CHAPTER 3
SOME PLANE TOPOLOGY

In this chapter we study in a deeper way the case of the plane $\mathbf{R}^2$.

1. Jordan's theorem for the circle.

Theorem 1.1. (Jordan's theorem for the circle). *Let $j : \mathbf{R}^2 \hookrightarrow \mathbf{S}^1$ be a continuous injective map. The set $\mathbf{R}^2 \backslash j(\mathbf{S}^1)$ has exactly two connected components, one bounded and the other one unbounded.*

Proof. We know already that $\mathbf{R}^2 \backslash j(\mathbf{S}^1)$ has at least two connected components one of them being unbounded. We will suppose that there exists at least three connected components U_1, U_2 and U_3 in $\mathbf{R}^2 \backslash j(\mathbf{S}^1)$ and we will obtain a contradiction. Using **2**.5.2, we can find two injective arcs $\alpha_1, \alpha_2 : [0,1] \to \mathbf{R}^2$ such that $\partial\alpha_i \subset j(\mathbf{S}^1), \alpha_i \backslash \partial\alpha_i \subset U_i$, for $i = 1, 2$, and the four points $A_1 = \alpha_1(0), A_2 = \alpha_2(0), B_1 = \alpha_1(1)$, and $B_2 = \alpha_2(1)$ are distinct and cyclically ordered on $j(\mathbf{S}^1)$ (see figure 1.1). Let us call $[A_1, A_2]$ and $[B_1, B_2]$ the two disjoint closed subarcs of $j(\mathbf{S}^1)$ determined by the four points A_1, A_2, B_1 and B_2. The set $\gamma = [A_1, A_2] \cup \alpha_2([0,1]) \cup [B_2, B_1] \cup \alpha_1([1,0])$ is homeomorphic to $\mathbf{S}^1$; in particular, we know that $\mathbf{R}^2\backslash\gamma$ is not connected. Remark that, by construction, we have $U_3 \subset \mathbf{R}^2\backslash\gamma$. Let V be the connected component of $\mathbf{R}^2\backslash\gamma$ containing the connected set U_3. Since $U_3 \cup]B_2, A_1[\cup]B_1, A_2[$ is connected—contained between U_3 and its closure $U_3 \cup j(\mathbf{S}^1)$—and it is disjoint from γ, it is contained in V. It follows that $j(\mathbf{S}^1) \subset V \cup \gamma$. Let W be a connected component of $\mathbf{R}^2\backslash\gamma$ distinct from V. The set W does not intersect $j(\mathbf{S}^1)$, but, since $\overline{W} \supset \alpha_1 \cap \alpha_2$, it intersects two distinct connected components of its complement, namely U_1 and U_2. This is a contradiction. □

Remark 1.2. This proof, due to Whyburn, is quite magic. We will give a more palatable proof later on.

Definition 1.3. (Interior and exterior). Let $\gamma \subset \mathbf{R}^2$ be a simple closed curve. Its interior component— or simply interior, if there no possible confusion— is the bounded component of $\mathbf{R}^2\backslash\gamma$, it is denoted by $\mathbb{I}(\gamma)$. Its exterior component— or simply exterior, if there no possible confusion— is the unbounded component of $\mathbf{R}^2\backslash\gamma$, it is denoted by $\mathbb{E}(\gamma)$.

2. The set of homotopy classes of maps to the circle.

We identify $\mathbf{R}^2$ to $\mathbf{C}$. In this identification, $\mathbf{S}^1$ becomes the unit circle in $\mathbf{C}$, and is naturally a group for the multiplication of complex numbers. In fact, $\mathbf{S}^1$ is a topological compact metric abelian group. If $f, g : X \to \mathbf{S}^1$ are two continuous maps the pointwise product $fg : X \to \mathbf{S}^1, x \mapsto f(x)g(x)$, is also a continuous map. In particular, the set of continuous maps from X to $\mathbf{S}^1$, denoted by $\mathcal{C}(X, \mathbf{S}^1)$, is an abelian group for this multiplication.

