



On a Class of Irrational Numbers

Author(s): M. R. Spiegel

Source: *The American Mathematical Monthly*, Vol. 60, No. 1 (Jan., 1953), pp. 27-28

Published by: [Mathematical Association of America](#)

Stable URL: <http://www.jstor.org/stable/2306475>

Accessed: 07/10/2010 04:16

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=maa>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



Mathematical Association of America is collaborating with JSTOR to digitize, preserve and extend access to *The American Mathematical Monthly*.

<http://www.jstor.org>

braic coefficients, and since $g(z)$ is not a polynomial by hypothesis, at least one number β, \dots, σ and the corresponding polynomial A, B, \dots, S are not zero. The Lindemann theorem applied to the equation $g(z) = w$ for algebraic $z \neq 0$ and w shows now that all of the coefficients B, C, \dots, S must vanish at this value of z . Since each of these polynomials has degree at most $m-1$, that situation can arise for at most $m-1$ points. Taking account of the case $z=0$, which is always exceptional, we obtain Theorem II.

Discussion. The fact that certain polynomial coefficients in (1) may be identically zero shows that *the m exceptional cases can actually occur*. If the coefficients ultimately form a periodic sequence, however, then the difference relation becomes simply $a_{n+p} = a_n$, so that (2) is $z^p = 1$. Thus the exponents β, \dots, σ in (1) are the p th roots of unity, the polynomials B, \dots, S are constants, and $z=0$ is the only exceptional value. Alternatively, it is easy to compute $f(z)$ in this case, and the denominator is found to have no multiple roots; so $m=1$. Hence *Theorem II contains Theorem I*.

Again, the Lindemann theorem follows from Theorem II when we choose $f(z) = a/(1-\alpha z) + b/(1-\beta z) + \dots + s/(1-\sigma z)$. Thus *Theorem II amounts to a rewording of the Lindemann theorem*, and an independent proof of the former would give the latter.

The theorem was suggested by [1], the proof by [3]-[5] and by conversation with Ernst Straus.

References

1. Dietrich and Rosenthal, Transcendence of Factorial Series with Algebraic Coefficients, Bull. Am. Math. Soc., v. 55, 1949, p. 55.
2. LeVeque, W. J., Note on the Transcendence of Certain Series, Proc. Am. Math. Soc., v. 2, 1951, No. 3, p. 401.
3. Pölya and Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. II, Dover Publications, N. Y., Ch. VII, Prob. 17.
4. *Ibid.*, Prob. 30.
5. *Ibid.*, Chap. VIII, Prob. 151.

ON A CLASS OF IRRATIONAL NUMBERS

M. R. SPIEGEL, Rensselaer Polytechnic Institute

The classical method of proving that e is an irrational number is well-known. The following theorem generalizes the approach in the classical method.

THEOREM

1. Let $a_n, n=1, 2, \dots$ be positive or negative integers or zero, provided only that there be an infinite number of a_n not equal to zero.
2. Let a number S be defined by the series

$$S = \sum_{n=1}^{\infty} \frac{a_n}{r^n (n!)^b}$$

where r and b are any positive integers.

3. Suppose that a positive constant C and a positive constant $\alpha < b$ exist, independent of n , such that

$$|a_n| < Cn^\alpha$$

for all n greater than some number N . Then the number S is irrational.

Proof. Suppose the number S represented by the series is rational and equal to p/q where p and q are taken to be relatively prime integers. Thus

$$\frac{p}{q} = \frac{a_1}{r(1!)^b} + \frac{a_2}{r^2(2!)^b} + \cdots + \frac{a_{Mq}}{r^{Mq}[(Mq)!]^b} + \cdots$$

Here M is an integer which we shall assume is larger than the number N described in 3.

Multiplying both sides by $r^{Mq}[(Mq)!]^b$ we obtain

$$F = G + H,$$

where

$$\begin{aligned} F &= \frac{p}{q} r^{Mq} [(Mq)!]^b \\ G &= \left\{ \frac{a_1}{r(1!)^b} + \cdots + \frac{a_{Mq}}{r^{Mq} [(Mq)!]^b} \right\} r^{Mq} [(Mq)!]^b \\ H &= \frac{a_{Mq+1}}{r^{Mq+1} (Mq+1)^b} + \frac{a_{Mq+2}}{r^{Mq+2} (Mq+2)^b (Mq+1)^b} + \cdots \end{aligned}$$

Quantities F and G are integers. However

$$\begin{aligned} |H| &\leq \frac{C}{r} \frac{(Mq+1)^\alpha}{(Mq+1)^b} + \frac{C}{r^2} \frac{(Mq+2)^\alpha}{(Mq+1)^b (Mq+2)^b} + \cdots \\ &< \frac{C}{r} \left[\frac{1}{(Mq+1)^{b-\alpha}} + \frac{1}{r(Mq+1)^{2b-\alpha}} + \cdots \right] \\ &\leq \frac{C}{r} \frac{(Mq+1)^\alpha}{(Mq+1)^b - 1}, \end{aligned}$$

which can be made less than one by choosing M large enough. Hence our supposition that the series represents a rational number is contradicted and so the theorem is proved.

Remark. The above theorem may be used to prove that $\sin 1$, $\cos 1$, $J_0(1)$ among many others, as well as e , are irrational numbers.