where it is assumed that u is continuous on $D+B$ and that u_x and $L(u)$ exist on D; *cf.* Gevrey [3], p. 363 and p. 369.

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CRITERIA FOR IRRATIONALITY OF CERTAIN CLASSES OF NUMBERS. II

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Introduction. In a paper with the same title published recently in this MONTHLY,* criteria were obtained for the irrationality of the number x given by the series (1) wherein the a_i are integers and the b_i are positive integers. In particular were considered the series (1) where a_i, b_i satisfy (2) and (3) below. These criteria depended on the limits of the sequence $c_i=a_i/b_i$. But the criteria there found did not cover the case when each of the limits of the sequence c_i is rational and not 0 and not 1. Theorem A below deals with this case.

The series (1) may be called Cantor's series since Cantor [1] showed that for given integers $b_i \geqq 2$, a real number x can be expanded uniquely (if irrational) in the form (1) with the a_i satisfying (3). (If x is rational two expansions are possible). Cantor showed that if the b_i satisfy the condition that the product $b_1b_2 \cdots b_i$ is divisible by an arbitrary integer q for some sufficiently large i (and

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therefore for all greater i), then x is irrational unless $a_i=0$ (all large i) or $a_i=b_i-1$ (all large i) in which cases x is rational. He also showed that when the b_i are ultimately periodic, x is rational if and only if the a_i are ultimately periodic.

In Theorem B below we give a criterion which includes both these criteria of Cantor.

1. We confine ourselves for simplicity to the series (1) subject to the conditions (2) and (3).

$$x = a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \cdots, \tag{1}$$

$$b_i \geqq 2, \qquad (i = 1, 2, 3, \cdots), \tag{2}$$

$$0 \leqq a_i \leqq b_i - 1, \qquad (i = 1, 2, \cdots). \tag{3}$$

Criteria for irrationality were obtained which depended on the limits of the sequence

$$c_i = a_i/b_i$$

when $i \to \infty$. It was proved that x is irrational in each of the following cases: for some subsequence (i_n)

$$\begin{array}{ll} c_{i_n} \to \xi & \text{(irrational)}, \\ c_{i_n} \to 1 & \text{provided } a_i < b_i - 1 \text{ infinitely often}, \\ c_{i_n} \to 0 & \text{provided } b_{i_n} \to \infty \text{ and } a_i > 0 \text{ infinitely often.} \end{array}$$

The theorem which follows covers the remaining case when *every* limit of the sequence (c_i) is a rational number h/k where $0<h<k$, $(h, k)=1$.

THEOREM A. *If every limit of* $c_i=a_i/b_i$ *is a rational number* h/k *where* $0<h<k$, $(h, k)=1$, *then* x *is irrational except possibly when*

$$a_i = [hb_i/k]$$

for all large i *in the subsequence for which* $c_i \to h/k$.

In the excepted case x *may be rational or irrational.*

Proof. By the hypothesis there exists a subsequence (i_n) such that $c_i \to h/k$ and either

$$\frac{a_i}{b_i} > \frac{h}{k} \qquad \text{infinitely often,}$$

or

$$\frac{a_i+1}{b_i} \leqq \frac{h}{k} \qquad \text{infinitely often}$$

in this subsequence. Hence $b_{i_n} \to \infty$.

Recalling that, by Lemma 2,

$$\frac{a_i}{b_i} \leqq x_i < \frac{a_i + 1}{b_i},$$

(for $a_i < b_i - 1$ infinitely often) we see that, for this subsequence,

$$x_i \to h/k, \qquad x_i \neq h/k.$$

If x is a rational number with denominator q, then each x_i is a rational number with denominator q. But a rational number with bounded denominator cannot tend to a rational h/k except by equality. The contradiction shows that x is irrational.

The excepted case is illustrated by two examples.

(i) $$a_i = i, \ b_i = 2i + 1 \ (i = 1, 2, \cdots): a_i/b_i \to \tfrac{1}{2}.$$

Here $x = \frac{1}{2}$; $a_i = [\frac{1}{2}b_i]$.

(ii) By Theorem 4 the number

$$y = \sum_{i=1}^{\infty} 1/b_1 \cdots b_i, \qquad (b_i = 3i + 2),$$

is irrational. Take $a_i = i$. Then

$$3x = \sum_{1}^{\infty} \frac{3i}{5 \cdots (3i + 2)} = 1 - y$$

is irrational so that x is irrational. Here plainly

$$c_i = a_i/b_i \to 1/3, \qquad a_i = [b_i/3].$$

2. Condensation. From the series (1), a series of precisely the same form may be obtained by grouping terms together thus:

$$\frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \cdots + \frac{a_{i_1}}{b_1 \cdots b_{i_1}} = \frac{A_1}{B_1}$$

where $B_1 = b_1 b_2 \cdots b_{i_1}$, $0 \leqq A_1 \leqq B_1 - 1$, these inequalities following at once from (3). By this process of condensation as it may be called we are led to

$$x = X = A_0 + \frac{A_1}{B_1} + \frac{A_2}{B_1 B_2} + \frac{A_3}{B_1 B_2 B_3} + \cdots$$

where

$$A_0 = a_0, \ B_1 = b_1 \cdots b_{i_1}, \ B_2 = b_{i_1+1} \cdots b_{i_1+i_2}, \cdots, \ B_i \geqq 2, \ 0 \leqq A_i \leqq B_i - 1,$$

a series of precisely the same form as (1).

THEOREM B. *A necessary and sufficient condition that* x *given by* (1), *subject to* (2) *and* (3), *shall be rational is this: coprime integers* h, k, $0 \leqq h \leqq k$, *an integer* N *and a condensation shall exist such that*

$$A_i = \frac{h}{k}(B_i - 1)$$

for all $i \geqq N$.

The sufficiency of the condition is obvious. That it is necessary may be seen as follows. If x is rational with denominator $q \geqq 1$, every x_i is rational with denominator $\leqq q$. Hence there exists a subsequence (j_n) such that

$$x_{j_n} = h/k, \qquad (0 \leqq h \leqq k,\ k \geqq 1,\ (h, k) = 1),$$

since $0 \leqq x_i \leqq 1$.

Use the condensation defined by

$$i_1 = j_1 - 1,\ i_1 + i_2 - 1 = j_2,\ i_1 + i_2 + i_3 - 1 = j_3, \cdots .$$

For this condensation

$$x_{j_1} = X_2, \qquad x_{j_2} = X_3, \cdots ,$$
$$B_r X_r = A_r + X_{r+1},$$

so that

$$A_r = X_r(B_r - 1) = \frac{h}{k}(B_r - 1), \qquad (r = 2, 3, \cdots),$$

which plainly implies that $k \mid B_r - 1$. Thus the condition is necessary.

The criterion given in Theorem B is interesting since it includes the two criteria given by Cantor: (i) when the b_i are such that, for every integer q, $q \mid b_1 b_2 \cdots b_n$ for all large n, (ii) when the b_i are periodic.

In the first case the criterion is this: x is rational if and only if $a_i = 0$ $(i > i_0)$ or if $a_i = b_i - 1$ (all $i > i_0$).

In the second case the criterion is this: x is rational if and only if the a_i are ultimately periodic.

It is unnecessary to give the details of the deduction of these criteria from Theorem B.

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