CHAPTER 5
THE ANNULUS THEOREM
AND ITS CONSEQUENCES

In this chapter, we generalize the Schoenflies theorem to annulii and disks with holes.

1. The annulus theorem.

Definition 1.1. (Annulus). An annulus is any compact space homeomorphic to $\mathbf{A}^2 = \mathbf{S}^1 \times [0,1]$.

Example 1.2. Any set of the form $\{x \in \mathbf{R}^2 \mid \alpha \leq \|x\| \leq \beta\}$, where $0 < \alpha < \beta < +\infty$, is an annulus.

Lemma 1.3. *If A is an annulus, its boundary ∂A consists of two disjoint simple closed curves. Any homeomorphism between two annulii sends the boundary of the first one to the boundary of the second one. The boundary of $\mathbf{S}^1 \times [0,1]$ is $\mathbf{S}^1 \times \{0\} \cup \mathbf{S}^1 \times \{1\}$.*

Proof. See **2**.4.9, **2**.4.10 and **2**.4.11. □

Proposition 1.4. *Let S (resp. S') be a connected component of the boundary of the annulus A (resp. A'). Any homeomorphism of S on S' can be extended to a homeomorphism of A on A'.*

Proof. Let $j : \mathbf{A}^2 \tilde{\to} A$ and $j' : \mathbf{A}^2 \tilde{\to} A'$ be homeomorphisms. By composing, if necessary, j (resp. j') with the homeomorphism $(s,t) \mapsto (s, 1-t)$ of $\mathbf{A}^2$ onto itself, we can assume that $S = j(\mathbf{S}^1 \times \{0\})$ (resp. $S' = j'(\mathbf{S}^1 \times \{0\})$). If $h : S \tilde{\to} S'$ is any homeomorphism, we can extend the homeomorphism $h' = j'^{-1} \circ h \circ j : \mathbf{S}^1 \times \{0\} \tilde{\to} \mathbf{S}^1 \times \{0\}$ to the homeomorphism

$H' : \mathbf{A}^2 \tilde{\to} \mathbf{A}^2, (s,t) \mapsto (h'(s), t)$. The homeomorphism $H = j' \circ H' \circ j^{-1} : A \tilde{\to} A'$ is an extension of h. $\square$

Theorem 1.5. (Annulus theorem). *Let γ be a simple closed curve contained in* $\mathrm{Int}(\mathbf{B}^2)$*. There exists a homeomorphism H of $\mathbf{B}^2$ on itself which is the identity on $\partial\mathbf{B}^2$ and takes γ to the circle $\frac{1}{2}\mathbf{S}^1 = \{x \in \mathbf{R}^2 | \|x\| = \frac{1}{2}\}$.*

Proof. Since $\mathrm{Int}(\mathbf{B}^2)$ is homeomorphic to $\mathbf{R}^2$, we can apply the Schoenflies theorem with compact support **4**.3.11 to obtain a homeomorphism with compact support of $\mathrm{Int}(\mathbf{B}^2)$ such that $H(\gamma) = \frac{1}{2}\mathbf{S}^1$. Since H has a compact support in $\mathrm{Int}(\mathbf{B}^2)$, it can be extended by the identity to $\partial\mathbf{B}^2$. $\square$

Corollary 1.6. (Annulus theorem). *Let $j : \mathbf{B}^2 \hookrightarrow \mathrm{Int}(\mathbf{B}^2)$ be an embedding. The set $\mathbf{B}^2 \backslash j(\mathrm{Int}(\mathbf{B}^2))$ is an annulus. In fact, we can find a homeomorphism $H : \mathbf{B}^2 \tilde{\to} \mathbf{B}^2$ such that H is the identity on $\mathbf{S}^1 = \partial\mathbf{B}^2$ and $H[j(\mathbf{B}^2)] = \frac{1}{2}\mathbf{B}^2 = \{x \in \mathbf{R}^2 | \|x\| \leq \frac{1}{2}\}$.*

2. Disks and spheres with holes.

Theorem 2.1. *Let $j_1, \ldots, j_k : \mathbf{B}^2 \hookrightarrow \mathrm{Int}(\mathbf{B}^2)$ (resp. ${j'}_1, \ldots, {j'}_k : \mathbf{B}^2 \hookrightarrow \mathrm{Int}(\mathbf{B}^2)$) be embeddings with disjoint images. There exists a homeomorphism $K : \mathbf{B}^2 \tilde{\to} \mathbf{B}^2$ such that $K[j_i(\mathbf{B}^2)] = j'_i(\mathbf{B}^2)$, for $i = 1, \ldots, k$. Moreover, we can choose K to be the identity on $\partial\mathbf{B}^2$.*

