CHAPTER 4
THE THEOREMS OF
CARATHÉODORY AND SCHOENFLIES

1. Length estimates and the Carathéodory theorem.

We will assume the reader familiar with the rudiments of the theory of holomorphic functions of one variable up to the Riemann mapping theorem. A good reference is Rudin's book "Real and complex analysis".

We will identify the Riemann sphere $\mathbf{S}^2$ with its complex structure to $\mathbf{R}^2 \cup \{\infty\}$ the one point compactification of $\mathbf{R}^2$.

The following theorem is a length-area estimate which is fundamental for our purposes.

Theorem 1.1. *Let F be a biholomorphic isomorphism $F : U \tilde{\to} V$, where U and V are open subsets of $\mathbf{R}^2$. Fix a point x_0 in $\mathbf{R}^2$ and define C_ρ as the circle of center x_0 and radius ρ. For each ρ define $l[F(U \cap C_\rho)]$ as the sum of the lengths of the connected components of $F(U \cap C_\rho)$ (each one of these connected component is a curve, of course $l[F(U \cap C_\rho)]$ may be $+\infty$ and $l(\emptyset) = 0$). We have the inequality:*

$$\int_0^\infty \frac{l[F(U \cap C_\rho)]^2}{\rho} \, d\rho \leq 2\pi A(V),$$

where $A(V)$ is the area of V for the Lebesgue measure on $\mathbf{R}^2$.

Proof. We can write $l[F(U \cap C_\rho)] = \int_{A_\rho} \rho |F'(\rho e^{i\theta})| \, d\theta$, where $A_\rho = \{\theta \in [0, 2\pi] | \rho e^{i\theta} \in U \cap C_\rho\}$. By the Cauchy-Schwarz inequality, we obtain:

$$l[F(U \cap C_\rho)] \leq \rho [\int_{A_\rho} |F'(\rho e^{i\theta})|^2 \, d\theta]^{1/2} [\int_{A_\rho} d\theta]^{1/2}.$$

This gives:

$$\int_0^\infty \frac{l[F(U\cap C_\rho)]^2}{\rho}\,d\rho \le \int_0^\infty \int_{A_\rho} |F'(\rho e^{i\theta})|^2 2\pi\rho\, d\theta d\rho = 2\pi A(V).$$

□

Corollary 1.2. *With the same hypothesis as in 1.1, if for some $R>0$ we have:*

$$A[F(U\cap\{x|\|x-x_0\|<R\})]<\infty,$$

then we can find a sequence $\rho_n\to 0$ such that:

$$\lim_{n\to\infty} l[F(U\cap C_{\rho_n})]=0.$$

Proof. If $\liminf_{\rho\to 0} l[F(U\cap C_\rho)]\neq 0$, we obtain:

$$\int_0^R \frac{l[F(U\cap C_\rho)]^2}{\rho}\,d\rho = +\infty.$$

But, by 1.1, we have:

$$\int_0^R \frac{l[F(U\cap C_\rho)]^2}{\rho}\,d\rho \le 2\pi A[F(U\cap\{x|\|x-x_0\|<R\})].$$

□

Corollary 1.3. *Let $F:\mathrm{Int}(\mathbf{B}^2)\tilde{\to}U$ be a biholomorphic isomorphism onto an open subset of $\mathbf{S}^2$ such that $F(0)=\infty$. If $x_0\in\partial\mathbf{B}^2$, let us call $\alpha_\rho(x_0)$ the (compact) arc $\mathbf{B}^2\cap\{x|\|x-x_0\|=\rho\}$. Given $x_0\in\partial\mathbf{B}^2$, we can find a sequence $\rho_n\to 0$ such that the map $F|\alpha_{\rho_n}(x_0)\backslash\partial\alpha_{\rho_n}(x_0)$ can be extended by continuity to $\alpha_{\rho_n}(x_0)$ and moreover, the length of $F[\alpha_{\rho_n}(x_0)]$ goes to 0 as n goes to ∞. Of course, since F is a homeomorphism, we have $F[\partial\alpha_{\rho_n}(x_0)]\subset\partial U$.*

proof. Exercise!

