CHAPTER 7
PRIME ENDS OF PLANAR OPEN SETS

In this chapter, we will find the prime ends of a planar open set of finite connectivity.

1. Some analytical preliminaries.

Lemma 1.1. *Let $R = [a,b] \times [c,d] \subset \mathbf{R}^2 = \mathbf{C}$ be a compact rectangle and let $F :]a,b[\times]c,d[\to C$ be a holomorphic injective map. If $\theta \in \mathbf{C}$, the set E of $y \in]c,d[$ such that $\lim_{x\to a} F(x+iy) = \theta$ has zero Lebesgue measure.*

Proof. Since F is injective, we can find $x_0 \in]a,b[$ such that $\theta \notin F\big(x_0\times]c,d[\big)$. Suppose now that $\epsilon > 0$ is fixed. By compactness of $F\big(x_0 \times [c+\epsilon, d-\epsilon]\big)$, there exists $\rho > 0$ such that for $r < \rho$ the ball $B(\theta, r)$ does not contain any point of $F\big(x_0 \times [c+\epsilon, d-\epsilon]\big)$. It follows that if $y \in E \cap [c+\epsilon, d+\epsilon]$, then for each $r < \rho$ we have $\big(]a,b[\times\{y\}\big) \cap F^{-1}\big(\partial B(\theta,r) \cap F(\mathrm{Int}(R))\big) \neq \emptyset$. Hence we obtain, $m\big(E \cap [c+\epsilon, d-\epsilon]\big) \leq m\big(F^{-1}\big(\partial B(\theta,r) \cap F(\mathrm{Int}(R))\big)\big) \leq$ length $F^{-1}\big(\partial B(\theta,r)\big)$, where m is the Lebesgue measure on $[c,d]$ and p is the projection of $[a,b] \times [c,d]$ on the second factor. We can apply **4**.2.1 to the map $F : F^{-1}\big(F(\mathrm{Int}(R))\big) \to \mathrm{Int}(R)$ to obtain a sequence of $r_n \to 0$ such that the length $l\big(F^{-1}\big(\partial B(\theta, r_n) \cap F(\mathrm{Int}(R))\big)\big) \to 0$. This gives that $m\big(E \cap [c+\epsilon, d-\epsilon]\big) = 0$ for each $\epsilon > 0$. □

Lemma 1.2. *Let $z_0 \in \partial\mathbf{B}^2$. For each $\rho \in]0,2[$, let be the intersection of $\partial B(z_0, r)$ with $\mathbf{B}^2$; and let $\gamma_-(r)$ and $\gamma_+(r)$ be the two endpoints of $\gamma(r)$, with $\gamma_-(r)$ before $\gamma_+(r)$ when we go along $\partial B(z_0; r)$ in the counterclockwise direction. Let $0 < \alpha < \beta < 2$ and let $F : \mathrm{Int}(\mathbf{B}^2) \cap \{z | \, \alpha < |z - z_0| < \beta\} \to C$ be a holomorphic injective map. If $\theta \in \mathbf{C}$, the two sets set $E_+(\theta) = \{r \in$*

$]\alpha,\beta[|\ \lim_{z\in\gamma(r),z\to\gamma_+(r)} F(z)=\theta\}$ *and* $E_-(\theta)=\{r\in]\alpha,\beta[|\ \lim_{z\in\gamma(r),z\to\gamma_-(r)} F(z)=\theta\}$ *have zero Lebesgue measure.*

Proof. The proof is very similar to the last one, it is left to the reader. □

Theorem 1.3. *Let* $F:\mathrm{Int}(\mathbf{B}^2)\to\mathbf{S}^2$ *be a conformal representation on a simply connected subset* U*, we will suppose that* $F(0)=\infty$*. Let* $z_0\in\partial\mathbf{B}^2$*, we can find a sequence* $(r_n!_{n\geq 1}$ *such that:*

(i) $\forall n\geq 1,\ 2^{n+1}\leq r_n\leq 2^n$*;*
(ii) $l(F(\gamma(r_n)))\to 0$ *as* $n\to\infty$*, where* $\gamma(\rho)=\mathrm{Int}(\mathbf{B}^2)\cap\{z|\,|z-z_0|=\rho\}$*.*
(iii) $\mathrm{Cl}_{\mathbf{S}^2}\big(F(\gamma(r_n))\big)\cap\mathrm{Cl}_{\mathbf{S}^2}\big(F(\gamma(r_{n'}))\big)=\emptyset$ *for* $n\neq n'$*.*

