combination of $A_1, A_2, \ldots, A_n$.

Combining these results we obtain:

A determinant vanishes if and only if the column vectors (or the row vectors) are linearly dependent.

Another way of expressing this result is:

The set of n linear homogeneous equations

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0 \qquad (i = 1,2,\ldots,n)$$

in n unknowns has a non-trivial solution if and only if the determinant of the coefficients is zero.

Another result that can be deduced is:

If $A_1, A_2, \ldots, A_n$ are given, then their linear combinations can represent any other vector B if and only if $D(A_1, A_2, \ldots, A_n) \neq 0$.

Or:

The set of linear equations

$$(19) \qquad a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \qquad (i = 1,2,\ldots,n)$$

has a solution for arbitrary values of the b_i if and only if the determinant of a_{ik} is $\neq 0$. In that case the solution is unique.

We finally express the solution of (19) by means of determinants if the determinant D of the a_{ik} is $\neq 0$.

We multiply for a given k the i-th equation with A_{ik} and add the equations. (15) gives

$$(20) \qquad D \cdot x_k = A_{1k}b_1 + A_{2k}b_2 + \cdots + A_{nk}b_n \qquad (k = 1,2,\ldots,n)$$

and this gives x_k. The right side in (12) may also be written as the determinant obtained from D by replacing the k-th column by $b_1, b_2, \ldots, b_n$. The rule thus obtained is known as Cramer's rule.

II FIELD THEORY

A. Extension Fields.

If E is a field and F a subset of E which, under the operations of addition and multiplication in E, itself forms a field, that is, if F is a subfield of E, then we shall call E an extension of F. The relation of being an extension of F will be briefly designated by $F \subset E$. If $\alpha, \beta, \gamma, \ldots$ are elements of E, then by $F(\alpha, \beta, \gamma, \ldots)$ we shall mean the set of elements in E which can be expressed as quotients of polynomials in $\alpha, \beta, \gamma, \ldots$ with coefficients in F. It is clear that $F(\alpha, \beta, \gamma, \ldots)$ is a field and is the smallest extension of F which contains the elements $\alpha, \beta, \gamma, \ldots$. We shall call $F(\alpha, \beta, \gamma, \ldots)$ the field obtained after the adjunction of the elements $\alpha, \beta, \gamma, \ldots$ to F, or the field generated out of F by the elements $\alpha, \beta, \gamma, \ldots$. In the sequel all fields will be assumed commutative.

If $F \subset E$, then ignoring the operation of multiplication defined between the elements of E, we may consider E as a vector space over F. By the degree of E over F, written (E/F), we shall mean the dimension of the vector space E over F. If (E/F) is finite, E will be called a finite extension.

THEOREM 6. If F, B, E are three fields such that $F \subset B \subset E$, then

$$(E/F) = (B/F)(E/B).$$

Let $A_1, A_2, \ldots, A_r$ be elements of E which are linearly independent with respect to B and let $C_1, C_2, \ldots, C_s$ be elements