $\mathbf{p} := P_0^{(n)}(K_\tau(B_\tau(1)))$ is nonzero (note that \mathbf{p} depends only on n). Set

$$f_\tau(\sigma) := P_0^{(n)}(K_\tau(\sigma))/\mathbf{p}.$$

By the proof of Lemma 3.4, we note that for any $k \geq 2$ there is a bijection $\phi_k : \mathcal{B}_n(1) \rightarrow \mathcal{B}_n(k)$ such that for any $\tau \in \mathcal{B}_n(1)$, we have $B_\tau(1) \cdot \overline{B_\tau(1)}^{-1} \overset{C_{n+1}}{\sim} B_{\phi_k(\tau)}(k) \cdot \overline{B_{\phi_k(\tau)}(k)}^{-1}$.

We can now prove the following stronger version of [26, proposition 4.5].

THEOREM 7.1. *Let B be a Brunnian n -string link. Then*

$$B \overset{C_{n+1}}{\sim} B_0 \cdot B',$$

where B_0 is determined by the Milnor invariants of B of length n as in (7.1), and where B' is given by

$$\prod_{\tau \in \mathcal{B}_n(1)} ((B_\tau(1))^{m_\tau} \cdot (\overline{B_\tau(1)})^{\mu_\tau(B) - \mu_\tau(B_0) - m_\tau}) \cdot \prod_{k=2}^n \prod_{\tau \in \mathcal{B}_n(k)} (B_\tau(k))^{\mu_\tau(B) - \mu_\tau(B_0)},$$

where $m_\tau = f_\tau(B) - f_\tau(B_0) - \sum_{k \geq 2} (\mu_{\phi_k(\tau)}(B) - \mu_{\phi_k(\tau)}(B_0))$ ($\tau \in \mathcal{B}_n(1)$).

Proof. By Lemma 3.4, we may assume that $n'_\tau(k) = 0$ in (7.2) for any $\tau \in \mathcal{B}_n(k)$ with $k \neq 1$. Hence the product $B_{(1)} \cdot \dots \cdot B_{(n)}$ is given by

$$\prod_{\tau \in \mathcal{B}_n(1)} ((B_\tau(1))^{n'_\tau(1)} \cdot (\overline{B_\tau(1)})^{n'_\tau(1)}) \cdot \prod_{k=2}^n \prod_{\tau \in \mathcal{B}_n(k)} (B_\tau(k))^{\mu_\tau(B) - \mu_\tau(B_0)}.$$

Let $\tau \in \mathcal{B}_n(1)$. It follows from the construction of $K_\tau(\sigma)$ and [17] that for any $\eta \in \mathcal{B}_n(1)$, we have $f_\tau(B_\eta(1)) = \delta_{\tau,\eta}$ and $f_\tau(\overline{B_\eta(1)}) = 0$, and that $f_\tau(B_{\phi_k(\eta)}(k)) = \delta_{\tau,\eta}$ for each $k \geq 2$.

By using a similar argument as Claim 4.3 and the multiplicativity of the HOMFLYPT polynomial, we thus have that for each $\tau \in \mathcal{B}_n(1)$

$$\begin{aligned} f_\tau(B) &= f_\tau(B_0) + f_\tau(B_{(1)} \cdot \dots \cdot B_{(n)}) \\ &= f_\tau(B_0) + n_\tau(1) + \sum_{k \geq 2} (\mu_{\phi_k(\tau)}(B) - \mu_{\phi_k(\tau)}(B_0)). \end{aligned}$$

Since $\mu_\tau(B) = \mu_\tau(B_0) + n_\tau(1) + n'_\tau(1)$, this completes the proof.

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