CHAPTER 6
PRIME ENDS

1. Definition and first properties.

We fix a metric space (X, d) and an open subset $U \subset X$.

Notation 1.1. If $A \subset U$ we denote by (A) (resp. $\mathrm{Fr}_U(A)$) the closure (resp. frontier) of A in U. Since U is open in X, we have $(A) = (A) \cap U$ (resp. $\mathrm{Fr}_U(A) = \mathrm{Fr}_X(A) \cap U$).

Definition 1.2. (Chain). A chain in U is a sequence $(\Omega_i)_{i \in \mathbf{N}}$ of subsets of U such that:

(i) $\forall i \in \mathbf{N}, \Omega_i$ is open and connected;
(ii) $\forall i \in \mathbf{N}, \mathrm{Fr}_U(\Omega_i)$ is connected and non empty;
(iii) $\forall i \in \mathbf{N}, (\Omega_{i+1}) \subset \Omega_i$.

Figure 1.a

Figure 1.b

Example 1.3. a) (Figure 1.a) The $\mathrm{Fr}_U(\Omega_i)$'s are all homeomorphic to $\mathbf{R}$ and accumulate on a subset of U which is homeomorphic to $\mathbf{R}$.
b) (Figure 1.b) Each $\mathrm{Fr}_U(\Omega_i)$ is a part of a circle that goes through x and y.

Definition 1.4. (Finer, equivalent chain). Let $(\Omega_i)_{i\in\mathbf{N}}$ and $(\Omega'_i)_{i\in\mathbf{N}}$ be two chains in U. We say that $(\Omega'_i)_{i\in\mathbf{N}}$ is finer (or refines) $(\Omega_i)_{i\in\mathbf{N}}$, if for each $i\in\mathbf{N}$ there exists $j\in\mathbf{N}$ such that $\Omega'_j\subset\Omega_i$. We denote this relation by $(\Omega'_i)_{i\in\mathbf{N}}\prec(\Omega_i)_{i\in\mathbf{N}}$. It is easy to check that the relation $\prec$ is a partial preorder on the set of chains in U. We say that $(\Omega_i)_{i\in\mathbf{N}}$ and $(\Omega'_i)_{i\in\mathbf{N}}$ are equivalent if $(\Omega_i)_{i\in\mathbf{N}}\prec(\Omega'_i)_{i\in\mathbf{N}}$ and $(\Omega'_i)_{i\in\mathbf{N}}\prec(\Omega_i)_{i\in\mathbf{N}}$. We denote by $\left[(\Omega_i)_{i\in\mathbf{N}}\right]$ or simply by $[\Omega_i]$ the equivalence class of all chains equivalent to $(\Omega_i)_{i\in\mathbf{N}}$. The ordering $\prec$ induces a partial order on the set of equivalence classes.

Example 1.5. If $(\Omega_i)_{i\in\mathbf{N}}$ is a chain and $(j(i))_{i\in\mathbf{N}}$ is a strictly increasing infinite sequence of integers, the chain $(\Omega_{j(i)})_{i\in\mathbf{N}}$ is an equivalent chain.

Up to now, the fact that U was an open subset was not used. We will use it in the next definition.

Definition 1.6. (Principal point). Let $(\Omega_i)_{i\in\mathbf{N}}$ be a chain in U and let x be a point in X. We say that x is a principal point of $(\Omega_i)_{i\in\mathbf{N}}$, if we have:

(i) $\mathrm{Fr}_U(\Omega_i)\to x$ as $i\to\infty$, this means that each neighborhood of x contains all but a finite number of the $\mathrm{Fr}_U(\Omega_i)$'s;
(ii) $x\notin(\mathrm{Fr}_U(\Omega_i))$ for all but a finite number of $i\in\mathbf{N}$.

Remark that, if x is a principal point of some chain in U, then $x\in\overline{U}$.