□

Theorem 1.4. (Carathéodory). *Let $F:\mathrm{Int}(\mathbf{B}^2)\tilde{\to}U$ be a biholomorphic isomorphism onto an open subset of $\mathbf{S}^2$. If ∂U is contained in a compact locally connected subset of $\mathbf{S}^2\backslash U$, then the map F extends continuously to a map $\overline{F}:\mathbf{B}^2\tilde{\to}\overline{U}$. Since F is a homeomorphism, we have $\overline{F}(\partial\mathbf{B}^2)=\partial U$.*

Proof. Without loss of generality, we can assume that $F(0)=\infty$. In particular, we have $\mathbf{S}^2\backslash U\subset\mathbf{R}^2$. Let us call C a compact locally connected subset of $\mathbf{R}^2$ containing ∂U. By a standard topological argument, given any $\varepsilon>0$, we can find a $\delta>0$ such that if two points $x,y\in C$ verify $\|x-y\|<\delta$, then we can find a compact connected subset K of C with diameter less than ε and containing x and y. Let x_0 be a point in $\partial\mathbf{B}^2$, we are going to show that the diameter of $F(\mathrm{Int}(\mathbf{B}^2)\cap\{x|\|x-x_0\|<\rho\})$ tends to 0 as $\rho\to 0$, this will prove that F can be extended continuously to $\mathbf{B}^2$. Let the ρ_n be given by 1.3. Let us put $A_n=\{x\in\mathrm{Int}(\mathbf{B}^2)|\|x-x_0\|<\rho_n\}$ and $B_n=\{x\in\mathrm{Int}(\mathbf{B}^2)|\|x-x_0\|>\rho_n\}$. For n big enough B_n contains 0,in particular $F(B_n\backslash\{0\})$ is unbounded in $\mathbf{R}^2$. We call a_n and b_n the ends of the arc $\alpha_{\rho_n}(x_0) = \{x \in \mathbf{B}^2 | \|x - x_0\| = \rho_n\}$. By 1.3, we can extend $F|\alpha_{\rho_n}(x_0)\backslash\{a_n, b_n\}$ to $\alpha_{\rho_n}(x_0)$ by continuity, we will still call F this extension. Since $F(a_n)$ and $F(b_n)$ are in $\partial U \subset C$ and the length of $F(\alpha_{\rho_n}(x_0))$ goes to 0, given $\varepsilon > 0$ we can find n_0 such that for each $n \geq n_0$ there exists a compact connected subset $K_n \subset \mathbf{R}^2\backslash U$ containing $F(a_n)$ and $F(b_n)$ and verifying $\operatorname{diam}(K_n \cup F[\alpha_{\rho_n}(x_0)]) < \varepsilon$. We can apply corollary **3**.3.8, to conclude that $F(A_n)$ and $F(B_n\backslash\{0\})$ are contained in distinct connected components of $\mathbf{R}^2\backslash(K_n \cup F[\alpha_{\rho_n}(x_0)])$—since F is a homeomorphism $F(A_n)$ and $F(B_n\backslash\{0\})$ are the two connected components of $F(U\backslash\{0\})\backslash F[\alpha_{\rho_n}(x_0)]$! Since $F(B_n\backslash\{0\})$ is unbounded, we obtain that $F(A_n)$ is contained in a bounded component of $\mathbf{R}^2\backslash(K_n \cup F[\alpha_{\rho_n}(x_0)])$ and therefore it must have a diameter less than ε. The rest of the theorem is an exercise for the reader. □

Corollary 1.5. (Carathéodory). *Let $F : \operatorname{Int}(\mathbf{B}^2) \tilde{\to} U$ be a biholomorphic isomorphism onto an open subset of $\mathbf{S}^2$. Then the map F can be extended to a continuous map $\overline{F} : \mathbf{B}^2 \tilde{\to} \overline{U}$ if and only if ∂U is locally connected. Moreover, if $\overline{F}$ exists, we have $\overline{F}(\partial \mathbf{B}^2) = \partial U$.*

