NOTES

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A Short Proof of $\zeta(2) = \pi^2/6$

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Since Euler's first proof (see, e.g., [2]), the famous sum $\zeta(2) = \sum 1/n^2 = \pi^2/6$ has been proved in many ways, some elementary but difficult or long, and others shorter, but using advanced methods such as Fourier series or complex analysis. One such proof is first to show that

$$\epsilon_2(w) = \sum_{n=-\infty}^{\infty} \frac{1}{(w-n)^2} = \frac{\pi^2}{\sin^2(\pi w)}, \tag{1}$$

and then set $w = 1/2$ to get

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}. \tag{2}$$

Since

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \zeta(2) - \frac{1}{4}\zeta(2) = \frac{3}{4}\zeta(2),$$

$\zeta(2) = \pi^2/6$ follows.

Ahlfors [1, pp. 188–189] proves (1) by showing that the poles of the right and left sides of (1) have the same principal parts, and hence that their difference is holomorphic; he then shows that this difference is also bounded and zero in the limit as $\Im(w) \to \pm\infty$, hence identically zero by Liouville's theorem.

Here we derive (1) by a slightly different argument, which avoids any analysis of principal parts, but instead uses the theorem that a function which is meromorphic on the extended complex plane must be rational [1, p. 140, Ex. 4]. For $z \in D := \mathbb{C} \setminus \{0, 1\}$, let

$$R(z) = \sum \frac{1}{\log^2(z)}. \tag{3}$$

Here the sum is taken over all branches of the logarithm. Each point in D has a neighborhood on which the branches of $\log(z)$ are analytic, and the series (3) converges uniformly away from $z = 1$; hence $R(z)$ is analytic on D. (Note that $R(z) = -\epsilon_2(\log(z)/(2\pi i))/(4\pi^2)$, so that we can also infer this from the properties of ϵ_2. However, we prefer to treat $R(z)$ as primitive.)

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We now observe:

(i) Since the convergence is uniform near $z = 0$, and each term of (3) approaches 0 as z approaches 0, the singularity at $z = 0$ is removable, and we can set $R(0) = 0$.

(ii) The only other singularity of R is a double pole at $z = 1$, contributed by the principal branch of $\log(z)$ in (3); further $\lim_{z\to 1}(z-1)^2 R(z) = 1$.

(iii) $R(1/z) = R(z)$.

(iv) By (i) and (iii), R is analytic at ∞, and hence is rational. Moreover $R(\infty) = 0$.

By (ii), the denominator of $R(z)$ is $(z-1)^2$, and since $R(0) = R(\infty) = 0$, the numerator must be of the form az; (ii) then gives $a = 1$, so that

$$R(z) = \frac{z}{(z-1)^2}. \tag{4}$$

Setting $z = e^{2\pi i w}$ gives (1) and hence (2). We may also derive (2) directly from (4) by setting $z = -1$.

REFERENCES

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2. R. E. Bradley, L. A. D'Antonio, and C. E. Sandifer, eds., *Euler at 300: An Appreciation*, Mathematical Association of America, Washington, DC, 2007.

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The Group of Symmetries of the Tower of Hanoi Graph

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The classical Tower of Hanoi puzzle, invented by the French mathematician Édouard Lucas in 1883, consists of 3 wooden pegs and n disks with pairwise different diameters. The n disks are initially stacked on a single peg in order of decreasing size, from the largest at the bottom to the smallest at the top (see Figure 1). The goal is to move the tower of disks to another peg, moving one topmost disk at a time while never stack-

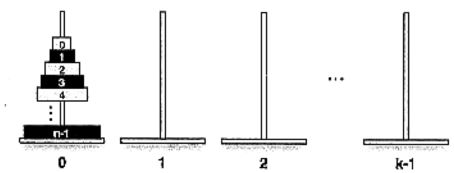

Figure 1. Convention for labeling k pegs and n disks in the Tower of Hanoi puzzle.

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