Proof. By **2**.2.5, we can find a homeomorphism F of $\mathbf{B}^2$ on itself which is the identity on $\partial\mathbf{B}^2$ and sends $j_i(0)$ on $j'_i(0)$, for $i = 1, \ldots, k$. So we can assume that $j_i(0) = j'_i(0)$. By corollary 1.6, we can extend $j_1, \ldots, j_k$ and ${j'}_1, \ldots, {j'}_k$ to embeddings $\mathbf{R}^2 \hookrightarrow \mathbf{B}^2$ which will be still denoted by $j_1, \ldots, j_k$ and ${j'}_1, \ldots, {j'}_k$. Since $j_1(\mathbf{B}^2), \ldots, j_k(\mathbf{B}^2)$ (resp. $j_1(\mathbf{B}^2), \ldots, j_k(\mathbf{B}^2)$) are disjoint, we can find $r > 1$ such that, if $r\mathbf{B}^2 = \{x \in \mathbf{R}^2 | \|x\| \leq r\}$, the sets $j_1(r\mathbf{B}^2), \ldots, j_k(r\mathbf{B}^2)$ (resp. $j'_1(r\mathbf{B}^2), \ldots, j'_k(r\mathbf{B}^2)$) are disjoint. Since $j_i(0) = j'_i(0)$, it is easy to find $\epsilon > 0$ such that $j_i(\epsilon\mathbf{B}^2) \subset j'_i(\mathrm{Int}(\mathbf{B}^2))$, for $i = 1, \ldots, k$. Let θ be a homeomorphism of $r\mathbf{B}^2$ on itself which is the identity on $\partial(r\mathbf{B}^2)$ and takes $\mathbf{B}^2$ on $\epsilon\mathbf{B}^2$. We can define a homeomorphism $G : \mathbf{B}^2 \tilde{\to} \mathbf{B}^2$ by $G =$identity outside $j_1(r\mathbf{B}^2) \cup \cdots \cup j_k(r\mathbf{B}^2)$ and $G | j_i(r\mathbf{B}^2) = j_i \circ \theta \circ j_i^{-1}$, for $i = 1, \ldots, k$. We have $G \circ j_i(\mathbf{B}^2) = j_i(\epsilon\mathbf{B}^2) \subset j'_i(\mathrm{Int}(\mathbf{B}^2))$. So we are reduced to the case where $j_i(\mathbf{B}^2) \subset j'_i(\mathrm{Int}(\mathbf{B}^2))$. Using a slight extension of 1.6, we can find, for each $i = 1, \ldots, k$, a homeomorphism h_i of $r\mathbf{B}^2$ on itself such that $h_i[{j'_i}^{-1} \circ j_i(\mathbf{B}^2)] = \mathbf{B}^2$ and $h_i =$identity on $\partial(r\mathbf{B}^2)$. It follows from this last condition that we can construct a homeomorphism

$H : \mathbf{B}^2 \tilde{\to} \mathbf{B}^2$ which is the identity outside $j'_1(r\mathbf{B}^2) \cup \cdots \cup j'_k(r\mathbf{B}^2)$, and $H|j'_i(r\mathbf{B}^2) = j'_i \circ h_i \circ j'^{-1}_i$. Of course $H[j_i(\mathbf{B}^2)] = j'_i(h_i[j'^{-1}_i \circ j_i(\mathbf{B}^2)]) = j'_i(\mathbf{B}^2)$. □

Definition 2.2. (Disk with holes). A disk with k-holes is a space homeomorphic to $\mathbf{B}^2 \setminus \bigcup_{i=1}^k j_i(\mathrm{Int}(\mathbf{B}^2))$, where $j_1, \ldots, j_k : \mathbf{B}^2 \hookrightarrow \mathrm{Int}(\mathbf{B}^2)$ are embeddings with disjoint images. If we do not want to precise the number k of holes, we will simply say a disk with holes. A disk with 0-holes is called a disk.

