Borromean surgery formula for the Casson invariant

JEAN-BAPTISTE MEILHAN

It is known that every oriented integral homology 3–sphere can be obtained from S^3 by a finite sequence of Borromean surgeries. We give an explicit formula for the variation of the Casson invariant under such a surgery move. The formula involves simple classical invariants, namely the framing, linking number and Milnor's triple linking number. A more general statement, for n independent Borromean surgeries, is also provided.

1 Introduction

A *Borromean surgery link* in a closed oriented 3–manifold M is an oriented framed link obtained by embedding in M a copy of the standard genus 3 handlebody containing the 6–component 0–framed oriented link depicted below. (Here and throughout this paper, blackboard framing convention is used.) Surgery along such a link is called a *Borromean surgery move*.
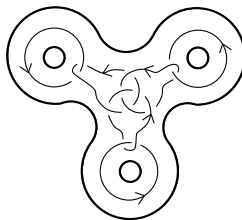
The notion of Borromean surgery was first introduced by Matveev [9] in the eighties, and was more recently used to define the Goussarov–Habiro finite type invariant theory for closed oriented 3–manifolds [2; 4].

Matveev showed that two closed oriented 3-manifolds are *Borromean equivalent*, ie are related by a sequence of Borromean surgery moves, if and only if they have the same first homology groups and isomorphic linking forms. In particular, any oriented integral homology 3–sphere is obtained from S^3 by a finite sequence of Borromean surgeries. It is thus a natural problem to study the behavior of classical invariants of

integral homology spheres, such as the Casson invariant, under this kind of surgery move. Our main result is Theorem 2.2, which expresses the variation of the Casson invariant under a Borromean surgery move in terms of simple classical invariants of the Borromean surgery link, namely framings, linking numbers and Milnor's triple linking number – see Section 2.3 for an explicit statement. A more general formula, for n independent Borromean surgeries on an integral homology sphere, is also provided in Theorem 2.3, which involves an additional cubic expression in some linking numbers of the surgery link.

The paper is organized as follows. In Section 2, we recall the definition and basic properties of the Casson invariant and state the above-mentioned surgery formulas. In Section 3 we prove a couple of preliminary results on Borromean surgery in S^3 using a crossing change formula for the Casson invariant due to Johannes [5]. In Section 4 we prove the formula of Theorem 2.2 for a single Borromean surgery using the results of Section 3 and Lescop's sum formula for the Casson invariant [7]. In Section 5 we use the theory of finite type invariants to prove the formula of Theorem 2.3 for multiple Borromean surgeries. In Section 6 we compare Theorem 2.2 with a similar formula derived from Lescop's global surgery formula, and we apply it to a realization problem for homology spheres of Mazur type.

2   Variation of the Casson invariant under Borromean surgery

2.1   The Casson invariant of integral homology spheres

In this paper, an *integral homology sphere* will always mean an oriented integral homology 3–sphere. We denote by $\mathbf{Z}HS$ the set of integral homology spheres, considered up to orientation-preserving diffeomorphisms.

Theorem 2.1   (Casson)   *There exists a unique function*

$$\lambda\colon \mathbf{Z}HS \longrightarrow \mathbf{Z}$$

such that, for every $M, N \in \mathbf{Z}HS$ and for every knot K in M :

(1) $\lambda(S^3) = 0$.

(2) *For $n \in \mathbf{Z}$, let M_{K_n} be the result of $(1/n)$–Dehn surgery on M along K. Then:*

$$\lambda(M_{K_{n+1}}) - \lambda(M_{K_n}) = \frac{1}{2}\Delta_K''(1),$$

where $\Delta_K(t)$ denotes the Alexander polynomial of K.

Furthermore,

(3) *$\lambda(M \sharp N) = \lambda(M) + \lambda(N)$, where $\sharp$ denotes the connected sum.*

(4) *The Rochlin invariant μ is the mod 2 reduction of λ. More precisely, $\mu(M) \equiv \lambda(M)$ (mod 2).*

This unique function is called the *Casson invariant* of integral homology spheres. This invariant counts, in some sense, the conjugacy classes of irreducible representations of $\pi_1(M)$ in $SU(2)$.

In [12], Walker extended the Casson invariant to a **Q**–valued invariant of rational homology spheres, called the Casson–Walker invariant. In this paper, we will use the following normalization of the Casson–Walker invariant

$$\lambda = \frac{\lambda_W}{2},$$

where λ_W denotes the normalization adopted by Walker in [12]. (Note that our notation are consistent, in the sense that if we restrict to integral homology spheres then $\lambda_W/2$ coincides with Casson's original invariant λ.)

The Casson–Walker invariant was extended by Lescop to all closed oriented 3–manifolds in [6].

2.2 Conventions

Let L be a Borromean surgery link in a rational homology sphere M.

