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CHARACTERIZATION OF Y_2-EQUIVALENCE FOR HOMOLOGY CYLINDERS

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ABSTRACT

For Σ a compact connected oriented surface, we consider homology cylinders over Σ: these are homology cobordisms with an extra homological triviality condition. When considered up to Y_2-equivalence, which is a surgery equivalence relation arising from the Goussarov-Habiro theory, homology cylinders form an Abelian group.

In this paper, when Σ has one or zero boundary component, we define a surgery map from a certain space of graphs to this group. This map is shown to be an isomorphism, with inverse given by some extensions of the first Johnson homomorphism and Birman-Craggs homomorphisms.

Keywords: Homology cylinder; finite type invariant; clover; clasper.

1. Introduction

1.1. *Homology cylinders*

Homology cylinders are important objects in the theory of finite type invariants of Goussarov-Habiro: they have thus appeared in both [6] and [4]. Let us recall the definition of these objects.

Let Σ be a compact connected oriented surface. A *homology cobordism* over Σ is a triple (M, i^+, i^-) where M is a compact oriented 3-manifold and $i^\pm : \Sigma \longrightarrow M$ are oriented embeddings with images $\Sigma^\pm$, such that:

(i) $i^\pm$ are homology isomorphisms;
(ii) $\partial M = \Sigma^+ \cup (-\Sigma^-)$ and $\Sigma^+ \cap (-\Sigma^-) = \pm\partial\Sigma^\pm$;
(iii) $i^+|_{\partial\Sigma} = i^-|_{\partial\Sigma}$.

Homology cobordisms are considered up to orientation-preserving diffeomorphisms. When $(i^-)_*^{-1} \circ (i^+)_* : H_1(\Sigma; \mathbf{Z}) \longrightarrow H_1(\Sigma; \mathbf{Z})$ is the identity, M is said to be a *homology cylinder*. The set of homology cobordisms is denoted here by $\mathcal{C}(\Sigma)$, and $\mathcal{HC}(\Sigma)$ denotes the subset of homology cylinders. If $M = (M, i^+, i^-)$ and

$N = (N, j^+, j^-)$ are homology cobordisms, we can define their *stacking product* by

$$M \cdot N := (M \cup_{i^- \circ (j^+)^{-1}} N, i^+, j^-) .$$

This product induces a monoid structure on $\mathcal{C}(\Sigma)$, with $\mathcal{HC}(\Sigma)$ a submonoid. The unit element is $1_\Sigma := (\Sigma \times I, Id, Id)$, where I is the unit interval $[0, 1]$ and where a collar of $\Sigma^\pm$ is stretched along $\partial\Sigma \times I$ so that the second defining condition for homology cobordisms is satisfied.

Habiro in [6, §8.5] outlined how homology cylinders can serve as a powerful tool in studying the mapping class groups of surfaces (see [3], [5], [12]). The connection lies on the homomorphism of monoids

$$\mathcal{T}(\Sigma) \xrightarrow{C} \mathcal{HC}(\Sigma)$$

sending each h in the Torelli group of Σ to the mapping cylinder $C_h = (\Sigma \times I, Id, h)$ (with, as above, a collar of $\Sigma^\pm$ stretched along $\partial\Sigma \times I$).

In the sequel, *we restrict ourselves* to the following two cases:

(i) $\Sigma = \Sigma_g$ is the standard closed oriented surface of genus $g \geq 0$, which here is referred to as the *closed case*;
(ii) $\Sigma = \Sigma_{g,1}$ is the standard compact oriented surface of genus $g \geq 0$ with one boundary component, which here is referred to as the *boundary case*.

The usual notations $\mathcal{T}_{g,1} = \mathcal{T}(\Sigma_{g,1})$ and $\mathcal{T}_g = \mathcal{T}(\Sigma_g)$ for the Torelli groups will be used. Also denote by H the first homology group of Σ with integer coefficients, by $\bullet$ the intersection form on H and by $(x_i, y_i)_{i=1}^g$ a symplectic basis for $(H, \bullet)$.

1.2. Y_k-*equivalence*

The theory of finite type invariants of Goussarov-Habiro has come equipped with a topological calculus toolbox: this was called *calculus of claspers* in [6] or alternatively *clovers* in [2]. We will assume a certain familiarity of the reader with these techniques.

In particular, let us recall that, for $k \geq 1$ an integer, the Y_k-*equivalence*[a] is the equivalence relation generated by surgery on connected clovers of degree k. Following Habiro in [6], we can then define a descending filtration of monoids

$$\mathcal{C}(\Sigma) \supset \mathcal{C}_1(\Sigma) \supset \mathcal{C}_2(\Sigma) \supset \cdots \supset \mathcal{C}_k(\Sigma) \supset \cdots$$

where $\mathcal{C}_k(\Sigma)$ is the submonoid consisting of the homology cobordisms which are Y_k-equivalent to the trivial cobordism 1_Σ. Note the following fact, a proof of which has been inserted in §4.

Proposition 1.1. *If* $\Sigma = \Sigma_g$ *or* $\Sigma_{g,1}$, *then* $\mathcal{HC}(\Sigma) = \mathcal{C}_1(\Sigma)$.

[a]This equivalence relation is called $(k-1)$-equivalence in [4], and A_k-equivalence in [6].

As mentioned by Habiro, we can show from the calculus of clovers that for every $k \geq 1$, the quotient monoid

$$\overline{\mathcal{C}}_k(\Sigma) := \mathcal{C}_k(\Sigma)/Y_{k+1}$$

is an Abelian group. In particular, $\overline{\mathcal{C}}_1(\Sigma)$ is the Abelian group of homology cylinders over Σ up to Y_2-equivalence. This group is the subject of the present paper.

For $k \geq 2$, Habiro gives a combinatorial upper bound for the Abelian group $\overline{\mathcal{C}}_k(\Sigma)$. Precisely, he defines $\mathcal{A}_k(H)$ to be the Abelian group (finitely) generated by unitrivalent graphs of internal degree k, with cyclic orientation at each trivalent vertex and whose univalent vertices are labelled by elements of H and are totally ordered. These graphs are considered modulo the well-known AS, IHX, multilinearity relations, and up to some "STU-like relations" dealing with the order of the univalent vertices. In the closed case, some relations of a symplectic type can be added. Then, there is a surjective *surgery map*

$$\mathcal{A}_k(H) \xrightarrow{\psi_k} \overline{\mathcal{C}}_k(\Sigma)$$

sending each graph G to $(1_\Sigma)_{\tilde{G}}$, where $\tilde{G}$ is a clover in the manifold 1_Σ with G as associated abstract graph, whose leaves are stacked from the upper surface $\Sigma \times 1$ according to the total order, framed along this surface and embedded according to the labels of the corresponding univalent vertices. The fact that ψ_k is well-defined also follows from the calculus of clovers.

As for the case $k = 1$, Habiro does not define any space of graphs but announces the following isomorphisms

$$\begin{cases} \overline{\mathcal{C}}_1(\Sigma_{g,1}) \simeq \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2 \\ \overline{\mathcal{C}}_1(\Sigma_g) \simeq \Lambda^3 H/(\omega \wedge H) \oplus \Lambda^2 H_{(2)}/\omega_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2 \end{cases} \tag{1.1}$$

where $H_{(2)} = H \otimes \mathbf{Z}_2$ and where

$$\omega = \sum_{i=1}^{g} x_i \wedge y_i \in \Lambda^2 H$$

is the *symplectic element*. This fact has been used afterwards in [12].

The goal of this paper is to prove these isomorphisms, in a diagrammatic way, by again defining a surgery map

$$\mathcal{A}_1(P) \xrightarrow{\psi_1} \overline{\mathcal{C}}_1(\Sigma)\,.$$

The space of graphs $\mathcal{A}_1(P)$ and the map ψ_1 appear to be meaningfully different from $\mathcal{A}_k(H)$ and ψ_k for $k > 1$, making thus the case $k = 1$ exceptional. Indeed, their definition will involve both the homology group H and *Spin*(Σ), the set of *spin structures* on Σ.

1.3. *The Abelianized Torelli group*

We denote by Ω_g the set of *quadratic forms* with $\bullet : H_{(2)} \times H_{(2)} \longrightarrow \mathbf{Z}_2$ as associated bilinear form, namely

$$\Omega_g = \left\{ H_{(2)} \xrightarrow{q} \mathbf{Z}_2 \ : \ \forall x, y \in H_{(2)},\ q(x+y) - q(x) - q(y) = x \bullet y \right\}.$$

Note that Ω_g is an affine space over $H_{(2)}$, with action given by

$$\forall q \in \Omega_g, \forall x \in H_{(2)}, \quad x \cdot q := q + x \bullet (-).$$

Thus, among the maps $\Omega_g \longrightarrow \mathbf{Z}_2$, there are the affine functions, and more generally there are the *Boolean polynomials* which are defined to be sums of products of affine ones (see [8, §4]). These polynomials form a $\mathbf{Z}_2$-algebra denoted by B_g, which is filtered by the *degree* (defined in the obvious way):

$$B_g^{(0)} \subset B_g^{(1)} \subset \cdots \subset B_g.$$

For instance, $B_g^{(1)}$ is the space of affine functions on Ω_g; the constant function $\overline{1} : \Omega_g \longrightarrow \mathbf{Z}_2$ sending each q to 1 and, for $h \in H$, the function $\overline{h}$ sending each q to $q(h)$ are affine functions. Note the following identity:

$$\forall h_1, h_2 \in H, \quad \overline{h_1 + h_2} = \overline{h_1} + \overline{h_2} + (h_1 \bullet h_2) \cdot \overline{1} \in B_g^{(1)}. \tag{1.2}$$

Another example of Boolean polynomial is the quadratic Boolean function

$$\alpha = \sum_{i=1}^{g} \overline{x_i} \cdot \overline{y_i},$$

which is known as the *Arf invariant*. For any basis $(e_i)_{i=1}^{2g}$ for H, there is an isomorphism of algebras:

$$B_g \simeq \frac{\mathbf{Z}_2[t_1, \ldots, t_{2g}]}{t_i^2 = t_i} \tag{1.3}$$

sending $\overline{1}$ to 1 and $\overline{e_i}$ to t_i.

Recall now from [8], that the many *Birman-Craggs homomorphisms* can be summed up into a single homomorphism

$$\mathcal{T}_{g,1} \xrightarrow{\beta} B_g^{(3)} \quad \text{or} \quad \mathcal{T}_g \xrightarrow{\beta} \frac{B_g^{(3)}}{\alpha \cdot B_g^{(1)}},$$

according to whether one is considering the boundary case or the closed case. Recall also from [9] that the *first Johnson homomorphism* is a homomorphism

$$\mathcal{T}_{g,1} \xrightarrow{\eta_1} \Lambda^3 H \quad \text{or} \quad \mathcal{T}_g \xrightarrow{\eta_1} \frac{\Lambda^3 H}{\omega \wedge H}.$$

Form the following pull-back:

$$\begin{tikzcd}
\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_g^{(3)} \arrow[d, "q"] \\
\Lambda^3 H \arrow[r, "- \otimes \mathbf{Z}_2"'] & \Lambda^3 H_{(2)},
\end{tikzcd}$$

where the map q is the canonical projection $B_g^{(3)} \longrightarrow B_g^{(3)}/B_g^{(2)}$ followed by the isomorphism $B_g^{(3)}/B_g^{(2)} \simeq \Lambda^3 H_{(2)}$ which identifies the cubic polynomial $\overline{h_1 h_2 h_3}$ with $h_1 \wedge h_2 \wedge h_3$ (this is well-defined because of (1.2) and (1.3)).

We denote by S the subgroup of this pull-back corresponding to $\omega \wedge H \subset \Lambda^3 H$ and $\alpha \cdot B_g^{(1)} \subset B_g^{(3)}$. Johnson has shown in [10] that, under the assumption $g \geq 3$, the homomorphisms η_1 and β induce isomorphisms

$$\frac{\mathcal{T}_{g,1}}{\mathcal{T}'_{g,1}} \xrightarrow[\simeq]{(\eta_1, \beta)} \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} \quad \text{and} \quad \frac{\mathcal{T}_g}{\mathcal{T}'_g} \xrightarrow[\simeq]{(\eta_1, \beta)} \frac{\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}}{S}.$$

Remark 1.2. Note that, because of (1.3), the codomains of these maps are respectively non-canonically isomorphic to $\Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$ and $\Lambda^3 H/(\omega \wedge H) \oplus \Lambda^2 H_{(2)}/\omega_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$.