Lemma 2.3. *The boundary of a disk with k-holes consists of $k+1$ disjoint closed curves. Two disks with holes are homeomorphic if and only if they have the same number of holes. Any homeomorphism between two disks with holes sends the boundary of the first one on the boundary of the second one.*

Theorem 2.4. *Let $j_1, \ldots, j_k : \mathbf{B}^2 \hookrightarrow \mathbf{S}^2$ (resp. ${j'}_1, \ldots, {j'}_k : \mathbf{B}^2 \hookrightarrow \mathbf{S}^2$) be embeddings with disjoint images. There exists a homeomorphism $K : \mathbf{S}^2 \tilde{\to} \mathbf{S}^2$ such that $K[j_i(\mathbf{B}^2)] = j'_i(\mathbf{B}^2)$, for $i = 1, \ldots, k$.*

Proof. We can assume that $k \geq 1$. By Schoenflies theorem, there exists a homeomorphism H of $\mathbf{S}^2$ on itself such that $H[j_1(\mathbf{B}^2)] = j'_1(\mathbf{B}^2)$. It is easy now to check that the theorem follows from its $\mathbf{B}^2$ version 2.1. □

Definition 2.5. (Sphere with holes). A sphere with k-holes is a space homeomorphic to $\mathbf{S}^2 \setminus \bigcup_{i=1}^k j_i(\mathrm{Int}(\mathbf{B}^2))$, where $j_1, \ldots, j_k$ are embeddings $\mathbf{B}^2 \hookrightarrow \mathbf{S}^2$ with disjoint images. If we do not want to precise the number k of holes, we will simply say a sphere with holes.

Lemma 2.6. *The boundary of a sphere with k-holes consists of k disjoint closed curves. Two spheres with holes are homeomorphic if and only if they have the same number of holes. Any homeomorphism between two spheres with holes sends the boundary of the first one on the boundary of the second one.*

Lemma 2.7. *A sphere with $(k+1)$-holes is a disk with k-holes.*

Theorem 2.8. *Let C be a boundary component of the disk (resp. sphere) with k-holes D. Let C' be a boundary component of the disk (resp. sphere) with k-holes D'. Any homeomorphism $h : C \tilde{\to} C'$ can be extended to a homeomorphism $H : D \tilde{\to} D'$. Moreover, if we have numberings $C = C_0, \ldots, C_k$ and $C' = {C'}_0, \ldots, {C'}_k$ of the connected components of ∂D and $\partial D'$, we can choose H such that $H(C_i) = C'_i$, for $i = 1, \ldots, k$.*

Proof. We will prove the theorem in the case $D = \mathbf{B}^2 \setminus \bigcup_{i=1}^k j_i(\mathbf{B}^2), C = \partial \mathbf{B}^2$ and $D' = \mathbf{B}^2 \setminus \bigcup_{i=1}^k j'_i(\mathbf{B}^2), C' = \partial \mathbf{B}^2$, the reader will deduce the general case from this one. By Alexander's trick **4**.3.7, we can extend h to a homeomorphism $F : \mathbf{B}^2 \tilde{\to} \mathbf{B}^2$. By 2.1, there exists a homeomorphism G of $\mathbf{B}^2$ on itself which is the identity on $\partial \mathbf{B}^2$ and such that $G[F \circ j_i(\mathbf{B}^2)] = j'_i(\mathbf{B}^2)$. The required extension of h is $G \circ F$. □

3. Compact subsets with a connected complement.

Theorem 3.1. *Let K be a compact subset in $\mathbf{R}^2$ (resp. $\mathbf{S}^2$ with $\mathbf{S}^2\backslash K \neq \emptyset$). The following conditions are equivalent:*

(i) $\mathbf{R}^2\backslash K$ *(resp. $\mathbf{S}^2\backslash K$) is connected;*
(ii) $[K, \mathbf{S}^1]$ *is trivial;*
(iii) *any neighborhood of K contains a neighborhood which is a finite disjoint union of disks.*