Proof. If ∂U is connected, theorem 1.4 shows that F has a continuous extension. Conversly, if there exists a continuous extension $\overline{F}$, we must have $\overline{F}(\partial \mathbf{B}^2) = \partial U$. But a Hausdorff image—under a continuous map—of a locally connected compact space must be locally connected. □

2. Characterisation of open subsets of the sphere with connected complement.

The following theorem is a form of the Riemann mapping theorem— see **3**.4.4 and theorems 13.11 and 14.8 of the second edition of Rudin's book "Real and complex analysis".

Theorem 2.1. *Let U be a non empty open connected subset of $\mathbf{S}^2$ such that $\mathbf{S}^2\backslash U$ is connected and contains at least two points. There exists a biholomorphic isomorphism $F : \operatorname{Int}(\mathbf{B}^2) \tilde{\to} U$. Moreover, if $x \in U$ is given, we can choose F such that $F(0) = x$.*

We obtain, using **3**.4.4, the following theorem.

Theorem 2.2. *Let U be a non empty open connected subset of $\mathbf{S}^2$ such that $\mathbf{S}^2\backslash U$ is connected, then U is homeomorphic either to $\mathbf{S}^2$ or to $\mathbf{R}^2$.*

Proof. If $\mathbf{S}^2\backslash U$ has at least two points this follows from 2.1. The rest of the theorem is trivial. □

Theorem 2.3. *Let U be an open connected subset of $\mathbf{S}^2$. The following statements are equivalent:*

(i) $\mathbf{S}^2 \backslash U$ *is connected;*
(ii) $[U, \mathbf{S}^1] = 0$*;*
(iii) U *is simply connected.*

Moreover, if U is non empty and is not the whole of $\mathbf{S}^2$, then it is homeomorphic to $\mathbf{R}^2$.

3.The Schoenflies theorems for curves.

Theorem 3.1. (Schoenflies). *Let γ be a simple closed curve in $\mathbf{S}^2$, and let V be a connected component of $\mathbf{S}^2 \backslash \gamma$. The closure $V \cup \gamma$ of V in $\mathbf{S}^2$ is homeomorphic to $\mathbf{B}^2$.*

In fact, we will prove the following stronger version of 3.1.

Theorem 3.2. *Let γ be a simple closed curve in $\mathbf{S}^2$, and let V be a connected component of $\mathbf{S}^2 \backslash \gamma$. There exists a biholomorphic isomorphism $F : \mathrm{Int}(\mathbf{B}^2) \tilde{\to} V$ Any such F extends to a homeomorphism $\overline{F} : \mathbf{B}^2 \tilde{\to} V \cup \gamma$.*

Proof. Remark that $\mathbf{S}^2 \backslash V$ is connected because it is the closure of the other connected component of $\mathbf{S}^2 \backslash \gamma$. Now, the existence of F follows from 2.1. By Carathéodory's theorem 1.4, we know that F extends continuously to a surjective map $\overline{F} : \mathbf{B}^2 \to V \cup \gamma$, with $\overline{F}(\partial \mathbf{B}^2) = \gamma$. It remains to show that $\overline{F}$ is injective on $\mathbf{B}^2$. Suppose there exists $x, y \in \mathbf{B}^2, x \neq y$, such that $\overline{F}(x) = \overline{F}(y)$. We choose α an arc contained in $\mathbf{B}^2$ joining x to y and intersecting $\partial \mathbf{B}^2$ only in its ends x and y (see Figure 3.1). The image $\overline{F}(\alpha)$ is in fact a simple closed curve contained in $V \cup \gamma$ and intersecting γ only in the point $\overline{F}(x) = \overline{F}(y)$. The connected set $\mathbf{S}^2 \backslash (V \cup \gamma)$ must be contained in one of the two connected components of $\mathbf{S}^2 \backslash \overline{F}(\alpha)$. Of course, the other connected component O of $\mathbf{S}^2 \backslash \overline{F}(\alpha)$ is contained in $V \cup \gamma$ and, since it is open, it must be contained in V. The set O is also a connected component of $V \backslash \overline{F}(\alpha)$. Since $F : \mathrm{Int}(\mathbf{B}^2) \to V$ is a homeomorphism, $F^{-1}(O)$ must be one of the two connected components of $\mathrm{Int}(\mathbf{B}^2) \backslash \alpha$, we call this connected component R. We have $\overline{F}(\overline{R} \cap \partial \mathbf{B}^2) = \overline{O} \cap \overline{F}(\partial \mathbf{B}^2) = (O \cup \overline{F}(\alpha)) \cap \gamma = \{\overline{F}(x)\}$. But $\overline{R} \cap \partial \mathbf{B}^2$ is one of the two segments delimited by x and y on $\partial \mathbf{B}^2$. Up to now, we have shown that if $\overline{F}$ is not injective on $\partial \mathbf{B}^2$, then it has to be constant on a non trivial segment of $\partial \mathbf{B}^2$. The next lemma will show that this is absurd. $\square$

