CHAPTER 4A
ON MAKING DIFFEOMORPHISMS THE IDENTITY NEAR A FIXED POINT

In this appendix we prove that we can change a diffeomorphism with positive jacobian near a fixed point to obtain a homeomorphism which is the identity near that fixed point.

1. Connectedness of sets of matrices.

Theorem 1.1. *If E is a real finite dimensional vector space, the group* $\mathrm{GL}_+(E)$ *of invertible linear maps with positive determinant is path connected. If E is a complex finite dimensional vector space, the group* $\mathrm{GL}(E)$ *of invertible linear maps is path connected.*

Remark 1.2. It follows from 1.1 that if E is a real finite dimensional vector space the group $\mathrm{GL}(E)$ of invertible linear maps has two connected components, namely $\mathrm{GL}_+(E)$ and $\mathrm{GL}_-(E)$ the subset of isomorphisms having negative determinant.

We are to prove 1.1 by induction on the dimension of the vector space. Suppose that E is a real (resp. complex) vector space with $\dim E = 1$, then $\mathrm{GL}_+(E) \cong \mathbf{R}_+^*$ (resp. $\mathrm{GL}(E) \cong \mathbf{C}^*$) which is obviously path connected.

Lemma 1.3. *Suppose E is a real (resp. complex) vector space with $\dim E \geq 2$. Given e and f in $E\backslash\{0\}$, there exists a path $t \in [0,1] \mapsto A_t \in \mathrm{GL}_+(E)$ (resp. $\mathrm{GL}(E)$) such that $A_0 = \mathrm{Id}$ and $A_1(e) = f$.*

Proof. We first suppose that e and f are independent. Call H the plane generated by e an f, and let H' be a complementary subspace of H in E, i.e. $E = H \oplus H'$. Let A_t be the linear map which is the identity on H', and satisfies $A_t(e) = e\cos(t\pi/2) + f\sin(t\pi/2)$ and $A_t(f) = -e\sin(t\pi/2) + f\cos(t\pi/2)$ — i.e. A_t is a "rotation"by angle $t\pi/2$ in the plane H. It is easy to verify that $\det(A_t) = 1, A_0 =$Id and $A_1(e) = f$. If e and f are colinear, we can, since $\dim E \geq 2$, find $g \in E\backslash\{0\}$ independent of e and f. Applying the independent case proven above we can find a continuous path B_t (resp. C_t) of linear maps with $\det(B_t) = 1, B_0 = \mathrm{Id}, B_1(e) = g$ (resp. $\det(C_t) = 1, C_0 = \mathrm{Id}, C_1(g) = f$). We can take $A_t = C_t \circ B_t$. □

End of the proof of 1.1. We will do the proof in the real case, the complex case is formally the same. We are doing the proof by induction on $n = \dim E$. Since the case $n = 1$ is already settled, we can assume $n \geq 2$. Let A be in $\mathrm{GL}_+(E)$. Pick a point $e \neq 0$ in E. By 1.3, we can find a path B_t in $\mathrm{GL}_+(E)$ with $B_0 = \mathrm{Id}$ and $B_1(A(e)) = e$. If we put $A_t = B_t \circ A$, we have a continuous path A_t in $\mathrm{GL}_+(E)$ from $A = A_0$ to a map A_1 verifying $A_1(e) = e$. Call $H = \mathbf{R}^2 e$ and let H' be a subspace of E complementary to H. With respect to the decomposition $E = H \oplus H'$, the map A_1 can be written $\begin{pmatrix} \mathrm{Id}_H & C \\ 0 & D \end{pmatrix}$, with $D \in \mathrm{GL}(H')$ and $C \in \mathrm{L}(H', H)$. Since $\det(A_1) = \det(D)$, we have $D \in \mathrm{GL}_+(H')$. Since $\dim H' = n-1$, we can apply our induction hypothesis to obtain a path D_t in $\mathrm{GL}_+(H')$ with $D_0 = D$ and $D_1 = \mathrm{Id}_{H'}$. The path $t \mapsto \begin{pmatrix} \mathrm{Id}_H & tC \\ 0 & D_t \end{pmatrix}$ gives a path in $\mathrm{GL}_+(E)$ between A_1 and Id_E. □