Up to isotopy one can always assume that there is a 3–ball B in M which intersects L as depicted below. The boundary of B intersects three components of L, called the *leaves* of L. We fix an order on the set of the leaves of L, and we denote them by F_1, F_2 and F_3 according to this order. Denote by f_i the framing of F_i, and by l_{ij} the linking number $\mathrm{lk}(F_i, F_j)$, $1 \leq i \neq j \leq 3$.

Observe that there is a canonical 3–component algebraically split link F_0 associated to the Borromean surgery link L as follows. For each pair $1 \leq i < j \leq 3$, consider the $(2, 2)$–tangle B_{ij} obtained by stacking $|l_{ij}|$ copies of the clasp-shaped $(2, 2)$–tangle $T_\pm$ depicted on the right-hand side of Figure 2 (depending on the sign of l_{ij}). If $l_{ij} = 0$,
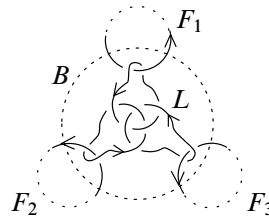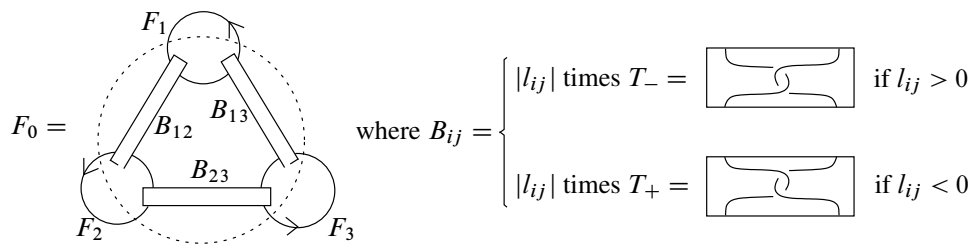
Figure 1

Figure 2: The algebraically split link F_0

B_{ij} is the trivial $(2, 2)$–tangle. Starting with the link $F_1 \cup F_2 \cup F_3$, insert for each $1 \le i < j \le 3$ the tangle B_{ij} into the leaves F_i and F_j so that their linking number is reduced to zero; see Figure 2. The resulting 3–component link is the algebraically split link F_0 associated to L.

Because F_0 has linking numbers zero, it has a well-defined *Milnor's triple linking number* $\overline{\mu}_{ijk}(F_0)$ in M (a short definition is given in Section 2.4 at the end of this section). In the following, we denote by $\mu_{123}(L; M)$ the triple linking number of the algebraically split link F_0 associated to L in M. We will usually simply write $\mu_{123}(L)$ when M is explicit from the context.

2.3 Statement of the main results

We can now state our two main results.

We first express the variation of the Casson–Walker invariant under surgery along a Borromean surgery link L in a rational homology sphere M, under certain restrictions.

Theorem 2.2 *Let L be a Borromean surgery link in a rational homology sphere M, such that the leaves of L have integral linking numbers and have self-linking numbers*

zero in $\mathbf{Q}/\mathbf{Z}$. Denote by M_L the result of surgery on M along L. Then the difference $\lambda(M_L) - \lambda(M)$ is given by the formula

$$-f_1.f_2.f_3 - 2.l_{12}.l_{13}.l_{23} - 2.\mu_{123}(L) + \sum_{\circlearrowleft_{1,2,3}} l_{23}.(l_{23}+1).f_1,$$

where the sum is over all cyclic permutations of the indices $(1, 2, 3)$.

Let us now consider the case of a disjoint union $L_1 \cup ... \cup L_n$ of n Borromean surgery links in an integral homology sphere M. We use the conventions of Section 2.2 for the orientation and ordering of the leaves of each link: the leaves of L_k are denoted by F_1^k, F_2^k and F_3^k $(1 \le k \le n)$. Denote by l_{ij}^{kl} the linking number $\mathrm{lk}(F_i^k, F_j^l)$; $1 \le i \ne j \le 3$, $1 \le k \ne l \le n$.

Theorem 2.3 *The difference $\lambda(M_{L_1 \cup ... \cup L_n}) - \lambda(M)$ is given by the formula*

$$\sum_{1 \le i \le n} \left(\lambda(M_{L_i}) - \lambda(M)\right) - 2 \sum_{1 \le k < l \le n} \sum_{\sigma \in S_3} s(\sigma) l_{1\sigma(1)}^{kl} l_{2\sigma(2)}^{kl} l_{3\sigma(3)}^{kl},$$

where, for an element σ in the symmetric group S_3, $s(\sigma) \in \{-1; +1\}$ denotes its signature.