Figure 3.1

Lemma 3.3. *Let α be a non trivial segment contained in $\partial\mathbf{B}^2$. A continuous map* $\mathrm{Int}(\mathbf{B}^2)\cup\alpha \to \mathbf{C}$*, which is constant on α and holomorphic on* $\mathrm{Int}(\mathbf{B}^2)$ *is in fact constant.*

Proof. Since there is a biholomorphic isomorphism of $\mathbf{S}^2$ which takes $\mathrm{Int}(\mathbf{B}^2)$ on the upper half plane $\mathbf{H} = \{x+iy \in \mathbf{C} | y > 0\}$, we can assume that we have a continuous map $h : \mathbf{H}\cup\alpha \to \mathbf{C}$, where α is a non trivial segment in $\mathbf{R} = \partial\mathbf{H}$. We want to show that h is constant if it is constant on α and holomorphic on $\mathbf{H}$. By multiplying h by a constant, we can assume that the value taken by α under h is real. By Schwarz reflection principle—see theorem 11.17 of Rudin's book—there exists a holomorphic extension $\overline{h}$ of h defined on $\mathbf{H} \cup (\alpha\backslash\partial\alpha) \cup \mathbf{H}_-$, where $\mathbf{H}_-$ is the lower half plane $\{x + iy \in \mathbf{C}| y > 0\}$. But this holomorphic function $\overline{h}$ is contant on the non discrete subset $\alpha\backslash\partial\alpha$, hence it must be constant. □

As a corollary of 3.2, we obtain:

Corollary 3.4. *Any simple closed curve in $\mathbf{S}^2$ can be approximated by an analytic simple closed curve.*

Corollary 3.5. *Let γ and γ' be simple closed curves in $\mathbf{S}^2$. Let V (resp. V') be a connected component of $\mathbf{S}^2\backslash\gamma$ (resp. $\mathbf{S}^2\backslash\gamma'$). There exists an analytic isomorphism $G : V \tilde{\to} V'$ Any such G extends to a homeomorphism $V \cup \gamma \tilde{\to} V' \cup \gamma'$. Moreover,if x (resp. x') is in V (resp. V'), we can find G such that $g(x) = x'$.*

Proof. By 2.1 and 3.2, there exists analytic isomorphisms $F : \mathrm{Int}(\mathbf{B}^2) \tilde{\to} V$ and $F' : \mathrm{Int}(\mathbf{B}^2) \tilde{\to} V$ such that $F(0) = x$ and $F'(0) = x'$. The map $F' \circ F^{-1}$ is an analytic isomorphism of V on V' with $F' \circ F^{-1}(x) = x'$. Let now $G : V \tilde{\to} V'$ be an arbitrary analytic isomorphism. By 3.2, the analytic isomorphisms F and $G\circ F$ extend respectively to homeomorphisms $\overline{F} : \mathbf{B}^2 \tilde{\to} V \cup \gamma$ and $\overline{G \circ F} : \mathbf{B}^2 \tilde{\to} V \cup \gamma'$. Of course, the required extension of G is $\overline{G \circ F} \circ \overline{F}^{-1}$. □