Remark that since $l(F(\gamma(r_n)))<+\infty$*, the restriction* $F|\gamma(r_n)$ *can be extended by continuity to* $\{\gamma_+(r_n),\gamma_-(r_n)\}=\{z\in\partial\mathbf{B}^2|\,|z-z_0|=r_n\}$*, and that* $\mathrm{Cl}_{\mathbf{S}^2}\big(F(\gamma(r_n))\big)$ *is the image of that extension. We denote by* $F(\gamma_+(r_n))$ *and* $F(\gamma_-(r_n))$ *the values of* $\gamma_+(r_n)$ *and* $\gamma_-(r_n)$ *under that extension.*

Proof. Call V_n the set $\{z\in\mathrm{Int}(\mathbf{B}^2)|\,2^{-n+1}<|z-z_0|<2^n\}$. Since $F(0)=\infty$, the union $\bigcup_{n\geq 1}V_n$ is included in the compact region $\mathbf{S}^2\backslash F\big(\mathrm{Int}(B(0,1/4))\big)$ of $\mathbf{R}^2$; hence, we have:

$$A\big(\bigcup_{n\geq 1}V_n\big)=\sum_{n\geq 1}A(V_n)<+\infty.$$

We are going to construct the sequence $(r_n)_{n\geq 1}$ by induction. Suppose that we have already defined $r_1,\dots,r_n$, with $l(F(\gamma(r_n)))^2\leq 2\pi A(V_n)/\log 2+1/n$. Define F_{n+1} as the set of numbers $r\in[2^{-(n+2)},2^{-(n+1)}]$ such that one of the two limits $\lim_{z\in\gamma(r),z\to\gamma_+(r)}F(z)$ or $\lim_{z\in\gamma(r),z\to\gamma_-(r)}F(z)$ exists and is equal to one of the $F(\gamma_+(r_i))$ or $F(\gamma_-(r_i)), i=1,\dots,n$.

By lemma 1.2, the set $F_{n+1}\subset[2^{-(n+2)},2^{-(n+1)}]$ has zero Lebesgue measure. Hence, the set $E_{n+1}=[2^{-(n+2)},2^{-(n+1)}]\backslash F_{n+1}$ has full Lebesgue measure in $[2^{-(n+2)},2^{-(n+1)}]$. By theorem 4.1.1, we have:

$$\int_{2^{-(n+2)}}^{2^{-(n+1)}}\frac{l[F(\gamma(\rho))]^2}{\rho}\,d\rho\leq 2\pi A(V_{n+1}).$$

By what we have obtained,; this implies:

$$\int_{E_{n+1}}\frac{l[F(\gamma(\rho))]^2}{\rho}\,d\rho\leq 2\pi A(V_{n+1}).$$

If we define L_{n+1} as $\inf\{l[F(\gamma(\rho))]|\,\rho\in E_{n+1}\}$, we have:

$$\begin{aligned}2\pi A(V_{n+1})\geq L_{n+1}^2\int_{E_{n+1}}\frac{d\rho}{\rho}&=L_{n+1}^2(\log 2^{-(n+2)}-\log 2^{-(n+1)})\\&=L_{n+1}^2\log 2.\end{aligned}$$

In particular, we can find $r_{n+1}\in E_{n+1}$ with:

$$l[F(\gamma(r_n))]^2\leq\frac{2\pi A(V_{n+1})}{\log 2}+\frac{1}{n+1}.$$

The reader will easily check that $(r_n)_{n\geq 1}$ has the required properties. □

2. Prime ends of a simply connected open set of the sphere.

Let us consider U a simply connected open subset of the sphere $\mathbf{S}^2$.

Proposition 2.1. *If U is $\mathbf{S}^2$ minus a point p, then $\hat{U} = \mathbf{S}^2$. The identification map $\mathbf{S}^2 \to \hat{U}$ d is defined as the identity on U andf sends p to the chain $\big(\mathrm{Int}(B(p, r_n))\backslash\{p\}\big)_{n\geq 1}$, where $(r_n)_{n\geq 1}$ is any sequence going to zero.*

Proof. The above defined map $\mathbf{S}^2 \to \hat{U}$ is clearly continuous. By 6.3.6, it has a dense image. Since, by 6.3.3, the space $\hat{U}$ is Hausdorff and since the sphere $\mathbf{S}^2$ is compact, this map is in fact a homeomorphism. □

We now consider the case where $\mathbf{S}^2\backslash U$ has at least two points. By the Riemann mapping theorem, there exists a biholomorphic map $F : \mathrm{Int}(\mathbf{B}^2)\tilde{\to} U$, we are going to show that F extends to a homeomorphism $\hat{F} : \mathbf{B}^2 \to \hat{U}$.

