braic coefficients, and since $g(z)$ is not a polynomial by hypothesis, at least one number $\beta, \cdots, \sigma$ and the corresponding polynomial $A, B, \cdots, S$ are not zero. The Lindemann theorem applied to the equation $g(z)=w$ for algebraic $z\neq 0$ and w shows now that all of the coefficients $B, C, \cdots, S$ must vanish at this value of z. Since each of these polynomials has degree at most $m-1$, that situation can arise for at most $m-1$ points. Taking account of the case $z=0$, which is always exceptional, we obtain Theorem II.

Discussion. The fact that certain polynomial coefficients in (1) may be identically zero shows that *the m exceptional cases can actually occur.* If the coefficients ultimately form a periodic sequence, however, then the difference relation becomes simply $a_{n+p}=a_n$, so that (2) is $z^p=1$. Thus the exponents $\beta, \cdots, \sigma$ in (1) are the pth roots of unity, the polynomials $B, \cdots, S$ are constants, and $z=0$ is the only exceptional value. Alternatively, it is easy to compute $f(z)$ in this case, and the denominator is found to have no multiple roots; so $m=1$. Hence *Theorem II contains Theorem I.*

Again, the Lindemann theorem follows from Theorem II when we choose $f(z)=a/(1-\alpha z)+b/(1-\beta z)+\cdots+s/(1-\sigma z)$. Thus *Theorem II amounts to a rewording of the Lindemann theorem,* and an independent proof of the former would give the latter.

The theorem was suggested by [1], the proof by [3]–[5] and by conversation with Ernst Straus.

References

1. Dietrich and Rosenthal, Transcendence of Factorial Series with Algebraic Coefficients, Bull. Am. Math. Soc., v. 55, 1949, p. 55.

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3. Pölya and Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. II, Dover Publications, N. Y., Ch. VII, Prob. 17.

4. *Ibid.*, Prob. 30.

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ON A CLASS OF IRRATIONAL NUMBERS

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The classical method of proving that e is an irrational number is well-known. The following theorem generalizes the approach in the classical method.

Theorem

1. *Let a_n, $n=1, 2, \cdots$ be positive or negative integers or zero, provided only that there be an infinite number of a_n not equal to zero.*

2. *Let a number S be defined by the series*

$$S=\sum_{n=1}^{\infty}\frac{a_n}{r^n(n!)^b}$$

where r and b are any positive integers.

3. *Suppose that a positive constant C and a positive constant $\alpha < b$ exist, independent of n, such that*

$$|a_n| < Cn^{\alpha}$$

for all n greater than some number N. Then the number S is irrational.

Proof. Suppose the number S represented by the series is rational and equal to p/q where p and q are taken to be relatively prime integers. Thus

$$\frac{p}{q} = \frac{a_1}{r(1!)^b} + \frac{a_2}{r^2(2!)^b} + \cdots + \frac{a_{Mq}}{r^{Mq}[(Mq)!]^b} + \cdots.$$

Here M is an integer which we shall assume is larger than the number N described in 3.

Multiplying both sides by $r^{Mq}[(Mq)!]^b$ we obtain

$$F = G + H,$$

where

$$F = \frac{p}{q} r^{Mq}[(Mq)!]^b$$

$$G = \left\{\frac{a_1}{r(1!)^b} + \cdots + \frac{a_{Mq}}{r^{Mq}[(Mq)!]^b}\right\} r^{Mq}[(Mq)!]^b$$

$$H = \frac{a_{Mq+1}}{r(Mq+1)^b} + \frac{a_{Mq+2}}{r^2(Mq+2)^b(Mq+1)^b} + \cdots.$$

Quantities F and G are integers. However

$$\begin{aligned} |H| &\leqq \frac{C}{r}\frac{(Mq+1)^{\alpha}}{(Mq+1)^b} + \frac{C}{r^2}\frac{(Mq+2)^{\alpha}}{(Mq+1)^b(Mq+2)^b} + \cdots \\ &< \frac{C}{r}\left[\frac{1}{(Mq+1)^{b-\alpha}} + \frac{1}{r(Mq+1)^{2b-\alpha}} + \cdots\right] \\ &\leqq \frac{C}{r}\frac{(Mq+1)^{\alpha}}{(Mq+1)^b - 1}, \end{aligned}$$

which can be made less than one by choosing M large enough. Hence our supposition that the series represents a rational number is contradicted and so the theorem is proved.

Remark. The above theorem may be used to prove that sin 1, cos 1, $J_0(1)$ among many others, as well as e, are irrational numbers.