Corollary 3.6. *Let γ and γ' be simple closed curves in $\mathbf{S}^2$. Let x be a point in $\mathbf{S}^2\backslash(\gamma\cup\gamma')$. Denote by V (resp. V') the connected component of x in $\mathbf{S}^2\backslash\gamma$ (resp. $\mathbf{S}^2\backslash\gamma'$). There exists a homeomorphism $H : V \cup \gamma \tilde{\to} V' \cup \gamma'$, which is the identity near x. Of course $H(V) = V'$ and $H(\gamma) = \gamma'$.*

Proof. By 3.5, there exists an analytic isomorphism $G : V \tilde{\to} V'$, which extends to a homeomorphism $\overline{G} : V\cup\gamma \tilde{\to} V\cup\gamma'$, and verifies $G(x) = x$. By **4A**.2.4, we can find a homeomorphism $H : V\cup\gamma \tilde{\to} V' \cup\gamma'$ equal to $\overline{G}$ outside a compact subset of V and equal to the identity near x. By construction, we have $H(V) = V'$ and $H(\gamma) = \gamma'$. But in fact, if $H : V \cup \gamma \tilde{\to} V' \cup \gamma'$ is any homeomorphism, by theorem **2**.4.10, we must have $H(V) = V'$ and $H(\gamma) = \gamma'$. □

Lemma 3.7. (Alexander's trick). *Let γ and γ' be simple closed curves in $\mathbf{S}^2$. Let U (resp. U') be a connected component of $\mathbf{S}^2\backslash\gamma$ (resp. $\mathbf{S}^2\backslash\gamma'$). Any homeomorphism of γ on γ' can be extended to a homeomorphism of $U \cup \gamma$ on $U' \cup \gamma'$.*

Proof. Let $g : \gamma \tilde{\to} \gamma'$ be a homeomorphism. Let $F : \mathbf{B}^2 \tilde{\to} U \cup \gamma$ and $F' : \mathbf{B}^2 \tilde{\to} U \cup \gamma'$ be given by 3.1 or 3.2. The homeomorphism $h = {F'}^{-1} \circ g \circ F : \mathbf{S}^1 \tilde{\to} \mathbf{S}^1$ can be extended to a homeomorphism $H : \mathbf{B}^2 \tilde{\to} \mathbf{B}^2$ by $H(0) = 0$ and $H(x) = \|x\| H(x/\|x\|), x \neq 0$— this is Alexander's trick. The desired extension of g is $F' \circ g \circ F^{-1}$. □

Theorem 3.8. (Schoenflies). *Let γ and γ' be simple closed curves in $\mathbf{S}^2$. Any homeomorphism of γ on γ' can be extended to a homeomorphism of $\mathbf{S}^2$ on itself.*

Proof. Let U_1, U_2 (resp. U_1', U_2') be the connected components of $\mathbf{S}^2 \backslash \gamma$ (resp. $\mathbf{S}^2 \backslash \gamma'$). Let $h : \gamma \tilde{\to} \gamma'$ be a homeomorphism. By 3.7, there exists, for $i = 1, 2$, an extension of h to a homeomorphism $H_i : U_i \cup \gamma \tilde{\to} U_i' \cup \gamma'$. It suffices to piece together H_1 and H_2 to obtain an extension of h to a homeomorphism of $\mathbf{S}^2$ on itself. □

Corollary 3.9. (Schoenflies). *Let γ and γ' be simple closed curves in $\mathbf{S}^2$. Let x be a point in $\mathbf{S}^2 \backslash (\gamma \cup \gamma')$. There exists a homeomorphism $H : \mathbf{S}^2 \tilde{\to} \mathbf{S}^2$ which is the identity near x and such that $H(\gamma) = \gamma'$.*