1.4. *Statement of the results*

In §2, we will construct the space of graphs $\mathcal{A}_1(P)$ and the surgery map $\psi_1 : \mathcal{A}_1(P) \longrightarrow \overline{\mathcal{C}}_1(\Sigma)$. Spin structures play a prominent role in their definitions.

Observe that, $\overline{\mathcal{C}}_1(\Sigma)$ being an Abelian group, the mapping cylinder construction induces a group homomorphism

$$\frac{\mathcal{T}(\Sigma)}{\mathcal{T}(\Sigma)'} \xrightarrow{C} \overline{\mathcal{C}}_1(\Sigma).$$

As pointed out by Garoufalidis and Levine in [3] and [12], Johnson homomorphisms and Birman-Craggs homomorphisms factor through $C : \mathcal{T}(\Sigma) \longrightarrow \mathcal{HC}(\Sigma)$. These extensions will be detailed in §3.

Next, we will specify in §4 an isomorphism $\rho : \mathcal{A}_1(P) \longrightarrow \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ and the following two theorems will be proved from the previous material.

Theorem 1.3. *In the boundary case, the diagram*

$$\begin{tikzcd}
\mathcal{A}_1(P) \arrow[r, "\psi_1"] \arrow[dr, "\rho"'] & \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "(\eta_1, \beta)"] & \frac{\mathcal{T}_{g,1}}{\mathcal{T}'_{g,1}} \arrow[l, "C"'] \arrow[dl, "(\eta_1, \beta)"] \\
& \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} &
\end{tikzcd}$$

commutes and all of its arrows are isomorphisms*, except for the two maps starting from* $\mathcal{T}_{g,1}/\mathcal{T}'_{g,1}$ *when* $g < 3$*.*

Theorem 1.4. *In the closed case, the diagram*

$$\begin{tikzcd}
\frac{\mathcal{A}_1(P)}{\rho^{-1}(S)} \arrow[r, "\psi_1"] \arrow[dr, "\rho"'] & \overline{\mathcal{C}}_1(\Sigma_g) \arrow[d, "{(\eta_1,\beta)}"] & \frac{\mathcal{T}_g}{\mathcal{T}'_g} \arrow[l, "C"'] \arrow[dl, "{(\eta_1,\beta)}"] \\
& \frac{\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}}{S} &
\end{tikzcd}$$

commutes and all of its arrows are isomorphisms*, except for the two maps starting from* $\mathcal{T}_g/\mathcal{T}'_g$ *when* $g < 3$*.*

Note that Theorems 1.3 and 1.4 together with Remark 1.2, give Habiro's isomorphisms (1.1), which are non-canonical. Also, we will easily deduce the following.

Corollary 1.5. *For* $\Sigma = \Sigma_{g,1}$ *or* Σ_g*, let* M *and* M' *be two homology cylinders over* Σ*. Then, the following assertions are equivalent*:

(a) M *and* M' *are* Y_2*-equivalent*;
(b) M *and* M' *are not distinguished by degree* 1 *Goussarov-Habiro finite type invariants*;
(c) M *and* M' *are not distinguished by the first Johnson homomorphism nor Birman-Craggs homomorphisms.*

Finally, if an embedding $\Sigma_{g,1} \hookrightarrow \Sigma_g$ is fixed, there is an obvious "filling-up" map $\mathcal{C}_1(\Sigma_{g,1}) \longrightarrow \mathcal{C}_1(\Sigma_g)$, through which the commutative diagrams of Theorems 1.3 and 1.4 are compatible. The reader is referred to §4 for a precise statement.

2. Definition of the Surgery Map ψ_1

In this section, we define the space of graphs $\mathcal{A}_1(P)$ and the surgery map ψ_1 announced in the introduction.

2.1. *Special Abelian groups and the $\mathcal{A}_1$ functor*

Let us denote by $\mathcal{A}b$ the category of Abelian groups. An *Abelian group with special element* is a pair (G, s) where G is an Abelian group and $s \in G$ is of order at most 2. We denote by $\mathcal{A}b_s$ the category of special Abelian groups whose morphisms are group homomorphisms preserving the special elements. We now define a functor

$$\mathcal{A}b_s \xrightarrow{\mathcal{A}_1} \mathcal{A}b$$

in the following way. For (G, s) an object in $\mathcal{A}b_s$, $\mathcal{A}_1(G, s)$ is the free Abelian group generated by Y-shaped unitrivalent graphs, whose trivalent vertex is equipped with a cyclic order on the incident edges and whose univalent vertices are labelled by G, subject to some relations. The notation

$$\mathsf{Y}[z_1, z_2, z_3]$$

will stand for the Y-shaped graph whose univalent vertices are colored by z_1, z_2 and $z_3 \in G$ in accordance with the cyclic order, so that our notation is invariant under cyclic permutation of the z_i's. The relations are the following ones:

$$\begin{aligned} \textbf{Antisymetry (AS)} \quad &: \quad \mathsf{Y}[z_1, z_2, z_3] = -\mathsf{Y}[z_2, z_1, z_3], \\ \textbf{Multilinearity of colors} \quad &: \quad \mathsf{Y}[z_0 + z_1, z_2, z_3] = \mathsf{Y}[z_0, z_2, z_3] + \mathsf{Y}[z_1, z_2, z_3], \\ \textbf{Slide} \quad &: \quad \mathsf{Y}[z_1, z_1, z_2] = \mathsf{Y}[s, z_1, z_2], \end{aligned}$$

where $z_0, z_1, z_2, z_3 \in G$. For $(G, s) \xrightarrow{f} (G', s')$ a morphism in $\mathcal{A}b_s$, $\mathcal{A}_1(f)$ maps each generator $\mathsf{Y}[z_1, z_2, z_3]$ of $\mathcal{A}_1(G, s)$ to $\mathsf{Y}[f(z_1), f(z_2), f(z_3)] \in \mathcal{A}_1(G', s')$.

Example 2.6. The map $[G \longmapsto (G, 0)]$ makes $\mathcal{A}b$ a (full) subcategory of $\mathcal{A}b_s$. It follows from the definitions that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}b \arrow[r, tail] \arrow[dr, "\Lambda^3(-)"'] & \mathcal{A}b_s \arrow[d, "\mathcal{A}_1"] \\ & \mathcal{A}b. \end{tikzcd}$$

Non-trivial examples will be given in the next paragraph. For future use, note that this category has an obvious pull-back construction extending that of $\mathcal{A}b$:

$$\begin{tikzcd} (G_1, s_1) \times_{(G,s)} (G_2, s_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (G_2, s_2) \arrow[d, "f_2"] \\ (G_1, s_1) \arrow[r, "f_1"'] & (G, s) \end{tikzcd}$$

where $(G_1, s_1) \times_{(G,s)} (G_2, s_2)$ is the subgroup of $G_1 \times G_2$ consisting of those (z_1, z_2) such that $f_1(z_1) = f_2(z_2)$, and with special element (s_1, s_2).

2.2. *Spin structures and the special Abelian group P*

In this paragraph, let M be a compact oriented 3-manifold endowed with a Riemannian metric, and let FM be its bundle of oriented orthonormal frames:

$$SO(3) \xrightarrow{i} E(FM) \xrightarrow{p} M\,.$$

Let $s \in H_1(E(FM); \mathbf{Z})$ be the image by i_* of the generator of $H_1(SO(3); \mathbf{Z}) \simeq \mathbf{Z}_2$. Recall that M is spinnable and that $Spin(M)$ can be defined as

$$Spin(M) = \left\{ y \in H^1(E(FM); \mathbf{Z}_2)\,,\ < y, s > \neq 0 \right\}\,,$$

which is essentially independent of the metric. The manifold M being spinnable, s is not 0 (and so is of order 2).

Now, $Spin(M)$ being an affine space over $H^1(M;\mathbf{Z}_2)$ with action given by

$$\forall x \in H^1(M;\mathbf{Z}_2), \forall \sigma \in Spin(M), \quad x \cdot \sigma := \sigma + p^*(x)\,,$$

we can consider the space

$$A\left(Spin(M), \mathbf{Z}_2\right)$$

of $\mathbf{Z}_2$-valued affine functions on $Spin(M)$. For instance, $\overline{1} \in A\left(Spin(M), \mathbf{Z}_2\right)$ will denote the constant map defined by $\sigma \longmapsto 1$.

There is a canonical map

$$A\left(Spin(M), \mathbf{Z}_2\right) \xrightarrow{\kappa} H_1(M;\mathbf{Z}_2)\,.$$

For $f \in A\left(Spin(M), \mathbf{Z}_2\right)$, the homology class $\kappa(f)$ is defined unambiguously by

$$\forall \sigma, \sigma' \in Spin(M), \quad f(\sigma') - f(\sigma) =< \sigma'/\sigma, \kappa(f) > \in \mathbf{Z}_2\,,$$

where $\sigma'/\sigma \in H^1(M;\mathbf{Z}_2)$ is defined by the affine action of $H^1(M;\mathbf{Z}_2)$ on $Spin(M)$. Another canonical map is

$$H_1\left(E(FM);\mathbf{Z}\right) \xrightarrow{e} A\left(Spin(M), \mathbf{Z}_2\right)$$

sending a x to the map defined by $\sigma \longmapsto\ < \sigma, x >$. Next lemma gives us a nice understanding of the special Abelian group $\left(H_1\left(E(FM);\mathbf{Z}\right), s\right)$.

Lemma 2.7. (a) *The following diagram of special groups is a pull-back diagram:*

$$\begin{tikzcd}
\left(H_1\left(E(FM);\mathbf{Z}\right), s\right) \arrow[r, "e"] \arrow[d, "p_*"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \left(A\left(Spin(M), \mathbf{Z}_2\right), \overline{1}\right) \arrow[d, "\kappa"] \\
\left(H_1\left(M;\mathbf{Z}\right), 0\right) \arrow[r, "- \otimes \mathbf{Z}_2"'] & \left(H_1\left(M;\mathbf{Z}_2\right), 0\right)\,.
\end{tikzcd}$$

(b) *Let t be the map*

$$\left\{\textit{Oriented framed knots in } M\right\} \xrightarrow{t} H_1\left(E(FM);\mathbf{Z}\right)$$

which adds to any oriented framed knot K an extra $(+1)$-twist, and next sends it to the homology class of its lift in FM. Then,

(i) *t is surjective*;

(ii) *$t_{K_1} = t_{K_2}$ if and only if K_1 and K_2 are cobordant as oriented knots in M and if their framings with respect to a surface with boundary $(K_1) \cup (-K_2)$ then differ from each other by an even integer*;

(iii) *if $K_1 \sharp K_2$ denotes the band connected sum of K_1 and K_2, then $t_{K_1 \sharp K_2} = t_{K_1} + t_{K_2}$*;

(iv) *the k-framed trivial oriented knot ($k \in \mathbf{Z}$) is sent by t to $k \cdot s$.*

Proof. We begin by proving (a). The commutativity of the diagram of special groups is easy to verify. By functoriality, we get a map

$$(H_1(E(FM);\mathbf{Z}), s) \xrightarrow{(p_*,e)} (H_1(M;\mathbf{Z}), 0) \times_{(H_1(M;\mathbf{Z}_2),0)} \left(A(Spin(M), \mathbf{Z}_2), \overline{1}\right).$$

The Serre sequence associated to the fibration FM gives for homology with integer coefficients:

$$0 \longrightarrow H_1(SO(3);\mathbf{Z}) \xrightarrow{i_*} H_1(E(FM);\mathbf{Z}) \xrightarrow{p_*} H_1(M;\mathbf{Z}) \longrightarrow 0.$$

The bijectivity of (p_*, e) follows from the exactness of this sequence.

We now prove (b) and we begin with assertion (iv). Let K be a trivial k-framed oriented knot, let $* \in K$ and let $e = (e_1, e_2, e_3) \in p^{-1}(*)$ be the framing of K at $*$. We denote by $\tilde{K}$ the lift of K to FM. Then, as a loop in $E(FM)$, $\tilde{K}$ is homotopic to the loop in the fiber $p^{-1}(*)$ defined by

$$[0,1] \ni t \longmapsto R_{2\pi(k+1)t}(e),$$

where R_θ (with $\theta \in \mathbf{R}$) denotes the rotation of oriented axis directed by e_3 and angle θ. From an appropriate description of the generator of $\pi_1(SO(3)) \simeq \mathbf{Z}_2$, it follows that $\left[\tilde{K}\right] = (k+1) \cdot s \in H_1(E(FM);\mathbf{Z})$, and assertion (iv) then follows.