By reducing the formula in Theorem 2.3 modulo 2 and using Theorem 2.1(4), we obtain a similar formula for the variation of Rochlin's μ–invariant of an integral homology sphere M under surgery along a disjoint union $L_1 \cup ... \cup L_n$ of Borromean surgery links.

Corollary 2.4

$$\mu(M_{L_1 \cup ... \cup L_n}) - \mu(M) = \sum_{1 \le i \le n} f_1^i.f_2^i.f_3^i \ (\mathrm{mod} \ 2).$$

This formula was already (essentially) established independently by Massuyeau in [8], and Auclair and Lescop in [1].

2.4 Milnor's triple linking number

Let $K = K_1 \cup K_2 \cup K_3$ be an algebraically split, oriented, ordered link in a rational homology sphere M. Pick a triple (S_1, S_2, S_3) of transverse Seifert surfaces such that each S_i is bounded by K_i $(i = 1, 2, 3)$ and does not intersect the two other components of K (this is possible since all linking numbers are zero). Then *Milnor's triple linking number $\bar{\mu}_{123}(K; M)$ of K in M* is defined as the negative of the algebraic intersection $\langle S_1, S_2, S_3 \rangle_M$, defined as follows.

Each surface S_i is equipped with a positive normal vector field, induced by its orientation and the orientation of M, and thus each triple intersection point $x \in S_1 \cap S_2 \cap S_3$ can be associated a sign, depending on whether the ordered basis of normal vectors to S_1, S_2 and S_3 at x agrees or not with the orientation of M. The algebraic intersection $\langle S_1, S_2, S_3 \rangle_M$ is the sum of these signs over all $x \in S_1 \cap S_2 \cap S_3$.

3   Preliminary results: Borromean surgery in S^3

In this section we prove a couple of preliminary lemmas on Borromean surgery in S^3.

3.1   Main tools

We recall here a self-crossing change formula for the Casson invariant due to Johannes, as well as a simple result derived from Kirby calculus.

3.1.1   Johannes' formula for the Casson invariant   In [5], Johannes expresses the difference of the Casson invariants of 3–manifolds presented by links in S^3 which differ by a crossing change within a component. This type of move on surgery links was studied by Matveev [9] under the name of *Whitehead surgery*.

For the purpose of the present paper, it is enough to consider the case of 2–component surgery links. Let $L^+ = L_1^+ \cup L_2$ and $L^- = L_1^- \cup L_2$ be two framed links in S^3, with same framing, which only differ by a crossing change on the first component. Let $L_a \cup L_b$ be the 2–component link obtained from $L_1^{\pm}$ by smoothing this crossing (see Figure 3).
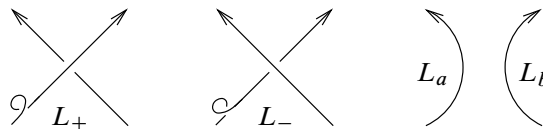

Figure 3

Then Johannes' formula [5] states that

$$(3\text{--}1) \qquad \lambda(S_{L+}^3) - \lambda(S_{L-}^3) = \frac{f_2 . l_{ab} - l_{a2} . l_{b2}}{f_1 . f_2 - l_{12}^2},$$

where f_1 (resp. f_2) denotes the framing of $L_1^{\pm}$ (resp. L_2), and l_{ab} (resp. l_{a2}, l_{b2}, l_{12}) denotes the linking number $\mathrm{lk}(L_a, L_b)$ (resp. $\mathrm{lk}(L_a, L_2)$, $\mathrm{lk}(L_b, L_2)$, $\mathrm{lk}(L_1^{\pm}, L_2)$).

Observe that the denominator in (3–1) is merely the determinant of the linking matrix of $L^{\pm}$.

Note that the formulas given in [5] are for the Casson–Walker invariant, using Walker's normalization (see Section 2.1). Note also that we fixed a sign error in [5] so that (3–1) agrees with Theorem 2.1.[1]

Remark 3.1 Surgery along a ± 1–framed knot and Whitehead surgery are two kinds of surgery moves relating all integral homology spheres, and Casson and Johannes respectively gave a formula for the variation of the Casson invariant under these moves. In this paper, we give such a formula for a third type of move which relates all integral homology spheres, namely Borromean surgery.

3.1.2 Borromean surgery link and Kirby equivalence In addition to Johannes' formula, we will need the following lemma, which follows directly from several applications of Kirby moves (see [2; 4]).

Lemma 3.2 Let K be a Borromean surgery link. K is Kirby-equivalent to the 2–component link $\widetilde{K}$ represented in Figure 4.
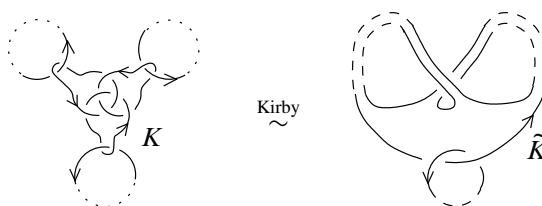

Figure 4: Two Kirby-equivalent links

Observe that the linking number of $\widetilde{K}$ is 1, and that one of its component has always framing 0. The determinant of the linking matrix of $\widetilde{K}$ is thus always -1.

