APPENDIX 3A
SOME PROPERTIES OF COMPACT OR OPEN SUBSETS IN A LOCALLY COMPACT AND LOCALLY CONNECTED SPACE

1. On connected components of a compact space.

Theorem 1.1. *Let K be a compact Hausdorff space. If C is a connected component of K, then it is the intersection of the open closed subsets of K which contain it.*

Proof. Let $(F_i)_{i\in I}$ be the family of closed and open subsets of K containing C. If the compact set $C' = \bigcap_{i\in I} F_i$ is not equal to C, then it is not connected. Hence we can write $C' = C_1 \cup C_2$, with C_1 and C_2 compact non empty and disjoint. Let U_1 and U_2 be disjoint open subsets of K containing respectively C_1 and C_2. Since the family $(F_i)_{i\in I}$ is stable under finite intersections and $\bigcap_{i\in I} F_i$ is contained in $U_1 \cup U_2$, we can find an index $i_0 \in I$ such that $F_{i_0} \subset U_1 \cup U_2$. Since C is connected and contained in $U_1 \cup U_2$, it cannot meet both U_1 and U_2. Suppose, for example, that $U_2 \cap C = \emptyset$; it follows that the subset $F_{i_0} \cap U_1 = F_{i_0} \backslash U_2$ is closed and open and contains C. In particular, we have $C' \subset F_{i_0} \cap U_1 \subset U_1$. This is impossible, since C_2 is a non empty subset of C' which does not meet U_1. □

Corollary 1.2. *Let F be a locally compact space. If C is a connected component of F which is compact, then it is the intersection of the compact open subsets of F which contain it.*

Proof. Let $K \subset F$ be a compact neighborhood of C. Applying 1.1 to the pair $C \subset K$, we obtain a family $(K_i)_{i\in I}$ of compact subsets of K, which are open relative to K and such that $C = \bigcap_{i\in I} K_i$. Since $\mathrm{Int}(K)$—the interior of K in F—contains C, we can find a finite subset $I_0 \subset I$ such that $\bigcap_{j\in I_0} K_j \subset \mathrm{Int}(K)$. It follows that for each $i \in I$, the compact subset $K_i' = K_i \cap (\bigcap_{i\in I_0} K_i)$ is open relative to $\mathrm{Int}(K)$ and hence is open in F. The intersection of the closed open subsets K_i' of F is precisely C. □

Corollary 1.3. *Let F be a locally compact space. If $C_1, \ldots, C_k$ are distinct compact connected component of F, we can find a partition of F into k open and closed subsets $F_1, \ldots, F_k$ such that $C_i \subset F_i$, for $i = 1, \ldots, k$.*

Proof. Exercise! □

Corollary 1.4. *Let F be a closed subset of the locally compact space X. If $C_1, \ldots, C_k$ are distinct compact connected component of F, we can find k disjoint open and relatively compact subsets $U_1, \ldots, U_k$ such that $C_i \subset U_i$ for $1, \ldots, k$, and ∂U_i is a compact subset of $X \backslash F$, for $i = 1, \ldots, k$.*

Proof. By 1.3, we can find $K_1, \ldots, K_k$ disjoint compact open subsets in F, with $C_i \subset K_i$, for $i = 1, \ldots, k$. By the definition of the relative topology, there exists $V_1, \ldots, V_k$ open subsets of X such that $V_i \cap F = K_i$, for $i = 1, \ldots, k$. By the local compactness of X, we can find $U_1, \ldots, U_k$ disjoint relatively compact open subsets in X, with $K_i \subset U_i \subset \overline{U}_i \subset V_i$, for $i = 1, \ldots, k$. Since $U_i \cap F = \overline{U}_i \cap F = K_i$, we must have $\partial U_i \cap F = \emptyset$. □

2. On neighborhoods of compact subsets.

Proposition 2.1. *Let K be a compact subset in a locally connected space X. Each neighborhood U of K in X contains an open neighborhood V of K which has a finite number of connected components. Moreover, by dropping out the components of V which do not intersect K, we can assume that each component of V intersects K.*

Proof. Since X is locally connected and K is compact, we can find a finite number $V_1, \ldots,$ V_n of open connected subsets of X such that $K \subset V = \bigcup_{i=1}^{k} V_i \subset U$, The open subset V has a finite number of connected components since it is a finite union of connected sets. □

Proposition 2.2. *Let K be a compact subset in a locally connected locally compact space X. Each neighborhood U of K in X contains a compact neighborhood C of K which has a finite number of connected components; moreover, we can assume that each component of C intersects K.*