Proof. We will consider, for example, the case $K \subset \mathbf{S}^2$. We have (i)$rightarrow$(ii) by **3**.3.1 and (iii)$arrow$(i) since (iii) implies that $\mathbf{S}^2\backslash K$ is an increasing union of disks with holes. We prove (i)$arrow$(iii). Let $U = \mathbf{S}^2\backslash K$, it is a non empty connected subset of $\mathbf{S}^2$. Let V be an open neighborhood of K, the complement $\mathbf{S}^2\backslash V$ is a compact subset of U. Since U is non empty locally compact and locally connected, we can find a compact connected subset $C \subset U$, with $\mathbf{S}^2\backslash V \subset C$. By **4**.2.2, each connected component of $\mathbf{S}^2\backslash C$ is an open set homeomorphic to $\mathbf{R}^2$. Since K is compact and contained in $\mathbf{S}^2\backslash C$, we can find a finite number $V_1, \ldots, V_n$ of connected components of $\mathbf{S}^2\backslash C$, with $K \subset V_1 \cup \cdots \cup V_n$. Since each V_i is homeomorphic to $\mathbf{R}^2$, it is easy, for $i = 1, \ldots, n$, to find a disk $D_i \subset V_i$ with $K \subset \bigcup_{i=1}^n \mathrm{Int}(D_i)$. We have $K \subset \bigcup_{i=1}^n D_i \subset \bigcup_{i=1}^n V_i \subset \mathbf{S}^2\backslash C \subset V$. □

Corollary 3.2. *Let K be a compact subset of $\mathbf{S}^2$ with $\mathbf{S}^2\backslash K$ non empty and connected. If U is an open connected neighborhood of K and V is a non empty open subset of U, there exists a homeomorphism with compact support $H : U \tilde{\to} U$ such that $H(K) \subset V$.*

Proof. By 3.1, there exists a finite number $j_1, \ldots, j_k$ of embeddings $\mathbf{R}^2 \hookrightarrow U$ with disjoint images and such that $K \subset \bigcup_{i=1}^k j_i(\mathbf{R}^2)$. By **2**.2.4, there exists a homeomorphism with compact support $h : U \tilde{\to} U$ such that $h(j_i(0)) \in V$, for $i = 1, \ldots, k$. It is easy to construct, for each $i = 1, \ldots, k$, a homeomorphism $g_i : h(j_i(\mathbf{R}^2)) \tilde{\to} h(j_i(\mathbf{R}^2))$ with $g_i(h[j_i(K \cap \mathbf{R}^2)]) \subset h(j_i(\mathbf{R}^2)) \cap V$. Piecing together $g_1, \ldots, g_k$ and extending the result by the identity outside $\bigcup_{i=1}^k h(j_i(\mathbf{R}^2))$, we obtain a homeomorphism with compact support $g : U \tilde{\to} U$. The homeomorphism H can be defined as the composition $g \circ h$. □

Corollary 3.3. *Let K be a compact subset of $\mathbf{S}^2$ with $\mathbf{S}^2\backslash K$ non empty and connected. If U is an open connected neighborhood of K, there exists an open subset $V \subset U$ homeomorphic to $\mathbf{R}^2$ and containing K.*

Definition 3.4. (Cellular compacta). A compact non empty subset K of $\mathbf{R}^2$ is cellular if K and $\mathbf{R}^2\backslash K$ are connected. A compact non empty subset of $\mathbf{S}^2$ is cellular if $K \neq \mathbf{S}^2$, and K and $\mathbf{S}^2\backslash K$ are connected.

Remark 3.5. If K is a compact subset of $\mathbf{R}^2$, it is a cellular subset of $\mathbf{R}^2$ if and only if it is a cellular subset of $\mathbf{S}^2 = \mathbf{R}^2 \cup \{\infty\}$.

Theorem 3.6. *The compact non empty subset K of $\mathbf{R}^2$ (or $\mathbf{S}^2$) is cellular if and only if there exists a basis of neighborhoods of K which are homeomorphic to $\mathbf{B}^2$.*

Proof. If K is cellular, by 3.1, each neighborhood of K contains a neighborhood which is a disjoint union of disks, since K is connected it is contained in one of these disks. Conversely, suppose that each neighborhood of K contains a neighborhood which is homeomorphic to

$\mathbf{B}^2$. This implies that $K \neq \mathbf{S}^2$. Moreover, K is connected as a decreasing sequence of compact connected sets and, by 3.1, $\mathbf{R}^2 \backslash K$ (or $\mathbf{S}^2 \backslash K$) is connected. $\square$

Theorem 3.7. *Let K be a compact subset of $\mathbf{R}^2$ with $\mathbf{R}^2 \backslash K$ connected. There exists a continuous map $H : \mathbf{R}^2 \to \mathbf{R}^2$ such that:*