Let us make an observation. Let K be any oriented framed knot in M; since the framing of K determines a trivialization of its normal bundle in M, it allows us to restrict any spin structure on M to K. Recall now that the cobordism group Ω_1^{Spin} is isomorphic to $\mathbf{Z}_2$ (with generator given by $\mathbf{S}^1$ endowed with the spin structure induced by its Lie group structure: see [11, pp. 35 and 36]). The following observation then makes sense:

$$\forall \sigma \in Spin(M), \quad e(t_K)(\sigma) = (K, \sigma|_K) \in \Omega_1^{Spin} \simeq \mathbf{Z}_2, \tag{2.4}$$

and can be derived from an appropriate characterization of the spin structures on the circle (see [11, pp. 35, 36]).

Let now K_1 and K_2 be some disjoint oriented framed knots in M. There is an obvious genus 0 surface with boundary $K_1 \sharp K_2 \dot\cup (-K_1) \dot\cup (-K_2)$. Then, according to (2.4), we have $e(t_{K_1 \sharp K_2}) = e(t_{K_1}) + e(t_{K_2})$. Also, $p_*(t_{K_1\sharp K_2}) = [K_1 \sharp K_2] = [K_1] + [K_2] = p_*(t_{K_1}) + p_*(t_{K_2})$, and so by (a), we obtain that assertion (iii) holds for K_1 and K_2.

We now justify assertion (ii). According to (a), $t_{K_1} = t_{K_2}$ if and only if $p_*(t_{K_1}) = p_*(t_{K_2})$ and $e(t_{K_1}) = e(t_{K_2})$. Also, the condition $p_*(t_{K_1}) = p_*(t_{K_2})$ holds if and only if K_1 and K_2 are homologous in M. In this case, let S be an embedded oriented surface in M such that $\partial S = K_1 \dot\cup (-K_2)$. Let k_i be the framing of K_i with respect to S and let K_i' be the oriented framed knot obtained from K_i by adding an extra $(-k_i)$-twist, so that the framing of K_i' is given by S. Then, according to (2.4), we have $e\left(t_{K_1'}\right) = e\left(t_{K_2'}\right)$. Moreover, applying assertions (iii) and (iv), we obtain: $e\left(t_{K_i'}\right) = e(t_{K_i}) + k_i \cdot s$. We conclude that $e(t_{K_1}) = e(t_{K_2})$ if and only if k_1 and k_2 are equal modulo 2, proving thus assertion (ii).

Let $x \in H_1(E(FM);\mathbf{Z})$, then $p_*(x) \in H_1(M;\mathbf{Z})$ can be realized by an oriented knot K in M: we give it an arbitrary framing. By construction, $p_*(t_K - x) = 0 \in H_1(M;\mathbf{Z})$, and so by exactness of the Serre sequence, $t_K - x = \varepsilon \cdot s$ with $\varepsilon \in \{0,1\}$. By possibly band-summing K with a trivial $(+1)$-framed knot when $\varepsilon = 1$, and according to assertion (iii) and (iv), the framed knot K can be supposed to be such that $t_K = x$; this proves assertion (i). $\square$

We now restrict ourselves to the 3-manifold $M = 1_\Sigma = \Sigma \times I$ where Σ can be Σ_g or $\Sigma_{g,1}$. The inclusion $i^+ : \Sigma \hookrightarrow 1_\Sigma$, with image Σ^+, induces an isomorphism between H and $H_1(M;\mathbf{Z})$ and a bijection between $Spin(\Sigma)$ and $Spin(M)$. As shown by Johnson in [7], there is an algebraic way to think of $Spin(\Sigma)$. Indeed, there exists a canonical affine isomorphism

$$Spin(\Sigma) \xrightarrow{\simeq} \Omega_g\,,$$

sending any spin structure σ to a quadratic form q_σ which can be defined as follows. Let $x \in H_{(2)} = H_1(\Sigma;\mathbf{Z}_2)$ be represented by an oriented simple closed curve on Σ^+; by framing it along Σ^+ and pushing it into the interior of 1_Σ, we get a framed oriented knot K in 1_Σ. Then,

$$q_\sigma(x) = e(t_K)\,(\sigma \times I) \ \in \mathbf{Z}_2. \tag{2.5}$$

Therefore, according to Lemma 2.7(a), $(H_1\left(E\left(F1_\Sigma\right);\mathbf{Z}\right), s)$ is canonically isomorphic to the special Abelian group defined by the pull-back construction

$$\begin{tikzcd}
(H,0) \times_{(H_{(2)},0)} \left(B_g^{(1)},\overline{1}\right) \arrow[r, "e"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & \left(B_g^{(1)},\overline{1}\right) \arrow[d, "\kappa"] \\
(H,0) \arrow[r, "-\otimes \mathbf{Z}_2"'] & \left(H_{(2)},0\right)
\end{tikzcd}$$

whose projections are denoted by p and e, and where κ is the composite

$$B_g^{(1)} \twoheadrightarrow B_g^{(1)}/B_g^{(0)} \xrightarrow{\simeq} H_{(2)}\,.$$

The last isomorphism here identifies $\overline{h}$ with $h_{(2)}$ for all $h \in H$ (this is well-defined by (1.2) and (1.3)). We define the special Abelian group P to be

$$\boxed{P = (H,0) \times_{(H_{(2)},0)} \left(B_g^{(1)},\overline{1}\right)\,,}$$

and $\mathcal{A}_1(P)$ is the space of graphs announced in the introduction.

Remark 2.8. Thus, any element z of P can be written as

$$z = \left(h, \overline{h} + \varepsilon \cdot \overline{1}\right) \in P\,,$$

with $h \in H$ and $\varepsilon \in \{0, 1\}$. Observe also the following. Suppose that there exists a simple oriented closed curve in Σ^+ with homology class h. Let K be the push-off of this curve, framed along Σ^+, with an extra ε-twist. Then, it follows from (2.5) that $t_K = z \in P \simeq H_1\left(E\left(F1_\Sigma\right); \mathbf{Z}\right)$.

Remark 2.9. According to the proof of Lemma 2.7, the Serre sequence for homology associated to the bundle $F1_\Sigma$ gives the following short exact sequence:

$$0 \longrightarrow \mathbf{Z}_2 \longrightarrow P \xrightarrow{\ p\ } H \longrightarrow 0,$$

where $\mathbf{Z}_2$ is injected into P by sending 1 to $(0, \overline{1})$. The map $s : H \longrightarrow P$ defined by $s(h) = \left(h, \overline{h}\right)$ is a set-theoretic section. According to (1.2), the associated 2-cocycle $H \times H \longrightarrow \mathbf{Z}_2$ is the mod 2 reduced intersection form of Σ. Thus, P is isomorphic to $H \rtimes \mathbf{Z}_2$ with crossed product defined by

$$(h_1, \varepsilon_1) \cdot (h_2, \varepsilon_2) = (h_1 + h_2, \varepsilon_1 + \varepsilon_2 + h_1 \bullet h_2)\,.$$

The element $\left(h, \overline{h} + \varepsilon \cdot \overline{1}\right) \in P$ corresponds to $(h, \epsilon) \in H \rtimes \mathbf{Z}_2$.

2.3. *The surgery map ψ_1*

In this paragraph, Σ is allowed to be Σ_g or $\Sigma_{g,1}$ and the surgery map $\psi_1 : \mathcal{A}_1(P) \longrightarrow \overline{\mathcal{C}}_1(\Sigma)$ is constructed by means of calculi of clovers.

Convention 2.10. Here, we *adopt* Goussarov's convention for the surgery meaning of Y-graphs and clovers [4, 2].

Denote by $\tilde{\mathcal{A}}_1(P)$ the free Abelian group generated by abstract Y-shaped graphs whose univalent vertices are labelled by P, and which are equipped with an orientation at their trivalent vertex: $\mathcal{A}_1(P)$ is a quotient of $\tilde{\mathcal{A}}_1(P)$. For each generator $\mathsf{Y}[z_1, z_2, z_3]$ of $\tilde{\mathcal{A}}_1(P)$, where $z_i \in P$, pick some disjoint oriented framed knots K_i in the interior of 1_Σ such that $t_{K_i} = z_i \in P \simeq H_1\left(E\left(F1_\Sigma\right); \mathbf{Z}\right)$; this is possible according to Lemma 2.7(b) (i). Next, pick an embedded 2-disk D in the interior of 1_Σ and disjoint from the K_i's, orient it in an arbitrary way, and connect it to the K_i's with some bands e_i. These band sums are required to be compatible with the orientations, and to be coherent with the cyclic ordering $(1, 2, 3)$. See Fig. 1 as an illustration. What we obtain in 1_Σ is precisely a *Y-graph*, as defined by Goussarov in [4]. We denote it by $\phi\big(\mathsf{Y}[z_1, z_2, z_3]\big)$. For example, as follows from Lemma 2.7(b) (iv), if z_1 is the special element s of P, the corresponding leaf K_1 of $\phi\big(\mathsf{Y}[s, z_2, z_3]\big)$ can be chosen to be unknotted and $(+1)$-framed; such a leaf is called *special* in [2].

We now put $\tilde{\psi}_1\big(\mathsf{Y}[z_1, z_2, z_3]\big)$ to be the Y_2-equivalence class of the surgered manifold $(1_\Sigma)_{\phi(\mathsf{Y}[z_1,z_2,z_3])}$, so that we get a map

$$\tilde{\mathcal{A}}_1(P) \xrightarrow{\ \tilde{\psi}_1\ } \overline{\mathcal{C}}_1(\Sigma).$$

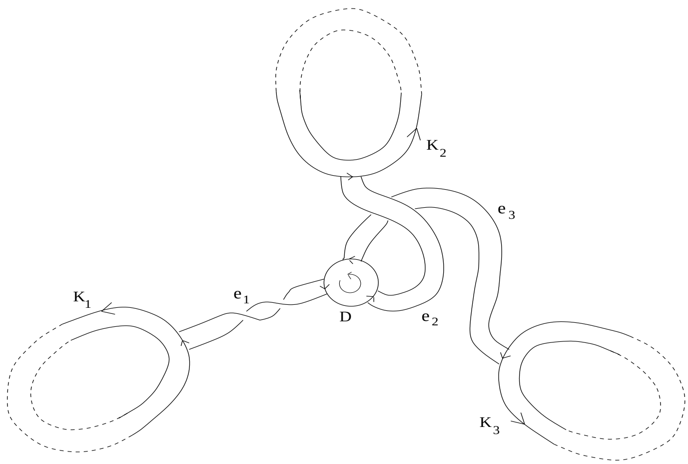

Fig. 1. Embedding the Y-graph.

Theorem 2.11. *The map $\tilde{\psi}_1$ does not depend on the choice of ϕ, and induces a surjective quotient map*

$$\mathcal{A}_1(P) \xrightarrow{\psi_1} \overline{\mathcal{C}}_1(\Sigma)\,.$$

Proof. The proof might be read with a copy of [2] in hand. Using the above notation, we begin with showing that $\tilde{\psi}_1\big(\mathsf{Y}[z_1, z_2, z_3]\big)$ does not depend on the choice of $\phi\big(\mathsf{Y}[z_1, z_2, z_3]\big)$. For this, we recall two facts concerning any Y-graph G in a homology cylinder M (see Remark 2.12 below):

Fact 1: the Y_2-equivalence class of M_G is not modified when an edge of G is band-summed with a (disjoint) oriented framed knot of M;

Fact 2: the Y_2-equivalence class of M_G is inverted when an edge of G is half-twisted.

Using these, the independance on the choice of the disk D, its orientation and the edges e_i is easily shown.