Proof. Let A be a compact neighborhood of K contained in U. If V is a neighborhood of K given by 2.1 and contained in A, the closure of V is the required neighborhood C. □

Proposition 2.3. *Let K be a compact subset in the connected locally connected locally compact space X. If U is a neighborhood of K in X, there is only a finite number of connected components of $X \backslash K$ which are not contained in U.*

Proof. We will assume that K is non empty. Since X is locally compact, we can assume that the boundary ∂U of U in X is compact. Let C be a connected component of $X \backslash K$. Since X is locally connected, the component C is open. This implies $\overline{C} \backslash C \neq \emptyset$, because X is connected and $X \backslash C \supset K \neq \emptyset$. In particular, we must have $\overline{C} \cap K \neq \emptyset$, and hence $C \cap U \neq \emptyset$. It follows that a connected component of $X \backslash K$ which is not contained in U must meet ∂U. The connected components of $X \backslash K$ which meet ∂U form in fact a covering of ∂U by disjoint open sets; since ∂U is compact, there is only a finite number of such components. □

Proposition 2.4. *Let X be a connected, locally connected and locally compact space. Let $K \subset U$ be respectively a compact and an open subset of X. We can find a compact subset K' of X such that:*

(i) *$K \subset K' \subset U$;*
(ii) *each connected component of K' intersects K;*
(iii) *$X \backslash K'$ has a finite number of connected components;*
(iv) *if C is a connected component of $X \backslash K'$ which is relatively compact in X, then it meets $X \backslash U$.*

Proof. Let W be a relatively compact subset of K in X with $\overline{W} \subset U$. Let $V_1, \dots, V_l$ be the connected components of $X \backslash K$ which are not contained in W—see 2.3. By renumbering the V_i's, we can find an integer k, with $0 \leq k \leq l$, such that for $i \leq k$ either V_i meets $X \backslash U$ or V_i is not relatively compact in X, and for $i > k$ the connected component V_i is relatively compact in X and is contained in U. The set $K' = X \backslash \bigcup_{i \leq k} V_i$ is compact, since it is closed and contained in the compact subset $\overline{W} \cup (\bigcup_{i>k} V_i)$. Since $\overline{W} \cup (\bigcup_{i \leq k} V_i) \subset U$, the set K' verifies (i); moreover, since $X \backslash K' = \bigcup_{i \leq k} V_i$, it verifies also (iii) and (iv). To see that K' verifies also (ii), we remark that K' is the union of K and of some connected components of $X \backslash K$ and that for a connected component C of $X \backslash K$, the connected set $\overline{C}$ is contained in $K \cup C$ and meets K. □

Corollary 2.5. *Let C be a compact subset of the connected locally connected locally compact space X. We can find a basis of compact neighborhoods C' of C in X such that:*

(i) *C' has a finite number of connected components, and each one of them meets C;*
(ii) *$X \backslash C'$ has a finite number of connected components.*

Proof. Let $U \supset C$ be open and let K be a compact neighborhood of C contained in U and having a finite number of connected components, each one of them meeting C—see 2.2. If we apply proposition 2.4 to the pair $K \subset U$, we obtain a neighborhood C' of C contained in U and having the required properties. □

Proposition 2.6. *Let U be an open subset of the connected locally connected locally compact space X. Given a compact subset $K \subset U$ and compact connected components $C_1, \dots, C_l$ of $X \backslash U$, there exists a compact subset K' of X such that:*

(i) *$K \subset K' \subset U$;*

(ii) *$X\backslash K'$ has a finite number of connected components;*
(iii) *the C_i's are contained in distinct relatively compact connected components of $X\backslash K'$;*
(iv) *a relatively compact connected component of $X\backslash K'$ contains a compact connected component of $X\backslash U$.*

Proof. By corollary 1.4, we can find $V_1, \ldots, V_l$ open subsets of X such that $V_i \supset C_i$, and ∂V_i is a compact subset of U. Let $A = K \cup (\bigcup_{i=1}^{l} \partial V_i) \subset U$. If we apply proposition 2.4, we obtain a compact set K' such that:

(i) $A \subset K' \subset U$;
(ii) $X\backslash K'$ has a finite number of connected components;
(iv) a relatively compact connected C component of $X\backslash K'$ meets $X\backslash U$. It follows that C contains a connected component of $X\backslash U$, which must be compact.

Since each V_i is relatively compact in X and $\partial V_i \subset K'$, the set $V_i \cap (X\backslash K')$ is a union of relatively compact components of $X\backslash K'$. □