(i) *H induces a homeomorphism between $\mathbf{R}^2 \backslash K$ and $\mathbf{R}^2 \backslash H(K)$;*
(ii) *H is the identity outside a compact subset of $\mathbf{R}^2$;*
(iii) *if $x \in H(K)$ its inverse image $H^{-1}(x)$ is exactly a connected component of K.*
Moreover, if A is a non empty compact subset of $\mathbf{R}^2$ with no isolated point, we can choose H such that:
(iv) *$H(K) \subset A$.*

Proof. We can assume that $K \cup A \subset \mathrm{Int}(\mathbf{B}^2)$. Using 3.1, we can find a fundamental system of neighborhoods $(V_i)_{i \geq 1}$ of K in $\mathrm{Int}(\mathbf{B}^2)$ such that for each $i \geq 1$, V_i is a disjoint union of a finite number of disks $V_i = \bigcup_{i=1}^{k_i} B_{i,j}$ and $V_{i+1} \subset \mathrm{Int}(V_i)$. We construct, by induction on $i \in \mathbf{N}$, a sequence of homeomorphisms $(H_i)_{i \in \mathbf{N}}$ of $\mathbf{R}^2$ such that:

(i) H_i is the identity outside $\mathbf{B}^2$;
(ii) $H_i = H_{i+1}$ outside V_i;
(iii) $\forall j = 1, \ldots, k_i, \mathrm{diam} H_i(B_{i,j}) < 1/2^i$ and $H_i(\mathrm{Int}(B_{i,j})) \cap A \neq \emptyset$.

To do this, we start with $H_0 = \mathrm{Id}_{\mathbf{R}^2}$. Suppose now that we have constructed H_i. For each $j = 1, \ldots, k_i$, let us consider the finite set $I_j = \{l | B_{i+1,l} \subset \mathrm{Int}(B_{i,j})\}$. Since $H_i(\mathrm{Int}(B_{i,j})) \cap A \neq \emptyset$ and A has no isolated point, we can find distinct points $x_k, k \in I_j$, with $x_k \in H_i(\mathrm{Int}(B_{i,j})) \cap A$. For each $k \in I_j$, let D_k be a small Euclidean ball centered on x_k contained in $H_i(\mathrm{Int}(B_{i,j}))$ and with diameter $< 1/2^{i+1}$. By theorem 2.1, for each j, we can find a homeomorphism h_j of $H_i(B_{i,j})$ on itself which is the identity on $H_i(\partial B_{i,j})$ and verifies $h_j(H_i(B_{i,j})) = D_k$, for each $k \in I_j$. We can obtain a homeomorphism $K_i : \mathbf{R}^2 \tilde{\to} \mathbf{R}^2$ by piecing the k_j's together and extending the result by the identity outside $H_i(V_i)$. We can set $H_{i+1} = K_i \circ H_i$. The required map H is the uniform limit of the H_i's. $\square$

Lemma 3.8. *Let A be a non empty compact space with no isolated point. If U is a non empty open subset of A, we can find a compact subset A_0 of U which is non empty and has no isolated points.*

Proof. Let V be a non empty open subset of U with $\overline{V} \subset U$. We claim that no point of $\overline{V}$ is isolated in $\overline{V}$. In fact, if $x \in V$, it cannot be isolated in V because V is open in A and no point of A is isolated in A; if $x \in \overline{V} \backslash V$, it is obvious that it is not isolated in $\overline{V}$. $\square$

Theorem 3.9. *Let K be a compact subset of $\mathbf{S}^2$ with $\mathbf{S}^2 \backslash K$ connected. If U is an open neighborhood of K in $\mathbf{S}^2$, there exists a continuous surjective map $H : \mathbf{S}^2 \to \mathbf{S}^2$ such that:*

(i) *H induces a homeomorphism between $\mathbf{S}^2 \backslash K$ and $\mathbf{S}^2 \backslash H(K)$;*
(ii) *H is the identity outside a compact subset of U;*
(iii) *if $x \in H(K)$ its inverse image $H^{-1}(x)$ is exactly a connected component of K.*
Moreover, if A is a non empty compact subset of $\mathbf{S}^2$ with no isolated point and such that $A \cap C \neq \emptyset$ for each connected component C of U, we can choose H such that:
(iv) *$H(K) \subset A$.*