3.2 Two lemmas in S^3

We can now establish the two above-mentioned preliminary lemmas. In both statements, we consider a local operation on a Borromean surgery link in S^3, and give a formula for the variation of the Casson invariant of the surgered manifolds.

---

[1]For example, consider the right-handed trefoil knot T in S^3. It can be easily verified by Theorem 2.1(2) that $\lambda(S^3_{T+1}) = 1$, but [5] would give -1.

3.2.1 Changing the framing   We first consider the operation of adding a kink to a leaf of a Borromean surgery link.

Lemma 3.3 *Let K be a Borromean surgery link in S^3, and let K_i be obtained by changing the framing of the i–th leaf of K by $+1$ as depicted below.*
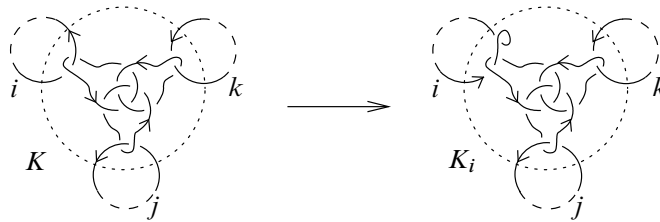
We have:
$$\lambda(S^3_K) = \lambda(S^3_{K_i}) - f_j . f_k + l_{jk}(1 + l_{jk}).$$

Proof   By Lemma 3.2, we have
$$\lambda(S^3_K) = \lambda(S^3_{\widetilde{K}}) \quad \text{and} \quad \lambda(S^3_{K_i}) = \lambda(S^3_{\widetilde{K}_i}),$$

where $\widetilde{K}$ and $\widetilde{K}_i$ are two 2–component links as depicted in Figure 5. Consider the
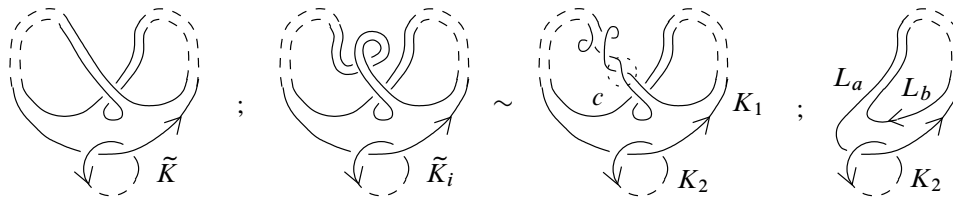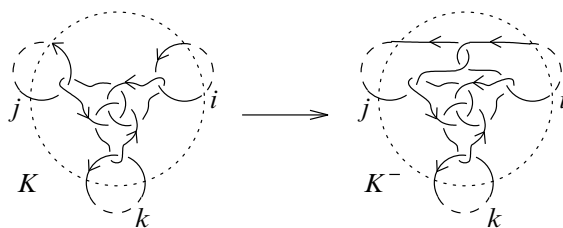

Figure 5

2–component link $K_1 \cup K_2$ isotopic to $\widetilde{K}_i$, and the positive crossing c in $K_1 \cup K_2$ as in Figure 5. Changing this crossing (without changing the framing) gives the link $\widetilde{K}$, and smoothing this crossing turns K_1 into a 2–component link $L_a \cup L_b$ satisfying $\mathrm{lk}(L_a, L_b) = -f_k$, $\mathrm{lk}(L_a, K_2) = 1 + l_{jk}$ and $\mathrm{lk}(L_b, K_2) = -l_{jk}$. The result then follows from Johannes' formula.   □

3.2.2 Changing the linking numbers   We now study the effect, at the level of the Casson invariant, of adding a clasp between two leaves of a Borromean surgery link.

Lemma 3.4 *Let K be a Borromean surgery link in S^3, and let K^- be obtained by adding a clasp between the leaves i and j of K as shown below.*

We have:
$$\lambda(S^3_K) = \lambda(S^3_{K^-}) - 2.l_{ik}.l_{jk} + 2.f_k.l_{ij}.$$