We now show the independance on the choice of the leaves K_i. Suppose for example that K_1' is another choice of K_1. Then, according to Lemma 2.7(b) (ii), there exists an embedded oriented surface F in 1_Σ such that $\partial F = K_1 \dot\cup (-K_1')$ and such that, if k (resp. k') is the framing of K_1 (resp. K_1') with respect to F, $(k - k')$ is even. We also assume transversality of F with the edges of the Y-graph, and with the two other leaves K_2 and K_3. Let $g(F)$ denote the genus of F, let m be the number of intersection points of F with the edges, and for $i = 2, 3$, let n_i be the number of intersection points of F with K_i. If all of the integers $g(F)$, $(k - k')$, m, n_2 and n_3 are zero, the two Y-graphs are isotopic and we are done. In the general case, recall from [2, §4.3] that there is a procedure for *simplifying the leaves.* The main tool for this is the following:

Fact 3: if G_1 and G_2 are two Y-graphs in 1_Σ obtained from a Y-graph G by *splitting a leaf*, then $(1_\Sigma)_G = (1_\Sigma)_{G_1} \cdot (1_\Sigma)_{G_2} \in \overline{\mathcal{C}}_1(\Sigma)$ (see Remark 2.12).

Splitting $\big(g(F)+|k-k'|/2+m+n_2+n_3\big)$ times the leaf K_1, splitting n_2 times the leaf K_2 and splitting n_3 times the leaf K_3, we see that the result $\tilde{\psi}_1\big(\mathsf{Y}[z_1,z_2,z_3]\big)$ in $\overline{\mathcal{C}}_1(\Sigma)$ defined by the choice of K_1 differs from the one defined by K_1' by some elements of the form $(1_\Sigma)_G$, where G satisfies one of the following conditions:

(i) G has a leaf which bounds a genus 1 surface disjoint from G and with respect to which the leaf is 0-framed;
(ii) G has a leaf which bounds a disk disjoint from G, and with respect to which the leaf is (± 2)-framed;
(iii) G has a leaf which bounds a disk with respect to which it is 0-framed, and this disk intersects G in exactly one point belonging to an edge;
(iv) G has two leaves which are linked as the Hopf link.

Let us now verify that all of these elements vanish in $\overline{\mathcal{C}}_1(\Sigma)$. If G is of type (i), the surgery effect of G is the same as a clover of degree 2 (apply [2, Lemma 5.1] and [2, Theorem 2.4]). If G is of type (ii), by again cutting its leaf we get $(1_\Sigma)_G = 2\cdot(1_\Sigma)_{G'}$ where G' has a special leaf; but $(1_\Sigma)_{G'} = -(1_\Sigma)_{G'}$ by Fact 2. If G is of type (iii), by applying Fact 1 the edge can be slid away from the leaf, we then get a Y-graph with a *trivial* leaf which has no surgery effect by the "blow-up move" of [2, Fig. 6]. If G is of type (iv), by applying [2, Theorem 2.4], we obtain a Y-graph with a looped edge, but this is stated to be 0 in $\overline{\mathcal{C}}_1(\Sigma)$ by the so-called *LOOP relation.* This relation is easily shown from [2, Lemma 2.3] and from Facts 1 and 2. This completes the proof of the independance of $\tilde{\psi}_1$ on ϕ.

The fact that $\tilde{\psi}_1$ is surjective follows immediately from the fact that the Abelian group $\overline{\mathcal{C}}_1(\Sigma)$ is generated by the homology cylinders $(1_\Sigma)_G$ where G is a single Y-graph (this is also proved by standard calculi of clovers).

We now show that the map $\tilde{\psi}_1$ factors through $\tilde{\mathcal{A}}_1(P) \longrightarrow \mathcal{A}_1(P)$. The AS relation is proved in $\overline{\mathcal{C}}_1(\Sigma)$ from Fact 2 and an isotopy of the Y-graph – see [2, Corollary 4.6].

The multilinearity relation follows from Fact 3. Indeed, let G be a Y-graph in 1_Σ with K as a leaf. Split the leaf K to K_1 and K_2, and let G_1 and G_2 be the corresponding new Y-graphs. Then, $(1_\Sigma)_G = (1_\Sigma)_{G_1} \cdot (1_\Sigma)_{G_2} \in \overline{\mathcal{C}}_1(\Sigma)$. Since K is the band connected sum of K_1 and K_2, we have by Lemma 2.7 (b) (iii): $t_K = t_{K_1} + t_{K_2} \in P$.

The slide relation is shown to be satisfied in $\overline{\mathcal{C}}_1(\Sigma)$ thanks to the "leaf slide" move of [2, Fig. 6]. For this, let G be a Y-graph in 1_Σ with some leaves K_1 and K_2 such that $t_{K_1} = -t_{K_2}$. By sliding the leaf K_2 along K_1, we obtain a new Y-graph G' with the same surgery effect as G, such that $K_1' = K_1$ and such that K_2' is the band connected sum of K_1 and K_2 with an extra (-1)-twist. So, by Lemma 2.7 (b) (iii) and (iv), we have $t_{K_2'} = t_{K_1} + t_{K_2} + s = s \in P$. This shows that the relation $\mathsf{Y}[z_1,-z_1,z_3] = \mathsf{Y}[z_1,s,z_3]$ $(z_1,z_3 \in P)$ is satisfied in $\overline{\mathcal{C}}_1(\Sigma)$. The slide relation, as stated in §2.1, follows then from the AS and multilinearity relations. □

Remark 2.12. The proofs of Facts 1, 2 and 3 use calculus of clovers and can respectively be obtained from the proof of Corollary 4.2, Lemma 4.4 and Corollary 4.3 in [2]. Alternatively, those facts can be considered as corollaries of these results in the following way. Denote by $\mathbf{Z}\mathcal{C}_1(\Sigma)$ the free Abelian group generated by the set $\mathcal{C}_1(\Sigma)$, and let

$$\mathbf{Z}\mathcal{C}_1(\Sigma) = \mathcal{F}_0^Y(1_\Sigma) \supset \mathcal{F}_1^Y(1_\Sigma) \supset \mathcal{F}_2^Y(1_\Sigma) \supset \cdots$$

be its Goussarov-Habiro filtration [2, §1.4]. Results are stated in [2] to hold in the graded space $\mathcal{G}_k(1_\Sigma) = \mathcal{F}_k^Y(1_\Sigma)/\mathcal{F}_{k+1}^Y(1_\Sigma)$. Consider also the homomorphism of Abelian groups

$$\mathbf{Z}\mathcal{C}_1(\Sigma) \xrightarrow{\ v\ } \overline{\mathcal{C}}_1(\Sigma)$$

which assigns to any homology cylinder its Y_2-equivalence class. The invariant v is primitive, in the sense that it restricts to $\mathcal{C}_1(\Sigma)$ to a monoid homomorphism, and is a degree 1 invariant[b] as follows from calculus of clovers. In particular, v induces a homomorphism $\mathcal{G}_1(1_\Sigma) \longrightarrow \overline{\mathcal{C}}_1(\Sigma)$, by which Facts 1, 2 and 3 are respectively the images of Corollary 4.2, Lemma 4.4 and Corollary 4.3.

3. Johnson Homomorphism and Birman-Craggs Homomorphisms for Homology Cylinders

In this section, the first Johnson homomorphism and the Birman-Craggs homomorphisms are extended to the monoid of homology cylinders.

3.1. *The first Johnson homomorphism for homology cylinders*

In [3] the notion of Johnson homomorphisms for homology cobordisms over $\Sigma_{g,1}$ was introduced. In this paragraph, we allow Σ to be Σ_g or $\Sigma_{g,1}$, and give the definition of the first Johnson homomorphism in both cases.

The fundamental group of Σ with base point $* \in \Sigma$ will be denoted by $\pi^{(*)}$, and $\pi_k^{(*)}$ will denote the k^{th} term of its lower central series, beginning at $\pi_1^{(*)} = \pi^{(*)}$. We denote by $(x_i, y_i)_{i=1}^g$ the based loops depicted in Fig. 2 or their corresponding images under an inclusion $\Sigma_{g,1} \subset \Sigma_g$. Then,

in the boundary case, $\pi^{(*)} = F(x_1, \ldots, x_g, y_1, \ldots, y_g)$,

and in the closed case, $\pi^{(*)} = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \mid \prod_{i=1}^g [x_i, y_i] = 1 \rangle$.

Given a homology cobordism $(M, i^+, i^-) \in \mathcal{C}(\Sigma)$, the map $i^\pm$ induces an isomorphism at the level of each nilpotent quotient (by Stallings [14]). We choose a

[b] In fact, v is a universal degree 1 primitive invariant for homology cylinders. See [6, §6.4] for a similar invariant, of any degree, for knots in the 3-sphere.

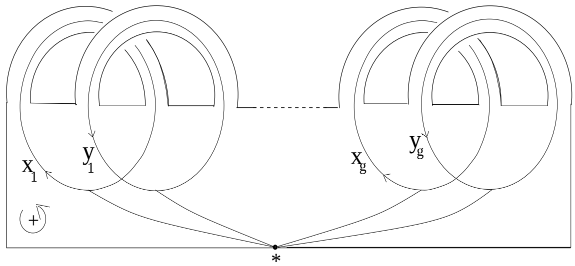

Fig. 2. The based curves $(x_i, y_i)_{i=1}^{g}$ on $\Sigma_{g,1}$.

path $\gamma \subset M$ going from $i^+(*)$ to $i^-(*)$, and then consider the following composite:

$$\frac{\pi^{(*)}}{\pi_3^{(*)}} \xrightarrow[\simeq]{i_3^+} \frac{\pi_1(M, i^+(*))}{\pi_1(M, i^+(*))_3} \xrightarrow[\simeq]{c_\gamma} \frac{\pi_1(M, i^-(*))}{\pi_1(M, i^-(*))_3} \xrightarrow[\simeq]{(i_3^-)^{-1}} \frac{\pi^{(*)}}{\pi_3^{(*)}}.$$

Up to inner automorphisms, this is independent on the choice of γ, so that there is a well-defined map

$$\mathcal{C}(\Sigma) \xrightarrow{\eta_1^{(*)}} Out\left(\frac{\pi^{(*)}}{\pi_3^{(*)}}\right),$$

satisfying $\eta_1^{(*)}(M \cdot N) = \eta_1^{(*)}(N) \cdot \eta_1^{(*)}(M)$. Let $\star$ be another base point in Σ, and γ an arbitrary path between $*$ and $\star$. Conjugation by γ induces an isomorphism $Out\left(\pi^{(*)}/\pi_3^{(*)}\right) \simeq Out\left(\pi^{(\star)}/\pi_3^{(\star)}\right)$. This isomorphism is independent on the choice of the path γ, and the maps $\eta_1^{(*)}$ and $\eta_1^{(\star)}$ are compatible through it. Therefore, we get a well-defined group denoted by $Out(\pi/\pi_3)$ and an anti-homomorphism of monoids

$$\mathcal{C}(\Sigma) \xrightarrow{\eta_1} Out\left(\frac{\pi}{\pi_3}\right). \tag{3.6}$$

If we restrict ourselves to homology cylinders, we are led to a map

$$\mathcal{C}_1(\Sigma) \xrightarrow{\eta_1} Ker\left(Out\left(\frac{\pi}{\pi_3}\right) \to Out\left(\frac{\pi}{\pi_2}\right)\right).$$

Observe the following exact sequence:

$$1 \longrightarrow Hom\left(H, \frac{\pi_2^{(*)}}{\pi_3^{(*)}}\right) \longrightarrow Aut\left(\frac{\pi^{(*)}}{\pi_3^{(*)}}\right) \longrightarrow Aut\left(\frac{\pi^{(*)}}{\pi_2^{(*)}}\right)$$

where any $f \in Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)$ is sent to the automorphism of $\pi^{(*)}/\pi_3^{(*)}$ which sends $\overline{x}$ to $\overline{x}f(\overline{x})$ (with $x \in \pi^{(*)}$). Hence we have the following exact sequence:

$$1 \longrightarrow \frac{Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)}{[H, -]} \longrightarrow Out\left(\frac{\pi}{\pi_3}\right) \longrightarrow Out\left(\frac{\pi}{\pi_2}\right).$$

Here, $[H,-]$ stands for the subgroup of $Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)$ consisting of those homomorphisms $[h,-]$ defined for any $h \in H$ by $x \longmapsto [h,x]$, where H is identified with $\pi_1^{(*)}/\pi_2^{(*)}$. Consequently, we have defined an anti-homomorphism of monoids

$$\mathcal{C}_1(\Sigma) \xrightarrow{\eta_1} \frac{Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)}{[H,-]}.$$

In the sequel, we denote by $\mathsf{L}(H) = \oplus_n \mathsf{L}_n(H)$, the free Lie $\mathbf{Z}$-algebra on the $\mathbf{Z}$-module H, and distinguish the boundary case from the closed case.