Lemma 2.1. *The group structure on $\mathcal{C}(X, \mathbf{S}^1)$ induces on $[X, \mathbf{S}^1]$ a structure of abelian group. The neutral element in $[X, \mathbf{S}^1]$ is the class of null homotopic maps.*

Proof. Exercise! □

Notation 2.2. The (surjective) exponential map $\mathbf{R} \to \mathbf{S}^1, x \mapsto e^{2i\pi x}$, will be denoted by exp. We have $\exp(x) = 1$ if and only if $x \in \mathbf{Z}$. Moreover, exp has a continuous inverse above any proper subset of $\mathbf{S}^1$.

Theorem 2.3. *A map $f : X \to \mathbf{S}^1$ is null homotopic if and only if there exists a lifting $\bar{f}$ of f to a continuous map $\bar{f} : X \to \mathbf{R}$ with $\exp \circ \bar{f} = f$.*

We need some lemmas to prove theorem 2.3.

Lemma 2.4. *Let K, X, Y be three spaces with K compact and Y metric. Endow the set $\mathcal{C}(K, Y)$ of continuous maps from K to Y with the topology of uniform convergence. There is a natural bijection between $\mathcal{C}(X \times K, Y)$ and $\mathcal{C}(X, \mathcal{C}(K, Y))$; it associates to the map $f : X \times K \to Y$ the map $f^{\#} : X \to \mathcal{C}(K, Y)$ defined by $f^{\#}(x)(k) = f(x, k)$.*

Proof. Exercise! □

Lemma 2.5. *The map $\varphi \mapsto \exp \varphi$ establishes a homeomorphism between the spaces $\mathcal{C}_0([0,1], \mathbf{R}) = \{\varphi \in \mathcal{C}([0,1], \mathbf{R}) | \varphi(0) = 0\}$ and $\mathcal{C}_1([0,1], \mathbf{S}^1) = \{\theta \in \mathcal{C}([0,1], \mathbf{S}^1) | \theta(0) = 1\}$ (the topology on set of maps is the topology of uniform convergence).*

Proof. (Sketch). The map is a continuous homomorphism of topological groups. It is injective, because $\exp \circ \varphi = 1$ implies $\varphi(x) \in \mathbf{Z}, \forall x \in [0,1]$. By the connectedness of $[0,1]$, this implies $\varphi(x) = \varphi(0) = 0, \forall x \in [0,1]$. Since the two groups are connected—exercise!— it suffices to check that there is a continuous section on a neighborhood of the constant map 1 in $\mathcal{C}_1([0,1], \mathbf{S}^1)$. Such a section is obtained by composition with a fixed continuous section of $\exp : \mathbf{R} \to \mathbf{S}^1$ above $\{z \in \mathbf{S}^1 || z - 1| < 2\}$. □

The following theorem is more general than 2.3.

Theorem 2.6. *Let $F : X \times [0,1] \to \mathbf{S}^1$ be a continuous map. If $f : X \to \mathbf{R}$ is a continuous map such that $F | X \times \{0\} = \exp \circ f$, then there exists a continuous map $\bar{F} : X \times [0,1] \to \mathbf{R}$ with $\exp \circ \bar{F} = F$ and $\bar{F} | X \times \{0\} = f$. In particular, $F | X \times \{1\}$ can be lifted to a continuous map $X \to \mathbf{R}$.*

Proof. Define $F' : X \times [0,1] \to \mathbf{S}^1$ by $F'(x,t) = F(x,t) \exp(-f(x))$. We have $F' | X \times \{0\} = 1$. It follows from 2.4 and 2.5, that we can lift F' to a continuous map $\bar{F}' : X \times [0,1] \to \mathbf{R}$, with $\bar{F}' | X \times \{0\} = 0$. The map $\bar{F}$ is defined by $\bar{F}(x,t) = \bar{F}'(x,t) + f(x)$. □

Proof of 2.3. Since a constant map can be lifted, by 2.6, a null homotopic map $f : X \to \mathbf{S}^1$ can be lifted to a continuous map $\bar{f} : X \to \mathbf{R}$ such that $\exp \circ \bar{f} = f$. Of course, since $\mathbf{R}$ is contractible, a map of the form $\exp \circ \bar{f} = f$ is null homotopic. □

Corollary 2.7. *For each topological space X, the group $[X, \mathbf{S}^1]$ has no torsion.*