Proof. Let U (resp. U') be the connected component of x in $\mathbf{S}^2 \backslash \gamma$ (resp. $\mathbf{S}^2 \backslash \gamma'$). Let V (resp. V') be the other connected component of x in $\mathbf{S}^2 \backslash \gamma$ (resp. $\mathbf{S}^2 \backslash \gamma'$). By 3.6, there exists a homeomorphism $H_1 : U \cup \gamma \tilde{\to} U' \cup \gamma'$ which is the identity near x. By 3.7, the restriction $H_1 | \gamma$ can be extended to a homeomorphism $H_2 : V \cup \gamma \tilde{\to} V' \cup \gamma'$. If we piece together H_1 and H_2, we obtain the desired homeomorphism. □

Warning 3.10. In 3.9, we cannot impose arbitrarily the map of γ on γ'.

Theorem 3.11. (Schoenflies with compact support). *Let γ and γ' be simple closed curves in $\mathbf{R}^2$. There exists a homeomorphism with compact support $H : \mathbf{R}^2 \tilde{\to} \mathbf{R}^2$ such that $H(\gamma) = \gamma'$.*

Proof. It suffices to remark that $\mathbf{R}^2 \subset \mathbf{S}^2 = \mathbf{R}^2 \cup \{\infty\}$ and to apply 3.9 with $x = \infty$. □

Theorem 3.12. (Schoenflies). *Let γ and γ' be simple closed curves in $\mathbf{R}^2$. Any homeomorphism of γ on γ' can be extended to a homeomorphism of $\mathbf{R}^2$ on itself.*

Proof. Let $g : \gamma \tilde{\to} \gamma$ be a homeomorphism. Chose $F : \mathbf{R}^2 \tilde{\to} \mathbf{R}^2$ (resp. $F' : \mathbf{R}^2 \tilde{\to} \mathbf{R}^2$) a homeomorphism such that $F(\mathbf{S}^1) = \gamma$ (resp. $F(\mathbf{S}^1) = \gamma'$). The homeomorphism $h = F'^{-1} \circ g \circ F : \mathbf{S}^1 \tilde{\to} \mathbf{S}^1$ can be extended to a homeomorphism H of $\mathbf{R}^2$ by $H(0) = 0$ and $H(x) = \|x\| H(x/\|x\|), x \neq 0$—another instance of Alexander's trick. The extension of g to $\mathbf{R}^2$ is $F' \circ H \circ F^{-1}$. □

Corollary 3.13. *Any imbedding $j : \mathbf{B}^2 \hookrightarrow \mathbf{R}^2$ can be extended to a homeomorphism of $\mathbf{R}^2$ on itself.*

Proof. By 3.12, the restriction $j | \mathbf{S}^1$ can be extended to a homeomorphism $J : \mathbf{R}^2 \tilde{\to} \mathbf{R}^2$. It follows from **2**.4.4 that $j(\mathbf{B}^2) = J(\mathbf{B}^2)$. This implies that we can piece together $j | \mathbf{B}^2$ and $J | \mathbf{R}^2 \backslash \mathrm{Int}(\mathbf{B}^2)$ to obtain a homeomorphism of $\mathbf{R}^2$. □

4.The Schoenflies theorems for an arc.

We now turn to the case of an arc in $\mathbf{S}^2$ or $\mathbf{R}^2$.

Lemma 4.1. *Let α be a simple arc in $\mathbf{S}^2$, there exists a simple closed curve γ in $\mathbf{S}^2$ containing α. Moreover, if $x_1, \ldots, x_n$ are in $\mathbf{S}^2 \backslash \alpha$, we can choose γ disjoint from $\{x_1, \ldots, x_n\}$.*

