CHAPTER 5A
KLEE'S TRICK

In this appendix we give an instance of Klee's trick.

Let x be in $\mathrm{Int}(\mathbf{B}^n)$. We call g_x the map $\mathbf{B}^n \to \mathbf{B}^n$ which is the identity on $\partial\mathbf{B}^n$ and maps each ray emanating from 0 linearly on the ray emanating from x.

Lemma 1. *The map $G : \mathrm{Int}(\mathbf{B}^n)\times\mathbf{B}^n \to \mathrm{Int}(\mathbf{B}^n)\times\mathbf{B}^n, (x,u) \mapsto (x, g_x(u))$ is a homeomorphism, in particular, the map $(x,u) \mapsto g_x^{-1}(u)$ is continuous on $\mathrm{Int}(\mathbf{B}^n) \times \mathbf{B}^n$.*

Proof. We have $g_x(y) = (1 - \|y\|)x + y$. It follows that G is continuous. Since for each $x \in \mathrm{Int}(\mathbf{B}^n)$, the map $y \mapsto g_x(y)$ is bijective, we obtain that G is bijective. For each $r < 1$, the map G induces a bijective continuous map from $r\mathbf{B}^n \times \mathbf{B}^n$ on itself, where $r\mathbf{B}^n = \{x \in \mathbf{R}^n | \|x\| \leq r\}$. It follows from the compactness of $r\mathbf{B}^n \times \mathbf{B}^n$ that G^{-1} is continuous on $r\mathbf{B}^n \times \mathbf{B}^n$. It is not difficult to conclude that G^{-1} is continuous on the whole of $\mathrm{Int}(\mathbf{B}^n) \times \mathbf{B}^n$. □

Theorem 2. *Let A be a closed subset of the metric space X, and let $\varphi : A \to \mathrm{Int}(\mathbf{B}^n)$ be a continuous function. If U is a neighborhood of A in X, we can define a homeomorphism $H : X \times \mathbf{B}^n \tilde{\to} X \times \mathbf{B}^n$ of the form $(x, u) \mapsto (x, h(x, u))$ such that H is the identity outside $U \times \mathrm{Int}(\mathbf{B}^n)$ and $H(x, 0) = (x, \varphi(x))$ if $x \in A$. This last condition means that H takes $A \times \{0\}$ on the graph of φ.*

Proof. By Tietze-Urysohn theorem **1**.4.4, we can extend φ to a continuous map $\bar{\varphi} : X \to \mathrm{Int}(\mathbf{B}^n)$ such that $\bar{\varphi}|X \backslash U = 0$. We define H by $(x, u) \mapsto (x, g_{\bar{\varphi}(x)}(u))$, its inverse is given by $(x, u) \mapsto (x, g^{-1}_{\bar{\varphi}(x)}(u))$. From lemma 1, we obtain that H is a homeomorphism. Moreover, we have $H(x, 0) = (x, g_{\bar{\varphi}(x)}(0)) = (x, \bar{\varphi}(x))$, and if $x \notin U$, we have $H(x, u) = (x, g_0(u)) = (x, u)$. □

Theorem 3. *Let A be a compact space and let $\varphi : A \hookrightarrow \mathrm{Int}(\mathbf{B}^k)$ and $\psi : A \hookrightarrow \mathrm{Int}(\mathbf{B}^k)$ be embeddings. There exists a homeomorphism $H : \mathbf{B}^k \times \mathbf{B}^n \tilde{\to} \mathbf{B}^k \times \mathbf{B}^n$ which is the identity on $\partial(\mathbf{B}^k \times \mathbf{B}^n)$ and such that $H(\varphi(x), 0) = (0, \psi(x))$.*

Proof. We can apply theorem 2 to obtain homeomorphisms $G, G' : \mathbf{B}^k \times \mathbf{B}^n \tilde{\to} \mathbf{B}^k \times \mathbf{B}^n$ such that $G(\varphi(x), 0) = (\varphi(x), \psi \circ \varphi^{-1}[\varphi(x)]) = (\varphi(x), \psi(x)), G'(0, \psi(x)) = (\varphi(x), \psi(x))$ and G, G' are the identity on $\partial(\mathbf{B}^k \times \mathbf{B}^n)$. We can define $H = G'^{-1} \circ G$. □