Proof. Let $f : X \to \mathbf{S}^1$ be a continuous map. If $n > 0$ is such that $f^n \sim 0$, by 2.3, there exists a continuous map $g : X \to \mathbf{R}$ with $\exp(g(x)) = f^n(x)$. It follows that the sets $X_j = \{x \in X | f(x) = \exp[g(x) + j]/n\}, j = 0, \ldots, n-1$, form a partition of X into closed sets. A lifting $\bar{f} : X \to \mathbf{R}$ is obtained by defining it on X_j by $\bar{f}(x) = [g(x) + j]/n$. □

Theorem 2.8. *The group* $[\mathbf{S}^1, \mathbf{S}^1]$ *is infinite cyclic. More precisely, the homomorphism* $\mathbf{Z} \to [\mathbf{S}^1, \mathbf{S}^1]$*, which sends* 1 *to the homotopy class of* $\mathrm{Id}_{\mathbf{S}^1}$*, is an isomorphism.*

Proof. Since $\mathrm{Id}_{\mathbf{S}^1}$ is essential by 2.7, the map is injective. Let $f : \mathbf{S}^1 \to \mathbf{S}^1$ be a continuous map. Since $[0,1]$ is contractible, we can lift $f \circ \exp : [0,1] \to \mathbf{S}^1$ to a map $\bar{f} : [0,1] \to \mathbf{R}$, this means that $\exp \circ \bar{f} = f \circ \exp$. In particular $n = \bar{f}(1) - \bar{f}(0) \in \mathbf{Z}$, since $\exp(\bar{f}(1)) = f(\exp(1) = f(\exp(0)) = \exp(\bar{f}(0))$. Remark now that the map $\bar{\varphi}(t) = \bar{f}(t) - nt$ verifies $\bar{\varphi}(0) = \bar{\varphi}(1)$, so we can define a continuous map $\mathbf{S}^1 \to \mathbf{R}$ by $\varphi \circ \exp|[0,1] = \bar{\varphi}$. The formula $\bar{\varphi}(t) = \bar{f}(t) - nt$ gives $\exp(\varphi(z)) = f(z)z^{-n}, \forall z \in \mathbf{S}^1$; this shows that f is homotopic to n times $\mathrm{Id}_{\mathbf{S}^1}$. □

Definition 2.9. (Degree). If $f : \mathbf{S}^1 \to \mathbf{S}^1$ is a continuous map, the degree of f, denoted by $\deg(f$, is the integer n such that f is homotopic to $z \mapsto z^n$.

Theorem 2.10. *Let* X *be a metric space which is a countable union of closed sets* $F_n, n \in \mathbf{N}$*, such that:*

(i) $\forall n \in \mathbf{N}, F_n$ *is connected,*
(ii) $\forall n \in \mathbf{N}, F_n \subset F_{n+1}$*,*
(iii) *each compact subset of* X *is contained in one of the* F_n*—this condition follows, for example, from the stronger one* $F_n \subset \mathrm{Int}(F_{n+1})$*.*

Then we have:

(a) *a continuous map* $f : X \to \mathbf{S}^1$ *is null homotopic if and only if for each* $n \in \mathbf{N}$ *the restriction* $f|F_n$ *is null homotopic,*
(b) *if, for each* $n \in \mathbf{N}$*, we have a continuous map* $f_n : F_n \to \mathbf{S}^1$*, with* $f_{n+1}|F_n \sim f_n$*, then there exists a continuous map* $f : X \to \mathbf{S}^1$ *with* $f|F_n \sim f_n, \forall n \in \mathbf{N}$*.*

In other terms, if we look at the projective system: $\cdots [F_{n+1}, \mathbf{S}^1] \stackrel{r_{n+1}}{\to} [F_n, \mathbf{S}^1] \stackrel{r_n}{\to} \cdots$*, where* $r_{n+1}([f]) = [f|F_n]$*, then* $[X, \mathbf{S}^1]$ *can be identified to the projective limit* $P = \varinjlim([F_n, \mathbf{S}^1], r_n)$*; the map* $[X, \mathbf{S}^1] \to P$ *is given by* $[f] \mapsto ([f|F_n[)_{n\in\mathbf{N}}$*.*