Proof. Since the connected open set $\mathbf{S}^2 \backslash \alpha$ has a connected complement, we can apply the Riemann mapping theorem 2.1 to find an analytic isomorphism $F : \mathrm{Int}(\mathbf{B}^2) \tilde{\rightarrow} \mathbf{S}^2 \backslash \alpha$. Since α is locally connected, by Carathéodory's theorem 1.4, we can extend F to a continuous surjective map $\overline{F} : \mathbf{B}^2 \rightarrow \mathbf{S}^2$. Let $a, b \in \partial \mathbf{B}^2$ be such that $\overline{F}(a)$ and $\overline{F}(b)$ are the endpoints of α. We can find an arc β in $\mathbf{B}^2$ joining a and b and such that $\beta \backslash \{a, b\} \subset \mathrm{Int}(\mathbf{B}^2) \backslash \{F^{-1}(x_1), \ldots, F^{-1}(x_n)\}$. We can take $\gamma = \overline{F}(\beta) \cup \alpha$. □

Lemma 4.2. *Let α be a simple arc in $\mathbf{R}^2$, there exists a simple closed curve γ in $\mathbf{R}^2$ containing α. Moreover, if $x_1, \ldots, x_n$ are in $\mathbf{R}^2 \backslash \alpha$, we can choose γ disjoint from $\{x_1, \ldots, x_n\}$.*

Theorem 4.3. (Schoenflies). *Let α and α' be simple arcs in $\mathbf{S}^2$. Any homeomorphism of α on α' can be extended to a homeomorphism of $\mathbf{S}^2$ on itself.*

Proof. Let γ and γ' be simple closed curves containing respectively α and α'. It is easy to show that a homeomorphism of α on α' can be extended to a homeomorphism of γ on γ'. Then we can apply theorem 3.8 to extend this last homeomorphism to $\mathbf{S}^2$. □

Our goal now is to prove that, in the case of $\mathbf{R}^2$, we can extend a homeomorphism between two arcs to a homeomorphism with compact support in $\mathbf{R}^2$.

Lemma 4.4. *If α is a simple arc in $\mathbf{R}^2$, there exists a homeomorphism with compact support taking α to a subarc of $\mathbf{S}^1 \subset \mathbf{R}^2$.*

Proof. By 4.2, there exists a simple closed arc $\gamma \subset \mathbf{R}^2$ and containing α. Schoenflies theorem with compact support 3.11 gives a homeomorphism with compact support of $\mathbf{R}^2$ taking γ to $\mathbf{S}^1$. □

Lemma 4.5. *If β and β' are two simple arcs in $\mathbf{S}^1$, there exists a homeomorphism with compact support of $\mathbf{R}^2$ taking β to β'.*

Proof. The proof is harder to say than to find, it is left as an Exercise! □

Corollary 4.6. *If α is a simple arc in $\mathbf{R}^2$, there exists a homeomorphism with compact support of $\mathbf{R}^2$ taking α to the arc $[0,1] \subset \mathbf{R} = \mathbf{R} \times 0 \subset \mathbf{R}^2$.*

proof. This is an easy consequence of 4.4 and 4.5. □

Lemma 4.7. *Any homeomorphism of $[0,1]$ can be extended to a homeomorphism with compact support of $\mathbf{R}^2$.*

Proof. Let h be a homeomorphism of $[0,1]$ on itself. We first consider the case $h(0) = 0$ and $h(1) = 1$. We extend h to a homeomorphism of $\mathbf{R}_+$ by defining it as the identity on $[1, \infty[$. If we define $H : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ by $H(0) = 0$ and $H(x) = [h(\|x\|)/\|x\|]x$, for $x \neq 0$, it can be checked that H is a homeomorphism of $\mathbf{R}^2$ which extends h and is the identity outside $\mathbf{B}^2$. To finish the proof, it suffices to construct a homeomorphism G with compact support of $\mathbf{R}^2$ such that $G([0,1]) = [0,1], G(0) = 1$ and $G(1) = 0$. This is left to the reader. □

We can sum up the results obtained in the following form

Theorem 4.8. *Let α and α' be two simple arcs in $\mathbf{R}^2$. Any homeomorphism between α and α' can be extended to a homeomorphism with compact support in $\mathbf{R}^2$. In particular, there exists a homeomorphism with compact support in $\mathbf{R}^2$ taking α to α'.*