Proof By Lemma 3.2, we have

$$\lambda(S^3_K) = \lambda(S^3_{\widetilde{K}}) \quad \text{and} \quad \lambda(S^3_{K^-}) = \lambda(S^3_{\widetilde{K}^-}),$$

where $\widetilde{K}$ and $\widetilde{K}^-$ are two 2–component links as depicted in Figures 5 and 6 respectively. Now observe that $\widetilde{K}^-$ is isotopic to the link $K_1 \cup K_2$ of Figure 6, which only differs from $\widetilde{K}$ by four crossing changes within the component K_1. The result thus follows
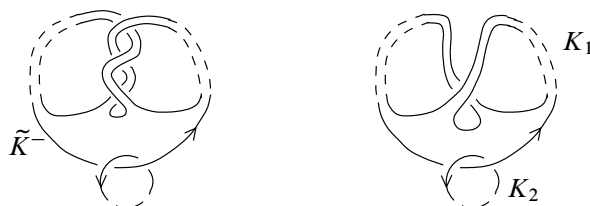

Figure 6

from 4 applications of Johannes' formula, similarly to the previous proof. □

4 Proof of Theorem 2.2

We can now prove Theorem 2.2. Our main tool here is Lescop's sum formula for the Casson invariant [7], which we briefly review in the next subsection.

4.1 Lescop's sum formula

Given a compact 3–manifold M with boundary, we denote by $\mathcal{L}_M$ the kernel of the map $H_1(\partial M; \mathbf{Q}) \to H_1(M; \mathbf{Q})$ induced by the inclusion. We call $\mathcal{L}_M$ the *Lagrangian* of M.

Let Σ be a closed oriented surface of genus g, and let A, B, A', B' be four rational homology handlebodies such that $\partial A = \partial A' = -\partial B = -\partial B' = \Sigma$. Assume that in

$H_1(\Sigma; \mathbf{Q})$ we have $\mathcal{L}_A = \mathcal{L}_{A'}$, $\mathcal{L}_B = \mathcal{L}_{B'}$, and $\mathcal{L}_A \cap \mathcal{L}_B = 0$. Then the main result of [7] states that $\lambda(A \cup B) - \lambda(A' \cup B) - \lambda(A \cup B') + \lambda(A' \cup B')$ is given by

$$-2 \sum_{\{i,j,k\} \subset \{1,\dots,g\}} \mathcal{I}_{AA'}(\alpha_i \wedge \alpha_j \wedge \alpha_k) \mathcal{I}_{BB'}(\beta_i \wedge \beta_j \wedge \beta_k),$$

where $\{\alpha_i\}_i$, resp. $\{\beta_i\}_i$, is a basis for $\mathcal{L}_A$, resp. $\mathcal{L}_B$, such that the intersection form $\cdot$ on Σ satisfies $\alpha_i \cdot \beta_j = \delta_{ij}$, and where $\mathcal{I}_{AA'}$ is the *intersection form on* $\Lambda^3 \mathcal{L}_A$ defined as follows. There is an isomorphism $H_2(A \cup_\Sigma -A'; \mathbf{Q}) \to \mathcal{L}_A$ coming from the Mayer–Vietoris sequence which, to the homology class of an oriented surface $S \subset A \cup_\Sigma -A'$, associates the class of the (oriented) boundary of $S \cap A$. Then $\mathcal{I}_{AA'}$ is the trilinear alternating form on $\mathcal{L}_A$ induced, *via* this isomorphism, by the triple intersection form on $H_2(A \cup_\Sigma -A'; \mathbf{Q})$.

Note that the above sum formula is for the Casson–Walker invariant, with the normalization specified in Section 2.1.

4.2 Proof of Theorem 2.2

Let L be a Borromean surgery link in a rational homology sphere M such that the linking numbers of the leaves are integers l_{ij}, and such that the leaves have self-linking numbers zero in $\mathbf{Q}/\mathbf{Z}$, and thus have integral framings f_i. We aim to compute $\lambda(M_L) - \lambda(M)$.

Consider the Borromean surgery link L_{st} of Figure 7 in the genus three handlebody A, standardly embedded in S^3. Denote by B the exterior of A in S^3, and set $A' = A_{L_{\text{st}}}$. The link L can be regarded as obtained by an embedding of A in M such that the
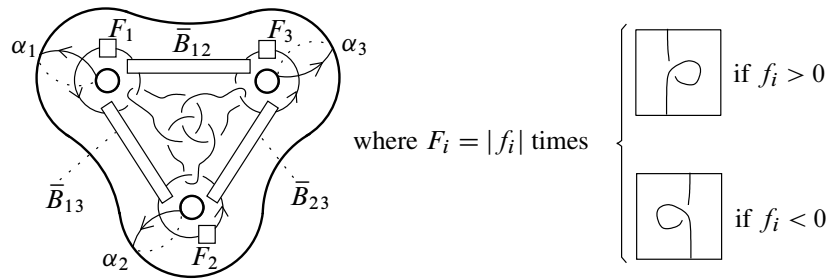

Figure 7: The link L_{st} in the handlebody A. Here $\bar{B}_{ij}$ is $|l_{ij}|$ times the $(2,2)$–tangle T_+ (resp. T_-) of Figure 2 if $l_{ij} > 0$ (resp. $l_{ij} < 0$).