Figure 2.a Figure 2.b

Example 1.7. a) In Figure 1.a and 1.b the chains do not have principal points.
b) (Figure 2) Let $V=\{(x,y)||x|<1,|y|<1\}\subset\mathbf{R}^2=X$. For $n\geq 1$, let $C_n=\{\left(\frac{2n-1}{2n},y\right)|-\frac{1}{2}\leq y\leq 1\}$ and $C'_n=\{\left(\frac{2n}{2n+1},y\right)|-1\leq y\leq\frac{1}{2}\}$. Define $U=V\setminus\bigcup_{n\geq 1}(C_n\cup C'_n)$. For $n\geq 1$, let Ω_n (resp. Ω'_n) be the component of $U\setminus\{\left(x,\frac{1}{2}\right)|\frac{2n-1}{2n}\leq x\leq\frac{2n}{2n+1}\}$ (resp. $U\setminus\{\left(x,-\frac{1}{2}\right)|\frac{2n}{2n+1}\leq x\leq\frac{2n+1}{2n+2}\}$) which does not contain $(0,0)$. The two chains $(\Omega_n)_{n\geq 1}$ and

$(\Omega'_n)_{n\geq 1}$ are equivalent. The point $\left(1, \frac{1}{2}\right)$ is a principal point of $(\Omega_n)_{n\geq 1}$ and the point $\left(1, -\frac{1}{2}\right)$ is a principal point of $(\Omega'_n)_{n\geq 1}$.

Lemma 1.8. *The point $x \in X$ is a principal point of the chain $(\Omega_i)_{i\in\mathbf{N}}$ in U if and only if we have:*

(i) *diam*$(\mathrm{Fr}_U(\Omega_i) \cup \{x\}) \to 0$, *as* $i \to \infty$;
(ii) $d(\mathrm{Fr}_U(\Omega_i) \cup \{x\}) > 0$, *for all but a finite number of* $i \in \mathbf{N}$.

Proof. Exercise! □

Proposition 1.9. *Suppose that X is complete for the metric d. Let $(\Omega_i)_{i\in\mathbf{N}}$ be a chain in U such that:*

(i) *diam*$\left(\bigcup_{i\geq n} \mathrm{Fr}_U(\Omega_i)\right) \to 0$, *as* $n \to \infty$;
(ii) *a point in X can belong to only a finite number of* $(\mathrm{Fr}_U(\Omega_i))$*'s — this is implied by the condition* $\forall i \neq j, (\mathrm{Fr}_U(\Omega_i)) \cap (\mathrm{Fr}_U(\Omega_i)) = \emptyset$.

Then $(\Omega_i)_{i\in\mathbf{N}}$ has a principal point.

Proof. Since $\mathrm{diam}\left(\bigcup_{i\geq n} \mathrm{Fr}_U(\Omega_i)\right) \to 0$, as $n \to \infty$ and X is complete, there exists $x \in X$ such that $x = \bigcap_{n\in\mathbf{N}}(\bigcup_{i\geq n} \mathrm{Fr}_U(\Omega_i))$. It is clear that $\mathrm{diam}(\mathrm{Fr}_U(\Omega_i) \cup \{x\}) \to 0$, as $i \to \infty$. Moreover, by condition (ii), the point x can be in only a finite number of $(\mathrm{Fr}_U(\Omega_i))$. □

Definition 1.10. (Topological chain). A chain $(\Omega_i)_{i\in\mathbf{N}}$ in U is topological — with respect to X — if it has a principal point in X.

Definition 1.11. (Prime point). A prime point of U (in X) is an equivalence class of chains which contains a topological chain. The set of prime points of U is denoted by $\hat{U}$.

Figure 3

Example 1.11. (Figure 3) Consider example 1.7.b. For each $n \geq 1$, call Ω''_n the connected component of $U \backslash \{\left(\frac{2n}{2n+1}, y\right) | \frac{1}{2} \leq y \leq 1\}$ which does not contain $(0,0)$. The chain $(\Omega''_n)_{n\geq 1}$ is equivalent to $(\Omega_n)_{n\geq 1}$ (or $(\Omega'_n)_{n\geq 1}$), but $\mathrm{diam}(\mathrm{Fr}_U(\Omega''_n)) \geq 1/2$, for each $n \in \mathbf{N}$.