Proof. To prove (a), it suffices to show that if a map $f : X \to \mathbf{S}^1$ is such that, for each $n \in \mathbf{N}$,there exists a continuous lift $\bar{f}_n : F_n \to \mathbf{R}$ with $f|F_n = \exp \circ \bar{f}_n$, then we can define a continuous lift $\bar{f} : X \to \mathbf{R}$ with $f = \exp \circ \bar{f}$. We construct $\bar{f}|F_n$ by induction. We start with $\bar{f}|F_0 = \bar{f}_0$. Suppose that we have constructed $\bar{f}|F_n$. Since $\bar{f}|F_n$ and $\bar{f}_{n+1}|F_n$ are two lifts of f, the map $\bar{f}|F_n - \bar{f}_{n+1}|F_n$ lifts a constant map. By the connectedness of F_n, we obtain $\bar{f}|F_n - \bar{f}_{n+1}|F_n = c_n$, where c_n is a constant. It suffices to define $\bar{f}|F_{n+1}$ as $\bar{f}_{n+1} + c_n$. this shows that we have a well defined lifting $\bar{f} : X \to \mathbf{R}$, with $\bar{f}|F_n$ continuous for each $n \in \mathbf{N}$. Condition (iii) implies that $\bar{f}$ is itself continuous. In fact, since X is metric, it suffices to check continuity on compact subsets, because a convergent sequence together with its limit is a compact subset. Continuity of $\bar{f}$ on compact subsets of X is forced by (iii). To prove (b), we construct $f|F_n$ by induction .We start with $f|F_0 = f_0$. Suppose that we have constructed $f|F_n \sim f_n$, then we have $f|F_n \sim f_{n+1}|F_n$. By the homotopy extension theorem **2**.1.10, we can find a continuous map $f|F_{n+1} : F_{n+1} \to \mathbf{S}^1$, homotopic to f_{n+1}, and whose restriction to F_n is precisely $f|F_n$. The continuity of f is proven in the same lines as we did it above for $\bar{f}$. □

Lemma 2.11. *Let X be a locally connected space, and let $(X_i)_{i\in I}$ be the family of its connected components. The restriction map, $f \mapsto (f|X_i)_{i\in I}$, induces an isomorphism between $[X, \mathbf{S}^1]$ and $\prod_{i\in I}[X_i, \mathbf{S}^1]$.*

Proof. Exercise! □

Corollary 2.12. *Let X be a locally connected metric space which is a countable union of closed sets $F_n, n \in \mathbf{N}$, such that:*

(i) *$\forall n \in \mathbf{N}, \forall C$ connected component of X, the set $C \cap F_n$ is connected,*
(ii) *$\forall n \in \mathbf{N}, F_n \subset F_{n+1}$,*
(iii) *each compact subset of X is contained in one of the F_n.*

Then $[X, \mathbf{S}^1]$ is isomorphic to the projective limit of the $[F_n, \mathbf{S}^1]$ under the restriction maps.

Proof. The proof is an easy exercise, using 2.10 and 2.11. □

3. Compact subsets in the plane.

Theorem 3.1. *Let K be a compact subset of the plane, and let $(U_i)_{i\in I}$ be the family of bounded connected components of $\mathbf{R}^2\backslash K$. For each $i \in I$, we choose a point $a_i \in U_i$. The group $[K, \mathbf{S}^1]$ is a free abelian group with a basis given by the Borsuk maps $(\theta_{a_i})_{i\in I}$, where $\theta_{a_i} : K \to \mathbf{S}^1, \theta_{a_i}(x) = (x - a_i)/\|x - a_i\|$.*