Theorem 2.2. *Let $F : \mathrm{Int}(\mathbf{B}^2)\tilde{\to} U$ be a conformal representation, it extends to a homeomorphism $\hat{F} : \mathbf{B}^2 \to \hat{U}$. In fact, the extension $\hat{F}$ can be thought as the natural extension $\hat{F} : \widehat{\mathrm{Int}(\mathbf{B}^2)} \to \hat{U}$.*

Proof. Without loss of generality, we can assume that $F(0) = \infty$. By proposition 6.3.8, each point z of $\partial\mathbf{B}^2$ can be identified to a point in $\widehat{\mathrm{Int}(\mathbf{B}^2)}$, by sending z to the class of any chain $\big(\mathrm{Int}(\mathbf{B}^2) \cap \mathrm{Int}(B(z, r_n))\big)_{n\geq 1}$, where $r_n \to 0$.

For each $z \in \partial\mathbf{B}^2$, let us choose a sequence $\big(r_n(z)\big)_{n\geq 1}$ as given by theorem 3.1. For each $z \in \partial\mathbf{B}^2$ and each $n \geq 1$, let us call $U_n(z)$ the connected component of $\mathrm{Int}(\mathbf{B}^2)\backslash\gamma(r_n(z))$ which does not contain 0. Of course, the sequence $\big(F(U_n(z))\big)_{n\geq 1}$ is a chain in U, since, for each $n \geq 1$, the frontier $\mathrm{Fr}_U\big(F(U_n(z))\big) = F(\gamma(r_n(z)))$ is connected and contained in $F(U_{n-1}(z))$. Moreover, by conditions (i) and (ii) of theorem 1.3, each $\big(F(U_n(z))\big)_{n\geq 1}$ is a topological chain.

We define $\hat{F}(z) = \big(F(U_n(z))\big)_{n\geq 1}$. It is clear that $\hat{F}\big(\{z \in \mathbf{B}^2 | \, |z - z_0| < r_n(z_0)\} \subset \hat{F}(U_n(z_0))$, since, for each $z \in \partial\mathbf{B}^2$ such that $|z - z_0| < r_n(z_0)$, we have $U_{n'}(z) \subset B(z, r_{n'}(z)) \subset B(z_0, r_{n'}(z_0))$ as soon as $r_{n'}(z) < r_n(z_0) - |z - z_0|$. This shows that $\hat{F}$ is continuous. If

$z \in \partial \mathbf{B}^2$, we have $\bigcap_{n\geq 1} U_n(z) = \emptyset$; since F is bijective, we obtain $\bigcap_{n\geq 1} F\big(U_n(z)\big) = \emptyset$. By 6.3.2, this shows that $\hat{F}(z) \in \hat{U}\backslash U$. To check that $\hat{F}$ is injective it now suffices to remark that for $z, z' \in \partial \mathbf{B}^2, z \neq z'$, we have, for n large, $F\big(U_n(z)\big) \cap F\big(U_n(z')\big) = \emptyset$ because F is bijective and $U_n(z) \bigcap U_n(z') = \emptyset$ for n large.

Up to now, we have shown that $\hat{F}$ is continuous and injective. By 6.3.3, it has also a dense image—it contains U. Moreover, since $\hat{U}$ is Hausdorff and $\mathbf{B}^2$ is compact, the extension $\hat{F}$ must be a homeomorphism. □

3. Prime ends of open subsets of finite connectivity.

Our goal is to prove the following theorem:

Theorem 3.1. *Let U be an open connected subset of $\mathbf{S}^2$ with finite connectivity. Let $F_1, \ldots, F_l$ be the connected components of $\mathbf{S}^2\backslash U$ which are not reduced to points and let $\{p_1, \ldots, p_k\}$ be the remaining part of $\mathbf{S}^2\backslash U$. The set of prime points $\hat{U}$ is homeomorphic to a sphere with l-holes. In fact, we have $\hat{U} = (\mathbf{S}^2\backslash(F_1 \bigcup \ldots \bigcup F_l))\hat{}$. Moreover, if for $i = 1, \ldots, l$, we choose $D_i \approx \mathbf{B}^2$ such that $\mathrm{Int}(D_i) \supset F_i$, and $D_i \cap D_j = \emptyset$ for $1 \leq i < j \leq l$, then there exists a homeomorphism of $\mathbf{S}^2\backslash \bigcup_{i=1}^{l} \mathrm{Int}(D_i)$ on $\hat{U}$. We can impose that that homeomorphism is the identity on any choosen compact subset of $\mathbf{S}^2\backslash \bigcup_{i=1}^{l} D_i$.*

We will need several lemmas.