In the boundary case, as $\pi^{(*)}$ is free and H is the Abelianized of $\pi^{(*)}$, $\mathsf{L}_2(H)$ is canonically isomorphic to $\pi_2^{(*)}/\pi_3^{(*)}$. Also, there is a sequence of isomorphisms $Hom\,(H, \mathsf{L}_2(H)) \simeq H^* \otimes \mathsf{L}_2(H) \simeq H \otimes \mathsf{L}_2(H)$, with last one induced by $\bullet$-duality. Through these, $[H,-] \subset Hom\,(H, \mathsf{L}_2(H))$ becomes $A_{g,1} \subset H \otimes \mathsf{L}_2(H)$ defined by

$$A_{g,1} = \left\{ \sum_{i=1}^{g} \left(x_i \otimes [h, y_i] - y_i \otimes [h, x_i]\right) \ \middle| \ h \in H \right\}.$$

Thus, η_1 takes values in

$$\frac{Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)}{[H,-]} \simeq \frac{H \otimes \mathsf{L}_2(H)}{A_{g,1}}.$$

The group $\Lambda^3 H$ can be seen as a subgroup of $H \otimes \mathsf{L}_2(H)$ in the following manner:

$$0 \longrightarrow \Lambda^3 H \xrightarrow{\nu} H \otimes \mathsf{L}_2(H) \xrightarrow{[-,-]} \mathsf{L}_3(H),$$

where ν is defined by $\nu(x \wedge y \wedge z) = x \otimes [y,z] + y \otimes [z,x] + z \otimes [x,y]$. Composing ν with the projection $H \otimes \mathsf{L}_2(H) \twoheadrightarrow H \otimes \mathsf{L}_2(H)/A_{g,1}$ still gives an injection

$$\Lambda^3 H \xrightarrow{\nu} \frac{H \otimes \mathsf{L}_2(H)}{A_{g,1}}.$$

This follows from the fact that

$$\forall h \in H, \quad [h, \omega] = 0 \in \mathsf{L}_3(H) \implies h = 0, \tag{3.7}$$

where $\omega = \sum_i [x_i, y_i] \in \mathsf{L}_2(H)$ corresponds *via* the canonical isomorphism $\mathsf{L}_2(H) \simeq \Lambda^2 H$ to the symplectic element ω, defined in the introduction.

We now prove that η_1 takes values in the subgroup $\Lambda^3 H$. Suppose for this that $f \in Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right) \subset Aut\left(\pi^{(*)}/\pi_3^{(*)}\right)$ is such that there exists a lift $\tilde{f} \in End(\pi^{(*)})$ of f fixing the boundary element $\partial := \prod_{i=1}^{g} [x_i, y_i]$ modulo $\pi_4^{(*)}$. Note that this property is verified by a representative for $\eta_1(M)$ if M is a homology cylinder, so that proving that $f \in Ker([-,-])$ will prove that $Im(\eta_1) \subset \Lambda^3 H$. Let $X_i = x_i^{-1} \tilde{f}(x_i) \in \pi_2^{(*)}$ and $Y_i = y_i^{-1} \tilde{f}(y_i) \in \pi_2^{(*)}$. We have

$$\begin{aligned} \tilde{f}(\partial) &= \textstyle\prod_i [\tilde{f}(x_i), \tilde{f}(y_i)] \\ &\equiv \textstyle\prod_i [x_i X_i, y_i Y_i] \\ &\equiv \textstyle\prod_i [x_i, y_i][X_i, y_i][x_i, Y_i] \mod \pi_4^{(*)}, \end{aligned}$$

which implies that $\prod_i [X_i, y_i][x_i, Y_i] \equiv 1 \bmod \pi_4^{(*)}$. Consequently,

$$\sum_i (x_i \otimes Y_i - y_i \otimes X_i) \in H \otimes \mathsf{L}_2(H)\,,$$

which essentially corresponds to f, goes to 0 by the bracketting map.

Let us now focus on the closed case. The canonical map $\mathsf{L}_2(H) \longrightarrow\!\!\!\!\!\to \pi_2^{(*)}/\pi_3^{(*)}$ induces an isomorphism between $\pi_2^{(*)}/\pi_3^{(*)}$ and $\mathsf{L}_2(H)/\omega$. Thus, in this case, η_1 takes values in

$$\frac{Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)}{[H, -]} \simeq \frac{H \otimes \mathsf{L}_2(H)}{A_g}$$

where $A_g = A_{g,1} + H \otimes \omega$. Since $\nu\,(\omega \wedge H) \subset A_g$, ν factors to give

$$\frac{\Lambda^3 H}{\omega \wedge H} \xrightarrow{\ \nu\ } \frac{H \otimes \mathsf{L}_2(H)}{A_g}.$$

It also follows from (3.7) that this new ν is still injective. Then, $\Lambda^3 H/\omega \wedge H$ can be seen as a subgroup of $Hom\left(H, \pi_2^{(*)}/\pi_3^{(*)}\right)/[H, -]$. Similarly to the boundary case, one shows that η_1 takes values in $\Lambda^3 H/\omega \wedge H$.

So far, we have defined some anti-homomorphisms of monoids

$$\mathcal{C}_1(\Sigma_{g,1}) \xrightarrow{\ \eta_1\ } \Lambda^3 H \quad \text{and} \quad \mathcal{C}_1(\Sigma_g) \xrightarrow{\ \eta_1\ } \frac{\Lambda^3 H}{\omega \wedge H},$$

but next lemma allows us to go a bit further.

Lemma 3.13. *Let (M, K) be a homology cylinder over Σ together with a loop K based on $* \in M$. Let also G be a degree 2 clover in M disjoint from K and let (M_G, K_G) be the result of the surgery along G. Then, there exists an isomorphism*

$$\frac{\pi_1(M, *)}{\pi_1(M, *)_3} \xrightarrow{\ \simeq\ } \frac{\pi_1(M_G, *)}{\pi_1(M_G, *)_3}$$

sending $[K]$ to $[K_G]$.

This lemma allows us to conclude with the following proposition-definition.

Proposition 3.14. *For homology cylinders over $\Sigma = \Sigma_{g,1}$ or Σ_g, there are some well-defined homomorphisms*

$$\overline{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{\ \eta_1\ } \Lambda^3 H \quad \textit{and} \quad \overline{\mathcal{C}}_1(\Sigma_g) \xrightarrow{\ \eta_1\ } \frac{\Lambda^3 H}{\omega \wedge H}.$$

Induced by the map (3.6)*, they are called the* first Johnson homomorphisms*.*

Remark 3.15. The composition of η_1 with the map $C : \mathcal{T}(\Sigma) \longrightarrow \overline{\mathcal{C}}_1(\Sigma)$ is the classical homomorphism defined in [9].

Proof of Lemma 3.13. Using [2, Lemma 5.1], one shows that

$$M_G \cong_+ M \setminus int(N(G)) \ \cup_{j|_\partial} \ (H_4)_L$$

where $H_4 \xhookrightarrow{j} M$ is an oriented embedding of the standard genus 4 handlebody onto $N(G)$, which is a regular neighborhood of G in M, and where L is the 2-component framed link shown[c] on Fig. 3. Through this diffeomorphism K_G goes to $K \subset M \setminus int(N(G))$.

Moreover, L is Kirby-equivalent to the 3-component link N drawn on the right part of Fig. 3. It turns out that N is a boundary link. More precisely, up to a (± 1)-

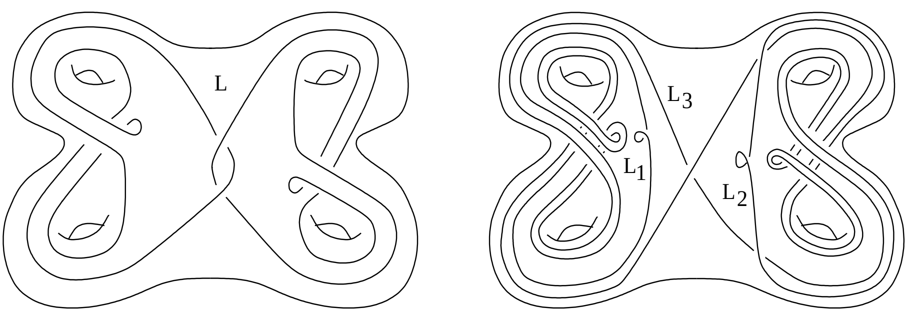

Fig. 3. The 2-component framed link L and a Kirby-equivalent boundary link N.

framing correction, one can push disjointly N_3, N_1 and then N_2 to the boundary of H_4. We obtain some simple closed curves on $\Sigma_4 = \partial H_4$, which are bounding curves. Therefore, twist along each of these curves induces the identity at the level of $\pi_1(\Sigma_4, *)/\pi_1(\Sigma_4, *)_3$. We then obtain the lemma by a Van-Kampen type argument. □

3.2. *Birman-Craggs homomorphisms for homology cylinders*

Birman-Craggs homomorphisms were defined in [1] and they were enumerated in [8]. Levine also outlined in [12] how they can be extended to homology cylinders. In this paragraph, we review Birman-Craggs homomorphisms in a self-contained way. For this, we use the spin refinement of the Goussarov-Habiro theory of finite type invariants, introduced by the first author in [13].

We first fix a few notation. If (M, σ) is a closed spin 3-manifold, let $R(M, \sigma) \in \mathbf{Z}_{16}$ denote its Rochlin invariant. If M is a homology sphere, we will denote its (unique) spin structure by σ_0. Recall from [13] that surgery along a Y-graph makes also sense among spin 3-manifolds:

$$\begin{pmatrix} \mathsf{Data}\text{: (i) } (M,\sigma), \text{ a closed spin 3-manifold} \\ \qquad\quad \text{(ii) } G, \text{ a } Y\text{-graph in } M \end{pmatrix} \rightsquigarrow \mathsf{Result}\text{: } (M_G, \sigma_G)\,.$$

[c]Blackboard framing convention is used.

The following lemma describes precisely how the Rochlin invariant is modified during surgery along a Y-graph.

Lemma 3.14. *Let (M,σ) be a closed spin 3-manifold, and let G be a Y-graph in M whose leaves are ordered, oriented and denoted by K_1, K_2 and K_3. Then,*

$$R(M_G,\sigma_G) - R(M,\sigma) = 8\cdot \prod_{k=1}^{3} e(t_{K_k})(\sigma) \in \mathbf{Z}_{16}, \tag{3.8}$$

where $8\cdot : \mathbf{Z}_2 \rightarrowtail \mathbf{Z}_{16}$ denotes the usual injection, and where $t_{K_k} \in H_1(E(FM);\mathbf{Z})$ and $e(t_{K_k}) \in A(Spin(M),\mathbf{Z}_2)$ have been defined in §2.2.

Proof. Let $j : H_3 \hookrightarrow M$ be the embedding of the genus 3 handlebody, determined (up to isotopy) by the Y-graph G in M. Then, it follows from [13, Proposition 1], that the variation $R(M_G,\sigma_G) - R(M,\sigma)$ only depends on $j^*(\sigma) \in Spin(H_3)$. Also, according to Eq. (2.4) from the proof of Lemma 2.7, the rhs of (3.8) is determined by $j^*(\sigma) \in Spin(H_3)$.