Proof. We first show the independence of the θ_{a_i}. Suppose that we have $\theta_{a_{i_1}}^{n_1} \sim \prod_{j=2}^{k} \theta_{a_{i_j}}^{n_j}$, with $n_1 \in \mathbf{Z}, n_1 \neq 0$, and $a_{i_j} \neq a_{i_1}, \forall j \geq 2$. Since $a_{i_j} \notin U_{i_1}, \forall j \geq 2$, the map $\prod_{j=2}^{k} \theta_{a_{i_j}}^{n_j}$ can be extended to $U_{i_1} \cup K$ by the same formula. By the homotopy extension property **2**.1.10, the map $\theta_{a_{i_1}}^{n_1}$ also extends to $U_{i_1} \cup K$. Since the formula defining θ_{a_1} makes sense outside $U_{i_1} \cup K$, we see that $\theta_{a_{i_1}}^{n_1}$ extends to $\mathbf{R}^2$. This implies that $\theta_{a_{i_1}}^{n_1}$ is null homotopic, and by 2.7, since $n_1 \neq 0$ it follows that θ_{a_1} is also null homotopic. This contradicts theorem **2**.3.3. We now prove that $(\theta_{a_i})_{i\in I}$ generates $[K, \mathbf{S}^1]$. Let $f : K \to \mathbf{S}^1$ be a continuous map. By **2**.3.10, we can find a finite set $a_{i_1}, \ldots, a_{i_k}$ and an extension $\bar{f} : \mathbf{R}^2\backslash\{a_{i_1}, \ldots, a_{i_k}\}$. We choose small disks $B_1, \ldots, B_k$ centered around $a_{i_1}, \ldots, a_{i_k}$ and contained in $U_{i_1}, \ldots, U_{i_k}$. We start with $f_0 = \bar{f}|\mathbf{R}^2\backslash \cup_{j=1}^{k} \mathrm{Int}(B_j)$. We construct, by induction on $j = 1, \ldots, k$, maps $f_j\mathbf{R}^2\backslash \bigcup_{l>j}^{k} \mathrm{Int}(B_j)$ and integers n_j, such that f_j is an extension of $f_{j-1}\theta_{a_{i_j}}^{-n_j}$. Suppose that we have already constructed $f_1, \ldots, f_j$ and $n_1, \ldots, n_j$. By 2.8, the map f_j restricted to S_{j+1} the frontier of B_{j+1} is homotopic to some power $\theta_{a_{i_{j+1}}}^{n_{j+1}}$ restricted to S_{j+1}; in particular, we can extend $f_j\theta_{a_{i_{j+1}}}^{-n_{j+1}}|S_{j+1}$ to a continuous map from B_{j+1} to $\mathbf{S}^1$. This allows us to extend $f_j\theta_{a_{i_{j+1}}}^{-n_{j+1}}$ to a map $f_{j+1} : \mathbf{R}^2\backslash \bigcup_{l>j+1}^{k} \mathrm{Int}(B_l) \to \mathbf{S}^1$. We have $f_k \sim 0$, since it is defined on $\mathbf{R}^2$; moreover $f_k|K = f \prod_{j=1}^{k} \theta_{a_{i_j}}^{-n_j}$. □

Corollary 3.2. (Jordan's theorem). *Let $j : \mathbf{S}^1 \hookrightarrow \mathbf{R}^2$ be an imbedding. The set $\mathbf{R}^2 \backslash j(\mathbf{S}^1)$ has exactly two connected components, one bounded and the other one unbounded. If x is a point in the bounded component of $\mathbf{R}^2 \backslash j(\mathbf{S}^1)$ then the degree of the composite map:*$\mathbf{S}^1 \xrightarrow{j} j(\mathbf{S}^1) \xrightarrow{\theta_x} \mathbf{S}^1$ *is* ± 1.

Proof. This follows from 3.1, since the map $j^* : [j(\mathbf{S}^1), \mathbf{S}^1] \to [\mathbf{S}^1, \mathbf{S}^1]$, which maps the homotopy class of f to that of $f \circ j$, is an isomorphism. □

Definition 3.3. Let x and y be two points in a topological space X. The subset A of X separates x and y (in X) if x and y belong to distinct connected components of $X \backslash A$.

The following proposition is easily seen to be a particular case of **2**.3.6.

Proposition 3.4. *Two points a, b in $\mathbf{R}^2$ are separated by the compact subset K if and only if the map $\theta_{a,b} : K \to \mathbf{S}^1, z \mapsto \frac{z-a}{\|z-a\|} \frac{\|z-b\|}{z-b}$ is essential.*

Theorem 3.5. (Janizewski lemma). *Let K be a compact subset in $\mathbf{R}^2$ such that $K = K_1 \cup K_2$, with K_1, K_2 compact and $K_1 \cap K_2$ connected. Two points in $\mathbf{R}^2 \backslash K$ are separated by K if and only if they are separated either by K_1 or by K_2.*

By proposition 3.4, the theorem is a consequence of the following lemma.