Definition 1.12. (Topology on $\hat{U}$). For each open subset O of U define $\hat{O} = \{[\Omega_i] \in \hat{U} | \exists i \in \mathbf{N}, \Omega_i \subset O\}$. It is obvious that $(O \cap O')\hat{} = \hat{O} \cap \hat{O}'$; hence the family $\{\hat{O} | O$ open subset in $U\}$ defines a basis for a topology. A subset $\Pi \subset \hat{U}$ is open if for each $\pi \in \Pi$ there exists an open subset $O \subset U$ with $\pi \in \hat{O}$.

Example 1.13. Let us look to the case $U = X = \mathbf{R}$. If $(\Omega_i)_{i \in \mathbf{N}}$ is a chain in $\mathbf{R}$, then each Ω_i has to be of the form $]-\infty, a[$ or $]a, +\infty[$ since it is a connected subset of $\mathbf{R}$ with connected frontier. It is easy to show that $\hat{\mathbf{R}}$ consists in two copies of $\mathbf{R}$, namely $\mathbf{R}_l$ and $\mathbf{R}_r$, where to $x \in \mathbf{R}_l$ (resp. $\mathbf{R}_r$) is associated the chain $\big(]-\infty, x + \frac{1}{n}[\big)_{n \in \mathbf{N}}$ (resp. $\big(]x - \frac{1}{n}, +\infty[\big)_{n \in \mathbf{N}}$). A set is open in $\hat{\mathbf{R}}$ if and only if it is of the form $]-\infty, a[_l \cup]b, +\infty[_r$, where $]-\infty, a[_l$ is an interval in $\mathbf{R}_l$ and $]b, +\infty[_r$ is an interval in $\mathbf{R}_r$.

2. Some general topology.

Lemma 2.1. *Let Y be an open subset of the metric space X. Suppose Y connected and locally connected. Given a and b in Y, we can find a connected subset F of Y closed in X and containing a and b*

Proof. Since X is metric and Y is open, for each $x \in Y$, we can find a closed neighborhood of x in X which is entirely contained in Y. In particular, every small enough neighborhood V of x verifies $(V) \subset Y$. It follows from the local connectedness of Y that we can find an open covering $(V_i)_{i \in I}$ of Y by connected subsets such that $(V_i) \subset Y$, for each $i \in I$. Since Y is connected, we can find a finite subset $\{i_1, \ldots, i_n\}$ of I such that $a \in V_{i_1}, b \in V_{i_n}$ and $V_{i_j} \cap V_{i_{j+1} \neq \emptyset}$, for $i = 1, \ldots, n-1$. We can take $F = \bigcup_{k=1}^{n} (V_{i_j})$. □

Lemma 2.2. *Let Y be a connected and locally connected metric space and let y be in Y. Any neighborhood V of y in Y intersects all the connected components of $Y \backslash \{y\}$.*

Proof. Let $(C_i)_{i \in I}$ be the family of connected components of $Y \backslash \{y\}$. Since Y is locally connected each C_i is open. We can write Y as a disjoint union of two open sets $Y = (\cup \{C_i | C_i \cap V = \emptyset\}) \cup (\cup \{C_i | C_i \cap V \neq \emptyset\} \cup V)$. Since Y is connected and $y \in V$, the open set $(\cup \{C_i | C_i \cap V = \emptyset\})$ must be empty. □

Corollary 2.3. *Let Y be a connected and locally connected metric space. If $y \in Y$ disconnects Y, then any neighborhood of y in Y has a non connected frontier.*