For $i_1, i_2, i_3 \in \{0,1\}$, we denote by $G_{i_1i_2i_3}$ the trivial Y-graph in $\mathbf{S}^3$ (with ordered and oriented leaves) and whose leaf number k is trivial and i_k-framed; we also denote by $j_{i_1i_2i_3} : H_3 \hookrightarrow \mathbf{S}^3$ the corresponding embedding. Then,

$$Spin(H_3) = \left\{ j^*_{i_1i_2i_3}(\sigma_0) | i_1, i_2, i_3 \in \{0,1\} \right\}.$$

Thus, it is enough to prove (3.8) when (M,σ) is $\left(\mathbf{S}^3,\sigma_0\right)$ and when G is a $G_{i_1i_2i_3}$, so that we now restrict ourselves to this case. By Lemma 2.7 (b) (iv), the rhs of Eq. (3.8) is 8 if $i_1 = i_2 = i_3 = 1$ and is 0 otherwise. The same holds for the lhs of Eq. (3.8). Indeed, surgery along a Y-graph with a trivial leaf has no effect (by the “blow-up move” of [2, Fig. 6]), and surgery on $\mathbf{S}^3$ along G_{111} gives the Poincaré sphere whose Rochlin invariant is $8 \in \mathbf{Z}_{16}$. It follows that Eq. (3.8) holds in these eight particular cases. $\square$

Let Σ be Σ_g or $\Sigma_{g,1}$. Let j be an oriented embedding of Σ in $\mathbf{S}^3$, and let $M = (M, i^+, i^-)$ be a homology cylinder over Σ. We can then cut $\mathbf{S}^3$ along $Im(j)$, and glue back M (using the identifications j, i^+ and i^-). We get a new homology sphere which is denoted by

$$\mathbf{S}^3(M,j).$$

It is shown in [13, Corollary 1] that the Rochlin invariant is a degree 1 invariant: in particular, it is preserved under a Y_2-surgery. Therefore, $R\left(\mathbf{S}^3(M,j),\sigma_0\right)$ only depends on the Y_2-equivalence class of M (and j). Suppose now we are given a surgery presentation of the Y_2-equivalence class of M on 1_Σ:

$$\psi_1\left(\sum_{i=1}^{n} \mathsf{Y}\left[z_1^{(i)}, z_2^{(i)}, z_3^{(i)}\right]\right) = M \in \overline{\mathcal{C}}_1(\Sigma).$$

Recall that the labels $z_k^{(i)}$ belong to P and thus give some $e\left(z_k^{(i)}\right) \in B_g^{(1)}$. We also put $\sigma = j^*(\sigma_0) \in Spin(\Sigma)$, which can be identified with the quadratic form $q_\sigma \in \Omega_g$ according to the Johnson construction (see §2.2). We then deduce from (3.8) the following *cubic* formula:

$$\frac{R\left(\mathbf{S}^3(M,j),\sigma_0\right)}{8} = \sum_{i=1}^{n}\prod_{k=1}^{3} e\left(z_k^{(i)}\right)(q_\sigma) \in \mathbf{Z}_2. \tag{3.9}$$

In particular, this shows that:

(i) $R\left(\mathbf{S}^3(M,j),\sigma_0\right)$ only depends on $\sigma = j^*(\sigma_0) \in Spin(\Sigma)$ (and the Y_2-equivalence class of M);
(ii) if N is another homology cylinder over Σ, then:

$$\frac{R\left(\mathbf{S}^3(M\cdot N,j),\sigma_0\right)}{8} = \frac{R\left(\mathbf{S}^3(M,j),\sigma_0\right)}{8} + \frac{R\left(\mathbf{S}^3(N,j),\sigma_0\right)}{8} \in \mathbf{Z}_2\,.$$

We now distinguish the case $\Sigma = \Sigma_g$ from the case $\Sigma = \Sigma_{g,1}$.

In the boundary case, any spin structure σ on $\Sigma_{g,1}$ can be realized as a $j^*(\sigma_0)$ for a certain embedding $j : \Sigma_{g,1} \hookrightarrow \mathbf{S}^3$. In fact, the specific embeddings of $\Sigma_{g,1}$ whose images are depicted in Fig. 4 do suffice.

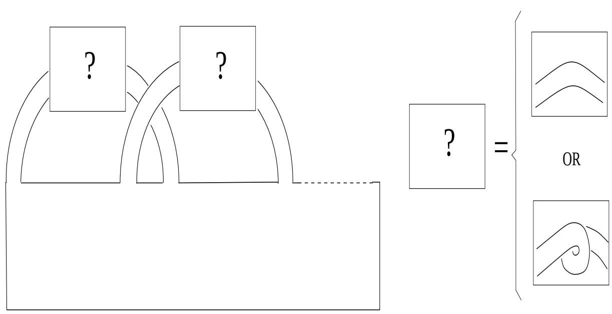

Fig. 4. Some particular embeddings of $\Sigma_{g,1}$ in $\mathbf{S}^3$.

As for the closed case, observe that any embedding $j : \Sigma_g \hookrightarrow \mathbf{S}^3$ is splitting, so that $\sigma = j^*(\sigma_0)$ is spin-bounding. Conversely, any spin structure on Σ_g which spin-bounds can be so realized: choose an appropriate embedding of Σ_g among the particular ones whose images are shown in Fig. 5.

Two other facts about these structures still have to be mentioned. First, $\sigma \in Spin(\Sigma_g)$ spin-bounds if and only if the Arf invariant $\alpha(q_\sigma)$ vanishes (see [11, p. 36]). Second, if f and f' are two cubic polynomials on Ω_g (namely $f, f' \in B_g^{(3)}$), then they are identical on the quadratic forms with trivial Arf invariant if and only if $f - f'$ is a multiple of α (see [8, Lemma 14] for a proof[d] of this algebraic fact).

[d]There, the proof is given for a genus $g \geq 3$, but the same arguments allow us to prove that this fact also holds for a genus $g = 0, 1$ or 2.

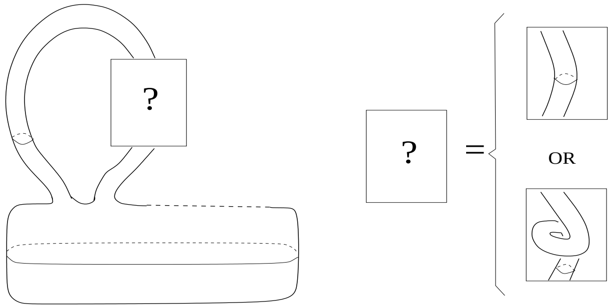

Fig. 5. Some particular embeddings of Σ_g in $\mathbf{S}^3$.

All of our present discussion leads to the following proposition-definition.

Proposition 3.17. *There exist some well-defined homomorphisms*

$$\overline{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{\beta} B_g^{(3)} \quad \textit{and} \quad \overline{\mathcal{C}}_1(\Sigma_g) \xrightarrow{\beta} \frac{B_g^{(3)}}{\alpha \cdot B_g^{(1)}},$$

such that, for M a homology cylinder over Σ and for $j : \Sigma \hookrightarrow \mathbf{S}^3$ an oriented embedding, we have

$$\beta(M)\left(q_{j^*(\sigma_0)}\right) = \frac{R\left(\mathbf{S}^3(M,j), \sigma_0\right)}{8} \in \mathbf{Z}_2\,.$$

Remark 3.18. By composing β with the map $C : \mathcal{T}(\Sigma) \longrightarrow \overline{\mathcal{C}}_1(\Sigma)$, we obtain the classical Birman-Craggs homomorphisms, as presented by Johnson in [8].

4. Proof of the Results

In this section, we prove the results announced in the introduction.

Convention 4.19. In the proofs, we will use some specific techniques of Habiro. Recall that its calculus of claspers developped in [6] is based on the definition of surgery along a *basic clasper*. So as to be consistent with our Convention 2.10, we define here a *basic clover C* in a 3-manifold M to be the embedding into M of the surface depicted on the left part of Fig. 6. *Surgery along C* is defined as the surgery along the 2-component framed link shown[e] in the right part of Fig. 6. Then, a basic clover is a basic clasper but with opposite surgery meaning. Consequently, before using one of the thirteen Habiro's moves, *we will have to take its mirror image.*

4.1. *Y-equivalence: proof of Proposition* 1.1

Since surgeries along clovers preserve homology, the inclusions $\mathcal{C}_1(\Sigma_g) \subset \mathcal{HC}(\Sigma_g)$ and $\mathcal{C}_1(\Sigma_{g,1}) \subset \mathcal{HC}(\Sigma_{g,1})$ are clear.

[e]Blackboard framing convention is used.

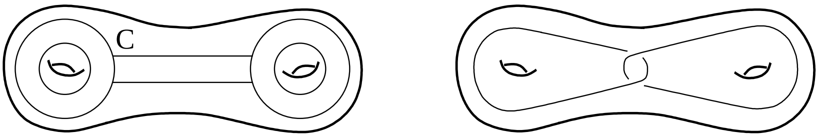

Fig. 6. A basic clover C and the associated framed link.

We now prove the inclusion $\mathcal{HC}(\Sigma_{g,1}) \subset \mathcal{C}_1(\Sigma_{g,1})$ using a result of Habegger. For this, we need the following definition. Let $k \geq 0$ be an integer, a *homology handlebody* of genus k is a pair (M, i) where

(i) M is a compact oriented 3-manifold whose integral homology groups are isomorphic to those of H_k, the standard genus k handlebody;
(ii) $i : \Sigma_k = \partial H_k \longrightarrow M$ is an oriented embedding with image ∂M.

Theorem 4.20. (Habegger, [5]) *Let (M_1, i_1) and (M_2, i_2) be genus k homology handlebodies such that*

$$Ker\left(H_1(\Sigma_k; \mathbf{Z}) \xrightarrow{i_{1,*}} H_1(M_1; \mathbf{Z})\right) = Ker\left(H_1(\Sigma_k; \mathbf{Z}) \xrightarrow{i_{2,*}} H_1(M_2; \mathbf{Z})\right).$$

Then, (M_1, i_1) and (M_2, i_2) are Y-equivalent.

In the sequel we identify H_{2g} with $\Sigma_{g,1} \times I$, and so Σ_{2g} with $\partial(\Sigma_{g,1} \times I)$. We also denote by (H_{2g}, j) the standard genus $2g$ handlebody, with inclusion $j : \Sigma_{2g} \hookrightarrow H_{2g}$. Any homology cobordism $M = (M, i^+, i^-)$ over $\Sigma_{g,1}$ produces a genus $2g$ homology handlebody (M, i), by defining $i : \Sigma_{2g} \longrightarrow M$ to be the diffeomorphism obtained from the gluing of i^+ with i^-. Suppose now that M is a homology cylinder. Proving that the homology handlebody (M, i) is Y-equivalent to (H_{2g}, j) will imply that the homology cylinder M is Y-equivalent to $(\Sigma_{g,1} \times I, Id, Id)$.

For this, let $x_1^*, \ldots, x_g^*, y_1^*, \ldots, y_g^*$ be some disjoint proper arcs in $\Sigma_{g,1}$, which are "dual" to the loops $x_1, \ldots, x_g, y_1, \ldots, y_g$ of Fig. 2, in the sense that x_k^* (resp. y_k^*) transversely intersects x_k (resp. y_k) once but does not intersect the other loops. For example, choose the first attaching region of each 1-handle. For each k, $X_k = x_k^* \times I$ and $Y_k = y_k^* \times I$ are discs in $\Sigma_{g,1} \times I$. The kernel of $j_* : H_1(\Sigma_{2g}) \longrightarrow H_1(\Sigma_{g,1} \times I)$ is spanned by $\partial X_1, \ldots, \partial X_g, \partial Y_1, \ldots, \partial Y_g$. On the other hand, observe that $\pm\partial Y_k$ (resp. $\pm\partial X_k$) is homologous to $x_k \times 0 - x_k \times 1$ (resp. to $y_k \times 0 - y_k \times 1$) in Σ_{2g}. Therefore, since M is a homology cylinder, $i(\partial X_k)$ and $i(\partial Y_k)$ are nul-homologous in M. As the kernel of $i_* : H_1(\Sigma_{2g}) \longrightarrow H_1(M)$ has to be of dimension $2g$, it is spanned by $\partial X_1, \ldots, \partial X_g, \partial Y_1, \ldots, \partial Y_g$. It follows from Theorem 4.20 that (M, i) is Y-equivalent to (H_{2g}, j), which proves the inclusion $\mathcal{HC}(\Sigma_{g,1}) \subset \mathcal{C}_1(\Sigma_{g,1})$.

Let us now justify the inclusion $\mathcal{HC}(\Sigma_g) \subset \mathcal{C}_1(\Sigma_g)$. Let $j : \Sigma_{g,1} \hookrightarrow \Sigma_g$ be an embedding and let $D \subset \Sigma_g$ be its complementary disk. Take a homology cobordism $M = (M, i^+, i^-)$ over $\Sigma_{g,1}$. Then, the embedding $(i^+)|_\partial \circ (j|_\partial)^{-1} = (i^-)|_\partial \circ (j|_\partial)^{-1} :$ $\partial D \hookrightarrow \partial M$ can be stretched to an embedding $\partial D \times I \hookrightarrow \partial M$. The latter allows us to attach the 2-handle $D \times I$ to M. This results in a homology cylinder over Σ_g. We have thus defined a *filling-up* map

$$\mathcal{C}(\Sigma_{g,1}) \xrightarrow{j} \mathcal{C}(\Sigma_g),$$

which is obviously surjective. Let $M \in \mathcal{HC}(\Sigma_g)$, and pick a $N \in \mathcal{C}(\Sigma_{g,1})$ such that M is a filling-up of N. Then, N has to be a homology cylinder and so is Y-equivalent to $1_{\Sigma_{g,1}}$. We conclude that $M \in \mathcal{C}_1(\Sigma_g)$, which completes the proof of Proposition 1.1.