Lemma 3.6. *Let A be a compact space which is a union of two compact subsets A_1 and A_2 with $A_1 \cap A_2$ connected. A map $f : A \to \mathbf{S}^1$ is null homotopic if and only if $f|A_1$ and $f|A_2$ are null homotopic.*

Proof. If $f|A_1$ and $f|A_2$ are null homotopic, by theorem 2.3, we can write $f|A_i = \exp \circ \varphi_i$, where $\varphi_i : A_i \to \mathbf{R}$, for $i = 1, 2$. Since $\exp \circ (\varphi_1 - \varphi_2)|A_1 \cap A_2 = (f/f)|A_1 \cap A_2 = 1$ and $A_1 \cap A_2$ is connected, the map $(\varphi_1 - \varphi_2)|A_1 \cap A_2$ is constant. If we call k its value, we see that we can define a lift $\varphi : A \to \mathbf{R}$ of f by $\varphi|A_1 = \varphi_1 + k$ and $\varphi|A_2 = \varphi_2$. The other half of the lemma is trivial. □

Corollary 3.7. (Janizewski lemma). *Let K and F be respectively a compact and a closed subset in $\mathbf{R}^2$ such that $K \cap F$ is connected. Two points in $\mathbf{R}^2 \backslash K \cup F$ are separated by $K \cup F$ if and only if they are separated either by K or by F.*

Proof. If two points in $\mathbf{R}^2 \backslash K \cup F$ are separated either by K or by F, they are of course separated by the larger set $K \cup F$. Suppose now that a and b are two points which are contained in the same connected component of $\mathbf{R}^2 \backslash F$; since this set is locally arcwise connected—as is any open set in $\mathbf{R}^2$—we can find a compact connected subset C of $\mathbf{R}^2 \backslash F$ which contains a and b. Let R be big enough to have $C \cup K$ contained in the interior of the ball $B(0, R)$. By construction a and b are not separated by $F \cup \partial B(0, R)$—the connected set C is disjoint from $F \cup \partial B(0, R)$. Suppose now that a and b are not separated by K. Since $K \cap [\partial B(0, R) \cup (F \cap B(0, R))] = K \cap F$ is connected, by 3.5 we obtain that a and b are not separated by the compact set $K \cup \partial B(0, R) \cup (F \cap B(0, R))$. In particular, there is a connected set A containing a and b and disjoint from $K \cup \partial B(0, R) \cup (F \cap B(0, R))$. Since a and b are in the interior of $B(0, R)$, the set A is also contained in $B(0, R)$ and is consequently disjoint from F. □

Corollary 3.8. *Let U be an open connected subset of $\mathbf{R}^2$ and let K be a compact subset in $\mathbf{R}^2$ such that $K\backslash U$ is connected. Two points in U are separated by K in U if and only if they are separated either by K in $\mathbf{R}^2$.*

Proof. Let $F = \mathbf{R}^2\backslash U$. Since U is connected any two points cannot be separated by F. By 3.7 two points are separated by $K \cup F$ if and only if they are separated by K. □

The following is a useful special case of 3.8.

Corollary 3.9. *Let U be an open connected subset of $\mathbf{R}^2$. Let α be an arc in $\mathbf{R}^2$ such that $\partial\alpha$ is contained in a compact connected subset C of $\mathbf{R}^2\backslash U$, and $\alpha\backslash\partial\alpha$ is contained in U. Distinct connected components of $U\backslash\alpha$ are contained in distinct connected components of $\mathbf{R}^2\backslash(\alpha \cup C)$.*

4. Open subsets of the plane.

Theorem 4.1. *Let U be a connected open subset in $\mathbf{R}^2$. We can find a family $(K_i)_{i\in\mathbf{N}}$ of compact subsets of U, such that:*