Proof. Let a, b be in distinct connected components C_a, C_b of $Y \backslash \{y\}$. Let U be a neighborhood of y in Y with $a, b \notin U$. By 2.2, $U \cap C_a$ (resp. $U \cap C_b$) is a non empty open subset of the connected set C_a (resp. C_b). Since it is not equal to C_a (resp. C_b), it must have a frontier point in C_a (resp. C_b). This means that at least the two connected components C_a and C_b of $Y \backslash \{y\}$ intersect $\mathrm{Fr}_Y(U)$, but the connected components of $Y \backslash \{y\}$ form a covering of $\mathrm{Fr}_Y(U)$ by disjoint open subsets. □

Theorem 2.4. *Let Y be a locally connected open subset of the metric space X. Suppose that each point in Y has a basis of connected neighborhoods each of which having a connected frontier. If O is an open connected subset of Y, we have:*

(i) *$\forall y \in Y, O \backslash \{y\}$ is connected;*
(ii) *if $a, b \in O$, we can find a connected subset $F \subset O$ closed in X and containing a and b.*

The following lemma is of fundamental importance. We decided to call it the "the 5-lemma". a better name would have been "the lemma on the five connected sets"— in French that sounds much better "le lemme des cinq connexes".

Lemma 2.5. (5-lemma). *Let X, P, Q, A and B be five connected sets with P, Q open subsets of X and A, B closed subsets of X. Suppose that $A \cap B = \emptyset, \mathrm{Fr}_X(P) \subset A$ and $\mathrm{Fr}_X(Q) \subset B$. We must have one of the following possibilities:*

(i) *$P \cup Q = X, A \subset Q, B \subset P$;*
(ii) *$(P \cup A) \cap (Q \cup B) = \emptyset$;*
(iii) *$P \cup A \subset Q \backslash B$;*
(iv) *$Q \cup B \subset P \backslash A$.*

Moreover if A and B are non empty, these four possibilities are mutually exclusive.

Proof. If A (resp. B) is empty, since P (resp. Q) is such that $\mathrm{Fr}_X(P) \subset A$ (resp. $\mathrm{Fr}_X(Q) \subset B$), we must have $P = \emptyset$ or $P = X$ (resp. $Q = \emptyset$ or $Q = X$). In the first case (iii) (resp. (iv)) holds and in the second case (iv) (resp. (iii)) holds. So we can suppose A and B non empty. Since A (resp. B) is connected and disjoint from $\mathrm{Fr}_X(Q)$ (resp. $\mathrm{Fr}_X(P)$), we must have either $A \subset Q$ or $A \cap Q = \emptyset$ (resp. $B \subset P$ or $B \cap P = \emptyset$). This gives us four mutually exclusive possibilities:

(a) $A \subset Q, B \subset P$;
(b) $A \cap Q = B \cap P = \emptyset$;
(c) $A \subset Q, B \cap P = \emptyset$;
(d) $B \subset P, A \cap Q = \emptyset$.

In case (a) we have also that $P \cup Q$ is closed since $P \cup Q \supset B \cup A \supset \mathrm{Fr}_X(Q) \cup \mathrm{Fr}_X(P)$. Moreover, $P \cup Q$ is open and non empty since $A \cup B \subset P \cup Q$. By the connectedness of X we must have $P \cup Q = X$, which shows that (i) holds. In case (b) we want to show that $P \cap Q = \emptyset$; this obviously implies that (ii) holds. Suppose $P \cap Q \neq \emptyset$. Since $\mathrm{Fr}_X(P \cap Q) \subset (\mathrm{Fr}_X(P) \cap Q) \cup (\mathrm{Fr}_X(Q) \cap P) \cup (\mathrm{Fr}_X(P) \cap \mathrm{Fr}_X(Q)) \subset (A \cap Q) \cup (B \cap P) \cup (A \cap B) = \emptyset$, by the connectedness of X, we must have $P \cap Q = X$ and hence $P = Q = X$. This is impossible, since A and B are non empty. In case (c) we have $P \neq X$, since B is non empty. By the connectedness of X, this implies that $\mathrm{Fr}_X(P) \neq \emptyset$. It follows that $P \cap Q \neq \emptyset$. Since

$P \cap \mathrm{Fr}_X(Q) = \emptyset$ and P is connected, we conclude that $P \subset Q$. From $A \subset Q$ and $(P \cup A) \cap B$, we obtain $P \cup A \subset Q \backslash B$. Similarly, case (d) implies (iv).