Lagrangians of B and $B' = M \setminus \text{Int}(A)$ (ie the curves of ∂A that bound in B and in

B') are the same. The main theorem of [7] gives

$$\lambda(S^3) - \lambda(S^3_{L_{st}}) - \lambda(M) + \lambda(M_L) = -2\mathcal{I}_{AA'}(\alpha_1 \wedge \alpha_2 \wedge \alpha_3)\mathcal{I}_{BB'}(\beta_1 \wedge \beta_2 \wedge \beta_3),$$

where the α_i are meridians of the three leaves of L_{st} (see Figure 7), and where the β_i are longitudes of the three leaves of the link F_0 of Figure 2.

For $i = 1, 2, 3$, let Σ^i_B (resp. $\Sigma^i_{B'}$) be oriented surfaces in B (resp. B') bounded by β_i, such that all surfaces are transverse to each other and to ∂B in $B \cup_{\partial B} -B'$. Denote by Σ_i the closed surface $\Sigma^i_B \cup_{\beta_i} \Sigma^i_{B'}$ in $B \cup_{\partial B} -B'$. Using the notation of Section 2.4, we have

$$\mathcal{I}_{BB'}(\beta_1 \wedge \beta_2 \wedge \beta_3) = \langle \Sigma_1, \Sigma_2, \Sigma_3 \rangle_{B \cup_{\partial B} -B'} = \langle \Sigma^1_B, \Sigma^2_B, \Sigma^3_B \rangle_B - \langle \Sigma^1_{B'}, \Sigma^2_{B'}, \Sigma^3_{B'} \rangle_{B'},$$

where $\langle \Sigma^1_B, \Sigma^2_B, \Sigma^3_B \rangle_B$ is zero, and where $-\langle \Sigma^1_{B'}, \Sigma^2_{B'}, \Sigma^3_{B'} \rangle_{B'}$ identifies directly with $\mu_{123}(L; M)$. It also follows from the above and a direct computation that $\mathcal{I}_{AA'}(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) = 1$. We thus obtain

$$\lambda(M_L) - \lambda(M) = -2\mu_{123}(L) + \lambda(S^3_{L_{st}}).$$

Now, we can use the two lemmas of Section 3 to compute $\lambda(S^3_{L_{st}})$ as follows. By applying Lemma 3.3 $|f_1| + |f_2| + |f_3|$ times to the link L_{st}, we can reduce the framings of its three leaves to zero, and the resulting Borromean surgery link L' satisfies

$$\lambda(S^3_{L_{st}}) = \lambda(S^3_{L'}) - f_1.f_2.f_3 + \sum_{\circlearrowleft_{1,2,3}} f_1.l_{23}.(l_{23} + 1).$$

Next, by suitably applying Lemma 3.4 $|l_{23}| + |l_{13}| + |l_{12}|$ times to the link L', we obtain a Borromean surgery link L_f in S^3 whose leaves are three 0–framed pairwise unlinked unknots, and such that

$$\lambda(S^3_{L'}) = \lambda(S^3_{L_f}) - 2.l_{12}.l_{13}.l_{23}.$$

Using Kirby calculus, one can easily check that $S^3_{L_f} \cong S^3$ (see for example [2, Section 2.1]). We thus have $\lambda(S^3_{L_f}) = 0$. It follows that

$$\lambda(S^3_{L_{st}}) = -f_1.f_2.f_3 - 2.l_{12}.l_{13}.l_{23} + \sum_{\circlearrowleft_{1,2,3}} l_{23}.(l_{23} + 1).f_1,$$

which concludes the proof.

5   Proof of Theorem 2.3

In order to prove Theorem 2.3, we will use the theory of finite type invariants.

5.1 Finite type invariants of integral homology spheres

The notion of finite type invariants of integral homology spheres is due to Ohtsuki, and is defined using algebraically split, unit-framed links [11]. For the purpose of the present paper, we use an alternative definition using the notion of Borromean surgery, due to Garoufalidis, Goussarov and Polyak [2], Goussarov [3] and Habiro [4].

Let $\mathcal{S}$ denote the free $\mathbf{Z}$–module generated by elements of $\mathbf{Z}HS$. For $k \geq 0$, let $\mathcal{S}_k$ denote the $\mathbf{Z}$–submodule of $\mathcal{S}$ generated by elements

$$[M;G] := \sum_{G' \subseteq G} (-1)^{|G'|} M_{G'},$$

where M is an integral homology sphere, and where $G = G_1 \cup ... \cup G_k$ is a disjoint union of k Borromean surgery links in M. The sum runs over all the subsets G' of G (regarded as the set of the links G_i) and $|G'|$ denotes the cardinality of G'. Observe that we have the equality

$$(5\text{--}1) \qquad\qquad M_G = \sum_{G' \subseteq G} (-1)^{|G'|}[M;G'].$$

A *finite type invariant of degree k* is a map $f : \mathbf{Z}HS \longrightarrow A$, where A is an abelian group, whose natural extension to $\mathcal{S}$ vanishes on $\mathcal{S}_{k+1}$.