- (i) *each K_i is connected;*
- (ii) *for each $i \in \mathbf{N}, K_i \subset \mathrm{Int}(K_{i+1})$;*
- (iii) *for each $i \in \mathbf{N}$, the set $\mathbf{R}^2\backslash K_i$ has a finite number of connected components, and each relatively compact component of $\mathbf{R}^2\backslash K_i$ contains a compact component of $\mathbf{R}^2\backslash U$;*
- (iv) *if x and y are contained in different connected components of $\mathbf{R}^2\backslash U$, and one of them is contained in a compact component of $\mathbf{R}^2\backslash U$, then there exists i_0 such that x and y are contained in different connected components of $\mathbf{R}^2\backslash K_{i_0}$.*

Corollary 4.2. *Let U be an open connected subset in $\mathbf{R}^2$. The group $[U, \mathbf{S}^1]$ is not reduced to zero if and only if $\mathbf{R}^2\backslash U$ has a compact component. Moreover, if c is in a compact component of $\mathbf{R}^2\backslash U$, the Borsuk map $\theta_c : U \to \mathbf{S}^1, x \mapsto (x-c)/\|x-c\|$ is not null homotopic and it is indivisible in $[U, \mathbf{S}^1]$.*

Proof. Suppose that $\mathbf{R}^2\backslash U$ has no compact component. Let $(K_n)_{n\in\mathbf{N}}$ be given by 4.1. Property (iii) of 4.1 implies that each $\mathbf{R}^2\backslash K_n$ has only unbounded components. By 3.1, we have $[K_n, \mathbf{S}^1] = 0$ for each $n \in \mathbf{N}$. By 2.10, this implies that $[U, \mathbf{S}^1] = 0$. Suppose now that C is a compact component of $\mathbf{R}^2\backslash U$. By **3A**.1.4, we can find an open relatively compact subset V of $\mathbf{R}^2$, with $C \subset V$ and $\mathrm{Fr}\, V \subset U$. of course, the compact set C is contained in a relatively compact component of $\mathbf{R}^2\backslash \mathrm{Fr}\, V$. This implies that if $c \in C$ the Borsuk map $\theta_c : \mathbf{R}^2\backslash\{c\} \to \mathbf{S}^1$ is such that $\theta_c|\,\mathrm{Fr}\, V$ is essential and indivisible. Since $\mathrm{Fr}\, V \subset U$, the restriction $\theta_c|U$ is also essential and indivisible □

Corollary 4.3. *Let U be an open connected subset in $\mathbf{R}^2$ such that $\mathbf{R}^2\backslash U$ has a finite number of compact components $C_1, \ldots, C_k$. Choose for each $i = 1, \ldots, k$, a point $c_i \in C_i$. Let $\theta_i : U \to \mathbf{S}^1, x \mapsto (x - c_i)/\|x - c_i\|$ be the Borsuk map with respect to c_i. The group $[U, \mathbf{S}^1]$ is the free abelian group generated by the homotopy classes of the θ_i.*

Proof.

Theorem 4.4. *Let U be a connected open subset in $\mathbf{R}^2$. The following statements are equivalent:*

(i) *$\mathbf{R}^2\backslash U$ has no compact connected component;*
(ii) *each continuous map $\varphi : U \to \mathbf{S}^1$ can be lifted to a continuous map $\bar{\varphi} : U \to \mathbf{R}$;*
(iii) *each continuous map $\theta : U \to \mathbf{C}\backslash\{0\}$ can be lifted to a continuous map $\bar{\theta} : U \to \mathbf{C}$, with $\theta = \exp \circ \bar{\theta}$;*
(iv) *each holomorphic map $\theta : U \to \mathbf{C}\backslash\{0\}$ can be lifted to a holomorphic map $\bar{\theta} : U \to \mathbf{C}$, with $\theta = \exp \circ \bar{\theta}$;*
(v) *each continuous map $\theta : U \to \mathbf{C}\backslash\{0\}$ has a continuous square root;*
(vi) *each holomorphic map $\theta : U \to \mathbf{C}\backslash\{0\}$ has a holomorphic square root.*

Proof. (i)⇔(ii) follows from 4.3. (ii)⇔(iii) follows from the fact that a continuous map with values in $]0, \infty[$ has a continuous logarithm. (iii)⇒(iv) is clear. (iv)⇒(i) follows from 4.2. (iii)⇒(v)⇒(vi) are easy. (vi)⇒(ii) follows from 4.2. □