3. A closer look at the space of prime points.

To avoid pathologies like example 1.13, we will suppose that we have a metric space X with an open subset U such that each point in U has a basis of open connected neighborhoods each of which having a connected U-frontier. This apparently inocuous hypothesis will allow us to prove a lot of things about the space of prime points.

Lemma 3.1. *Let $(\Omega_i)_{i\in\mathbf{N}}$ be a topological chain in U, and let $(\Omega'_i)_{i\in\mathbf{N}}$ be a not necessarily topological chain in U such that $\forall i \neq j, (\mathrm{Fr}_U(\Omega'_i)) \cap (\mathrm{Fr}_U(\Omega'_j)) = \emptyset$. If $\Omega_i \cap \Omega'_j \neq \emptyset$, for all $i, j \in \mathbf{N}$, then $(\Omega_i)_{i\in\mathbf{N}} \prec (\Omega'_i)_{i\in\mathbf{N}}$;*

Proof. Let x be a principal point of $(\Omega_i)_{i\in\mathbf{N}}$. By disjointness, there is at most one $j \in \mathbf{N}$ such that $x \in (\mathrm{Fr}_U(\Omega'_j))$ so by forgetting one of the Ω_j's, we can assume that $\forall j \in \mathbf{N}, x \notin (\mathrm{Fr}_U(\Omega'_j))$.Let us choose $y \in \mathrm{Fr}_U(\Omega_2) \subset (\Omega_2) \subset \Omega_1$. Since $\Omega_1 \cap \Omega'_1$ is non empty, the union $\Omega_1 \cup \Omega'_1$ is connected. Using 2.4, we can find, for each $j \geq 2$, a connected set $F_j \subset (\Omega_1 \cup \Omega'_1)\backslash\{x\}$ such that $y \in F_j, F_j \cap \mathrm{Fr}_U(\Omega'_j) \neq \emptyset$ and F_j is closed in X. We fix now a $j \geq 2$. Since $x \notin (\mathrm{Fr}_U(\Omega'_j))$ and $x \notin F_j$, we have for i big enough $\mathrm{Fr}_U(\Omega_i) \cap \left(\mathrm{Fr}_U(\Omega'_j) \cup F_j\right) = \emptyset$. We can apply the 5-lemma 2.5 to the five connected sets $\Omega_1 \cup \Omega'_1, \Omega_i, \Omega'_j, \mathrm{Fr}_U(\Omega_i)$ and $\mathrm{Fr}_U(\Omega'_j) \cup F_j$. The first and fourth possibilities of the 5-lemma are excluded because $y \in F_j$ and $y \notin \Omega_i$. The second possibility is excluded because $\Omega_i \cap \Omega'_j \neq \emptyset$. So we must have $\Omega_i \cup \mathrm{Fr}_U(\Omega_i) \subset \Omega'_j$. □

corollary 3.2. *Let $(\Omega_i)_{i\in\mathbf{N}}$ and $(\Omega'_i)_{i\in\mathbf{N}}$ be topological chains in U. If $\forall i \in \mathbf{N}, \Omega_i \cap \Omega'_i \neq \emptyset$, then $[\Omega_i] = [\Omega'_i]$*