Proof. Let $V_1, \dots, V_k$ be disjoint open subsets of U homeomorphic to $\mathbf{R}^2$ with $K \subset \bigcup_{i=1}^k V_i$. Since A intersects each connected component of U, we can use theorem **2**.2.4 to obtain a homeomorphism $h : \mathbf{S}^2 \tilde{\to} \mathbf{S}^2$ with compact support contained in U and such that $h(V_i) \cap A \neq \emptyset$, for $i = 1, \dots, k$. By lemma 3.8, we can find a compact set $A_i \subset h(V_i) \cap A$ with no isolated point. Let us define $K_i = K \cap V_i$ for $i = 1, \dots, k$; it is easy to see that the K_i's are open and closed in K and that they satisfy $[K_i, \mathbf{S}^1] = 0$—any continuous map on K_i can be extended continuously to K as a constant map outside K_i! It follows from 3.7 that, for $i = 1, \dots, k$, we can find a map $g_i : h(V_i) \to h(V_i)$ such that:

(i) g_i induces a homeomorphism between $h(V_i) \backslash h(K_i)$ and $h(V_i) \backslash g_i[h(K_i)]$;
(ii) g_i is the identity outside a compact subset of $h(V_i)$;
(iii) if $x \in g_i(h(K_i))$ its inverse image $g_i^{-1}(x)$ is exactly a connected component of $h(K_i)$;
(iv) $g_i(h(K_i)) \subset A_i$.

We define $g : \mathbf{S}^2 \to \mathbf{S}^2$ by piecing together $g_1, \dots, g_k$ and extending the result by the identity outside $\bigcup_{i=1}^k V_i$. We can now set $H = g \circ h$. □

Corollary 3.10. *Let K be a cellular subset of $\mathbf{S}^2$ and let V be a neighborhood of K. There exists a map $H : \mathbf{S}^2 \to \mathbf{S}^2$ such that H is the identity outside V, the image $H(K)$ is reduced to one point x_0 and H induces a homeomorphism between $\mathbf{S}^2 \backslash K$ and $\mathbf{S}^2 \backslash \{x_0\}$.*

Corollary 3.11. *Let K be a totally disconnected compact subset of $\mathbf{S}^2$. If U is an open connected subset of $\mathbf{S}^2$ containing K and A is a compact subset of U with no isolated point,then there exists a homeomorphism with compact support $H : U \tilde{\to} U$ such that $H(K) \subset A$.*

Corollary 3.12. *Let U be a non empty connected open subset of $\mathbf{S}^2$, then U is homeomorphic to $\mathbf{S}^2 \backslash K$ where K is some totally disconnected compact subset of $\mathbf{S}^2$. Moreover, if A is a compact subset of $\mathbf{S}^2$ with no isolated point, we can choose $K \subset A$.*

Definition 3.13. Let U be an open connected subset of $\mathbf{S}^2$, its connectivity is the number of connected components of $\mathbf{S}^2 \backslash U$ if there is a finite number of such components and is ∞ otherwise. By corollary **3**.4.3, the open set U has connectivity $p \geq 1$ if and only if the group $[U, \mathbf{S}^1]$ is a free abelian group on $p - 1$ generators. We will say that U is p-connected (resp. simply connected, doubly connected) if its connectivity is p (resp. 1,2).

Corollary 3.14. *Let p be an integer ≥ 1, any two p-connected open connected subset of $\mathbf{S}^2$ are homeomorphic. In particular, they are homeomorphic to $\mathbf{S}^2$ minus p distincts points.*

4. Extending homeomorphisms between totally disconnected sets.

Theorem 4.1. *Let K and K' be two totally disconnected compact subsets of $\mathbf{S}^2$,and let U be an open connected subset containing $K \cup K'$.Any homeomorphism $h : K \tilde{\to} K'$ can be extended to a homeomorphism $H : U \tilde{\to} U$ with compact support.*

Proof. Since $K \cup K'$ is totally disconnected, by 3.3, we can assume that U is homeomorphic to $\mathbf{R}^2$, hence we can suppose that $U = \mathbf{R}^2$. By 3.11, we can find homeomorphisms $G, G' : \mathbf{R}^2 \tilde{\to} \mathbf{R}^2$ with compact support such that $G(K) \subset]-1,1[\times \{0\}$ and $G'(K') \subset \{0\} \times]-1,1[$. By Klee's trick **5A**.3, there exists a homeomorphism H_1 of $\mathbf{R}^2$ with support in $[-1,1] \times [-1,1]$ such that $H_1 | G(K) = G' \circ h \circ G^{-1}$. We can define H as ${G'}^{-1} \circ H_1 \circ G$. □