Proposition 5.1 [11; 2] *The Casson invariant is a degree 2 finite type invariant.*

So we can now use the theory of finite type invariants to study the behavior of the Casson invariant under Borromean surgery.

5.2 Proof of Theorem 2.3

Let $L = L_1 \cup ... \cup L_n$ be a disjoint union of Borromean surgery links in an integral homology sphere M. By using (5–1) and Proposition 5.1, one can check that

$$\lambda(M_L) - \lambda(M) = \sum_{i=1}^{n} \big(\lambda(M_{L_i}) - \lambda(M)\big) + \sum_{1 \leq i < j \leq n} \lambda([M; Y_i \cup Y_j]).$$

So we are left with the computation of $\lambda([M; Y_i \cup Y_j])$ for all $i < j$. By Theorem 2.2, we have

$$\lambda([M; Y_i \cup Y_j]) = \big(\lambda((M_{Y_j})_{Y_i}) - \lambda(M_{Y_j})\big) - \big(\lambda(M_{Y_i}) - \lambda(M)\big)$$
$$= -2\mu_{123}(Y_i; M_{Y_j}) + 2\mu_{123}(Y_i; M),$$

where we also use the fact that a Borromean surgery preserves the framings and linking numbers [9]. So, if we denote by Σ_r a Seifert surface for F_r^i ($r = 1, 2, 3$), we see that $\lambda([M; Y_i \cup Y_j])$ counts, with signs, the triple intersection points between Σ_1, Σ_2 and Σ_3 created when doing surgery along Y_j.

For each leaf F_s^j of Y_j ($s = 1, 2, 3$), denote by B_s the non-leaf component of Y_j having linking number 1 with F_s^j, oriented as in Section 2.2. Using Kirby moves, one sees that each positive (resp. negative) intersection point between Σ_r and F_s^j ($r, s = 1, 2, 3$) contributes, *via* surgery, to a band sum of F_r^i with a parallel copy of B_s with opposite (resp. same) orientation. So surgery along Y_j results in band summing copies of B_1, B_2 and B_3 with the leaves of Y_i, with the number and orientation of these copies being completely determined by the linking numbers of the leaves of Y_i and Y_j. Each nontrivial contribution to μ_{123} comes from a triple of band sums of pairwise distinct leaves of Y_i with copies of pairwise distinct B_s's. It thus follows from an easy counting argument that

$$\lambda([M; Y_i \cup Y_j]) = -2 \sum_{\sigma \in S_3} s(\sigma) l_{1\sigma(1)}^{ij} l_{2\sigma(2)}^{ij} l_{3\sigma(3)}^{ij},$$

which concludes the proof.

Remark 5.2 The theory of finite type invariants, that we used here to prove Theorem 2.3, could actually also be used to give an alternative (and quite simple) proof of Theorem 2.2. Such a proof, however, is only valid for integral homology spheres and is therefore not included in this paper.

6 A remark and an application

6.1 Comparison to Lescop's surgery formula

One can also use Lescop's global surgery formula to express the variation of the Casson invariant under a Borromean surgery, in terms of the multivariable Alexander polynomial Δ [6].

Consider the Borromean surgery link L as in Figure 1 and denote by B the standard Borromean link. Then [6, 1.4.8] gives that $\lambda(M_L) - \lambda(M)$ equals

$$-\begin{vmatrix} f_1 & l_{12} & l_{13} \\ l_{12} & f_2 & l_{23} \\ l_{13} & l_{23} & f_3 \end{vmatrix}.\zeta(B) - \sum_{\mathfrak{S}_{1,2,3}} \begin{vmatrix} f_1 & l_{12} \\ l_{12} & f_2 \end{vmatrix}.\zeta(B \cup F_3) - \sum_{\mathfrak{S}_{1,2,3}} f_1.\zeta(B \cup F_2 \cup F_3) - \zeta(L),$$

where, for an n–component link K, $\zeta(K) = \partial^n/(\partial t_1...\partial t_n)\Delta_K(1,...,1)$. For the Borromean link B we know that $\zeta(B) = 1$, and one can easily compute the values of ζ in the two sums of the above formula: $\zeta(B \cup F_i) = 0$ and $\zeta(B \cup F_i \cup F_j) = -l_{ij}$ for all $1 \le i \ne j \le 3$. It follows in particular that

$$\frac{\partial^6}{\partial t_1...\partial t_6}\Delta_L(1,...,1) = 2.\mu_{123}(L).$$

6.2 An application

As a conclusion, we give an application of our main theorem. Namely we show how Theorem 2.2 provides an alternative proof of a realization theorem for the Casson invariant of homology spheres of Mazur type.