Proof. This follows from 3.1, since $\Omega_i \cap \Omega'_j \supset \Omega_k \cap \Omega'_k$, where $k = \max(i, j)$. □

theorem 3.3. *The space $\hat{U}$ is Hausdorff.*

Proof. Let $(\Omega_i)_{i\in\mathbf{N}}$ and $(\Omega'_i)_{i\in\mathbf{N}}$ be non equivalent topological chains in U. By 3.2, there exist $i \in \mathbf{N}$ such that $\Omega_i \cap \Omega'_i = \emptyset$. This implies that the neighborhoods $\hat{\Omega}_i$ and $\hat{\Omega}'_i$ of $(\Omega_i)_{i\in\mathbf{N}}$ and $(\Omega'_i)_{i\in\mathbf{N}}$ are disjoint. □

Definition 3.4. (Inclusion of U in $\hat{U}$). If $x \in U$, we denote by $\pi(x)$ the equivalence class of the chain which defines a basis of neighborhood of x. The prime point $\pi(x)$ is well defined because x has a basis of neighborhoods each of which being an open connected set with connected frontier, this chain is obviously topological with x as a principal point, and all such chains are equivalent.

Corollary 3.5. *Let $(\Omega_i)_{i\in\mathbf{N}}$ be a topological chain in U, let $(\Omega'_i)_{i\in\mathbf{N}}$ be a not necessarily topological chain in U equivalent to $(\Omega_i)_{i\in\mathbf{N}}$, and let x be a point in U. The following statements are equivalent:*

(i) $[\Omega_i] = \pi(x)$;
(ii) $x \in \bigcap_{i\in\mathbf{N}} \Omega'_i$;
(iii) $x \in \bigcap_{i\in\mathbf{N}} (\Omega'_i)$;
(iv) $x \in \bigcap_{i\in\mathbf{N}} (\Omega'_i)$;
(v) $\{x\} = \bigcap_{i\in\mathbf{N}} (\Omega'_i)$;
(vi) $\{x\} = \bigcap_{i\in\mathbf{N}} (\Omega'_i)$.

Proof. We have obviously (i)⇒(ii)⇒(iii)⇒(iv), (i)⇒(v)⇒(iii), and (i)⇒(vi)⇒(iv). It suffices to show that (iv)⇒(i). Since $x \in \bigcap_{i\in\mathbf{N}} (\Omega'_i)$, each neighborhood of x intersects Ω'_i, for each $i \in \mathbf{N}$. It follows from 3.2 that $(\Omega'_i)_{i\in\mathbf{N}}$ is equivalent to any topological chain defining $\pi(x)$. □

Theorem 3.6. *The map $\pi : U \to \hat{U}$ is a homeomorphism onto its image. The set $\pi(U)$ is open and dense in $\hat{U}$.*

Proof. the fact that π is injective follows easily from the fact that two points in U have disjoint neighborhoods. If O is an open subset of U, we have obviously $\pi^{-1}(\hat{O}) = O$. This shows that π is a homeomorphism on its image. The fact that $\pi(U)$ is dense follows from the inclusion $\pi(O) \subset \hat{O}$. It remains to show that $\pi(U)$ is open. Let O be an open subset of U with $(O) \subset U$. Let $(\Omega_i)_{i\in\mathbf{N}}$ be a topological chain with $[\Omega_i] \in \hat{O}$, we have, for i large enough $\Omega_i \subset O$ If x is a principal point of $(\Omega_i)_{i\in\mathbf{N}}$,we obtain $x \in \bigcap_{i\in\mathbf{N}} (\Omega_i) \subset (O) \subset U$. By 3.5, this gives $\pi(x) = [\Omega_i]$ and $x \in \bigcap_{i\in\mathbf{N}} \Omega_i \subset O$, i.e. $[\Omega_i] \in \pi(O)$. Since $[\Omega_i]$ is an arbitrary point in $\hat{O}$, we have $\pi(O) = \hat{O}$. Obviously each point in U has an open neighborhood O with $(O) \subset U$. □

By 3.6, we can identify U and $\pi(U)$.

Definition 3.7. (Prime end). A prime end of U is a prime point in $\hat{U}$ which is not in U.

Proposition 3.8. $(\mathrm{Int}\,(\mathbf{B}^2))\hat{}= \mathbf{B}^2$.