Corollary 1.4. *Let E be a complex vector space. If we consider $\mathrm{GL}(E, \mathbf{C})$ as a subset of the real linear maps, we have $\mathrm{GL}(E, \mathbf{C}) \subset \mathrm{GL}_+(E, \mathbf{R})$.*

Proof. Since $\mathrm{GL}(E, \mathbf{C})$ is connected, its image under $\mathrm{GL}(E, \mathbf{R}) \xrightarrow{\det} \mathbf{R}^*$ must also be connected; hence it must be contained in $\mathbf{R}^*_+$. □

2. Blowing up a fixed point.

We consider in the following a map $H : U \to \mathbf{R}^n$, where U is an open subset of $\mathbf{R}^n$ containing 0. We will suppose that H is a homeomorphism between U and the open set

$H(U) \subset \mathbf{R}^n$. Remark that, by "Invariance of domain" **2**.4.5, it suffices to know that H is continuous and injective. We will also suppose that $H(0) = 0$ and that H has a derivative at 0 denoted by A. We will assume that A is invertible. In particular, there exists a map $\varepsilon : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\lim_{r \to 0} \epsilon(r) = 0$ and $\|H(x) - A(x)\| \leq \|x\|\varepsilon(\|x\|)$. Since A is invertible, there exists $K > 0$ such that $\|x\|/K \leq \|A(x)\| \leq K\|x\|$. We introduce polar coordinates $p : \mathbf{S}^{n-1} \times \mathbf{R}_+ \to \mathbf{R}^n$ by $p(x, r) = rx$. Of course $p^{-1}(0) = \mathbf{S}^{n-1} \times \{0\}$ and p induces a homeomorphism of $\mathbf{S}^{n-1} \times]0, \infty[$ on $\mathbf{R}^n \backslash \{0\}$.

Lemma 2.1. *The homeomorphism* $\tilde{H} = p^{-1} \circ H \circ p : p^{-1}(U \backslash \{0\}) \tilde{\to} p^{-1}(H(U) \backslash \{0\})$ *extends continuously to* $\mathbf{S}^{n-1} \times \{0\}$ *by the formula* $\tilde{H}(x, 0) = (A(x)/\|A(x)\|, 0)$. *The resulting extension is a homeomorphism between* $p^{-1}(U)$ *and* $p^{-1}(H(U))$.

Proof. The map $\tilde{A} = p^{-1} \circ A \circ p : \mathbf{S}^{n-1} \times]0, \infty[\to \mathbf{S}^{n-1} \times]0, \infty[$ is given by the formula $\tilde{A}(x, r) = (A(x)/\|A(x)\|, r\|A(x)\|)$. It is clear that the same formula gives an extension to $\mathbf{S}^{n-1} \times [0, \infty[$ which is a homeomorphism. We want to estimate the distance between $\tilde{H}(x, r)$ and $\tilde{A}(x, r)$ as $r > 0$ goes to 0. If we write $\tilde{H}(x, r) = (x', r')$, we have $x' = H(rx)/r'$ and $r' = \|H(rx)\|$. This gives:

$$\text{(i)} \qquad \begin{aligned} |r\|A(x)\| - r'| &= |\|A(rx)\| - \|H(rx)\|| \\ &\leq \|A(rx) - H(rx)\| \leq r\varepsilon(r), \end{aligned}$$

$$\text{(ii)} \qquad \begin{aligned} \|\frac{H(rx)}{r'} - \frac{A(x)}{\|A(x)\|}\| &\leq \frac{\|H(rx) - A(rx)\|}{r'} + \|(\frac{1}{r'} - \frac{1}{\|A(rx)\|})A(rx)\| \\ &\leq \frac{1}{r'}[r\varepsilon(r) + |\|A(rx)\| - r'|] \leq \frac{2r\varepsilon(r)}{r'}. \end{aligned}$$