Recall that a *homology sphere of Mazur type* is obtained by surgery on S^3 along a 2–component link $K_1 \cup K_2$ such that $\mathrm{lk}(K_1, K_2) = \pm1$, K_1 is a trivial knot with framing 0 and K_2 has framing $r \in \mathbf{Z}$. Such an integral homology sphere bounds a contractible 4–manifold, and thus its Casson invariant is an even number. Mizuma showed the following.

Theorem 6.1 [10] *For any even integer k, there is a homology sphere of Mazur type M such that $\lambda(M) = k$.*

Mizuma describes this homology sphere in terms of double branched cover of S^3 along so-called knots of 1–*fusion*. Theorem 2.2 allows us to give explicit examples in terms of Borromean surgery links. Indeed, consider the link $L_{\pm n}$ depicted in the left part of Figure 8. There, the three leaves have framing zero. Clearly, by Theorem 2.2, surgery on S^3 along $L_{\pm n}$ produces an integral homology sphere with Casson invariant $\mp2n$. On the other hand, by Lemma 3.2, the link $L_{\pm n}$ is Kirby-equivalent to a 2–component link $K_1 \cup K_2$ as depicted on the right-hand side of the figure, which satisfies the above-mentioned conditions. Therefore $S^3_{L_{\pm n}} \cong S^3_{K_1 \cup K_2}$ is a homology sphere of Mazur type.

References

[1] **E Auclair**, **C Lescop**, *Clover calculus for homology 3–spheres via basic algebraic topology*, Algebr. Geom. Topol. 5 (2005) 71–106   MR2135546

[2] **S Garoufalidis**, **M Goussarov**, **M Polyak**, *Calculus of clovers and finite type invariants of 3–manifolds*, Geom. Topol. 5 (2001) 75–108   MR1812435
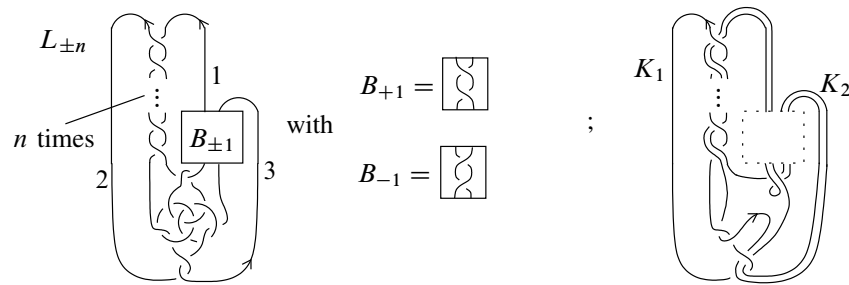
Figure 8

[3]   **M Goussarov**, *Finite type invariants and n–equivalence of 3–manifolds*, C. R. Acad. Sci. Paris Sér. I Math. 329 (1999) 517–522   MR1715131

[4]   **K Habiro**, *Claspers and finite type invariants of links*, Geom. Topol. 4 (2000) 1–83 MR1735632

[5]   **J Johannes**, *A type 2 polynomial invariant of links derived from the Casson–Walker invariant*, J. Knot Theory Ramifications 8 (1999) 491–504   MR1697385

[6]   **C Lescop**, *Global surgery formula for the Casson–Walker invariant*, Annals of Math. Studies 140, Princeton University Press (1996)   MR1372947

[7]   **C Lescop**, *A sum formula for the Casson–Walker invariant*, Invent. Math. 133 (1998) 613–681   MR1645066

[8]   **G Massuyeau**, *Spin Borromean surgeries*, Trans. Amer. Math. Soc. 355 (2003) 3991–4017   MR1990572

[9]   **S V Matveev**, *Generalized surgeries of three-dimensional manifolds and representations of homology spheres*, Mat. Zametki 42 (1987) 268–278, 345   MR915115

[10]   **Y Mizuma**, *On the Casson invariant of Mazur's homology spheres*, preprint

[11]   **T Ohtsuki**, *Finite type invariants of integral homology 3–spheres*, J. Knot Theory Ramifications 5 (1996) 101–115   MR1373813

[12]   **K Walker**, *An extension of Casson's invariant*, Annals of Math. Studies 126, Princeton University Press (1992)   MR1154798

CTQM - Department of Mathematical Sciences, University of Aarhus
Ny Munkegade, bldg 1530, 8000 Aarhus C, Denmark

meilhan@imf.au.dk