4.2. *The boundary case: proof of Theorem* 1.3

Recall from Example 2.6 that the Abelian group $\mathcal{A}_1(H,0)$ can be identified with $\Lambda^3 H$, and likewise $\mathcal{A}_1\left(H_{(2)},0\right)$ with $\Lambda^3 H_{(2)}$. The following lemma will allow us to identify $\mathcal{A}_1\left(B_g^{(1)},\overline{1}\right)$ with $B_g^{(3)}$.

Lemma 4.21. *Let* $\gamma : \mathcal{A}_1\left(B_g^{(1)},\overline{1}\right) \longrightarrow B_g^{(3)}$ *be the map given by multiplying the labels of the abstract Y-graphs:* $\gamma(\mathsf{Y}[z_1,z_2,z_3]) = z_1 z_2 z_3$. *Then, γ is a well-defined isomorphism.*

Proof. The fact that γ is well-defined is clear. In order to show that γ is an isomorphism, it suffices to construct an epimorphism $B_g^{(3)} \xrightarrow{\epsilon} \mathcal{A}_1\left(B_g^{(1)},\overline{1}\right)$ such that $\gamma \circ \epsilon$ is the identity.

By choosing a basis $(e_j)_{j=1}^{2g}$ for H, one determines an isomorphism between $B_g^{(3)}$ and $\mathbf{Z}_2 \oplus H_{(2)} \oplus \Lambda^2 H_{(2)} \oplus \Lambda^3 H_{(2)}$: for $k = 1,2,3$ and $j_1,\ldots,j_k \in \{1,\ldots,2g\}$ pairwise distinct, the monomial $\prod_{i=1}^k \overline{e_{j_i}}$ is identified with the wedge product $\wedge_{i=1}^k e_{j_i}$, and $\overline{1}$ with $1 \in \mathbf{Z}_2$. Since $B_g^{(1)}$ is a period 2 group, so is $\mathcal{A}_1\left(B_g^{(1)},\overline{1}\right)$ by the multilinearity relation. Then, it suffices to define ϵ on the above mentioned $\mathbf{Z}_2$-basis of $\mathbf{Z}_2 \oplus H_{(2)} \oplus \Lambda^2 H_{(2)} \oplus \Lambda^3 H_{(2)} \simeq B_g^{(3)}$. We put $\epsilon(1) = \mathsf{Y}\left[\overline{1},\overline{1},\overline{1}\right]$, $\epsilon(e_j) = \mathsf{Y}\left[\overline{e_j},\overline{1},\overline{1}\right]$, $\epsilon(e_{j_1} \wedge e_{j_2}) = \mathsf{Y}\left[\overline{e_{j_1}},\overline{e_{j_2}},\overline{1}\right]$ (with $j_1 \neq j_2$) and $\epsilon(e_{j_1} \wedge e_{j_2} \wedge e_{j_3}) = \mathsf{Y}\left[\overline{e_{j_1}},\overline{e_{j_2}},\overline{e_{j_3}}\right]$ (with j_1,j_2,j_3 pairwise distinct). The map ϵ is surjective by the multilinearity and slide relations, and obviously satisfies $\gamma \circ \epsilon = Id$. □

Recall from §2.2 that the maps

$$P \xrightarrow{p} (H,0) \quad \text{and} \quad P \xrightarrow{e} \left(B_g^{(1)},\overline{1}\right)$$

are the canonical projections of the pullback of special Abelian groups

$$P = (H,0) \times_{(H_{(2)},0)} \left(B_g^{(1)},\overline{1}\right).$$

They happen to be surjective.

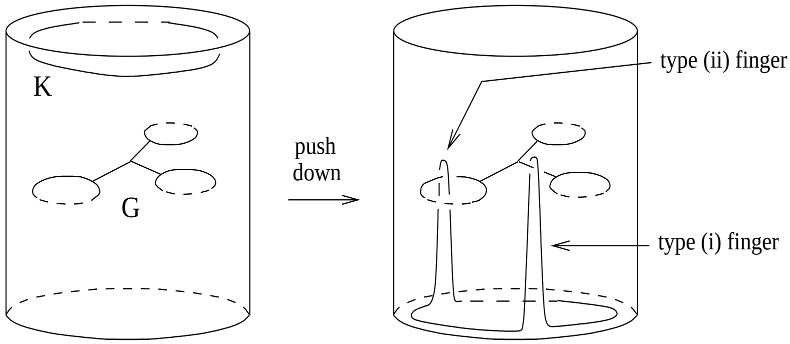

Fig. 7. Pushing the curve K down the cylinder.

Lemma 4.22. *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{A}_1(P) \arrow[r, "\psi_1"] \arrow[dr, "\mathcal{A}_1(p)"'] & \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "\eta_1"] \\ & \mathcal{A}_1(H,0). \end{tikzcd}$$

Proof. Let us verify that $\eta_1(\psi_1(Y)) = \mathcal{A}_1(p)(Y)$ for a generator $Y = \mathsf{Y}[z_1, z_2, z_3]$ of $\mathcal{A}_1(P)$. We put $M = \psi_1(Y)$, so that $M = (1_{\Sigma_{g,1}})_G$ where G is an appropriate Y-graph as described in §2.3. Its leaves are in particular ordered and oriented, they are denoted by K_1, K_2 and K_3: $[K_i] = p(z_i) \in H$. Set $\pi = \pi_1(\Sigma_{g,1}, *)$ and let $\overline{y} \in \pi/\pi_3$ be represented by $y \in \pi$: we want to compute $\eta_1(M)$ on $\overline{y}$. This goes as follows: choose an immersed based curve k in $\Sigma^+_{g,1}$ representing y (*via* the identification of $\Sigma_{g,1}$ with $\Sigma^+_{g,1}$), pick an oriented based knot $K \subset M$ in a collar of $\Sigma^+_{g,1}$ which is a push-off of k, and find another based knot $K' \subset M$ in a collar of $\Sigma^-_{g,1}$ such that the pairs (M, K) and (M, K') are Y_2-equivalent. Then (*via* the identification of $\Sigma_{g,1}$ with $\Sigma^-_{g,1}$), this knot K' determines a $y' \in \pi$, and by Lemma 3.13, the result $\eta_1(M)(\overline{y})$ is then $\overline{y'} \in \pi/\pi_3$. We now explain the procedure how to construct K' from K.

In $1_{\Sigma_{g,1}} \setminus G$, K can be pushed down in a collar of $\Sigma^-_{g,1}$ *up to* some "fingers" which are of two types (see Fig. 7):

(i) the finger is pointing on an edge of G,
(ii) the finger is pointing on an leaf K_i of G.

But, each finger of type (i) can be isotoped along the corresponding edge towards its leaf and so can be replaced by two fingers of type (ii), so that up to some isotopy of the immersed curve k in $\Sigma^+_{g,1}$, we can suppose each finger to be of type (ii). Since K_i has been oriented, each finger comes with a sign. Let k_i be an immersed curve on $\Sigma^+_{g,1} \subset 1_{\Sigma_{g,1}}$ such that $[k_i] = p(z_i) \in H$. We can suppose that K_i is a push-off of k_i (with possibly an additional twist): there are then as many fingers as intersection points of k_i with k in $\Sigma^+_{g,1}$; the sign of the finger corresponds with the sign of the intersection point contributing to $[k] \bullet [k_i] \in \mathbf{Z}$.

A finger move can be realized by surgery on a basic clover. Let K' be a copy of K in a collar of $\Sigma^-_{g,1} \subset 1_{\Sigma_{g,1}} \setminus G$. There is then a family of basic clovers $(C_j^{(i)})_{j=1,\dots,n_i}^{i=1,2,3}$ in $1_{\Sigma_{g,1}} \setminus G$, such that each $C_j^{(i)}$ has a simple leaf which laces K' and another simple leaf wich laces the leaf K_i, and such that:

$$(M,K) \text{ is diffeomorphic to } \left(1_{\Sigma_{g,1}}, K'\right)_{\left(\cup_{i,j} C_j^{(i)}\right) \cup G}.$$

According to the sign of the corresponding finger, each basic clover comes with a sign denoted by $\varepsilon(i,j)$. *Cutting* the leaf K_1 (see [2, Corollary 4.3]) n_1 times, we obtain n_1 new Y-graphs $G_j^{(1)}$ ($j \in \{1,\dots,n_1\}$): two leaves of $G_j^{(1)}$ are copies of K_2 and K_3, and the third leaf forms with a leaf of $C_j^{(1)}$ the Hopf-link. Hence, by applying Habiro's move 2 (or [2, Theorem 2.4]) to $C_j^{(i)} \cup G_j^{(i)}$ we obtain a new Y-graph still denoted by $G_j^{(i)}$. We do the same for $i = 2$ and $i = 3$, therefore:

$$(M,K) \text{ is } Y_2\text{-equivalent to } \left(1_{\Sigma_{g,1}}, K'\right)_{\left(\cup_{i,j} G_j^{(i)}\right) \cup G}.$$

Up to Y_2-equivalence of the pair $\left(1_{\Sigma_{g,1}}, K'\right)_{G \cup \left(\cup_{i,j} G_j^{(i)}\right)}$, one can suppose that, for each (i,j), the whole of $G_j^{(i)}$ lies in a collar neighborhood of $\Sigma^-_{g,1} \subset 1_{\Sigma_{g,1}}$. We now do the surgery along G, and then along each of the $G_j^{(i)}$: the latter does not modify the 3-manifold M but changes the knot. The new knot we obtain is still denoted by K' and satisfies the announced required properties.

We now calculate the $y' \in \pi$ defined by K'. In view of Habiro's move 10, the contribution of each Y-graph $G_j^{(1)}$ to the modification of K' is in π the commutator $\left[k_2, k_3^{-1}\right]^{\varepsilon(1,j)}$. Therefore, we obtain

$$y' \cdot y^{-1} = \prod_{i \in \mathbf{Z}_3} \left[k_{i+1}, k_{i+2}^{-1}\right]^{[k] \bullet [k_i]} \in \frac{\pi_2}{\pi_3}. \tag{4.10}$$

Then, as a homomorphism $H \longrightarrow \pi_2/\pi_3 = L_2(H)$, $\eta_1(M)$ sends any $h \in H$ to

$$-\sum_{i \in \mathbf{Z}_3} (h \bullet p(z_i)) \cdot [p(z_{i+1}), p(z_{i+2})] \in L_2(H).$$

which corresponds to $\sum_{i \in \mathbf{Z}_3} p(z_i) \otimes [p(z_{i+1}, p(z_{i+2})]$ in $H \otimes L_2(H)$, to $p(z_1) \wedge p(z_2) \wedge p(z_3)$ in $\Lambda^3 H$, and so to $\mathcal{A}_1(p)(Y)$. $\square$

Lemma 4.23. *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{A}_1(P) \arrow[r, two heads, "\psi_1"] \arrow[dr, "\mathcal{A}_1(e)"'] & \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "\beta"] \\ & \mathcal{A}_1\left(B_g^{(1)}, \overline{1}\right). \end{tikzcd}$$

Proof. According to the definition of β we gave in Proposition 3.17, this is a direct consequence of Eq. (3.9). $\square$

We still denote by

$$(H,0) \xrightarrow{-\otimes \mathbf{Z}_2} \left(H_{(2)},0\right) \text{ and } \left(B_g^{(1)},\overline{1}\right) \xrightarrow{\kappa} \left(H_{(2)},0\right),$$

the maps which appear in the pullback diagram for P (see §2.2). Then, as a consequence of the two preceeding lemmas, $\mathcal{A}_1(\kappa)\beta\psi_1 = \mathcal{A}_1(\kappa e) = \mathcal{A}_1\left((-\otimes \mathbf{Z}_2)p\right) = \mathcal{A}_1(-\otimes \mathbf{Z}_2)\eta_1\psi_1$. Since ψ_1 is an epimorphism, we get: $\mathcal{A}_1(\kappa)\beta = \mathcal{A}_1(-\otimes \mathbf{Z}_2)\eta_1$. Construct the following pull-back:

$$\begin{tikzcd} \mathcal{A}_1(H,0) \times_{\mathcal{A}_1(H_{(2)},0)} \mathcal{A}_1\left(B_g^{(1)},\overline{1}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}_1\left(B_g^{(1)},\overline{1}\right) \arrow[d, "\mathcal{A}_1(\kappa)"] \\ \mathcal{A}_1(H,0) \arrow[r, "\mathcal{A}_1(-\otimes \mathbf{Z}_2)"'] & \mathcal{A}_1(H_{(2)},0) \end{tikzcd}$$

which, through the above mentioned identifications, is essentially the pull-back diagram for $\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ appearing in §1.3. By the universal property of the pull-backs, there is then a homomorphism

$$\overline{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{(\eta_1,\beta)} \mathcal{A}_1(H,0) \times_{\mathcal{A}_1(H_{(2)},0)} \mathcal{A}_1\left(B_g^{(1)},\overline{1}\right) \simeq \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}.$$

Moreover, we also have by functoriality another natural map

$$\mathcal{A}_1\big(\underbrace{(H,0) \times_{(H_{(2)},0)} (B_g^{(1)},\overline{1})}_{P}\big) \xrightarrow{\rho} \mathcal{A}_1(H,0) \times_{\mathcal{A}_1(H_{(2)},0)} \mathcal{A}_1\left(B_g^{(1)},\overline{1}\right).$$

Lemmas 4.22 and 4.23 can then be summarized in the commutativity of the following diagram:

$$\begin{tikzcd} \mathcal{A}_1(P) \arrow[r, two heads, "\psi_1"] \arrow[dr, "\rho"'] & \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "(\eta_1,\beta)"] \\ & \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}. \end{tikzcd}$$

The following lemma will be the final step in proving Theorem 1.3.