Using (i) and the fact that $\|A(x)\|$ is bounded away from 0 on $\mathbf{S}^{n-1}$, it is easy to show that r/r' is bounded for r near 0. It follows then from (i) and (ii) that for r small enough $d(\tilde{H}(x, r), \tilde{A}(x, r)) \leq C\varepsilon(r)$, where C is a constant. The lemma is a routine consequence of this estimation. □

Corollary 2.2. *Let* $V \subset U$ *be an open neighborhood of 0. If* $\rho > 0$ *is small enough so that* $B(0, \rho) = \{x \in \mathbf{R}^n | \|x\| \leq \rho\} \subset V \bigcap H(U)$, *we can find a homeomorphism* $\tilde{G}$ *between* $U \backslash \mathrm{Int}(B(0, \rho))$ *and* $H(U) \backslash \mathrm{Int}(B(0, \rho))$ *which is equal to* $\tilde{H}$ *outside* V *and to the map* $x \mapsto [\|x\|/\|A(x)\|]A(x)$ *on* $\partial B(0, \rho) = \{x \in \mathbf{R}^n | \|x\| = \rho\}$.

Proof. Since $V \bigcap H(U)$ is open, we can find $\rho' > \rho$ such that $B(0, \rho') \subset V \bigcap H(U)$. Let $\theta : \mathbf{R}_+ \tilde{\to} [\rho, \infty[$ be a homeomorphism such that $\theta(r) = r$, for $r \geq \rho'$. We define a homeomorphism $\Theta : \mathbf{S}^{n-1} \times \mathbf{R}_+ \tilde{\to} \mathbf{S}^{n-1} \times [\rho, \infty[$ by $\Theta(x, r) = (x, \theta(r))$. It is easy to check that, if we extend $\tilde{H}$ to $\mathbf{S}^{n-1} \times \{0\}$ as provided by lemma 2.1, the map $G = p \circ \Theta \circ \tilde{H} \circ \Theta^{-1} \circ p^{-1}$ is well defined on $U \backslash \mathrm{Int}(B(0, \rho))$ and has the required properties. □

Corollary 2.3. *If* $\det A > 0$, *then given any neighborhood* V *of 0, we can find a homeomorphism* $U \tilde{\to} H(U)$ *which is equal to* H *outside* V *and equal to the identity on a neighborhood of 0.*

Proof. By 2.2, it suffices to show that if $\det A > 0$, then the map $\partial B(0, \rho) \to \partial B(0, \rho), x \mapsto [\|x\|/\|A(x)\|]A(x)$ can be extended to a homeomorphism of $B(0, \rho)$ on itself which is equal to the identity near 0. By 1.1, we can find a continuous map $[0,\rho] \to \mathrm{GL}_+(\mathbf{R}^n), r \mapsto A_r$ such that $A_\rho = A$ and $A_r = \mathrm{Id}$, for $r \in [0,\rho/2]$. The desired homeomorphism can be defined by the formula $x \mapsto [r/\|A_r(x)\|]A_r(x)$, where $r = \|x\|$. □

The following theorem sums up the results of this appendix.

Theorem 2.4. *Let $H : U \to \mathbf{R}^n$ be a homeomorphism of the open subset U of $\mathbf{R}^n$ onto the open subset $H(U)$ of $\mathbf{R}^n$. Suppose that for some x in U we have $H(x) = x$ and that the derivative of H at x exists, is invertible, and has a positive determinant. If $V \subset U$ is a neighborhood of x, there exists a homeomorphism $G : U \tilde{\to} H(U)$ which is equal to H outside V and is equal to the identity in a neighborhood of x.*