The Irrationality of e^x For Nonzero Rational x

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The standard method to prove the irrationality of e^x for nonzero rational x is by obtaining a rational approximation using the differential and integral properties of e^x and the differential properties of $x^n(1-x)^n/n!$ [1]. Here we give a simple proof using the theory of continued fractions. We write a continued fraction as

$$\cfrac{b_1}{a_1+\cfrac{b_2}{a_2+\dots+\cfrac{b_n}{a_n+\dots}}} = \left[\frac{b_1}{a_1},\frac{b_2}{a_2},\dots,\frac{b_n}{a_n},\dots\right] = \left[\frac{b_i}{a_i}\right]_{i=1}^{\infty};\ a_i \neq 0. \quad (1)$$

The following theorem is in the classical text [2]; we present it with a new, more concise proof.

Theorem. *The value of the continued fraction* (1) *is irrational if there exists a positive integer n such that* (*a*) a_i, b_i *are rational and* $a_i \neq 0$ *for* $i = 1,\dots,n$, *and* (*b*) a_i, b_i *are positive integers with* $a_i > b_i$ *for all* $i \geq n+1$.

Proof: Since

$$\left[\frac{b_i}{a_i}\right]_{i=1}^{\infty} = \left[\frac{b_1}{a_1},\frac{b_2}{a_2},\dots,\frac{b_n}{a_n},\alpha\right], \quad \text{where } \alpha = \left[\frac{b_i}{a_i}\right]_{i=n+1}^{\infty},$$

it is sufficient to show that α is irrational. If α is rational, define

$$f_k = \left[\frac{b_{n+i}}{a_{n+i}}\right]_{i=k+1}^{\infty} \quad \text{so } f_k = \frac{b_{n+k+1}}{a_{n+k+1}+f_{k+1}} \quad \text{for } k = 0,1,\dots \quad (2)$$

The hypotheses ensure that $0 < f_k < 1$, for $k = 0,1,\dots$ Since $f_0 = \alpha$, and the rationality of f_k implies f_{k+1} is also rational, we write $f_k = p_k/q_k$, where p_k, q_k are relatively prime positive integers with $q_k > p_k$. From (2) we get

$$f_{k+1} = \frac{p_{k+1}}{q_{k+1}} = \frac{q_k b_{n+k+1} - p_k a_{n+k+1}}{p_k},$$

so $p_{k+1} < q_{k+1} \leq p_k$, i.e., $p_0, p_1, p_2, \dots$ is a strictly decreasing infinite sequence of positive integers. This is not possible and hence α is irrational. ∎

A continued fraction expansion for e^x can be obtained from an expansion for $\tan x$; we follow Schlömilch [3]. Define $y_0 = \cos\sqrt{x}$ and let y_n denote the n^{th} derivative of y_0. Then $2\sqrt{x}y_1 = -\sin\sqrt{x}$ and

$$\frac{y_1}{\sqrt{x}} + 2\sqrt{x}y_2 = -\frac{y_0}{2\sqrt{x}}, \qquad \text{so } 4xy_2 + 2y_1 + y_0 = 0. \quad (3)$$

Differentiating (3) n times gives

$$4xy_{n+2} + (4n+2)y_{n+1} + y_n = 0. \quad (4)$$

Denote y_{m+1}/y_m by $u^{(m+1)}$. Then $u^{(1)} = -\tan\sqrt{x}/(2\sqrt{x})$, and dividing (4) by y_{n+1} yields $u^{(n+1)} = -1/[(4n+2) + 4xu^{(n+2)}]$ for $n = 0, 1, \ldots$ Replacing $\sqrt{x}$ by x in this recursion relation gives the continued fraction representation

$$\tan x = \left[\frac{2x}{2}, \frac{-4x^2}{6}, \frac{-4x^2}{10}, \ldots, \frac{-4x^2}{4n+2}, \ldots\right], \tag{5}$$

which converges for all finite complex x except at the poles of the function $\tan x$ [3]. The identity

$$e^x = \cfrac{1}{1 - \cfrac{2}{1 - \cfrac{1}{i\tan\left(\cfrac{ix}{2}\right)}}} \tag{6}$$

is valid for all finite complex x. Replace x by $ix/2$ in (5) and substitute into (6) to get Euler's continued fraction expansion

$$e^x = \left[\frac{1}{1}, \frac{-2x}{2+x}, \frac{x^2}{6}, \frac{x^2}{10}, \ldots, \frac{x^2}{4n+2}, \ldots\right], \tag{7}$$

which converges for all finite complex x.

Let p and q be integers, $q > 0$. The value of a continued fraction (1) does not change if any triple (b_i, a_i, b_{i+1}) is multiplied by a nonzero constant, so in (7) we multiply (b_i, a_i, b_{i+1}) by q for $i = 2, 3, \ldots$ to get

$$e^x = \left[\frac{1}{1}, \frac{-2xq}{(2+x)q}, \frac{q^2x^2}{6q}, \frac{q^2x^2}{10q}, \ldots, \frac{q^2x^2}{(4n+2)q}, \ldots\right].$$

The choice $x = p/q$ gives

$$e^{p/q} = \left[\frac{1}{1}, \frac{-2p}{p+2q}, \frac{p^2}{6q}, \frac{p^2}{10q}, \ldots, \frac{p^2}{(4n+2)q}, \ldots\right],$$

which satisfies the hypotheses of the theorem for all $n > (p^2 - 2q)/4q$ except for $p = 0$, where its value is 1. Thus, *e^x is irrational for all rational $x \neq 0$.*

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