Lemma 3.2. *Let $C_1, \ldots, C_m$ be the connected components of $\mathbf{S}^2\backslash V$, where V is an open subset of finite connectivity in $\mathbf{S}^2$. Let $W_1, \ldots, W_m$ be open disjoint neighborhood of respectively $C_1, \ldots, C_m$. If $[\Omega_i]_{i\in\mathbf{N}}$ is a topological chain defining a prime end of V, then there exists $j \in \{1, \ldots, m\}$ such that $\Omega_i \subset W_j$ for i large enough. Hence, we obtain that $\hat{V} = bigl(\mathbf{S}^2\backslash \bigcup_{j=1}^{m} W_j\big) \bigcup \big(\bigcup_{j=1}^{m} (W_j\backslash C_j)\hat{}\big)$, where, here $(W_j\backslash C_j)\hat{} = \{[\Omega_i] \in \hat{V} \mid \exists i \in \mathbf{N}, \Omega_i \subset W_j\}$.*

Proof. Since $[\Omega_i]_{i\in\mathbf{N}}$ defines a prime end of V, we have $I = \bigcap_{i\in\mathbf{N}} \mathrm{Cl}_{\mathbf{S}^2}(\Omega_i) \subset \bigcup_{i=1}^{m} C_i$. But I is connected, because it is the decreasing intersection of the compact connected sets $\mathrm{Cl}_{\mathbf{S}^2}(\Omega_i)$. This implies that $I \subset C_j$ for some j. By compactness, we must have $\mathrm{Cl}_{\mathbf{S}^2}(\Omega_i) \subset W_j$ for i big enough. □

Lemma 3.3. *Under the hypothesis of 3.2, if we call U_j the simply connected set $\mathbf{S}^2\backslash C_j$, we have:*

$$\{[\Omega_i] \in \hat{U}_j \mid \exists i \in \mathbf{N}, \Omega_i \subset W_j\} = \{[\Omega_i] \in \hat{V} \mid \exists i \in \mathbf{N}, \Omega_i \subset W_j\}.$$

Proof of theorem 3.1. Consider the open connected set U. Using 3.3 and 1.1, we can replace U by $U \bigcup \{p_1, \ldots, p_l\}$, and then suppose that each connected component of $\mathbf{S}^2\backslash U$ has at least two points; of course, we call $F_1, \ldots, F_m$ these components. Let $D_1, \ldots, D_m$ be topological disks such that $\mathrm{Int}(D_i) \supset F_i$ and $D_i \bigcap D_j = \emptyset, i \neq j$, the existence of such disks follows from ??. Call U_i the open subset $\mathbf{S}^2\backslash F_i$. It is easy to obtain from 1.2 that $\{[\Omega_i] \in \hat{U}_j \mid \exists i \in \mathbf{N}, \Omega_i \subset D_j\}$ is homeomorphic to an annulus with one component of the boundary equal to $\mathrm{Fr}_{\mathbf{S}^2}(D_j)$ and the other one equal to the set of prime ends of U_j. It is easy to finish the proof. □

4. Impression, set of principal points and accessible prime end.

We establish some results on impression, set of principal points and accessible prime end for planar open subsets of finite connectivity.

Theorem 4.1. *Let U be a simply connected open subset of $\mathbf{S}^2$ with $\mathbf{S}^2\backslash U$ having at least two points. Let $F : \mathrm{Int}(\mathbf{B}^2) \tilde{\to} U$ be a biholmorphic homeomorphism. Call $\hat{F} : \mathbf{B}^2 \tilde{\to} \hat{U}$ its continuous extension. We have:*

$$\mathrm{P}\big(\hat{F}(e^{i\theta_0})\big) = \{x \mid \exists r_j \to 1, x = F(r_j e^{i\theta_0})\}$$
$$\mathrm{I}\big(\hat{F}(e^{i\theta_0})\big) = \{x \mid \exists z_j \to e^{i\theta_0}, F(z_j) \to x\}.$$

We will need the following lemma:

Lemma 4.2. *Fix $\theta_0 \in \mathbf{R}$. Define:*

$$C_n^\theta = \{\rho e^{i\theta} + e^{i\theta_0} \mid \frac{1}{2^{n+1}} \leq \rho \leq \frac{1}{2^n} \text{ and } \rho e^{i\theta} + e^{i\theta_0} \in \mathbf{B}^2\}.$$

We have:

$$\forall \alpha \leq \beta \in \mathbf{R},$$
$$\int_\alpha^\beta \sum_{n=1}^{\infty} l\big(F(C_n^\theta)\big)^2 \, d\theta \leq A\big(\big(F\big(\mathrm{Int}(\mathbf{B}^2)\backslash \frac{1}{2}\mathbf{B}^2\big)\big).$$

Proof. Since we have:

$$l\big(F(C_n^\theta)\big) = \int_{2^{-n-1}}^{2^{-n}} |F'(\rho e^{i\theta} + e^{i\theta_0})| \, d\rho.$$

By the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} l\big(F(C_n^\theta)\big)^2 &\leq [\frac{1}{2^n} - \frac{1}{2^{n+1}}] \int_{2^{-n-1}}^{2^{-n}} |F'(\rho e^{i\theta} + e^{i\theta_0})|^2 \, d\rho \\ &= \frac{1}{2^{n+1}} \int_{2^{-n-1}}^{2^{-n}} |F'(\rho e^{i\theta} + e^{i\theta_0})|^2 \, d\rho \\ &\leq \int_{2^{-n-1}}^{2^{-n}} |F'(\rho e^{i\theta} + e^{i\theta_0})|^2 \rho \, d\rho \text{ because } \rho \geq \frac{1}{2^{n+1}}. \end{aligned}$$

This gives:

$$\sum_{n=1}^{\infty} l\big(F(C_n^\theta)\big)^2 = \int_0^{\frac{1}{2}} |F'(\rho e^{i\theta} + e^{i\theta_0})|^2 \rho \, d\rho.$$

From which it follows:

$$\begin{aligned} \int_\alpha^\beta \sum_{n=1}^{\infty} l\big(F(C_n^\theta)\big)^2 \, d\theta &\leq \int_\alpha^\beta \int_0^{\frac{1}{2}} |F'(\rho e^{i\theta} + e^{i\theta_0})|^2 \rho \, d\rho \\ &= A\big(\big(F\big(\mathrm{Int}(\mathbf{B}^2)\backslash \frac{1}{2}\mathbf{B}^2\big)\big). \end{aligned}$$

□

Corollary 4.3. *Under the hypothesis of 4.2, we have:*

$$\lim_{n\to\infty} l\big(F(C_n^\theta)\big) = 0, \ \textit{for almost all } \theta.$$

Corollary 4.4. *Given $0 < \alpha < \beta < \pi$, we can find $r_0 > 0$ and a sequence of closed simple curves $C_l \subset \mathrm{Int}(\mathbf{B}^2)$ such that:*

(i) $\mathbb{I}(C_l) \bigcap \mathbb{I}(C_k) = \emptyset$ *for* $k \neq l$;
(ii) $\mathrm{diam}(C_l) \to 0$;
(iii) $\bigcup_{l\in\mathbf{N}} [\mathbb{I}(C_l) \bigcup C_l] \supset \{\rho e^{i\theta} + e^{i\theta_0} \,|\, 0 < \rho < r_0, \alpha \leq \theta \leq \beta\}$.

Proof. Put together 4.3 and 1.3. □

Definition 4.5. Stoltz domain. A domain of the form $\{\rho e^{i\theta} + e^{i\theta_0} \,|\, 0 < \rho < r_0, \alpha \leq \theta \leq \beta\}$ with $0 < \alpha < \beta < \pi$ is called a Stoltz domain with vertex $e^{i\theta_0}$.

Corollary 4.6. *Let U be a simply connected open subset of $\mathbf{S}^2$ with $\mathbf{S}^2 \backslash U$ containing at least two points. Let $F : \mathrm{Int}(\mathbf{B}^2) \widetilde{\to} U$ be a biholomorphic automorphism. Given $e^{i\theta_0} \in \partial \mathbf{B}^2$, the set of accumulation points of F along a Stoltz domain with vertex at $e^{i\theta_0}$ is independent of the Stoltz domain. It is also the set of accumulation points of F along any arc ending at $e^{i\theta_0}$ and contained in a Stoltz domain. It is also equal to $\mathcal{P}\big(\hat{F}(e^{i\theta_0})\big)$.*