Lemma 4.24. *The map* $\rho : \mathcal{A}_1(P) \longrightarrow \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ *is an isomorphism.*

Assume Lemma 4.24. Then, from the previous commutative diagram, it follows that ψ_1 is injective, and so is an isomorphism: as a consequence, the same holds for (η_1, β). The commutativity of

$$\begin{tikzcd} \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[d, "(\eta_1,\beta)"', "\simeq"] & \frac{\mathcal{T}_{g,1}}{\mathcal{T}'_{g,1}} \arrow[l, "C"'] \arrow[dl, "(\eta_1,\beta)"] \\ \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} & \end{tikzcd}$$

follows from Remarks 3.15 and 3.18. In particular, when $g \geq 3$, C is an isomorphism because $(\eta_1, \beta) : \mathcal{T}_{g,1}/\mathcal{T}'_{g,1} \longrightarrow \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ is so by [10].

Proof of Lemma 4.24. We proceed as in Lemma 4.21. It suffices to construct an epimorphism

$$\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} \overset{\epsilon}{\twoheadrightarrow} \mathcal{A}_1(P)$$

such that $\rho \circ \epsilon$ is the identity.

Pick a basis $(e_i)_{i=1}^{2g}$ of H: we have seen in the proof of Lemma 4.21 that this choice determines an isomorphism between $B_g^{(3)}$ and $\Lambda^3 H_{(2)} \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$. Thus, it also defines an isomorphism between $\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ and $\Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$. We now define ϵ by putting

(i) $\epsilon(e_i \wedge e_j \wedge e_k) = \mathsf{Y}\left[(e_i, \overline{e_i}), (e_j, \overline{e_j}), (e_k, \overline{e_k})\right]$, with $1 \leq i < j < k \leq 2g$,
(ii) $\epsilon(e_i \wedge e_j) = \mathsf{Y}\left[(e_i, \overline{e_i}), (e_j, \overline{e_j}), (0, \overline{1})\right]$, with $1 \leq i < j \leq 2g$,
(iii) $\epsilon(e_i) = \mathsf{Y}\left[(e_i, \overline{e_i}), (0, \overline{1}), (0, \overline{1})\right]$, with $1 \leq i \leq 2g$,
(iv) and $\epsilon(1) = \mathsf{Y}\left[(0, \overline{1}), (0, \overline{1}), (0, \overline{1})\right]$.

Here, elements of P are denoted as in Remark 2.8. This assignation well defines ϵ because (i) determines ϵ on a basis of the free group $\Lambda^3 H$, while (ii),(iii) and (iv) assign elements of $\mathcal{A}_1(P)$ of order at most 2 to each element basis of the $\mathbf{Z}_2$-vector space $\Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbf{Z}_2$. Obviously, ϵ followed by ρ gives the identity. Take now any generator $\mathsf{Y}[z_1, z_2, z_3]$ of $\mathcal{A}_1(P)$. For $i = 1, 2, 3$, $z_i \in P$ can be written as a linear combination of some $(e_j, \overline{e_j})$ and $(0, \overline{1})$. The multilinearity, AS and slide relation allow us to conclude that $\mathsf{Y}[z_1, z_2, z_3]$ is realized by ϵ. Thus, ϵ is surjective. $\square$

4.3. *The closed case: proof of Theorem* 1.4

An isomorphism

$$\mathcal{A}_1(P) \xrightarrow{\rho} \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$$

is defined formally in the same way as in the boundary case (see Lemma 4.24). Recall that S stands for the subgroup of the pullback $\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}$ corresponding to $\omega \wedge H \subset \Lambda^3 H$ and $\alpha \cdot B_g^{(1)} \subset B_g^{(3)}$. Then, $\rho^{-1}(\mathrm{S})$ is the subgroup of $\mathcal{A}_1(P)$ comprising the elements

$$\sum_{i=1}^{g} \mathsf{Y}\left[(x_i, \overline{x_i}), (y_i, \overline{y_i}), z\right], \quad \text{where } z \text{ is any element of } P\,.$$

Lemma 4.25. *In the closed case, the surgery map ψ_1 defined in* §2.3 *vanishes on the subspace $\rho^{-1}(S)$.*

As mentioned in the introduction, these symplectic relations $\rho^{-1}(\mathrm{S})$ appears in [6] for higher degrees.

Proof of Lemma 4.25. Let $z \in P$, we aim to show that

$$\sum_{i=1}^{g} \psi_1\left(\mathsf{Y}\left[(x_i, \overline{x_i}), (y_i, \overline{y_i}), z\right]\right) = 0 \in \overline{\mathcal{C}}_1(\Sigma_g)\,. \tag{4.11}$$

Consider in 1_{Σ_g} a basic clover G with one trivial leaf f, and the other leaf f' satisfying $t_{f'} = z \in P$. Then, f being trivial, $\left(1_{\Sigma_g}\right)_G$ is diffeomorphic to 1_{Σ_g}. Furthermore, f can be seen as a push-off of ∂D where D is a 2-disk in Σ_g^+: in particular, f bounds the push-off of $\Sigma_g^+ \setminus D$ which is an embedded genus g surface. By applying Habiro's moves 7 and 5, f can be split in g pieces so that G is equivalent to the union of g basic clovers denoted by $G_1, \ldots, G_g$. See Fig. 8. Each clover G_i has

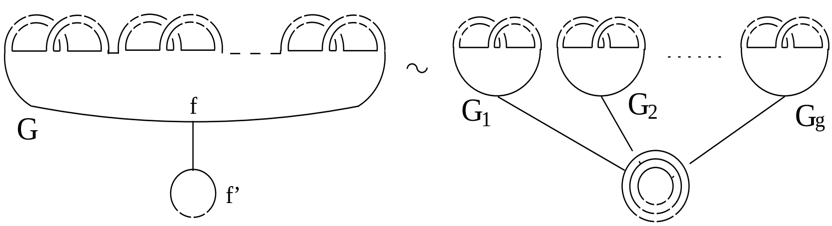

Fig. 8. Splitting the null-homologous leaf f.

a leaf which bounds a genus 1 surface; by applying Habiro's move 10, it is seen to be equivalent to a Y-graph G_i'. According to Remark 2.8, the leaves of G_i' represent $(x_i, \overline{x_i})$, $(y_i, \overline{y_i})$ and z in P, so that $\left(1_{\Sigma_g}\right)_{G_i'} = \psi_1\left(\mathsf{Y}\left[(x_i, \overline{x_i}), (y_i, \overline{y_i}), z\right]\right) \in \overline{\mathcal{C}}_1(\Sigma_g)$. Equation (4.11) then follows. □

By the same arguments, appropriate versions of Lemmas 4.22 and 4.23 hold in the closed case: $\mathcal{A}_1(p) = \eta_1 \circ \psi_1$ and $\mathcal{A}_1(e) = \beta \circ \psi_1$. This leads us to a commutative diagram

$$\begin{tikzcd}
\frac{\mathcal{A}_1(P)}{\rho^{-1}(\mathrm{S})} \arrow[r, two heads, "\psi_1"] \arrow[dr, "\rho"', "\simeq"] & \overline{\mathcal{C}}_1(\Sigma_g) \arrow[d, "(\eta_1,\beta)"] \\
& \frac{\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}}{\mathrm{S}} \simeq \frac{\Lambda^3 H}{\omega \wedge H} \times_{\left(\frac{\Lambda^3 H_{(2)}}{\omega_{(2)} \wedge H_{(2)}}\right)} \frac{B_g^{(3)}}{\alpha \cdot B_g^{(1)}}
\end{tikzcd}$$

from which it follows that ψ_1, and then (η_1, β), are isomorphisms. The commutativity of the right triangle in Theorem 1.4 is still given by Remarks 3.15 and 3.18.

4.4. *Finite type invariants of degree* 1*: proof of Corollary* 1.5

The equivalence (a)$\Leftrightarrow$(b) immediately results from the existence of the universal degree one primitive invariant υ introduced in Remark 2.12. The equivalence (c)$\Leftrightarrow$(a) is a direct consequence of Theorems 1.4 and 1.3.

4.5. *From the boundary case to the closed case*

In this last paragraph, we *fix* an isomorphism

$$H_1(\Sigma_{g,1}; \mathbf{Z}) \xrightarrow{\phi} H_1(\Sigma_g; \mathbf{Z}).$$

It allows us to identify the sets H, $\Omega_g \simeq Spin(\Sigma)$, B_g and P corresponding to $\Sigma_{g,1}$ with those of Σ_g.

Moreover, let $j : \Sigma_{g,1} \hookrightarrow \Sigma_g$ be an embedding such that $j_* = \phi$ at the level of $H_1(-; \mathbf{Z})$. Recall from §4.1 the filling-up map, which can be restricted to

$$\mathcal{C}_1(\Sigma_{g,1}) \xrightarrow{j} \mathcal{C}_1(\Sigma_g).$$

Note that it is compatible with the "extending by the identity" map $\mathcal{T}_{g,1} \longrightarrow \mathcal{T}_g$ defined by j, and that it induces a group homomorphism $\overline{\mathcal{C}}_1(\Sigma_{g,1}) \longrightarrow \overline{\mathcal{C}}_1(\Sigma_g)$. The latter can be verified to be independent on the choice of the embedding j such that $j_* = \phi$, and so can be denoted by

$$\overline{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{\phi} \overline{\mathcal{C}}_1(\Sigma_g).$$

The commutativity of the following diagram is easily proved from the various definitions:

$$
\begin{tikzcd}
\mathcal{A}_1(P) \arrow[rrr, "\psi_1"] \arrow[dr, "\rho"'] \arrow[dd] & & & \overline{\mathcal{C}}_1(\Sigma_{g,1}) \arrow[dll, "(\eta_1,\beta)"'] \arrow[dd, "\phi"] & \\
& \Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)} \arrow[dd] & & & \frac{\mathcal{T}_{g,1}}{\mathcal{T}'_{g,1}} \arrow[ul, "C"'] \arrow[lll, "(\eta_1,\beta)"] \arrow[dd] \\
\frac{\mathcal{A}_1(P)}{\rho^{-1}(\mathrm{S})} \arrow[rrr, "\psi_1"] \arrow[dr, "\rho"'] & & & \overline{\mathcal{C}}_1(\Sigma_g) \arrow[dll, "(\eta_1,\beta)"'] & \\
& \frac{\Lambda^3 H \times_{\Lambda^3 H_{(2)}} B_g^{(3)}}{\mathrm{S}} & & & \frac{\mathcal{T}_g}{\mathcal{T}'_g} \arrow[ul, "C"'] \arrow[lll, "(\eta_1,\beta)"]
\end{tikzcd}
$$

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